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Market Power and Welfare in Asymmetric Divisible Good Auctions*

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Abstract

We analyze a divisible good uniform-price auction that features two groups each with a finite number of identical bidders. At equilibrium the relative market power (price impact) of a group increases with the precision of its private information and declines with its transaction costs. An increase in transaction costs and/or a decrease in the precision of a bidding group's information induces a strategic response from the other group, which thereafter attenuates its response to both private information and prices. A "stronger" bidding group -which has more precise private information, faces lower transaction costs, and is more oligopsonistic- has more price impact and so will behave competitively only if it receives a higher per capita subsidy rate. When the strong group values the asset no less than the weak group, the expected deadweight loss increases with the quantity auctioned and also with the degree of payoff asymmetries. Price impact and the deadweight loss may be negatively associated. The results are consistent with the available empirical evidence.

KEYWORDS: demand/supply schedule competition, private information, liquidity auctions, Treasury auctions, electricity auctions, market integration.

JEL: D44, D82, G14, E58

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1 Introduction

Divisible good auctions are common in many markets, including government bonds, liquidity (refinancing operations), electricity, and emission markets.[1] In those auctions, both market power (price impact) and asymmetries among the participants are important; asymmetries can make price impact relevant even in large markets. However, theoretical work in this area has been hampered by the difficulties of dealing with bidders that are asymmetric, have market power, and are competing in terms of demand or supply schedules in the presence of private information. This paper helps to fill that research gap by analyzing asymmetric uniform-price auctions in which there are two groups of bidders. Our aims are to characterize the equilibrium, to perform comparative statics and welfare analysis (from the standpoint of revenue and deadweight loss), and finally to derive implications for policy.

Divisible good auctions are typically populated by heterogenous participants in a concentrated market, and often we can distinguish a core group of bidders together with a fringe. The former are *strong* in the sense that they have better information, endure lower transaction costs, and are more oligopolistic (or oligopsonistic) than members of the fringe. As examples we discuss Treasury and liquidity auctions in addition to wholesale electricity markets. Uniform-price auctions are often used in Treasury, liquidity and electricity auctions.

Treasury auctions have bidders with significant market shares. That will be the case in most systems featuring a primary dealership, where participation is limited to a fixed number of bidders (this occurs, for example, in 29 out of 39 countries surveyed by Arnone and Iden 2003). A prime example are US Treasury auctions, which are uniform-price auctions since 1998. In these auctions, the top five bidders typically purchase close to half of US Treasury issues.[2] Primary dealers went from 46 in 1998 to 23 presently. Those account for a very substantial portion of volume (from 69% to 88% of tendered quantities in the sample of Hortaçsu et al. (2016) for the years 2009-2013). Indirect bidders place their bids through the primary dealers and other direct bidders tender from 6 to 13%.

Primary dealers enjoy an information advantage because they aggregate the information of indirect bidders and face lower transaction costs.[3] Primary dealers bid systematically lower prices than the other participants in the auction, according to Hortaçsu et al. (2016), not

[1] See Lopomo et al. (2011) for examples of such auctions.

[2] According to Euromoney, the top five commanded 58% of the market in 2016, up from 43% in 2006; and the top ten dealers 85%. See also Malvey and Archibald (1998).

[3] A proxy for the information advantage of a primary dealer is the number of indirect bidders that go through this dealer. For evidence from Canadian Treasury auctions, see Hortaçsu and Kastl (2012); for a theoretical model see Boyarchenko et al. (2015).

because they have a lower valuation of the securities but because they exercise market power.[4]

Armantier and Sbaï (2006) test for whether the bidders in French Treasury auctions are symmetric; these authors conclude that such auction participants can be divided into two distinct groups as a function of (a) their level of risk aversion and (b) the quality of their information about the value of the security to be sold. One small group consists of large financial institutions, which possess better information and are willing to take more risks. Kastl (2011) also finds evidence of two distinct groups of bidders in (uniform-price) Czech Treasury auctions. Other papers that report asymmetries between bidders in Treasury auctions include, among others, Umlauf (1993) for Mexico, Bjonnes (2001) for Norway, and Hortaçsu and McAdams (2010) for Turkey.

Bindseil et al. (2009) and Cassola et al. (2013) find that the heterogeneity of bidders in liquidity auctions is relevant. Cassola et al. (2013) analyze the evolution of bidding data from the European Central Bank's weekly refinancing operations before and during the early part of the financial crisis. The authors show that effects of the 2007 subprime market crisis were heterogeneous among European banks, and they conclude that the significant shift in bidding behavior after 9 August 2007 may reflect a change in the cost of short-term funding on the interbank market and/or a strategic response to other bidders. In particular, Cassola et al. (2013) find that one third of bidders experienced no change in their costs of short-term funds from alternative sources; this means that their altered bidding behavior was mainly strategic: bids were increased as a best response to the higher bids of rivals.[5]

Concentration is high also in other markets, such as wholesale electricity. This issue has attracted attention from academics and policymakers alike. A number of empirical studies have concluded that sellers have exercised significant market power in wholesale electricity markets (see, e.g., Green and Newbery 1992; Wolfram 1998; Borenstein et al. 2002; Joskow and Kahn 2002).[6] Most wholesale electricity markets prefer using a uniform-price auction to using a pay-as-bid auction (Cramton and Stoft 2006, 2007). In several of these markets (e.g., California, Australia), generating companies bid to sell power and wholesale customers bid to buy power. In such markets, asymmetries are prevalent. For example, some generators of wholesale electricity rely heavily on nuclear technology, which has flat marginal costs, whereas others rely mostly

[4] Experimental work has found also substantial demand reduction in uniform-price auctions (see e.g. Kagel and Levin 2001; Engelbrecht-Wiggans et al. 2006).

[5] Bidder asymmetry has also been found in procurement markets, including school milk (Porter and Zona 1999; Pesendorfer 2000) and public works (Bajari 1998).

[6] European Commission (2007) has asserted that "at the wholesale level, gas and electricity markets remain national in scope, and generally maintain the high level of concentration of the pre-liberalization period. This gives scope for exercising market power" (Inquiry pursuant to Article 17 of Regulation (EC) No 1/2003 into the European gas and electricity sectors (Final Report), Brussels, 10.1.2007).

on fuel technologies, which have steep marginal costs. Holmberg and Wolak (2015) argue that, in wholesale electricity markets, information on suppliers' production costs is asymmetric. For evidence on the effect of cost heterogeneity on bidding in wholesale electricity markets, see Crawford et al. (2007) and Bustos-Salvagno (2015).

Our paper makes progress within the linear-Gaussian family of models by incorporating bidders' asymmetries with regard to payoffs and information. We model a uniform-price auction where asymmetric strategic bidders compete in terms of demand schedules for an inelastic supply (we can easily accommodate supply schedule competition for an inelastic demand). Bidders may differ in their valuations, transaction costs, and/or the precision of their private information. For simplicity and with an empirical basis, we reduce heterogeneity to two groups; within each group, agents are identical.[7] We seek to identify the conditions under which there exists a linear equilibrium with symmetric treatment of agents in the same group (i.e., we are looking for equilibria such that demand functions are both linear and identical among individuals of the same type). After showing that any such equilibrium must be unique, we derive comparative statics results.

More specifically, our analysis establishes that the number of group members, the transactions costs, the extent to which bidders' valuations are correlated, and the precision of private information affect the sensitivity of traders' demands to private information and prices. When valuations are more correlated, traders react less to the private signal and to the price. We also find that the relative price impact of a group increases with the precision of its private information and decreases with its transaction costs. Furthermore, if the transaction costs of a group increase, then the traders of the other group respond strategically by diminishing their reaction to private information and submitting steeper schedules. This result is consistent with the empirical findings of Cassola et al. (2013) in European post-crisis liquidity auctions.

If a group of traders is stronger in the sense described previously (i.e., if its private information is more precise, its transaction costs are lower, and it is more oligopolistic), then the members of that group react more (than do the bidders of the other group) to the private signal and also to the price. This result may help explain the finding of Hortaçsu and Puller (2008) for the Texas balancing market where, there is no accounting for private information on costs that, small firms use steeper schedules than the theory would predict.[8]

[7]All traders in a group receive the same signal. In case one group does not receive a signal we reproduce the information structure in Grossman and Stiglitz (1980) of uninformed and informed traders.

[8]Linear supply function models have been used extensively for estimating market power in wholesale electricity auctions. Holmberg et al. (2013) provide a foundation for the continuous approach as an approximation to the discrete supply bids in a spot market. In their experimental work, Brandts et al. (2014) find that observed behavior is more consistent with a supply function model than with a discrete multi-unit auction model.

We also find that, when the expected valuations between groups differ, the auction's expected revenue needs *not* be decreasing in the transaction costs of bidders, the noise in their signals, or the correlation of values. These findings contrast with the results obtained when groups are symmetric. We bound the expected revenue of the auction between the revenues of auctions involving extremal yet symmetric groups.

In this paper we consider large markets and find that, if there is both a small and a large group of bidders, then the former (oligopsonistic) group has more price impact and yet even the latter (large) group does not behave competitively since it retains some price impact due to incomplete information. We also prove that the equilibrium under imperfect competition converges to a price-taking equilibrium in the limit as the number of traders (of both groups) becomes large.

