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A Sequential Allocation Problem: The Asymptotic Distribution of Resources

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Abstract

In this paper we consider a sequential allocation problem with n individuals. The first individual can consume any amount of some endowment leaving the remaining for the second individual, and so on. Motivated by the limitations associated with the cooperative or non-cooperative solutions we propose a new approach. We establish some axioms that should be satisfied, representativeness, impartiality, etc. The result is a unique asymptotic allocation rule. It is shown for $n = 2, 3, 4$, and a claim is made for general n. We show that it satisfies a set of desirable properties.

Key words: Sequential allocation rule, River sharing problem, Cooperative and non-cooperative games, Dictator and ultimatum games.
JEL classification: C79, D63, D74.

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1. Introduction

We analyze the sequential allocation of a divisible resource among agents who are ordered linearly. A well known example of this particular situation is the stylized river sharing problem.[1] The river flow is equivalent to a resource

[1]In spite of our sequential allocation problem can be seen as a particular case of a general river sharing and that we frequently refer to it for intuition and to contextualize our argumentation, we would like to keep our problem independent by its own as it might be extended in directions different than a river sharing can possibly be. A river sharing problem poses different challenges and a solution may employ specific information that we

or endowment and the countries, states or cities through which it passes are the individuals. The first individual (in the upstream) can consume any amount of the available endowment leaving the remaining for the second individual, and so on.[2] Since the property rights are not well-defined, it is difficult to apply the Coase (1960) principle. Therefore, we have an allocation problem.

Often, a solution is enforced by third parties, but it can also be the result of negotiations between the individuals. Failures in negotiations are common and eventually will end up in international courts. Therefore, it is interesting to discuss the law perspective on river sharing disputes.[3] The *absolute territorial sovereignty* (ATS) principle states that a country has absolute sovereignty over the flow on its territory regardless of any harm it may cause to other downstream countries. This prior appropriation principle is compatible with non-cooperative and strictly self-interested behavior and is widely recognized as unfair.[4] Another principle is the *unlimited territorial integrity* (UTS) which states that upstream countries cannot affect the natural flow of the water into downstream countries. This principle applied to our setting has no meaning and it is inconsistent. An upstream individual cannot consume any endowment without damaging other downstream individual (zero-sum problem). *Limited territorial sovereignty* (LTS) has become the most important principle in international water law. Countries must respect each other's rights. The doctrine of equitable resources utilization applied to our setting includes the equal allocation as a particular case. This allocation coincides with the Adams's (1963) notion of equity.[5]

The sequential nature of the problem limits to a great extent the coalition

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do not have or assume in our framework.

[2]There are similarities with the well-known dictator game of Kahneman et al. (1986) or the ultimatum game of Güth et al. (1982). We do not want to distinguish too much between these two, as they deliver asymptotically similar equilibria and real life veto power situations might be ambiguously enforceable. Therefore, it is not clear which we one we are closer to.

[3]Ambec, Dinar and McKinney (2013) point out for the vulnerability and monitoring difficulties associated with the compliance of existing water sharing arrangements.

[4]Carraro, Marchiori and Sgobbi (2007) and Ambec and Ehlers (2008a) survey the literature on non-cooperative and cooperative solutions for the river sharing problem. Parrachino, Dinar and Patrone (2006) for a general reviews on the literature.

[5]It is also the allocation that results from the application of the Shapley (1953) value to a simultaneous version of our problem (addictive characteristic function).

possibilities. For example, a coalition between the second and the third agent that ignores the first agent is innocuous. This is the case because the flow passes first through the latter. Similarly, a coalition between the first and the third agents that promises something to the latter, depends crucially on the consumption of the second agent. Moreover, since the first individual can consume all the endowment, there may be no incentives to negotiate something with the second individual. Another difficulty is that these disputes or series of negotiations are often deadlocked, i.e., no agreement can be reached. The questions is what equity agreement would the parties agree to sign?

We do not assume explicitly the existence of a third party that can enforce a particular allocation. Contrary to most of the literature in allocation problems, we have a consensus maximizing objective, rather than a welfare or other maximizing objective. Our goal is to present a practical solution built on strong and realistic arguments that cannot be rejected by the involved parties. We search for a compromise between a game theoretic non-cooperative and cooperative agreement (a compromise between the ATS and LTS principles). Therefore, we do not restrict excessively the solution design. At the same time we do not want to induce a particular result. We achieve it through a set of axioms that imply an admissible set of allocations. The solution must be impartial and representative. It results to be unique. In order to express these concepts mathematically we consider a discrete action space. As a result, the sum of each individual payoffs becomes easier to find. This sum over the total gives the share of each individual on the total endowment. In the process, the discretization becomes finer and finer and the relative difference between allocations vanish. Therefore, we do not ignore any admissible allocation profile that could possibly be built with a continuous action space. The result is a unique asymptotic distribution of the endowment that receives as input any possible individual claim or concern that we consider to be admissible. We show that it satisfies a set of desirable properties.

Our results are supported by Engel (2011). For instance, if there are more than one recipient, together they receive substantially more than in a situation with a single receiver. Therefore, the dictator accepts as natural a lower allocation. Bahr and Requate (2013) perform an experimental design with a structure similar to our sequential allocation problem. They found that when the share of the first individual is in between 50% and 66% the share of the second individual is between 21% and 31%. They also found that the sharing behavior of individual 2 with respect to individual 3 is not significantly different from the dictator's behavior in the usual two player dictator game. These results are close to the ones obtained in the present paper. Bonein and Serra (2007) performed a similar experiment in a sequential dictator game. In one treatment the individuals 2 and 3 played a ultimatum game while in the other treatment they played a dictator game. Individual 2's offer as a percentage of player 1's offer was around 40%, in the former case, and around 30%, in the latter case. Our allocation predicts a 40% offer. Empirical support is always important, but in our case it assumes extra importance because we have a consensus maximizing objective. If the allocation rule in the present paper replicates the average or representative human behavior, even in very tricky and subtle situations, then it is more likely that this objective is achieved.[6]

Approaches based on cooperative game theory have been extensively applied to sequential allocation problems such as the river sharing problem. One that is sufficiently representative and has received some attention in the literature is Ambec and Sprumont (2002). Based on the first two principles, Ambec and Sprumont define,[7] respectively, a core lower bound and an aspiration upper bound on the welfare of a coalition of agents. Welfare is derived from quasilinear preferences over water and money. They show that these bounds uniquely determine the "downstream incremental distribution" to allocate the total welfare among the agents. The marginal contribution of each member of the coalition determines its share. The compensation between individuals is guaranteed with monetary transfers. On the contrary, we do not explicitly consider transfers or any other trading mechanism. For the sake of generality we do not explicitly define an utility function.[8] This is a complex and subjective issue with profound implications on the results. Therefore, their approach is not applicable to our setting. In fact, this is true for any

[6] Cason and Mui (1998) also consider a sequential dictator game, however, their setting has no immediate translation to ours.

[7] See Ambec and Ehlers (2008b) for an extension of downstream incremental distribution for single-peaked preferences. See also Kilgour and Dinar (2001) and Wang (2011), among others.

[8] Dinar, Ratner and Yaron (1992) critique the use of game theoretical based transfers that are not related to market prices, and the representation of the problem in the "utility-space". More recently, the same position is defended in Houba (2008).

coalition approach that we can think about. In addition, our approach is not a compromise between the ATS and the UTS principles but it is more similar to a compromise between the ATS and LTS. In other words, we search for a compromise in between the non-cooperative and the most cooperative outcome.

Ansink and Weikard (2012) transform a river sharing problem in a sequence of two-agent river sharing problems, and show mathematical equivalence to bankruptcy problems. In spite of possible axiomatization, an application of bankruptcy methods eliminates from the river sharing problem any strategic consideration associated with position, i.e., being the most upstream or downstream is irrelevant. Actually, even claims are not well defined. In a real life situation, the simple task of validating any claim is a harder problem than the river sharing problem itself. Rationing problems, as in Moulin (2000), follow a sequential structure that can be adapted to our setting. Priority rules with ordered individuals, first allocate resources to these ones (on the upstream) until their claim is satisfied. In our setting this implies that individual 1 consumes the full endowment.

Herings and Predtetchinski (2012) consider a sequential bargaining protocol in which each individual share in the endowment is sequentially determined. The sequential structure can be adapted to our setting. Alternating offer bargaining have in common the threat of delay and the equilibrium unanimity requirement. Translated to our setting with no delay, all individuals obtain the same payoff independently of their location because of unanimity. Therefore, the sequential nature of the problem is lost. We do not impose unanimity, instead we search for a proposal that maximizes individuals consensus and reduces the potential of a bargaining impasse. Delay in negotiations is implicitly in the final allocation. Moreover, in real life situations the individuals veto power might have enforcement limitations.

The paper is organized as follows. Sections 2 and 3 present the model and motivates our approach, respectively. Sections 4 and 5, define and represent a set of axioms that we want to be satisfied by our rule. Sections 6 and 7, present our results and investigate their properties. Finally, Section 8 concludes with some extensions and practical issues.

2. The Sequential Allocation Model

Consider a divisible endowment $E \in \mathbb{R}_+$ to be allocated sequentially to a group of individuals, whose set is denoted by $N = \{1, ..., n\}$. Individuals are identified with respect to their relative position. If $i < j$ we say that i is upstream from j or that j is downstream from i.[9] In other words, individual 1 is the first to have access to the endowment and to consume an amount $c_1 \in [0, E]$. The remaining endowment, $E - c_1$, is passed to individual 2, which consumes $c_2 \in [0, E - c_1]$ and passes the remaining to individual 3, and so on. The process ends with the individual n, which consumes the remaining endowment $c_n = E - \sum_{i=1}^{n-1} c_i$. We denote $c \in [0, E]^n$ as the vector of consumptions.

3. Intuition and Rationale for the Approach

The allocation method that we propose in this paper is new to the literature. For that reason this section is devoted to justify its existence. We also address some issues that cannot be ignored in the design of a consensual allocation proposal.

3.1. ***The non-cooperative equilibrium is unfair***

In a non-cooperative context, without any type of punishment, rational behavior implies that individual 1 consumes the full endowment $c_1 = E$, and passes nothing to the other individuals. The structure is similar to the well-known dictator game, Kahneman et al. (1986), the difference is the existence of multiple steps. This zero sum decision problem has been used to test the altruism of individuals and equity concerns about the well being of others. Empirical studies and controlled experiments show that individual 1 does not consume the full endowment, $c_1 < E$, but passes some non-negligible endowments to the other individual, contradicting the theory (see Engel (2011) and Camerer (2003) for surveys).

A more realistic non-cooperative setting is the well known ultimatum game, Güth et al. (1982). In this case individual 2 has veto power over the allocation proposed by individual 1, in which case both individuals obtain a zero payoff. In terms of our setting, this is equivalent to an impasse in the negotiation process. However, in reality things are not so strict and further negotiations may take place. Again, in our setting the main difference is the existence of multiple steps, that is, individual 3 also has veto power over the allocation proposed by individual 2, and so on. With a continuous

[9] We can think on E as a river flow that passes through a number of countries, regions or cities, the individuals in our context.

consumption space the unique subgame perfect equilibrium is asymptotic similar to the one in the dictator game. Individual 1 passes some infinitesimal amount to individual 2 and this one passes some infinitesimal amount to individual 3, an so on. Since these amounts are infinitesimally small we have $c \to (3, 0, 0)$. However, empirical evidence shows that individual 2 receives some non-infinitesimal and measurable amount contradicting the theory (see Camerer and Thaler (1995) for a survey).

The theoretical result is a consequence of the location advantage of the upstream over the downstream individuals and the "more the better" property of the utility function. This result is very unequal and hard to defend. In spite of it the payoffs in any allocation proposal must reflect that individual 1 has at least a weak advantage over the subsequent individual, and individual 2 has at least a weak advantage over individual 3, an so on. Actually, this feature distinguishes this problem from others in the literature.

From an equity point of view it seems consensual that every involved individual must receive something. What is not clear is the value of this something. Moreover, the empirical results in the dictator and ultimatum games point to the existence of altruistic and equity concerned behaviors. It suggests that allocations that are more equitable than the strictly self-interested non-cooperative allocation receive a natural support from the individuals. This is also true in our context in which decisions are expected to be more carefully thought-out (or taken by groups - countries, governments, etc.), which are often more rational (in game theoretic sense) and self-interested than individual decisions (Charness and Sutter, 2012).

Note also that the individuals that are farther downstream have less bargaining power because of their worst strategic location. Actually, individual n has the least bargaining power to impose any agreement. Consequently, the only way to protect its position is to rely on the individuals' sense of equity and justice. As we will see below, this individual final allocation is going to have a unique property specific to this fact.

Note that we do not state whether our setting is closer to a multistage dictator or ultimatum game, rather we leave this as an open issue. In real life allocation problems of this kind, individual 2 might be forced to conform with what individual 1 has proposed or it might have power to veto against it. This issue becomes even more complex when we add more individuals to the problem.

3.2. ***Limitations of the cooperative approach***

Our notion of equity follows the "equity theory" of social psychology, Adams (1963). Therefore, since the ratio of inputs to outcomes is the same for every individual, each one should be treated in the same way. Therefore, the most equitable allocation would be $c = (E/n, ..., E/n)$.