Finally, we provide a welfare analysis. Toward that end, we characterize the deadweight loss at the equilibrium and show how a subsidy scheme may induce an efficient allocation. We find that if one group is stronger (as previously defined), then it should garner a higher per capita subsidy rate; the reason is that traders in the stronger group will behave more strategically and so must be compensated more to become competitive. The paper also underscores how the bidder heterogeneity (in terms of information, preferences, or group size documented in previous work) may increase deadweight losses. In particular, when the strong group values the asset at least as much as the weak group, the deadweight loss increases with the quantity auctioned and also with the degree of payoff asymmetries.

Our work is related to the literature on divisible good auctions. Results in symmetric pure common value models have been obtained by Wilson (1979), Back and Zender (1993), and Wang and Zender (2002), among others.[9] Kastl (2011) extends the Wilson model to consider discrete bids in an independent values context. This model is extended in Hortaçsu and Kastl (2012) and Hortaçsu et al. (2016).

Results in interdependent values models with symmetric bidders are obtained by Vives (2011,

Ciarreta and Espinosa (2010) use Spanish data in finding more empirical support for the smooth supply model than the discrete-bid auction model.

[9] Wilson (1979) compares a uniform-price auction for a divisible good with an auction in which the good is treated as an indivisible good; he finds that the price can be significantly lower if bidders are allowed to submit bid schedules rather than a single bid price. That work is extended by Back and Zender (1993), who compare a uniform-price auction with a discriminatory auction. These authors demonstrate the existence of equilibria in which the seller's revenue in a uniform-price auction can be much lower than the revenue obtained in a discriminatory auction. According to Wang and Zender (2002), if supply is uncertain and bidders are risk averse, then there may exist equilibria of a uniform-price auction that yield higher expected revenue than that from a discriminatory auction.

2014) and Ausubel et al. (2014), for example.[10] Vives (2011), while focusing on the tractable family of linear-Gaussian models, shows how increased correlation in traders' valuations increases the price impact of those traders. Bergemann et al. (2015) generalize the information structure in Vives (2011) while retaining the assumption of symmetry. Rostek and Weretka (2012) partially relax that assumption and replace it with a weaker "equicommonality" assumption on the matrix correlation among the agents' values.[11] Du and Zhu (2015) consider a dynamic auction model with ex post equilibria. For the case of complete information, progress has been made in divisible good auction models by characterizing linear supply function equilibria (e.g., Klemperer and Meyer 1989; Akgün 2004; Anderson and Hu 2008). An exception that incorporates incomplete information is the paper by Kyle (1989), who considers a Gaussian model of a divisible good *double* auction in which some bidders are privately informed and others are uninformed.[12] Sadzik and Andreyanov (2016) study the design of robust exchange mechanisms in a two-type model similar to the one we present here.

Despite the importance of bidder asymmetry, results in multi-unit auctions have been difficult to obtain. As a consequence, most papers that deal with this issue focus on auctions for a single item. In sealed-bid, first-price, single-unit auctions, an equilibrium exists under quite general conditions (Lebrun 1996; Maskin and Riley 2000a; Athey 2001; Reny and Zamir 2004). Uniqueness is explored in Lebrun (1999) and Maskin and Riley (2003). Maskin and Riley (2000b) study asymmetric auctions, and Cantillon (2008) shows that the seller's expected revenue declines as bidders become less symmetric. On the multi-unit auction front, progress in establishing the existence of monotone equilibria has been made by McAdams (2003, 2006); those papers address uniform-price auctions characterized by multi-unit demand, interdependent values and independent types.[13] Reny (2011) stipulates more general existence conditions that allow for infinite-dimensional type and action spaces; these conditions apply to uniform-price, multi-unit auctions with weakly risk-averse bidders and interdependent values (and where

[10] Ausubel et al. (2014) find that, in symmetric auctions with decreasing linear marginal utility, the seller's revenue is greater in a discriminatory auction than in a uniform-price auction. Pycia and Woodward (2016) demonstrate that a discriminatory pay-as-bid auction is revenue-equivalent to the uniform-price auction provided that supply and reserve prices are set optimally.

[11] This assumption states that the sum of correlations in each column of this matrix (or, equivalently, in each row) is the same and that the variances of all traders' values are also the same. Unlike our model, Rostek and Weretka's (2012) model maintains the symmetry assumption as regards transaction costs and the precision of private signals. The equilibrium they derive is therefore still symmetric because all traders use identical strategies.

[12] Lambert et al. (2016) extend the Kyle (1985) model to general correlation patterns among random variables.

[13] McAdams (2006) uses a discrete bid space and atomless types to show that, with risk neutral bidders, monotone equilibria exist. The demonstration is based on checking that the single-crossing condition used by Athey (2001) for the single-object case extends to multi-unit auctions.

bids are restricted to a finite grid).

The rest of our paper is organized as follows. Section 2 outlines the model. Section 3 characterizes the equilibrium, analyzes its existence and uniqueness, and derives comparative statics results. We address large markets in Section 4 and develop the welfare analysis in Section 5. Section 6 concludes. Proofs are gathered in Appendix A and complementary material in Appendix B.

2 The model

Traders, of whom there are a finite number, face an inelastic supply for a risky asset. Let Q denote the aggregate quantity supplied in the market. In this market there are buyers of two types: type 1 and type 2. Suppose that there are n_i traders of type i, $i = 1, 2$. In that case, if the asset's price is p, then the profits of a representative type-i trader who buys x_i units of the asset are given by

$$\pi_i = (\theta_i - p)\, x_i - \lambda_i x_i^2/2.$$

So, for any trader of type i, the marginal benefit of buying x_i units of the asset is $\theta_i - \lambda_i x_i$, where θ_i denotes the valuation of the asset and $\lambda_i > 0$ reflects an adjustment for transaction costs or opportunity costs (or a proxy for risk aversion). Traders maximize expected profits and submit demand schedules, after which the auctioneer selects a price that clears the market. The case of supply schedule competition for inelastic demand is easily accommodated by considering negative demands ($x_i < 0$) and a negative inelastic supply ($Q < 0$). In this case, a producer of type i has a quadratic production cost $-\theta_i x_i + \lambda_i x_i^2/2$.

We assume that θ_i is normally distributed with mean $\overline{\theta}_i$ and variance σ_θ^2, $i = 1, 2$. The random variables θ_1 and θ_2 may be correlated, with correlation coefficient $\rho \in [0, 1]$. Therefore, $cov(\theta_1, \theta_2) = \rho\sigma_\theta^2$.[14] All type-$i$ traders receive the same noisy signal $s_i = \theta_i + \varepsilon_i$, where ε_i is normally distributed with null mean and variance $\sigma_{\varepsilon_i}^2$. Error terms in the signals are uncorrelated across groups ($cov(\varepsilon_1, \varepsilon_2) = 0$) and are also uncorrelated with valuations of the asset ($cov(\varepsilon_i, \theta_j) = 0$, i, $j = 1, 2$).

In our model, two traders of distinct types may differ in several respects:

- different willingness to possess the asset ($\theta_1 \neq \theta_2$),
- different transaction costs ($\lambda_1 \neq \lambda_2$), and/or
- different levels of precision of private information ($\sigma_{\varepsilon_1}^2 \neq \sigma_{\varepsilon_2}^2$).

[14] The value of ρ will depend of the type of security. In this sense, Bindseil et al. (2009) argue that the common value component is less important in a central bank repo auction than in a T-bill auction.

Applications of this model are Treasury auctions and liquidity auctions. For Treasury auctions, θ_i is the private value of the securities to a bidder of type i; that value incorporates not only the resale value but also idiosyncratic elements as different liquidity or portfolio immunization needs of bidders in the two groups. Financial intermediaries may assign different values to the Treasury instruments according to their use as collateral. In particular, primary dealers may attach a value to a bond beyond resale value to be used as collateral in operations with the Fed. For liquidity auctions, θ_i is the price (or interest rate) that group i commands in the secondary interbank market (which is over-the-counter). Here λ_i reflects the structure of a counterparty's pool of collateral in a repo auction. A bidder bank prefers to offer illiquid collateral to the central bank in exchange for funds; as allotments increase, however, the bidder must offer more liquid types of collateral which have a higher opportunity cost. This yields a declining marginal valuation.[15]

3 Equilibrium

Denote by X_i the strategy of a type-i bidder, $i = 1, 2$, which is a mapping from the signal space to the space of demand functions. Thus, $X_i(s_i, \cdot)$ is the demand function of a type-i bidder that corresponds to a given signal s_i. Given her signal s_i, each bidder in a Bayesian equilibrium chooses a demand function that maximizes her conditional profit (while taking as given the other traders' strategies). We assume that there is a unique market clearing price given the submitted demand schedules and signal realizations.[16] Our attention will be restricted to anonymous linear Bayesian equilibria in which strategies are linear and identical among traders of the same type (for short, equilibria).

Definition. An *equilibrium* is a linear Bayesian equilibrium such that the demand functions for traders of type i, $i = 1, 2$, are identical and equal to

$$X_i(s_i, p) = b_i + a_i s_i - c_i p,$$

where b_i, a_i, and c_i are constants.

3.1 Equilibrium characterization

Consider a trader of type i. If rival's strategies are linear and identical among traders of the same type and if the market clears, that is, if $(n_i - 1)X_i(s_i, p) + x_i + n_j X_j(s_j, p) = Q$,

[15] See Ewerhart et al. (2010).

[16] If there is no market clearing price the market shuts down and if there are multiple clearing prices the auctioneer choses the one that maximizes volume traded.

for $j = 1, 2$ and $j \neq i$, then this trader faces the residual inverse supply $p = I_i + d_i x_i$, where $I_i = ((n_i - 1)(b_i + a_i s_i) + n_j (b_j + a_j s_j) - Q) / ((n_i - 1) c_i + n_j c_j)$ and $d_i = 1/((n_i - 1) c_i + n_j c_j)$. The slope of residual supply (d_i) is an index of the trader's market power or price impact.[17] Indeed, by putting one more unit in the market a trader of type i will move the price by d_i. A competitive trader would face a flat residual supply ($d_i = 0$). The slope d_i increases, and the residual supply becomes less elastic, the steeper are the demand functions submitted by the other traders (i.e., the lower are c_i and c_j).

As a consequence, this trader's information set (s_i, p) is informationally equivalent to (s_i, I_i). Note that only the intercept of the residual supply for a trader of type i is informative about the signal of traders of type j. The bidder therefore chooses x_i to maximize

$$\mathbb{E}[\pi_i | s_i, p] = (\mathbb{E}[\theta_i | s_i, I_i] - I_i - d_i x_i) x_i - \lambda_i x_i^2 / 2.$$

The first-order condition (FOC) is given by $\mathbb{E}[\theta_i | s_i, I_i] - I_i - 2 d_i x_i - \lambda_i x_i = 0$, or equivalently,

$$X_i(s_i, p) = (\mathbb{E}[\theta_i | s_i, p] - p) / (d_i + \lambda_i). \tag{1}$$

The second-order condition (SOC) that guarantees a maximum is $2d_i + \lambda_i > 0$, which implies that $d_i + \lambda_i > 0$.

Using the expression for I_i and assuming that $a_j \neq 0$, we can show that (s_i, p) is informationally equivalent to (s_1, s_2). Therefore, since $\mathbb{E}[\theta_i | s_i, p] = \mathbb{E}[\theta_i | s_i, I_i]$, it follows that

$$\mathbb{E}[\theta_i | s_i, p] = \mathbb{E}[\theta_i | s_1, s_2]. \tag{2}$$

According to Gaussian distribution theory,

$$\mathbb{E}[\theta_i | s_i, s_j] = \overline{\theta}_i + \Xi_i \left(s_i - \overline{\theta}_i\right) + \Psi_i \left(s_j - \overline{\theta}_j\right), \tag{3}$$

where

$$\Xi_i = \frac{1 - \rho^2 + \widehat{\sigma}^2_{\varepsilon_j}}{\left(1 + \widehat{\sigma}^2_{\varepsilon_i}\right)\left(1 + \widehat{\sigma}^2_{\varepsilon_j}\right) - \rho^2} \text{ and } \Psi_i = \frac{\rho \widehat{\sigma}^2_{\varepsilon_i}}{\left(1 + \widehat{\sigma}^2_{\varepsilon_i}\right)\left(1 + \widehat{\sigma}^2_{\varepsilon_j}\right) - \rho^2},$$

with $\widehat{\sigma}^2_{\varepsilon_i} = \sigma^2_{\varepsilon_i} / \sigma^2_\theta$ and $\widehat{\sigma}^2_{\varepsilon_j} = \sigma^2_{\varepsilon_j} / \sigma^2_\theta$. We remark that Equation (3) has the following implications.

1. The private signal s_i is useful for predicting θ_i whenever $1 - \rho^2 + \widehat{\sigma}^2_{\varepsilon_j} \neq 0$, that is, when either the liquidation values are not perfectly correlated ($\rho \neq 1$) *or* type-j traders are imperfectly informed about θ_j ($\sigma^2_{\varepsilon_j} \neq 0$).

[17] We assume that $(n_i - 1) c_i + n_j c_j \neq 0$.

2. The private signal s_j is useful for predicting θ_i whenever $\rho\widehat{\sigma}^2_{\varepsilon_i} \neq 0$, that is, when the private liquidation values are correlated ($\rho \neq 0$) *and* type-i traders are imperfectly informed about θ_i ($\sigma^2_{\varepsilon_i} \neq 0$).

Our first proposition summarizes the previous results. It shows the relationship between a_i and c_i in equilibrium and also indicates that these coefficients are positive (see Lemma A1 and A2 in Appendix A for more details).

Proposition 1. *Let $\rho < 1$. If equilibrium exists, then it is unique and the demand function of a trader of type i ($i = 1, 2$), $X_i(s_i, p) = b_i + a_i s_i - c_i p$, is given by $X_i(s_i, p) = (\mathbb{E}[\theta_i|s_i, p] - p) / (d_i + \lambda_i)$, with $d_i = 1/((n_i - 1)c_i + n_j c_j)$. In addition, we have that $a_i = \Delta_i c_i > 0$, where $\Delta_i = 1 / \left(1 + (1+\rho)^{-1} \widehat{\sigma}^2_{\varepsilon_i}\right)$, $c_i = (1 - \Lambda_i)/(d_i + \lambda_i)$, where $\Lambda_i \equiv \Psi_i \left(\frac{n_i c_i}{n_j c_j} + 1\right) / \Delta_j$, and the ratio c_1/c_2 is the unique positive solution of a cubic polynomial.*

The term Λ_i in the price coefficient in the equilibrium demand c_i is the information-sensitivity weight of the price (i.e., the coefficient of the price in $\mathbb{E}[\theta_i|s_i, p]$). The larger Λ_i, the lower the price coefficient in demand will be (lower c_i). From the perspective of a bidder in group i a high price conveys the news that s_j is high and, therefore, that the value θ_i will tend to be high. If the price is more informative about θ_i, then a bidder in group i is more cautious and submits then a steeper schedule. Furthermore, Λ_i vanishes when $\Psi_i = 0$, that is, when either the valuations are uncorrelated ($\rho = 0$) or the private signal s_i is perfectly informative ($\sigma^2_{\varepsilon_i} = 0$), since in those cases the price conveys no additional information to a trader of type i. In this case $c_i = 1/(d_i + \lambda_i)$ and since $d_i = 1/((n_i - 1)c_i + n_j c_j)$ we observe, therefore, a basic *strategic complementarity* in the *slopes* of the demands submitted by the traders. According to this *strategic effect* if the rivals of a trader of type i, say traders of type j, bid a steeper demand function with a lower c_j, then the slope of the (inverse) residual supply d_i for this trader increases and so he also has an incentive to bid a steep demand function (lower c_i). However, if $\Psi_i > 0$ then there is also an *inference effect* from the information conveyed by the price. A lower c_j increases Λ_i and also will tend to depress c_i. This is so because a lower reaction to the price and signal by rivals induces a trader of type i to give a higher weight to the price in the estimation of his value and, hence,the magnitude of the inference effect raises.

Since $a_i > 0$ and $c_i > 0$, for $i = 1, 2$, it follows that in equilibrium the higher the value of the trader's observed private signal (or the lower the price), the higher the quantity she will demand. When $\sigma^2_{\varepsilon_i} > 0$ we have $a_i < c_i$, since $\Delta_i < 1$ in this case; when $\sigma^2_{\varepsilon_i} = 0$, we have $\Delta_i = 1$ and $a_i = c_i$. Observe that we can write the demand as $X_i(s_i, p) = b_i + c_i(\Delta_i s_i - p)$.

Because p is a linear function of s_1 and s_2, for $i = 1, 2$ we have $\mathbb{E}[\theta_i|s_i, p] = \mathbb{E}[\theta_i|s_1, s_2]$ (i.e., Equation (2) holds). The equilibrium price is therefore *privately revealing*, in other words, the private signal and the price enable a type-i trader to learn as much as about θ_i if she had access to all the information available in the market, (s_1, s_2).

If $\rho = 0$ or if both signals, $i = 1, 2$, are perfectly informative ($\sigma^2_{\varepsilon_i} = 0$), or useless (or $\sigma^2_{\varepsilon_i} = \infty$), then bidders do not learn about θ_i from prices. Hence, $\mathbb{E}[\theta_i|s_i] = \mathbb{E}[\theta_i|s_i, p] = \mathbb{E}[\theta_i|s_1, s_2]$ for $i = 1, 2$. The demand functions are given by

$$X_i(s_i, p) = (\mathbb{E}[\theta_i|s_i] - p) / (d_i + \lambda_i), \ i = 1, 2.$$

Hence, $c_i = 1/(d_i + \lambda_i)$ and in this case, the equilibrium coincides with the full-information equilibrium (denoted by superscript f).[18] Furthermore, when $\rho = 0$, d_i is independent of $\sigma^2_{\varepsilon_i}$, $i = 1, 2$; and when $\sigma^2_{\varepsilon_i} = 0$ or $\sigma^2_{\varepsilon_i} = \infty$, for $i = 1, 2$, d_i is independent of ρ.

Our next proposition summarizes when equilibrium exists. If an equilibrium does exist, then Proposition 1 implies that it is unique.

Proposition 2. *Equilibrium exists iff* $c_i > 0$, $i = 1, 2$.

Complete information. When $\rho = 0$, *or* $\sigma^2_{\varepsilon_i} = 0$, *or* $\sigma^2_{\varepsilon_i} = \infty$, $i = 1, 2$, *equilibrium exists iff* $n_1 + n_2 \geq 3$.