This definition ignores strategic issues such as the position of the individuals in the sequence. Therefore, if we would consider a simultaneous and non-sequential setting this allocation matches the well-known Shapley value. The linearity of the utility function implies that the characteristic function is additive but not superadditive. The value of the union of two coalitions is the same as the sum of the coalitions' separate values. In this hypothetical scenario the problem could be approached through the Shapley value, for instance. However, our setting is sequential.

There are some other difficulties when considering coalition games in our sequential setup. For instance, the coalition between individuals 2 and 3 with the allocation $c = (0, E/2, E/2, 0, ..., 0)$ is not possible without the individual 1 agreement. Actually, the upstream individual 1 has all bargaining power. In spite of possible altruistic and equity concerns, it is hard to expect that individual 1 accepts something different than $c_1 \in [E/n, E]$. In the best scenario individual 1 may consume $c_1 = E/n$ in order to induce the most equity allocation. However, individual 2 does not necessarily need to follow this implicit or explicit recommendation and instead chooses $c_2 = E(n-1)/n$. Therefore, reasoning backwards, individual 1 would never chooses $c_1 = E/n$ without a guarantee that all individuals would do the same. Otherwise, its objective would have been in vain. These difficulties and the associated type of reasoning lead us to conclude that any allocation proposal different from the non-cooperative equilibrium must be sufficiently consensual (i.e., obtain the agreement of the involved parties) in order to later be naturally enforced. A similar conclusion holds for the coalition between individuals 1 and 3 and the allocation $(E/2, 0, E/2, 0, ..., 0)$, which is not possible without the agreement of individual 2.

4. Properties of the solution design

We have pointed out the limitations of game theory as a tool to deal with sequential allocation problems. We do not assume explicitly the existence of a third party that can enforce a particular allocation. Instead, our approach is a hybrid between a non-cooperative and a cooperative agreement. Therefore, we do not restrict excessively the solution design in order to not remove the non-cooperative nature of the problem or to impossibility a potential equity oriented agreement. On the same time we do not want to induce a particular result.

So far, we have conclude that individual $i \in N$ cannot get more than individual $i-1$ and no less than individual $i+1$, and that every individual independently of its position must receive something. The former is an implication of the positional disadvantage and advantage of individual i with respect to individual $i-1$ and $i+1$, respectively, but without ignoring equity issues. The latter is based on the idea of fairness and justice. Formally,

Axiom 1 (Strategic Advantage). *If $i < j$ then $c_i \geq c_j$ for all $i, j \in N$.*

The strategic advantage of agent $i < j$ over agent j is respected (at least weakly) in every payoff profile that contributes to the final solution. Individual i knows that is better positioned than agent j. Therefore, a proposal that does not reflect it in terms of payoffs is unacceptable from its perspective. Note that we do not restrict the equity objective. However, to reach a consensual agreement among the involved parties we have to be realistic about the requirements that we impose, as they influence the final solution.

Axiom 2 (Non-zero Payoff Right). *$c_i > 0$ for all $i \in N$.*

This statement is trivial. However, if we look to the non-cooperative side of the problem (as a sequential dictator or ultimatum game) the equilibrium payoffs are $c_1 \to E$ and $c_i \to 0$ for all $i \in N \backslash 1$. Therefore, we are imposing that every individual obtains a measurable share of the total endowment.

Definition 1 (Admissible Profile). *An allocation profile that simultaneously satisfies axioms 1 and 2 is called admissible. The set of such allocation profiles is called the admissible set.*[10]

Axioms 1 and 2 impose the following payoff bounds,

$$c_1 \in [E/n, E), c_n \in (0, E/n] \text{ and } c_i \in (0, E/i), \tag{1}$$

[10] Table 1 presents the set of admissible payoff profiles for $n = 3$ and $m = 1, 2, 3, 4$.

for $i \in N \setminus \{1, n\}$. The converse is not true, since the bounds do not imply axioms 1 and 2. Note that the set of admissible payoff profiles that satisfy these bounds is uncountable. This aspect leads to some technical issues that are addressed later.

We can imagine an uncountable set of individuals suggesting different final allocations. Some of those might be more self-interested while others are more equity oriented. Among these suggestions there might exist allocations inside and outside the admissible set.

Axiom 3 (Representativeness). *An allocation is context-representative if it receives as input every allocation in the context.*

This principle limits the set of allocation profiles that we consider for the solution design. The context is the settings that frames the background under which the solution expresses its representativeness. Based on the arguments that motivated Definition 1 we exclude from consideration allocations that are not admissible. Our context is the set of admissible allocations. In technical terms, representativeness means that each allocation on the admissible set receives at least one unit of the endowment (strictly positive weight).

We also want the final solution to respect the principle of impartial treatment. Once the decision to consider a given allocation profile is taken, this must be equally weighted. In other words, there is no payoff profile that is more or less important than any other.

Axiom 4 (Impartiality). *A solution is impartial if every input is uniformly weighted.*

Impartiality is an important concept and it is fundamental to justice. We consider it in order to remove from the proposed allocation any potential bias, prejudice, or any individual preference that is not properly founded.

Contrary to most of the literature in allocation problems, we do not have a utility or welfare maximizing objective, rather a consensus maximizing objective. Our goal is to present a practical solution built on strong and realistic arguments that cannot be rejected by the involved parties. If there are several of these solutions this objective is at risk as individuals may split between the available alternatives. Therefore uniqueness is also a desired property.

Definition 2 (Final Solution). *The final solution must be admissible - representative, impartial and unique.*

Note that as in the computation of the Shapley (1953) value the allocations are equally weighted. The difference is that Shapley constructs allocations over coalitions while we construct allocations over an admissible set that satisfies a number of properties. This way we go around the impossibility of forming meaningful coalitions in sequential settings.

5. The procedure in detail

In this section we describe in detail the construction of our allocation proposal. In particular, the mathematical representation of the principles presented in the previous section.

5.1. ***Continuous versus discrete action space***

In a continuous action space, between two admissible allocation profiles that satisfy the bounds in (1) there is an uncountable set of possible allocations. Actually, the meaning of "between" is not well defined, as the comparison between two profiles always imply that at least some individual gets better off at expenses of another individual. In order to express these concepts mathematically we consider a discrete action space. This makes it easier to account for all admissible allocation profiles of Definition 1 because this set is countably infinite and has no implications in the objectives of axioms 3 and 4.[11] In other words we move from the usual continuous action space in which endowments and allocations are values in $\mathbb{R}_+$, to a discrete action space in which endowments and allocations are values in $\mathbb{N}_1$. Note that in the end we obtain an asymptotic distribution that is valid under general action spaces.

The discretized set of admissible allocations is given in Table 1 for $n = 3$ and $E = 3, 6, 9, 12$.