Incomplete information. When $\rho\sigma^2_{\varepsilon_i} > 0$ *and* $\sigma^2_{\varepsilon_j} \geq 0$, $j \neq i$, *and* $\rho < 1$, *we find a necessary and sufficient condition for* $c_i > 0$, $i = 1, 2$ (*see Proposition 2A in Appendix A*). *It follows that equilibrium exists if:* (i) n_1, n_2 *are large enough;* (ii) *given* n_i, n_j *is large enough and* ρ *low enough, for* $i, j = 1, 2$, $j \neq i$, (iii) $\sigma^2_{\varepsilon_j} = 0$ *and* $n_j \geq 2$.

Remark 1. Equilibrium does not exist for ρ close to 1 and low n_i. This is so because in those cases the market power of traders explodes and the demand schedules would become vertical (with $c_i \to 0$, $i = 1, 2$). As ρ increases the informational component of the price is more important and traders submit steeper demands (see Proposition 3 below). Neither does an equilibrium exist when $\rho = 1$. If the price reveals a sufficient statistic for the common valuation, then no trader has an incentive to place any weight on her signal. But if traders put no weight on signals, then the price contains no information about the common valuation. This conundrum is related to the Grossman-Stiglitz (1980) paradox. In fact, $\rho < 1$ and $n_1 + n_2 \geq 3$ are necessary conditions for the existence of equilibrium with incomplete information (in line with Kyle 1989; Vives 2011).[19]

[18] In the full (shared) information setup, traders can access (s_1, s_2). In this framework the price does not provide any useful information.

[19] Du and Zhu (2016) consider ex post nonlinear equilibria in a bilateral divisible double auction and show that with more than three symmetric traders there are no nonlinear equilibria in the class of smooth demands downward sloping in price and upward sloping in signals.

To develop a better understanding of the equilibrium and the condition that guarantees its existence, we consider three particular cases of the model: a monopsony competing with a fringe, symmetric groups, and an informed and an uninformed group.

Monopsony competing with fringe

Corollary 1. *For* $n_2 = 1$ *the equilibrium exists if* $1 - \rho^2 > (2\rho - 1)\widehat{\sigma}^2_{\varepsilon_1}$ *and* $n_1 > \bar{n}_1\left(\rho, \widehat{\sigma}^2_{\varepsilon_1}, \widehat{\sigma}^2_{\varepsilon_2}\right)$, *where* $\bar{n}_1$ *increases with* ρ, $\widehat{\sigma}^2_{\varepsilon_1}$, *and* $\widehat{\sigma}^2_{\varepsilon_2}$. *If, also,* $\lambda_2 = 0$ *and* $\sigma^2_{\varepsilon_2} = 0$, *then* $\bar{n}_1\left(\rho, \widehat{\sigma}^2_{\varepsilon_1}, 0\right) = 1 + \rho\widehat{\sigma}^2_{\varepsilon_1} \big/ \left(1 - \rho^2 - (2\rho - 1)\widehat{\sigma}^2_{\varepsilon_1}\right)$ *and* $x_2 = c_2(\theta_2 - p)$, *with* $c_2 = n_1 c_1$.

An equilibrium with linear demand functions exists provided there is a sufficiently competitive trading environment (n_1 high enough, and with the threshold $\bar{n}_1$ increasing with ρ and $\widehat{\sigma}^2_{\varepsilon_i}$ ($i = 1, 2$), which raise the informational component of the price). In the particular case where $\lambda_2 = 0$ and $\sigma^2_{\varepsilon_2} = 0$, expressions for the equilibrium coefficients can be characterized explicitly (see Appendix A). From the expressions for c_i ($i = 1, 2$) it follows that, if $n_1 = \bar{n}_1$, then the equilibrium cannot exist because in this case the demand functions would be completely inelastic ($c_i = 0$, $i = 1, 2$).

Symmetric groups

Consider the following symmetric case: $n_i = n$, $\lambda_i = \lambda$, and $\sigma^2_{\varepsilon_i} = \sigma^2_{\varepsilon}$, $i = 1, 2$. We find that equilibrium exists iff $n > 1 + \rho\widehat{\sigma}^2_{\varepsilon} \big/ \left((1 - \rho)\left(1 + \rho + \widehat{\sigma}^2_{\varepsilon}\right)\right)$, where $\widehat{\sigma}^2_{\varepsilon} = \sigma^2_{\varepsilon}/\sigma^2_{\theta}$. Therefore, the equilibrium's existence is guaranteed provided either that n is high enough or that ρ or $\widehat{\sigma}^2_{\varepsilon}$ is low enough.

Vives (2011) also analyzes divisible good auctions with symmetric bidders, but in his model the bidders receive different private signals. The condition that guarantees existence of an equilibrium in Vives' setup is $n > 1 + n\rho\widehat{\sigma}^2_{\varepsilon} \big/ \left((1 - \rho)\left(1 + (2n - 1)\rho + \widehat{\sigma}^2_{\varepsilon}\right)\right)$. Direct computation yields that the condition derived in the model of Vives is more stringent than the condition derived in our setup. The reason is that, in Vives (2011), the degree of asymmetry in information (and induced market power) is greater because each of the $2n$ traders receives a private signal.

Informed and uninformed groups

Consider the case where group 1 is uninformed (with no signal) while group 2 is informed ($\sigma^2_{\varepsilon_2} < \infty$). This is akin to the information structure considered by Grossman and Stiglitz (1980). Then equilibrium exists if and only if $\rho < \frac{n_2(n_1 + n_2 - 2)}{2n_1 + n_2 - 2}$, which again holds for ρ small (the threshold is increasing in n_2 and decreasing in n_1 for $n_2 > 2$). We have that $a_2 = \left(1+\widehat{\sigma}^2_{\varepsilon_2}\right)^{-1} c_2$.[20] This expression is different from the case in which group 1 is informed ($\sigma^2_{\varepsilon_1} < \infty$) since then $a_2 = \left(1+\frac{\widehat{\sigma}^2_{\varepsilon_2}}{1+\rho}\right)^{-1} c_2$. Therefore, the equilibrium that we obtain here is not the limit of the general case when $\sigma^2_{\varepsilon_1} \to \infty$ for $\rho > 0$. This is so, since when $\rho > 0$, no matter how large $\sigma^2_{\varepsilon_1}$ is, but finite, there is learning from the price by the two groups. By contrast, when group 1 does not receive any private signal, there is no learning from the price by group 2. Furthermore, in this last case, we have that c_1 and c_2 (d_1 and d_2) are independent of $\sigma^2_{\varepsilon_2}$, and b_2 is independent of $\overline{\theta}_1$ and Q.

3.2 Comparative statics

We start by considering how the model's underlying parameters affect the equilibrium and, in particular, price impact (Proposition 3). We then explore how the equilibrium is affected when there are two distinct groups of traders, that is, a strong group and a weak group (Corollary 2).

Proposition 3. *Let $\rho\sigma^2_{\varepsilon_1}\sigma^2_{\varepsilon_2} > 0$. Then, for $i = 1, 2$, $i \neq j$, the following statements hold.*

(i) *An increase in $\overline{\theta}_i$ or Q, or a decrease in $\overline{\theta}_j$, raises the demand intercept b_i.*

(ii) *An increase in λ_i, λ_j, $\sigma^2_{\varepsilon_i}$, $\sigma^2_{\varepsilon_j}$, or ρ makes demand less responsive to private signals and prices (lower a_i and c_i) and increases price impact (d_i).*

(iii) *If $\sigma^2_{\varepsilon_i}$ and/or λ_i increase, then d_i/d_j decreases.*

(iv) *If n_i and/or n_j increase, then d_i decreases.*

Remark 2. In the complete information setting ($\rho = 0$ or when $\sigma^2_{\varepsilon_i} = 0$, or $\sigma^2_{\varepsilon_i} = \infty$, $i = 1, 2$) prices do not convey information, d^f_i and d^f_j, $j \neq i$, are independent of $\sigma^2_{\varepsilon_i}$; and comparative statics of d^f_i, d^f_j on λ_i, n_i hold as in the proposition. Indeed, if $\rho = 0$, then: (a) both c_i and d_i (as well as c_j and a_j, $j \neq i$) are independent of $\sigma^2_{\varepsilon_i}$; (b) a_i decreases with $\sigma^2_{\varepsilon_i}$; and (c) b_i is independent of both Q and $\overline{\theta}_j$. If $\sigma^2_{\varepsilon_i} = 0$ for $i = 1, 2$, then $b_i = 0$, and c_i, c_j, a_i, a_j, d_i, and d_j, $i = 1, 2$, $j \neq i$, are independent of ρ. That is, for the information parameters to matter for price impact, it is necessary that prices convey information. Proposition 3(ii) implies that, if $\rho\sigma^2_{\varepsilon_1}\sigma^2_{\varepsilon_2} > 0$, then $d^f_i < d_i$, $i = 1, 2$. Thus, asymmetric information increases the price impact of traders in both groups beyond the full-information level.

Remark 3. When groups are symmetric then results (ii) and (iv) hold when $\lambda_i = \lambda_j$, $\sigma^2_{\varepsilon_i} = \sigma^2_{\varepsilon_j}$, and $n_i = n_j$ move together.

The only equilibrium coefficient affected by the quantity offered in the auction (Q) and by the prior mean of the valuations ($\overline{\theta}_i$ and $\overline{\theta}_j$) is b_i. Proposition 3(i) indicates that if Q increases,

[20] In the case where $\sigma^2_{\varepsilon_1} < \infty$ and $\sigma^2_{\varepsilon_2} = 0$, then $b_2 = 0$ and $a_1 = c_1$.

then all the bidders will increase their demand (higher b_1 and b_2). Moreover, if the prior mean of the valuation of group i increases, then the bidders in this group will demand a greater quantity of the risky asset (higher b_i). Then the intercept I_j of the inverse residual supply for the group j bidder rises in response to a higher $\overline{\theta}_i$. That reaction leads the traders in group j to reduce their demand for the risky asset (and lower b_j). We see thus here a strategic substitutability in demand intercepts.