5.2. ***Construction of the procedure***

We start by considering the following example. Suppose that $n = 3$ and $E = 3$. In this case there is one admissible allocation that satisfies our requirements, $c = (1, 1, 1)$. Any other allocation either fails axiom 1 or axiom 2, or both, and for that reason not admissible. The allocations $c = (2, 1, 0)$ and $c = (3, 0, 0)$ fail the non-zero requirement of axiom 2. Now, suppose that

[11] The need of a discrete action space is also motivated by the problem of defining infinitesimal small or large values on a real numbers system.

$E=3$	$E=6$	$E=9$	$E=12$
$1,1,1$	$4,1,1$	$7,1,1$	$10,1,1$
	$3,2,1$	$6,2,1$	$9,2,1$
	$2,2,2$	$5,3,1$	$8,3,1$
		$5,2,2$	$8,2,2$
		$4,4,1$	$7,4,1$
		$4,3,2$	$7,3,2$
		$3,3,3$	$6,5,1$
			$6,4,2$
			$6,3,3$
			$5,5,2$
			$5,4,3$
			$4,4,4$
$sum(total)$	$sum(total)$	$sum(total)$	$sum(total)$
$1,1,1(3)$	$9,5,4(18)$	$34,18,11(63)$	$81,40,23(144)$

Table 1: The set of admissible allocation profiles for n=3 and m=1,2,3,4.

$E=6$, in this case there are three allocations that satisfy our admissible requirements, $c=(4,1,1)$, $c=(3,2,1)$ and $c=(2,2,2)$. This process of generating admissible allocations can be generalized by letting $E=nm$, where $m=1,2,...$, and n is the number of individuals (see Remark 1 in the end of this section for a detailed explanation). For the case that $n=3$ and $m=1,2,3,4$, the set of admissible allocations is given in Table 1.

Following the discussion, with a discrete action space, we can rewrite the bounds in (1) as

$$c_1 \in [m, nm-1], c_n \in [1, m] \text{ and } c_i \in [1, nm/i - 1],$$

for $i \in N \backslash \{1, n\}$. Subsequently, for a given n and m, we sum vertically the individual $i \in N$ payoffs and denote this sum as $sum_i^n(m)$. Therefore, every admissible allocation contributes equally for the final allocation, the impartiality principle of Definition 2. As m grows these sums become more complex. The objective is to find a general expression or recursion that characterizes the sum of the values in the sequence for any m. In the bottom of Table 1 we show each individual sum of payoffs over all admissible allocations. We are interested in the asymptotic ratio r_i^n of the individual i sum of admissible payoffs with respect to the total sum, i.e., $r_i^n(m) \equiv sum_i^n(m) / \sum_{i=1}^n sum_i^n(m)$ as $m \uparrow \infty$. The result is the share of individual i on the total endowment. For instance, in the case $n = 3$, the ratio of individual 1 grows from $r_1^3(1) = 1/3$ to $r_1^3(4) = 9/16$ (see Table 1), and converges asymptotically to $r_1^3 = r_1^3(\infty) = 11/18$ (see Proposition 2 below). In other words, when there are three individuals our allocation rules states that the most upstream individual must receive 61.1(1)% percent of the total endowment under dispute (see also Section 8.1 for applied issues). Formally,

Definition 3. *Given n individuals, the individual $i \in N$ admissible asymptotic allocation is defined as*

$$\phi_i^n \equiv r_i^n E \equiv \lim_{m \to \infty} \frac{sum_i^n(m)}{\sum_{i=1}^n sum_i^n(m)} E, \tag{2}$$

where r_i^n represents the share on the total endowment.

Note the compromise between cooperative (preferred by the most downstream individuals) and non-cooperative behavior (preferred by the most upstream individuals). For instance, if $E = 9$ (see Table 1) we are considering admissible allocations that can be regarded as the result of a more cooperative agreement, $(3, 3, 3)$ or $(4, 3, 2)$, and admissible allocations that seem to be the result of a more non-cooperative agreement, $(7, 1, 1)$ or $(6, 2, 1)$. In between we also consider admissible allocations that may not fall in any of these more extreme sets, i.e., $(5, 3, 1)$, $(5, 2, 2)$ and $(4, 4, 1)$. Consequently, as $m \uparrow \infty$ (or equivalently $E \uparrow \infty$) the relative discretization becomes finer and finer and the relative difference between allocations vanishes. Therefore, we do not ignore any admissible allocation profile that could possible be built with a continuous action space.

Recall that we started the discussion justifying the passage from a continuous to a discrete consumption space. Now, asymptotically, we move back from the discrete to the continuous space. The result is a unique distribution of consumptions.

The uniqueness, the equal weight or impartiality and the contribution of every allocation that can be defended as a final agreement under reasonable arguments (the set of admissible allocations) are the strongest aspects of our solution.

Remark 1. *Note that we consider $E = nm$ instead of $E = 1, ..., n-1, n, n+1, ...,$ (for $E < n$ the defined admissible set is empty) that is, the discrete* *endowment grows n units per unit increment on m. We do it to simplify the computation of the general expressions that characterize the sum of each individual payoff as a function of m. This way, we always consider the most equity profile* $(m, ..., m)$ *. Asymptotically, that is for* $m \uparrow \infty$*, both approaches are equivalent.*

6. Allocation Results

Our goal is to present a practical solution built on strong and realistic arguments that cannot be rejected by the involved parties. Such a solution is unique, impartial and representative. Contrary to most of the literature in allocation problems, we have a consensus maximizing objective, rather than a welfare or other maximizing objective.

6.1. ***Two Individuals***

This case is particularly simple.

Proposition 1. *Suppose that* $n = 2$*. The admissible asymptotic endowment allocation is*

$$\phi^2 = \left(\frac{3}{4}, \frac{1}{4}\right) E.$$

The literature on the dictator and ultimatum game is extensive, see Camerer (2003), Camerer and Thaler (1995). Engel (2011) aggregates information of 129 published papers on the topic and found that dictators on average keep a share of 71.65%, which is very closed to the 75% proposed by our allocation rule. We compare our results with this particular paper because of the benefits associated with the use of a large sample size. Other papers predict different but close values, depending on which treatment is used.

6.2. ***Three Individuals***

The set of admissible payoff profiles for the case of three individuals are presented in Table 1. The sum of payoff profiles for each individual and the aggregate sum for $m = 1, 2, 3, 4$, are shown in the last row.