All these results provide testable predictions. Furthermore, if we have estimates of transactions costs, precision of the signals and correlation of values we can predict how changes in these parameters lead to changes in the slopes of submitted demands and the price impact of the two groups.

Part (ii) of Proposition 3 shows how the response to private information and price varies with several parameters. If the transaction costs for a bidder of type i, λ_i, increase, then the bidder sets lower a_i and c_i. Moreover, any increase in a group's transaction costs also affects the behavior of traders in the other group. If λ_i increases, then a_i and c_i decrease, in which case the slope of the inverse residual supply for group j increases (higher d_j) as well as the weight on the price Λ_j in the estimation of θ_j. That is, both the strategic and the inference effects work in the same direction. This change leads group-j traders to reduce their demand sensitivity to signals and prices (lower a_j and c_j). We can therefore see how an increase in the transaction costs for group-i traders (say, a deterioration of their collateral in liquidity auctions that raises λ_i) leads not only to steeper demands for bidders in group i but also, as a reaction, to steeper demands for group-j traders.

We also analyze how the response to private information and price varies with a change in the precision of private signals. If the private signal of type-i bidders is less precise (higher $\sigma^2_{\varepsilon_i}$), then their demand is less sensitive to private information and prices. A private signal of reduced precision also gives the type-i bidder more incentive to consider prices when predicting θ_i (higher Λ_i),[21] which leads in turn to this bidder having a steeper demand function (lower c_i). The same can be said for a bidder of type j because of strategic complementarity in the slopes of demand functions (the decrease in c_i leads to increased d_j and to lower a_j and c_j in turn).[22] This result (in the supply competition model) may help explain why, in the Texas balancing market, small firms use steeper supply functions than predicted by theory (Hortaçsu and Puller 2008). Indeed, smaller firms may receive lower-quality signals owing to economies of scale in information gathering.

We also find that the more highly the valuations are correlated (higher ρ), the less is trader

[21] It is easy to check that Ψ_i/Δ_j, $i \neq j$, $i = 1, 2$, is increasing in $\sigma^2_{\varepsilon_i}$.

[22] An increased $\sigma^2_{\varepsilon_i}$ leads to lower inference component in the price for a bidder of type j, Λ_j, but this effect is overpowered by the increase in d_j in its influence on c_j.

responsiveness to private signals (lower a_i, $i = 1, 2$) and the steeper are inverse demand functions (lower c_i, $i = 1, 2$). We can explain these results by recalling that, when the valuations are correlated ($\rho > 0$), a type-i trader learns about θ_i from prices. In fact, the information-sensitivity weight on the price (Λ_i, $i = 1, 2$) is higher when ρ is larger,[23] in which case demand is less sensitive to private information. The rationale for the relationship between the correlation coefficient (ρ) and the slopes of demand functions is as follows. An increment in the price of the risky asset makes an agent more optimistic about her valuation, which leads to less of a reduction in demand quantity than in the case of uncorrelated valuations.[24]

Cassola et al. (2013) show how distressed bidders after the August 2007 shock suffered a large decline in the valuation of their collateral in the interbank market, which in terms of our model shows up in an increased λ_i. Those banks had also an increase in the valuation for liquidity (which in our model shows up as an increased $\overline{\theta}_i$).[25] In Cassola et al. (2013) it is assumed that the private valuations of the traders are independent since the common component is known. This means that there are no information effects. However, if the common value component is not known, as it plausible to believe, and if in the crisis the signals of the groups become noisier, in particular for those of the group hit by the shock, and the correlation of valuations increases, then all these effects reinforce the steepening of the demand schedules (as found in Cassola et al. (2013)).

Figure 1 illustrates the case of initially identical groups that become differentiated after a shock induces a higher λ_1, noise in the signal for group 1 ($\sigma^2_{\varepsilon_1}$) and correlation ρ as well as the groups' willingness to pay for liquidity (both $\overline{\theta}_1$ and $\overline{\theta}_2$, which affect the intercepts of the demand functions). This corresponds to the case of a group of banks being hit in the crisis and the quality of their information deteriorating (or their perceived uncertainty increasing) as well as the correlation of valuations raising. Note that the expected marginal valuation of liquidity for the safe group need not change a lot (consistent with findings of Cassola et al. (2013)).

[23] It is easy to check that Ψ_i/Δ_j, $i \neq j$, $i = 1, 2$,is increasing in ρ.

[24] A high price conveys the good news that the private signal received by other group's traders is high. When valuations are positively correlated, a bidder infers from the high private signal of the other group that her own valuation is high.

[25] The marginal valuation of a bidder of type i is $\theta_i - \lambda_i x_i$. This is akin to the marginal valuation in Figure 4 in Cassola et al. (2013). There a decreased collateralized borrowing capacity of a bidder (K) will make the slope of the marginal valuation steeper.

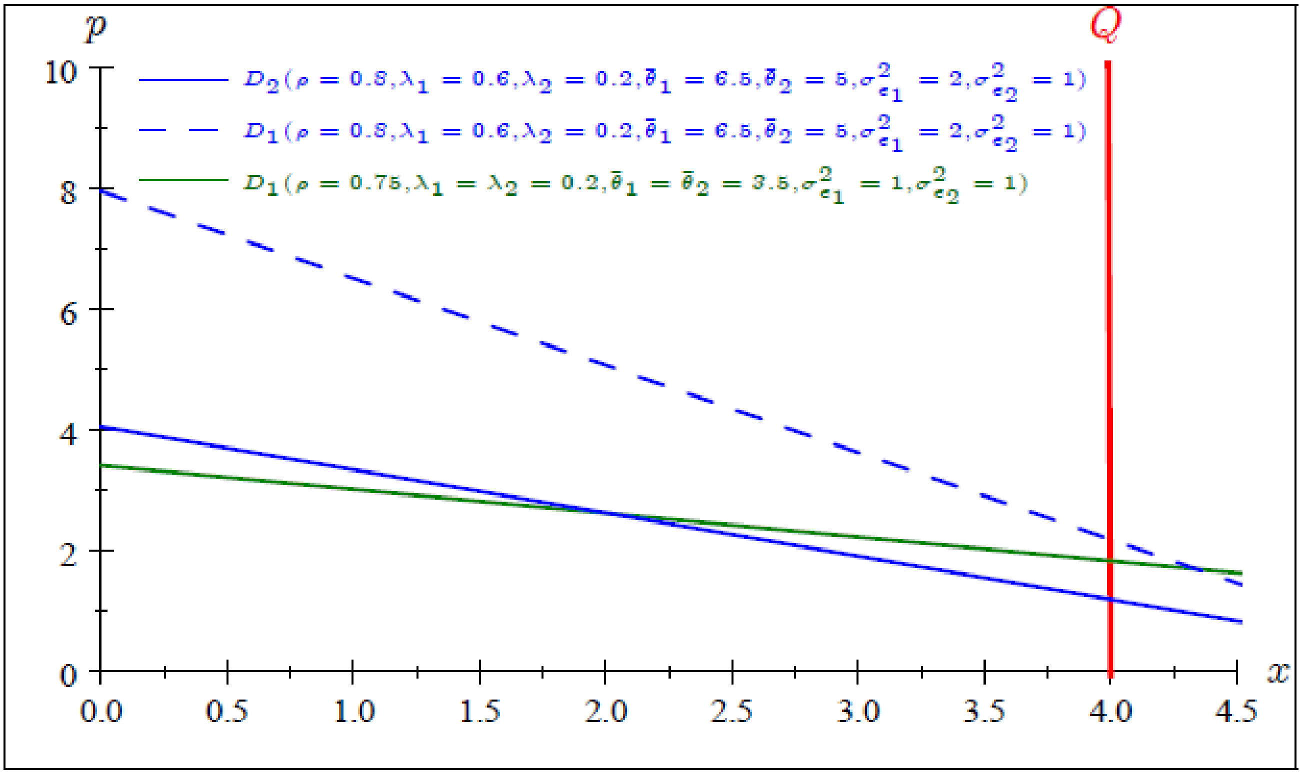

Figure 1: Comparative statics on demand functions $n_1 = 5; n_2 = 5; \sigma^2_\theta = 5; Q = 4$.

Proposition 3(iii) states that any increase in the signal's noise or in group i's transaction costs has the effect of reducing its relative price impact, since then the ratio d_i/d_j $(i \neq j)$ decreases. Finally, part (iv) formalizes the anticipated result that an increase in the number of auction participants (higher n_i or n_j) reduces the price impact of traders in both groups.

Corollary 2 (Strong and weak groups). *Suppose that $\sigma^2_{\varepsilon_1} \geq \sigma^2_{\varepsilon_2}$, $\lambda_1 \geq \lambda_2$, and $n_1 \geq n_2$, and suppose that at least one of these inequalities is strict. Then, in equilibrium, the following statements hold.*

(i) *The stronger group (here, group 2) reacts more both to private information and to prices $(a_1 < a_2,\ c_1 < c_2)$ and has more price impact $(d_1 < d_2)$ than does the weaker group.*

(ii) *The value of the difference $d_1 + \lambda_1 - (d_2 + \lambda_2)$ is, in general, ambiguous:*

- $d_1 + \lambda_1 > d_2 + \lambda_2$ *whenever the differences between groups mainly stem from the transaction costs and λ_1/λ_2 is high enough (with complete information, $d_1 + \lambda_1 > d_2 + \lambda_2$ iff $\lambda_1 > \lambda_2$);*

- $d_1 + \lambda_1 < d_2 + \lambda_2$ *with $\lambda_1 > \lambda_2$ whenever ρ is large.*

Part (i) of this corollary shows that if a group of traders is less informed, has higher transaction costs, and is more numerous, then it reacts less both to private signals and to prices. Observe in particular that group-1 traders, having less precise private information, rely more on the price for information (higher Λ_1); as a result, their overall price response $(c_1 = (1 - \Lambda_1) / (d_1 + \lambda_1))$

is smaller. Similarly, group-1 traders, for whom n_1 is larger, put more information-sensitivity weight on the price (which depends more strongly on s_1).

The precise conditions for Corollary 2(ii), which is useful for comparing allocations across groups, are given in Corollary 2A(ii) in Appendix A. When we have that $d_1 + \lambda_1 < d_2 + \lambda_2$ with $\lambda_1 > \lambda_2$ it means that traders in the group with lower transaction costs are more cautious in order to diminish price impact.

3.3 Equilibrium quantities and expected revenue

3.3.1 Equilibrium quantities

Let $t_i = \mathbb{E}[\theta_i | s_1, s_2]$, $i = 1, 2$, be the predicted values with full information (s_1, s_2). After some algebra, it follows that equilibrium quantities are functions of the vector of predicted values $t = (t_1, t_2)$:

$$x_i(t) = \underbrace{\frac{n_j(t_i - t_j)}{u}}_{x_i^V(t)} + \underbrace{\frac{d_j + \lambda_j}{u}Q}_{x_i^C(t)}, \; i = 1, 2, \; j \neq i, \tag{4}$$

$$\text{where } u = n_i(d_j + \lambda_j) + n_j(d_i + \lambda_i). \tag{5}$$

Observe that, according to these expressions, the equilibrium quantities can be decomposed into two terms: a *valuation* trading term and a *clearing* trading term, which we denote by (respectively) $x_i^V(t)$ and $x_i^C(t)$ for group i, $i = 1, 2$. With regard to the information trading term, it vanishes when $t_1 = t_2$, but has a positive (resp. negative) value for the group with the higher (resp. lower) value of t_i. Higher "total" transaction costs $(d_i + \lambda_i)$ lower the response to valuation differences $t_i - t_j$. Moreover, $n_1 x_1^V(t) + n_2 x_2^V(t) = 0$. As for the clearing trading term, we remark that it vanishes when $Q = 0$; otherwise, it is positive for both groups yet lower (resp. higher) for the group with higher (resp. lower) $d_i + \lambda_i$. In addition, $n_1 x_1^C(t) + n_2 x_2^C(t) = Q$.

Taking expectations in Equation (4), we have

$$\mathbb{E}[x_1(t)] - \mathbb{E}[x_2(t)] = \frac{(n_1 + n_2)\left(\overline{\theta}_1 - \overline{\theta}_2\right) + (d_2 + \lambda_2 - (d_1 + \lambda_1))Q}{n_1(d_2 + \lambda_2) + n_2(d_1 + \lambda_1)}.$$

Group 1 trades more when it values the asset more highly $(\overline{\theta}_1 > \overline{\theta}_2)$ and when its traders are less cautious $(d_2 + \lambda_2 > d_1 + \lambda_1)$ than group 2. By combining Corollary 2 with the equation just displayed, we obtain the following remarks.

Remark 4. If Q is low enough, then $\mathbb{E}[x_1(t)] > \mathbb{E}[x_2(t)]$ whenever $\overline{\theta}_1 > \overline{\theta}_2$. In contrast, if Q is high enough, then $\mathbb{E}[x_1(t)] > \mathbb{E}[x_2(t)]$ whenever $d_2 + \lambda_2 > d_1 + \lambda_1$. Under the assumptions of Corollary 2, this latter inequality is satisfied if (34) holds or if λ_1/λ_2 is sufficiently low.

Remark 5. When $Q = 0$ (i.e., the so-called double auction case), then $\mathbb{E}[x_2(t)] < 0 < \mathbb{E}[x_1(t)]$ iff $\overline{\theta}_1 > \overline{\theta}_2$. Then group 1 consists of buyers and group 2 of sellers.

3.3.2 Expected price and revenue

Our aim here is to identify factors that affect expected price and revenue in the auction (see Appendix B for an analysis of bid shading). Let $\tilde{t} = (n_1 t_1 + n_2 t_2) / (n_1 + n_2)$. From the demand of bidders it follows that $p(t) = t_i - (d_i + \lambda_i) x_i(t)$, $i = 1, 2$. Therefore,

$$p(t) = \tilde{t} - ((d_1 + \lambda_1) n_1 x_1(t) + (d_2 + \lambda_2) n_2 x_2(t)) / (n_1 + n_2).$$

From the above expression we can derive the expected price:

$$\mathbb{E}[p(t)] = \left(\frac{n_1}{d_1 + \lambda_1} \overline{\theta}_1 + \frac{n_2}{d_2 + \lambda_2} \overline{\theta}_2 - Q \right) \Big/ \left(\frac{n_1}{d_1 + \lambda_1} + \frac{n_2}{d_2 + \lambda_2} \right). \tag{6}$$

It is worth noting that, in the double auction case ($Q = 0$), $\mathbb{E}[p]$ is a convex combination of $\overline{\theta}_1$ and $\overline{\theta}_2$. Also, for symmetric groups (except possibly with respect to the means) we have $\mathbb{E}[p] = \left(\overline{\theta}_1 + \overline{\theta}_2\right)/2$.

Proposition 4. *Let $\rho \sigma^2_{\varepsilon_1} \sigma^2_{\varepsilon_2} > 0$. In equilibrium, the following statements hold.*

(i) *If $\overline{\theta}_1 = \overline{\theta}_2$, then the expected price is increasing in n_i but is decreasing in λ_i, $\sigma^2_{\varepsilon_i}$, and ρ, $i = 1, 2$. Otherwise, if $\left|\overline{\theta}_1 - \overline{\theta}_2\right|$ is large enough, then these results need not hold.*

(ii) *The seller's expected revenue $\mathbb{E}[p] Q$:*

- *increases with $\overline{\theta}_i$ for $i = 1, 2$;*
- *is maximum when $Q = \left(\frac{n_1}{d_1+\lambda_1} \overline{\theta}_1 + \frac{n_2}{d_2+\lambda_2} \overline{\theta}_2 \right) \Big/ 2$, which increases with n_i and $\overline{\theta}_i$ and decreases with ρ, λ_i and $\sigma^2_{\varepsilon_i}$, $i = 1, 2$.*

Corollary 3. *The expected revenue is between* (a) *the larger expected revenue of the auction in which both groups are ex ante identical with a large number of bidders* (*each group with* $\max\{n_1, n_2\}$), *high expected valuation* ($\max\left\{\overline{\theta}_1, \overline{\theta}_2\right\}$), *low transaction costs* ($\min\{\lambda_1, \lambda_2\}$) *and precise signals* ($\min\left\{\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}\right\}$) *and* (b) *the smaller expected revenue of the auction in which both groups are ex ante identical but with the opposite characteristics* (i.e., $\min\{n_1, n_2\}$, $\min\left\{\overline{\theta}_1, \overline{\theta}_2\right\}$, $\max\{\lambda_1, \lambda_2\}$, *and* $\max\left\{\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}\right\}$).

Remark 6. If $\rho = 0$, then $\mathbb{E}[p]$ is independent of $\sigma^2_{\varepsilon_i}$ ($i = 1, 2$), and if $\sigma^2_{\varepsilon_i} = 0$, $i = 1, 2$, then $\mathbb{E}[p]$ is independent of ρ. The reason is that in both cases, d_i is independent of $\sigma^2_{\varepsilon_i}$ and ρ. If $\sigma^2_{\varepsilon_1} = \infty$, then $\mathbb{E}[p]$ is independent of $\sigma^2_{\varepsilon_2}$.

Proposition 4 indicates that the relationship between expected price (on the one hand) and λ_i, $\sigma^2_{\varepsilon_i}$, and ρ, $i = 1, 2$ (on the other hand) is potentially ambiguous. For example, if $\overline{\theta}_2 - \overline{\theta}_1$ is high enough, then $\mathbb{E}[p]$ is decreasing in n_1; yet, if $\overline{\theta}_1 = \overline{\theta}_2$, then the derived results are in line with those in the symmetric case, where $\mathbb{E}[p] = \overline{\theta} - (d + \lambda) Q/2n$ (see Vives 2010, Prop. 2).