Proposition 2. *Suppose that* $n = 3$*. The admissible asymptotic endowment allocation is*

$$\phi^3 = \left(\frac{11}{18}, \frac{5}{18}, \frac{2}{18}\right) E.$$

Bahr and Requate (2013) perform an experimental design with a structure similar to our sequential allocation problem. They found that when individual 1 share is in between 50% and 66% the share of individual 2 is between 21% and 31%. Our allocation rule suggests approximately 61% and 28%, respectively. Bonein and Serra (2007) performed similar experiments in a sequential dictator game. In one treatment the individuals 2 and 3 played a ultimatum game while in the other treatment they played a dictator game. The offer of individual 2, as a percentage of the offer of individual 1 was around 40%, in the former case, and around 30%, in the latter case. Our allocation rules predicts a 40% offer.

6.3. ***Four Individuals***

The four individuals case is more complex. There are more payoff profiles to consider and the expressions for the general sum, $sum_i^4(m)$ for $i = 1, 2, 3, 4$, are given by non-trivial recursions.

Proposition 3. *Suppose that $n = 4$. The admissible asymptotic endowment allocation is*

$$\phi^4 = \left(\frac{25}{48}, \frac{13}{48}, \frac{7}{48}, \frac{3}{48}\right) E.$$

6.4. ***General:*** n ***Individuals***

This paper is the first step in what we believe to be a new class of allocation rules for sequential problems. The ultimate objective is to derive a general expression for the asymptotic share ϕ_i^n, as a function of the number of individuals n and the identity of the individual $i \in N$. In the cases $n = 2, 3, 4$, we obtain the value of each individual asymptotic allocation. However, the construction of general recursions $sum_i^n(m)$ with $i = 1, 2, ..., n$, for $n = 5$ or larger becomes impracticable and beyond the scope of this paper. In spite of it, using the information that we have obtained so far and based on a method similar to one usually employed to construct the Lorenz curve we found that is possible to obtain directly the n asymptotic share values in the vector r^n. The result is a system of $2n-1$ equations and $2n$ unknowns (Figure 1 provides an illustration, see the proof of Conjecture 1 for a more detailed explanation), which cannot be solved without an additional linearly independent equation. This is achieved by noting that the most downstream individual obtains a share of the total endowment equal to $r_2^2 = 1/4$, $r_3^3 = 1/9$, $r_4^4 = 1/16$, for

$n = 2, 3, 4$, respectively. Based on this observation we claim that individual n share is given by

$$r_n^n = \frac{1}{n^2}. \tag{3}$$

The refered method holds for $n = 2, 3, 4$. We claim that it holds for general n. The result is stated as a claim because we want to be careful on deriving general conclusions from sequences with three numbers.

Conjecture 1 (Claim). *The admissible asymptotic endowment allocation is,*

$$\phi_i^n = (n-1)\,\frac{\sum_{k=i}^{n-1} r_k^{n-1} - \sum_{k=i+1}^{n} r_k^n}{n-i}E$$

for $i = 1, ..., n$ and $n = 2, 3, ...,$ where r_n^n is given by (3).

7. Properties of the Allocation Rule

In this section we study some of the basic properties of the solution. We show that it satisfies a set of desirable properties. We conclude with cross-comparison comments on the Herings and Predtetchinski (2012) allocation rule (for general discount factor on the unit interval).

P1: (monotonic decreasing with i) *$\phi_i^n > \phi_{i+1}^n$, for $i = 1, 2, ..., n-1$.*

It is the most natural property. The higher the individual in the stream, the larger its share on the total endowment. This property is connected with the strategic advantage principle of axiom 1.

P2: (monotonic decreasing with n) *$\phi_1^n > \phi_1^{n+1}$ and $\phi_i^n > \phi_{i+1}^{n+1}$, for $i = 1, 2, ..., n$.*

The allocation of the most upstream and downstream individuals always decreases as the number of individuals increases. This property should be natural for every individual allocation. The difficulty is that there is some ambiguity in the comparisons. For instance, when passing from $n = 3$ to $n = 4$ it is not clear whether we should compare the individual $i = 2$ in $n = 3$ with the individual $i = 2$ or $i = 3$ in $n = 4$. On the contrary, the position of the two most extreme individuals is unambiguous. A monotonic decreasing relation with n holds for downward diagonal comparisons. In this sense the

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$i=1$	0.75000	0.6111(1)	0.5208(3)	0.4566(6)	0.4083(3)
$i=2$	0.25000	0.2777(7)	0.2708(3)	0.2566(6)	0.2416(6)
$i=3$		0.1111(1)	0.1458(3)	0.1566(6)	0.1583(3)
$i=4$			0.06250	0.09000	0.1027(7)
$i=5$				0.04000	0.0611(1)
$i=6$					0.0277(7)

Table 2: Individual asymptotic shares of the total endowment for n=2,3,4,5,6.

comparison uses the bottom as reference, for instance, between the worst individuals, the second worst individuals, etc. Table 2 provides a numeric illustration. Note also the existence of a monotonic increasing relation with n in the upward diagonal, i.e., $\phi_{i+1}^{n} < \phi_{i}^{n+1}$, *for* $i = 1, 2, ..., n-1$.

P3: **(monotonic decreasing relative bargaining power with** i**)** $\phi_i^n/\phi_{i+1}^n > \phi_{i+1}^n/\phi_{i+2}^n$, *for* $i = 1, 2, ..., n-3$.

The result states that the individual i allocation is not only larger than that of individual $i+1$ (see P1) but it is relatively much larger than the one that $i+1$ obtains with respect to $i+2$. The allocation of the upstream individuals allocation is increasing larger with respect to that of the downstream individuals. In other words, as we decrease from $i = n$ to $i = 1$ the individual allocations increase convexly. However, the result is not valid for $i = n-2$ as stated in the following property.

The last three properties are empirically supported by Bahr and Requate (2013), Bonein and Serra (2007) and Engel (2011).

P4: **(individual** n **weak relative bargaining power)** $\phi_{n-2}^n/\phi_{n-1}^n < \phi_{n-1}^n/\phi_n^n$.

The monotonic relation of $P3$ is interrupted in the last comparison. The bargaining power of individual $i = n-2$ over $i = n-1$ is lower than the bargaining power of individual $i = n-1$ over $i = n$. This is because individual n is the last in the sequence. Consequently, the decisions of individual n has no influence over the decisions of the others. Virtually, individual n has no bargaining power. Note that there are two factors that play a role in each individual allocation. The first factor is the equity concern of the other individuals. The second factor is the strategic position of each individual.