We should like to understand how ex ante differences among bidders affect the seller's expected revenue. Suppose that group 2 is our strong group; it has lower transaction costs ($\lambda_2 < \lambda_1$), is less numerous ($n_2 < n_1$), and is better informed ($\sigma^2_{\varepsilon_2} < \sigma^2_{\varepsilon_1}$). If this group values the asset less, $\overline{\theta}_2 < \overline{\theta}_1$ (resp., values it more, $\overline{\theta}_2 > \overline{\theta}_1$), then expected revenue is lower (resp., higher) than in the case where $\overline{\theta}_1 = \overline{\theta}_2$. If $\overline{\theta}_1 \approx \overline{\theta}_2$, then Proposition 4(i) suggests that group 2's relatively small size ($n_2 < n_1$) reduces the seller's expected revenue, although both its relatively low transaction costs ($\lambda_2 < \lambda_1$) and its relatively precise signals ($\sigma^2_{\varepsilon_2} < \sigma^2_{\varepsilon_1}$) have the opposite effect. So in general, the ex ante differences between the two groups have an ambiguous effect on the seller's expected revenue. Nonetheless, part (ii) of Proposition 4 directly follows from part (i).

4 Large markets

Our objective in this section is to determine whether (or not) the equilibrium under imperfect competition converges to a price-taking equilibrium in the limit as the number of traders becomes large. We examine two possible scenarios: in the first, only group 1 is large; in the second, both groups of bidders are large. The *per capita* supply (denoted by q) is assumed to be inelastic, that is, $Q = (n_1 + n_2)q$.

4.1 Oligopsony with competitive fringe

Proposition 5. *Let $\rho\sigma^2_{\varepsilon_1} > 0$. Suppose that $n_1 \to \infty$ and $n_2 < \infty$. Then an equilibrium exists iff $n_2 > \bar{n}_2\left(\rho, \widehat{\sigma}^2_{\varepsilon_1}, \widehat{\sigma}^2_{\varepsilon_2}\right)$, where $\bar{n}_2$ is increasing in ρ and $\widehat{\sigma}^2_{\varepsilon_1}$ and where $\bar{n}_2$ is decreasing in $\widehat{\sigma}^2_{\varepsilon_2}$ whenever $(2\rho - 1)\widehat{\sigma}^2_{\varepsilon_1} < 1 - \rho^2$. An agent in the large group absorbs the inelastic per capita supply in the limit ($\lim_{n_1\to\infty} b_1 = q$, $\lim_{n_1\to\infty} a_1 = \lim_{n_1\to\infty} c_1 = 0$) and retains some price impact ($\lim_{n_1\to\infty} d_1 > 0$), while an agent in the small group commands a higher degree of market power ($\lim_{n_1\to\infty} d_2 > \lim_{n_1\to\infty} d_1$).*

When $n_2 = 1$, the existence condition stated in Proposition 5 boils down to $(2\rho - 1)\widehat{\sigma}^2_{\varepsilon_1} < 1 - \rho^2$ from Corollary 1. Equation (38) in Appendix A shows that, when $n_2 = \bar{n}_2\left(\rho, \widehat{\sigma}^2_{\varepsilon_1}, \widehat{\sigma}^2_{\varepsilon_2}\right)$, the demand functions for bidders in group 2 would be completely inelastic $\left(\lim_{n_1\to\infty} c_2 = 0\right)$. This explains why the inequality $n_2 > \bar{n}_2\left(\rho, \widehat{\sigma}^2_{\varepsilon_1}, \widehat{\sigma}^2_{\varepsilon_2}\right)$ is required for the existence of equilibrium. Neither group 1 nor group 2 has flat *aggregate* demand in the limit, and each group has some price impact. We see that an agent in the large group just absorbs the inelastic per capita supply, behaving like a "Cournot quantity setter", and keeping some price impact ($\lim_{n_1\to\infty} d_1 > 0$), while bidders in the small group command relatively more market power ($\lim_{n_1\to\infty} d_2 > \lim_{n_1\to\infty} d_1$). It is worth to remark that the large group retains price impact in the limit only if this group learns from the price (incomplete information and correlation of values, $\rho\sigma^2_{\varepsilon_1} > 0$). In this case the aggregate demand of group 1 does not become flat, $\lim_{n_1\to\infty} n_1 c_1 < \infty$. If $\rho\sigma^2_{\varepsilon_1} = 0$, $\lim_{n_1\to\infty} n_1 c_1 = \infty$ and there is no price impact in the limit: $\lim_{n_1\to\infty} d_i = 0$, $i = 1, 2$. It is easy to see also that, in the limit, the price depends only on the valuations and price impact of agents in the competitive fringe: $\lim_{n_1\to\infty} p = \mathbb{E}\left[\theta_1|s_1, s_2\right] - \left(\lim_{n_1\to\infty} d_1 + \lambda_1\right) q$.[26]

If the small group is fully informed ($\sigma^2_{\varepsilon_2} = 0$) and the large group is entirely uninformed ($\sigma^2_{\varepsilon_1} \to \infty$), then: $\bar{n}_2 = 2\rho$; an equilibrium always exists for $n_2 > 2$; and the equilibrium coefficients for group 2 are $\lim_{n_1\to\infty} b_2 = 0$, and $\lim_{n_1\to\infty} a_2 = \lim_{n_1\to\infty} c_2 = (n_2 - 2\rho) / ((n_2 - \rho)\lambda_2)$. In this case, the groups' relative price impact is given by $\lim_{n_1\to\infty} (d_2/d_1) = 1 + \rho/(n_2 - \rho)$. As ρ increases the relative price impact of group 2 increases also. This accords with the general theme that increased correlation in the presence of asymmetric information raises price impact.

4.2 A large price-taking market

Consider now the following setup. There is a continuum of bidders along the interval $[0, 1]$, and we let q denote the aggregate (average) quantity supplied in the market. Suppose that a fraction μ_i ($0 < \mu_i < 1$) of these bidders are traders of type i, $i = 1, 2$. Then the following proposition characterizes the equilibrium of this continuum economy and shows that it is the limit of a finite economy's equilibrium.

Proposition 6. *Let $Q = (n_1 + n_2)q$. Suppose that n_1 and n_2 both approach to infinity and that $n_i/(n_1 + n_2)$ converges to μ_i ($0 < \mu_i < 1$) for $i = 1, 2$. Then, the equilibrium coefficients converge to the equilibrium coefficients of the equilibrium of the continuum economy setup, which are given by*

$$\begin{aligned} b_i &= \frac{\widehat{\sigma}^2_{\varepsilon_i}\left(\mu_j\left(\overline{\theta}_i - \rho\overline{\theta}_j\right) + \rho\lambda_j q\right)}{\mu_i\rho\widehat{\sigma}^2_{\varepsilon_i}\lambda_j + \mu_j\left(1 - \rho^2 + \widehat{\sigma}^2_{\varepsilon_i}\right)\lambda_i}, \; a_i = \frac{\mu_j\left(1 - \rho^2\right)}{\mu_i\rho\widehat{\sigma}^2_{\varepsilon_i}\lambda_j + \mu_j\left(1 - \rho^2 + \widehat{\sigma}^2_{\varepsilon_i}\right)\lambda_i}, \; \textit{and} \\ c_i &= \frac{\mu_j(1 - \rho)\left(1 + \rho + \widehat{\sigma}^2_{\varepsilon_i}\right)}{\mu_i\rho\widehat{\sigma}^2_{\varepsilon_i}\lambda_j + \mu_j\left(1 - \rho^2 + \widehat{\sigma}^2_{\varepsilon_i}\right)\lambda_i}, \; \textit{where } i, j = 1, 2, \; j \neq i. \end{aligned}$$

[26] The limit expected quantity of a bidder of group 2 is given by $\lim_{n_1\to\infty} \mathbb{E}\left[x_2(t)\right] = \left(\overline{\theta}_2 - \overline{\theta}_1 + \left(\lim_{n_1\to\infty} d_1 + \lambda_1\right) q\right) / \left(\lim_{n_1\to\infty} d_2 + \lambda_2\right)$.

5 Welfare analysis

This section focuses on the welfare loss at the equilibrium. We characterize the equilibrium and efficient allocations in Subsection 5.1 and analyze deadweight losses in Subsection 5.2.

5.1 Characterizing the equilibrium and efficient allocations

Recall that $t_i = \mathbb{E}[\theta_i | s_1, s_2]$, $i = 1, 2$, that is, the predicted values with full information (s_1, s_2) and $t = (t_1, t_2)$. The strategies in the equilibrium induce outcomes as functions of the realized vector of predicted values t and are given in Equation (4). One can easily show that the equilibrium outcome solves the following *distorted* benefit maximization program:[27]

$$\max_{x_1, x_2} \mathbb{E}\left[n_1\left(\theta_1 x_1 - (d_1 + \lambda_1) x_1^2/2\right) + n_2\left(\theta_2 x_2 - (d_2 + \lambda_2) x_2^2/2\right) \middle| t\right]$$

$$s.t.\ n_1 x_1 + n_2 x_2 = Q,$$

where d_1 and d_2 are the equilibrium parameters. The efficient allocation would obtain if we set $d_1 = d_2 = 0$, which corresponds to a price-taking equilibrium (denoted by superscript o). The equilibrium strategy of a type-i bidder ($i = 1, 2$) will be of the form $X_i^o(s_i, p) = b_i^o + a_i^o s_1 - c_i^o p$, $i = 1, 2$, and is derived by maximizing the following program:

$$\max_{x_i} \left(\mathbb{E}[\theta_i | s_i, p] - p\right) x_i - \lambda_i x_i^2/2,$$

while taking prices as given. The FOC of this optimization problem yields

$$\mathbb{E}[\theta_i | s_i, p] - p - \lambda_i x_i = 0.$$

After identifying coefficients and solving the corresponding system of equations, we find that there exists a unique equilibrium in this setup. The equilibrium coefficients coincide with those in Proposition 6 for the continuum market.