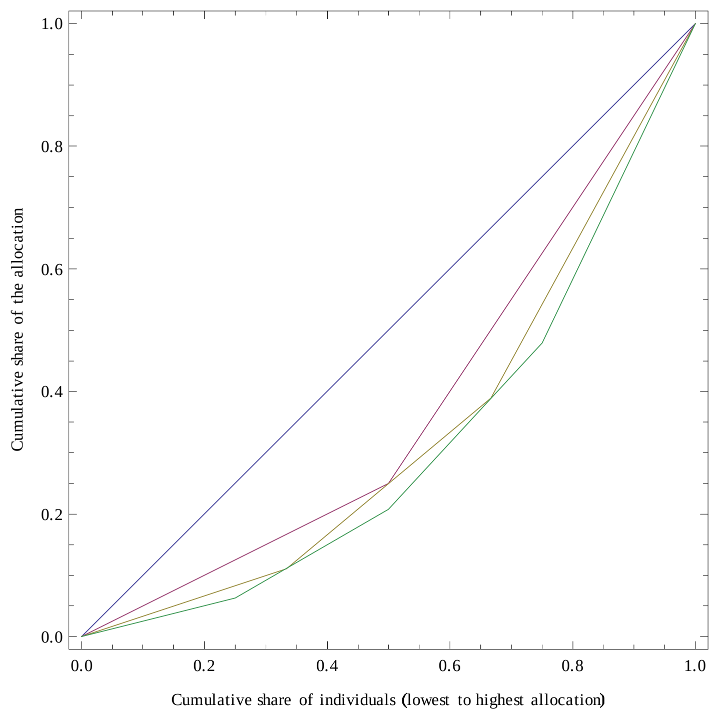

Figure 1: Lorenz curve (perfect equality curve in blue, $n = 2$ in red, $n = 3$ in brown, $n = 4$ in green)

The last individual does not benefit from the latter. For that reason its allocation falls abruptly relatively to the allocation of individual $i = n - 1$.[12]

Bahr and Requate (2013) tested the individual 2 sharing behavior with respect to individual 3 against the behavior in the usual two players dictator game and found no significant difference. These observations are important. Recall that we have a consensus maximizing objective. If the allocation rule in the present paper replicates the average or representative human behavior, even in very tricky and subtle situations, then it is more likely that this objective is achieved.

[12] It seems clear that the equity concerns of the others toward individual $i = 1$ is negative and equal to $\phi_1^n - 1$ while the strategic positioning value is equal to 1. On the other hand, the strategic positioning value of individual n is equal to 0 but the benefits from positive equity concerns is equal to ϕ_n^n. The distinction between the equity concerns and the strategy positioning components, for each individual allocation, seems to be an interesting topic for further research.

P5: (monotonic decreasing relative bargaining power with n) $\phi_i^n/\phi_{i+1}^n > \phi_i^{n+1}/\phi_{i+1}^{n+1}$, *for* $i = 1, 2, ..., n-1$.

Similar to property $P3$, as the number of involved individuals increases, the relative individual bargaining power decreases. The relative bargaining power is measured by the ratio between allocations.

P6: (Lorenz inequality increases with n)

$$\frac{\sum_{k=i}^{n}\phi_k^n - \sum_{k=i}^{n-1}\phi_k^{n-1}}{\frac{n+1-i}{n} - \frac{n-i}{n-1}} \;\left|\begin{array}{c} > \\ < \end{array}\right|\; 1 \; if \; \frac{n-i}{n} \;\left|\begin{array}{c} > \\ < \end{array}\right|\; \frac{1}{2}\,, \; for \; i = 2, ..., n.$$

The larger the number of individuals, the larger the area between the line of perfect equality and the Lorenz curve. In spite of the allocations distribution being adjusted for the increasing number of individuals and that everybody obtains less, see P2, the more upstream individuals concessions to the more downstream ones decreases in relative terms.

Herings and Predtetchinski (2012) allocation rule does not satisfy properties $P3$, $P4$ and $P5$. The relative bargaining power is constant for varying i and n. An implication is that on the contrary to our allocation proposal their rule satisfies consistency, see Thomson (2011) or Moulin (2000) among others. In sequential allocation problems of the kind presented in this paper, consistency imposes that pairwise allocations must be linked in a predetermined (linear) order, which is mathematically convenient in some class of allocation problems, see for example Thomson (2003). In our perspective, whether a rule is consistent or not cannot be regarded as a good or bad property. For instance, in our setting if we add an individual to a $n = 3$ problem the relative relation position between the allocation of individuals 1 and 2 varies. Clearly, in both cases individual 1 maintains the largest allocation and bargaining power. However, when we pass from the case $n = 3$ to $n = 4$ the relative allocation of individual 1 with respect to individual 2 decreases. Both individuals lose bargaining power, but individual 2 has loses less than individual 1. This effect slows down the Lorenz inequality effect for increasing n.

8. Extensions

Our approach is particularly flexible in the sense that the reader is more or less free to define the admissible set. However, these choices may have implications on the final solution. On the other hand, we are less flexible with axioms 3 and 4, or Definition 1, as they characterize our approach. Therefore, we consider possible extensions associated with relaxations of the axioms 1 and 2 of Section 4 that were used to define the admissible set. These will necessarily result in allocations that are less equitable in terms of the Lorenz curve. Other extensions associated with variations of the original sequential allocation problem (non-constant endowments, unequal weights, asymmetric individuals, satiation levels, etc.) are also possible.

One possibility is to keep axiom 2 but replace axiom 1 by the following strict version. In other words, an upstream individual does not have an allocation with a strict advantage over a downstream individual.

Axiom 5 (Strict Strategic Advantage). *If $i < j$ then $c_i > c_j$ for all $i, j \in N$.*

The reverse possibility is to maintain axiom 1 but replace axiom 2 by the following relaxed version. In this case, we do not exclude from consideration allocation profiles in which one or more individuals obtains a zero payoff.

Axiom 6 (No Non-zero Rights). *$c_i \geq 0$ for all $i \in N$.*

We can also consider the strict version of axiom 1 and the relaxed version of axiom 2 simultaneously, i.e., replace these by axioms 5 and 6, respectively.

In the three cases considered the distribution tends to favor the upstream with respect to the downstream individuals. In other words, they lead to distributions of the total endowment that are less equitable in Lorenz sense for $n \geq 3$. In the case $n = 2$ the asymptotic distribution remains the same as in Proposition 1. It happens because if axiom 1 is replaced by axiom 5 we simply remove the payoff profile (m, m) from the admissible set, which appears only once for all m. If axiom 2 is replaced by Axiom 6 we add the payoff profile $(2m, 0)$. Asymptotically single terms become irrelevant. However, for $n \geq 3$ we must expect different asymptotic distributions, since the removed and/or added allocation profiles increase with m. To see it, consider the following example.

Example 1. *Let $n = 3$ and $m = 2$. If axioms 1 and 2 hold we have the above defined admissible set, i.e., $(4, 1, 1)$, $(3, 2, 1)$ and $(2, 2, 2)$, with the respective vector of individual shares $r^3(2) = \frac{1}{18}(9, 5, 4)$. If we replace axiom*

1 by axiom 5, the admissible set is composed of a single payoff profile, i.e., $(3,2,1)$*, with the respective vector of individual shares* $r^3(2) = \frac{1}{18}(9,6,3)$*. If instead we relax axiom 2 by axiom 6, we have a larger admissible set, i.e.,* $(6,0,0)$*,* $(5,1,0)$*,* $(4,2,0)$*,* $(4,1,1)$*,* $(3,3,0)$*,* $(3,2,1)$ *and* $(2,2,2)$*, with the respective vector of individual shares* $r^3(2) = \frac{1}{18}(11.6, 4.7, 1.7)$*. If we simultaneously strict axiom 1 and relax axiom 2 the admissible set of payoff profiles is* $(5,1,0)$*,* $(4,2,0)$ *and* $(3,2,1)$*, with the respective vector of individual shares* $r^3(2) = \frac{1}{18}(12,5,1)$*.*

From the example, it is clear that the share of the total endowment to individual 3 is always smaller with respect to the admissible set defined by us in Section 4. The opposite conclusion holds for individual 1, which never gets into a worst situation. Mixed results are observed for individual 2. These conclusions remain valid for larger values of m and in particular for $m \uparrow \infty$.

8.1. ***A Note for Practitioners***

Some situations justify that prior to the distribution of the total endowment among the involved parties every individual receives a minimum amount. This amount can be used for consumption or not. For example in a river sharing problem, observations of this kind make sense when a minimum flow is required to keep the habitat of certain species protected. Therefore, from the total river flow only a part of it can be used for consumption. Other situations require, for instance, that every individual receives an equal amount, E_a/n, and only the remaining endowment, E_b, can be allocated according to the methods suggested in the present paper. In this case, individual $i \in N$ obtains $E_a/n + r_i^n E_b$. When justified, this kind of procedure allows for distributions that are less asymmetric and more equitable in Lorenz sense.

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Appendix

Proof of Proposition 1. Let the endowment be $E = 2m$ for each $m = 1, 2, ...,$ and $i = 1, 2$, we sum all admissible payoffs until a pattern emerges. The general expression for the sequence 2, 8, 18, 32, 50, 72, ..., that represents the aggregate sum of payoffs over all agents and profiles is $2m^2$. The expression for the sequence 1, 5, 12, 22, 35..., that represents the individual 1 sum of payoffs over all profiles is $sum_1^2(m) = m(3m-1)/2$. Therefore, by (2) the asymptotic fraction of the total endowment is,

$$r_1^2(m) = \frac{m(3m-1)/2}{2m^2} \to \frac{3}{4}.$$

The expression for the sequence 1, 3, 6, 10, 15, ..., that represents the individual 2 sum of payoffs over all profiles is $sum_2^2(m) = m(m+1)/2$. Similarly, the asymptotic fraction of the total endowment is,

$$r_2^2(m) = \frac{m(m+1)/2}{2m^2} \to \frac{1}{4}.$$

■

Proof of Proposition 2. Similarly, let the endowment be $E = 3m$, we proceed for $m = 1, 2, ...,$ until a pattern emerges. The general expression for the sequence 3, 18, 63, 144, 285, ...,[13] that represents the aggregate sum of payoffs over all agents and profiles is

$$sum^3(m) = 3m\left(m^2 - \lfloor m^2/4 \rfloor\right) = \left(18m^3 + 3m(1-(-1)^m)\right)/8,$$

where $\lfloor . \rfloor$ denotes the floor function. The expression for the sequence 1, 9, 24, 81, 163, 282, ..., that represents the individual 1 sum of payoffs over all profiles is

$$\begin{aligned} sum_1^3(3,m) &= 3m\left(m^2 - \lfloor m^2/4 \rfloor\right) - \sum\nolimits_{k=1}^{m}\left(2k^2 - \lfloor k^2/4 \rfloor\right) \\ &\quad - \sum\nolimits_{k=1}^{\lfloor (m-1)/2 \rfloor}(m+k)(m-2k) \\ &= \left(22m^3 - 6m^2 - (m+1) - (3m-1)(-1)^m\right)/16. \end{aligned}$$

Therefore, by (2) the asymptotic fraction of the total endowment is

$$r_1^3(m) = \frac{\left(22m^3 - 6m^2 - (m+1) - (3m-1)(-1)^m\right)/16}{\left(18m^3 + 3m(1-(-1)^m)\right)/8} \to \frac{11}{18}.$$

[13] The sequences in this paper have been found by the authors and are registered in the "OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, http://oeis.org."

The expression for the sequence 1, 5, 18, 40, 80, 135, ..., that represents the individual 2 sum of payoffs over all profiles is

$$\begin{aligned} sum_2^3(m) &= \sum\nolimits_{k=1}^{m} k^2 + \sum\nolimits_{k=1}^{\lfloor (m-1)/2 \rfloor} (m+k)(m-2k) \\ &= \left(10m^3 + 3m\left(1-(-1)^m\right)\right)/16, \end{aligned}$$

and the asymptotic fraction of the total endowment is

$$r_2^3(m) = \frac{\left(10m^3 + 3m\left(1-(-1)^m\right)\right)/16}{\left(18m^3 + 3m\left(1-(-1)^m\right)\right)/8} \to \frac{5}{18}.$$

Similarly, the expression for the individual 3 sum sequence 1, 4, 11, 23, 42, 69, ..., is given by,

$$sum_3^3(m) = \sum\nolimits_{k=1}^{m} \left(k^2 - \lfloor k^2/4 \rfloor\right) = \left(4m^3 + 6n^2 + 4m + \left(1-(-1)^m\right)\right)/16,$$

and the asymptotic fraction of the total endowment is

$$r_3^3(m) = \frac{\left(4m^3 + 6n^2 + 4m + \left(1-(-1)^m\right)\right)/16}{\left(18m^3 + 3m\left(1-(-1)^m\right)\right)/8} \to \frac{2}{18}.$$

■

Proof of Proposition 3. Similarly, let the endowment be $E = 4m$, we proceed for $m = 1, 2, ...,$ until a pattern emerges. The general expression for the sequence 4, 40, 180, 544, 1280, 2592, ..., that represents the aggregate sum of payoffs over all agents and profiles is given by the recursion,

$$\begin{aligned} sum^4[m] &= \frac{m}{m-1} sum^4[m-1] \\ &+4m \sum_{k=0}^{2m} \left(\left\lfloor \frac{4m-2-k}{2} \right\rfloor - k \right) \left\lfloor \frac{sgn\left(\left\lfloor \frac{4m-2-k}{2} \right\rfloor - k\right) + 2}{2} \right\rfloor, \end{aligned} \tag{4}$$

where $sum^4[m] = \sum_{i=1}^{4} sum_i^4[m]$, $sum^4[1] = 4$ and sgn denotes the sign function. The total number of admissible profiles is $sum^4[m]/nm$. The expression $sum_1^4(m)$ for the individual 1 sum sequence 1, 17, 84, 262, 629, 1289, ..., is given by the recursion,

$$\begin{aligned} sum_1^4[m] &= sum_1^4[m-1] + \frac{sum^4[m-1]}{4m-4} \\ &+ \sum_{k=0}^{2m} \sum_{l=k+1}^{\left\lfloor \frac{4m-2-k}{2} \right\rfloor} (4m-2-l-k) \left\lfloor \frac{sgn\left(\left\lfloor \frac{4m-2-k}{2} \right\rfloor - k\right) + 2}{2} \right\rfloor, \end{aligned}$$

where $sum_1^4[1] = 1$ and $sum^4[m-1]$ is given by (4). The asymptotic fraction of the total endowment is obtained numerically and is given by,

$$r_1^4(m) = \frac{sum_1^4(m)}{sum^4(m)} \to \frac{25}{48}.$$

The expression $sum_2^4(m)$ for the individual 2 sequence 1, 10, 46, 141, 334, 680, ..., is given by the recursion,

$$\begin{aligned} sum_2^4[m] &= sum_2^4[m-1] + \frac{sum^4[m-1]}{4m-4} \\ &+ \sum_{k=0}^{2m} \sum_{l=k+1}^{\left\lfloor \frac{4m-2-k}{2} \right\rfloor} l \left\lfloor \frac{sgn\left(\left\lfloor \frac{4m-2-k}{2} \right\rfloor - k\right) + 2}{2} \right\rfloor, \end{aligned}$$

where $sum_2^4[1] = 1$ and $sum^4[m-1]$ is given by (4). The asymptotic fraction of the total endowment is obtained numerically and is

$$r_2^4(m) = \frac{sum_2^4(m)}{sum^4(m)} \to \frac{13}{48}.$$

For simplicity, we consider the individual 4 expression $sum_4^4(m)$ for the sum sequence 1, 6, 21, 55, 119, 227, ..., which is given by the following recursion,

$$sum_4^4[m] = sum_4^4[m-1] + sum^4[m]/(4m)$$

where $sum_4^4[1] = 1$ and $sum^4[m-1]$ is given by (4). Note that $sum_4^4(m) = \sum_{k=1}^{m} sum^4[k]/k$. The asymptotic fraction of the total endowment is obtained numerically and is

$$r_4^4(m) = \frac{sum_4^4(m)}{sum^4(m)} \to \frac{3}{48}.$$

Finally, the expression $sum_2^4(m)$ for the individual 3 sum sequence 1, 7, 29, 86, 198, 396, ..., and the asymptotic fraction of the total endowment are obtained as the residual difference. ■

Proof of Conjecture 1. From the cases $n = 2, 3, 4$, we claim that $r_n^n = 1/n^2$. The following method is found to hold for $n = 2, 3, 4$. It is obtained with a method similar to one used to construct the Lorenz curve. In the proof we also show how to obtain general expressions for r_i^n. Let

$(n+1-i)/n$ be the cumulative share of the individuals $i, i+1, ..., n$, and let y_i^n be the cumulative share of the allocation shares, $r_n^n + r_{n-1}^n + ... + r_i^n$. Start from the individual n, which represents a share of the total individuals and endowment equal to $1/n$ and $1/n^2$, respectively. The expression for the line that passes through the points $(0,0)$ and $(1/n, y_n^n)$ is the solution of the system,

$$\begin{cases} y_n^n = a_0 + b_0 \frac{1}{n} \\ 0 = a_0 + b_0 0 \end{cases},$$

where a_0 is the intercept and b_0 is the slope. We have a system of two equations and three unknowns, but since we claim that $y_n^n = r_n^n$, we can solve the system. Similarly, the line that passes through the points $(1/n, y_n^n)$ and $\left(2/n, y_{n-1}^n\right)$ is the solution of the system,

$$\begin{cases} y_{n-1}^n = a_1 + b_1 \frac{2}{n} \\ y_n^n = a_1 + b_1 \frac{1}{n} \end{cases},$$

where $y_{n-1}^n = r_n^n + r_{n-1}^n$. Since $y_n^n = r_n^n$, we have a system of two equations and three unknowns. Now, we have a new equation that is obtained from observing that the line equation passes tangent to the point $\left(1/(n-1), y_{n-1}^{n-1}\right)$, where $y_{n-1}^{n-1} = r_{n-1}^{n-1} = 1/(n-1)^2$. Figure 1 provides an illustration. Therefore, in addition we have the equation $y_{n-1}^{n-1} = a_1 + b_1 \frac{1}{n-1}$. The relevant solution is

$$y_{n-1}^n = \frac{(n-1)\, y_{n-1}^{n-1} - (n-2)\, y_n^n}{1} = \frac{3n-2}{(n-1)\, n^2}, \tag{5}$$

which implies

$$r_{n-1}^n = y_{n-1}^n - y_n^n = (n-1)\, \frac{y_{n-1}^{n-1} - y_n^n}{1} = \frac{2n-1}{(n-1)\, n^2}.$$

Similarly, the line that passes through the point $\left(2/n, y_{n-1}^n\right)$ and $\left(3/n, y_{n-2}^n\right)$ is the solution of the system,

$$\begin{cases} y_{n-2}^n = a_2 + b_2 \frac{3}{n} \\ y_{n-1}^n = a_2 + b_2 \frac{2}{n} \end{cases},$$

where $y_{n-2}^n = r_n^n + r_{n-1}^n + r_{n-2}^n$. We have a system of two equations and three unknowns. Again, we have an equation that is obtained from observing that the line equation passes tangent to the point $\left(2/(n-1), y_{n-2}^{n-1}\right)$, where from (5) we know that $y_{n-2}^{n-1} = r_{n-1}^{n-1} + r_{n-2}^{n-1} = \frac{3(n-1)-2}{(n-2)(n-1)^2}$. Figure 1 provides an illustration. Therefore, in addition we have the equation $y^{n-1}_{n-2} = a_2 + b_2 \frac{2}{n-1}$. The relevant solution is

$$y^n_{n-2} = \frac{(n-1)\, y^{n-1}_{n-2} - (n-3)\, y^n_{n-1}}{2} = \frac{2\left(3n^2 - 7n + 3\right)}{(n-2)\,(n-1)\, n^2},$$

which implies

$$r^n_{n-2} = y^n_{n-2} - y^n_{n-1} = (n-1)\, \frac{y^{n-1}_{n-2} - y^n_{n-1}}{2} = \frac{3n\,(n-2) + 2}{(n-2)\,(n-1)\, n^2}.$$

We continue until we reach $i = 1$, i.e., when we have obtained the n elements in the vector r^n. ■

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