Proposition 7. *Let $Q = (n_1 + n_2)q$ and let $\mu_i = n_i/(n_1 + n_2)$ for $i = 1, 2$. Then there exists a unique price-taking equilibrium, and the equilibrium coefficients coincide with the equilibrium coefficients of the continuum setup (whose expressions are given in the statement of Proposition 6).*

Our next corollary provides some comparative statics results.

Corollary 4. *Let $\rho\sigma_{\varepsilon_1}^2\sigma_{\varepsilon_2}^2 > 0$. Then the only equilibrium coefficients affected by Q, $\overline{\theta}_i$, and $\overline{\theta}_j$ are the intercepts of the demand functions (with b_i^o increasing in $\overline{\theta}_i$ and Q and decreasing in*

[27] See Lemma A3 in Appendix A.

$\overline{\theta}_j)$ *for* $i,j=1,2$ *and* $i\neq j$*. Furthermore, the demands of group* i *are less sensitive to private signals and prices (lower* a_i *and* c_i*) in response to an increase in* λ_i*,* λ_j*,* ρ*,* $\sigma^2_{\varepsilon_i}$*, and* μ_i*, and to a decrease in* μ_j*; however, group* i*'s demands are not affected by* $\sigma^2_{\varepsilon_j}$*.*

Observe that, under competitive behavior, we can derive an additional comparative statics result: the relationship between the equilibrium coefficients and the proportion of individuals in group 1. In particular, increasing the proportion μ_1 of type-1 traders leads, for those traders, to an increased information-sensitivity weight of the price (higher $\Psi_1\left(n_1c_1^o+n_2c_2^o\right)\left(n_2a_2^o\right)^{-1}$) and so a lower overall response to the price ($c_1^o=\lambda_1^{-1}\left(1-\Psi_1\left(n_1c_1^o+n_2c_2^o\right)\left(n_2a_2^o\right)^{-1}\right)$); the opposite holds for type-2 traders.

Thus the auction outcome can be obtained as the solution to a maximization problem with a more concave objective function than the expected total surplus, which suggests that inefficiency may be eliminated by quadratic subsidies ($\kappa_i x_i^2/2$, $i=1,2$) that compensate for the distortions. The per capita subsidy rate (κ_i) to a trader of type i must be such that it compensates for the distortion $d_i\left(\kappa_i\right)$ while accounting for the subsidy. Since the aim is to induce competitive behavior, the trader should be led to respond with c_i^o to the price. This means that the exact amount of κ_i must be $d_i(c_1^o,c_2^o)$, since that would be the distortion arising when traders use the competitive linear strategies. The following proposition shows that, if subsidies are selected properly, then bidders behave competitively and so the equilibrium allocation is efficient.

Proposition 8. *Let* $i=1,2$ *and* $i\neq j$*. Then the efficient allocation is induced by the quadratic subsidies* $\kappa_i x_i^2/2$*, where* $\kappa_i=d_i(c_i^o,c_j^o)=1/\left(\left(n_i-1\right)c_i^o+n_jc_j^o\right)$*. If* $\rho\sigma^2_{\varepsilon_i}\sigma^2_{\varepsilon_j}>0$*, then the per capita subsidy rates (*κ_i*,* $i=1,2$*) increase with* ρ*,* $\widehat{\sigma}^2_{\varepsilon_1}$*,* $\widehat{\sigma}^2_{\varepsilon_2}$*,* λ_1*, and* λ_2 *but decrease with* n_1 *and* n_2*. We have that* $\kappa_1<\kappa_2$ *iff* $c_1^o<c_2^o$*.*

Combining Propositions 7 and 8 now yields closed-form expressions for the optimal subsidy rates:

$$\kappa_i=\frac{1}{n_j\left(1-\rho\right)}\left(\frac{\left(n_i-1\right)\left(1+\widehat{\sigma}^2_{\varepsilon_i}+\rho\right)}{n_i\lambda_j\rho\widehat{\sigma}^2_{\varepsilon_i}+n_j\lambda_i\left(1-\rho^2+\widehat{\sigma}^2_{\varepsilon_i}\right)}+\frac{n_i\left(1+\widehat{\sigma}^2_{\varepsilon_j}+\rho\right)}{n_i\lambda_j\left(1-\rho^2+\widehat{\sigma}^2_{\varepsilon_j}\right)+n_j\lambda_i\rho\widehat{\sigma}^2_{\varepsilon_j}}\right)^{-1},$$

$i=1,2$, $i\neq j$. If $\rho=0$ (or, with full information, if $\sigma^2_{\varepsilon_i}=0$, $i=1,2$), then $\kappa_i^f=1/\left(\left(n_i-1\right)\lambda_i^{-1}+n_j\lambda_j^{-1}\right)$, $i=1,2$. Proposition 8 implies that the optimal subsidy rates with incomplete information and learning from prices are higher than with full information: $\kappa_i>\kappa_i^f$ if (a) $\rho>0$ and (b) at least one of $\widehat{\sigma}^2_{\varepsilon_1}$ or $\widehat{\sigma}^2_{\varepsilon_2}$ is strictly positive. If $\sigma^2_{\varepsilon_1}=\infty$ then κ_i, $i=1,2$, is independent of $\sigma^2_{\varepsilon_2}$.

Remark 7. With symmetric groups we obtain symmetric optimal subsidies which change as described in Proposition 8 with changes in $\lambda_i=\lambda_j$, $\sigma^2_{\varepsilon_i}=\sigma^2_{\varepsilon_j}$, and $n_i=n_j$.

Note that the optimal subsidy rates are decreasing in the number of traders because, when there are many agents, competitive behavior is already being approached in the market without subsidies. Moreover, $\text{sgn}\{\kappa_1 - \kappa_2\} = \text{sgn}\{c_1^o - c_2^o\}$. Hence $\kappa_1 < \kappa_2$ iff $c_1^o < c_2^o$. The implication is that the bidders who require a higher per capita subsidy rate are the ones whose demands are more sensitive to price. Corollary 3 allows us to conclude that, if there is a group with more precise private information, with lower transaction costs, and that is less numerous, then it is the group meriting a higher per capita subsidy rate. The reason is that the stronger group's strategic behavior is more pronounced and so it must receive more compensation in order to become competitive.

The expected (total) optimal subsidy for group i is $\kappa_i \mathbb{E}\left[(x_i^o(t))^2\right]/2$.[28] If $\overline{\theta}_2 \geq \overline{\theta}_1$, then $\mathbb{E}\left[(x_2^o(t))^2\right] > \mathbb{E}\left[(x_1^o(t))^2\right]$, from which it follows that the bidders from the stronger group (group 2) should receive the higher expected subsidy. However, if $\overline{\theta}_1 > \overline{\theta}_2$, then there are parameter configurations under which bidders from the weaker group (group 1) should receive the higher expected subsidy, even though $\kappa_1 < \kappa_2$. These conclusions would have to be revised if redistributive considerations come into play.[29]

Our result has policy implications. It implies, for example, that a central bank seeking an efficient distribution of liquidity among banks should relax collateral requirements (i.e., provide a larger subsidy) to the strong group. This prescription sounds counterintuitive because the efficiency motive may conflict with the central bank's function as lender of last resort, which often involves shoring up weak banks (e.g., the European Central Bank relaxing the collateral requirements for Greek banks to avoid a meltdown of that country's banking system). Another example is that of a wholesale electricity market characterized by a small (oligopolistic) group and a fringe; in this case, a regulator looking to improve productive efficiency should set a higher subsidy rate for the oligopolistic group. This could be accomplished by offering differential subsidies to renewable energy technologies, for instance, that lower the marginal cost of production.

It is worth noting that primary dealers in the US Treasury are required to bid at least the pro-rate share of those dealers present in the auction ("demonstrate substantial presence") and in exchange enjoy privileges such as exclusive intermediation of OMO, and in the crisis period access to the QE auction mechanism as well as to the Primary Dealer Credit Facility. This may be interpreted as a subsidy that lowers the effective transaction cost of the dealers since they

[28] We have that $x_i^o(t) = (n_j(t_i - t_j) + \lambda_j Q) / (n_i \lambda_j + n_j \lambda_i)$.

[29] Athey et al. (2013) find with regard to US Forest Service timber auctions that restricting entry increases small business participation but substantially reduces efficiency and revenue. In contrast, subsidizing small bidders directly increases revenue and the profits of small bidders without much cost in efficiency. See also Loertscher and Marx (2016) and Pai and Vohra (2012).

have the obligation to bid a minimum amount.

5.2 Deadweight loss

The expected deadweight loss, $\mathbb{E}[DWL]$, at an anonymous allocation $(x_1(t), x_2(t))$ is the difference between expected total surplus at the efficient allocation, ETS^o, and at the baseline allocation, denoted simply by ETS. Lemma A4 in Appendix A shows that

$$\mathbb{E}[DWL] = \frac{1}{2}\lambda_1 n_1 \mathbb{E}\left[(x_1(t) - x_1^o(t))^2\right] + \frac{1}{2}\lambda_2 n_2 \mathbb{E}\left[(x_2(t) - x_2^o(t))^2\right],$$

where $(x_1^o(t), x_2^o(t))$ corresponds to the price-taking equilibrium. We can use the equilibrium expressions for $(x_1(t), x_2(t))$ and $(x_1^o(t), x_2^o(t))$ to show that

$$\mathbb{E}[DWL] = \phi \mathbb{E}\left[((n_2 d_1 + n_1 d_2)(t_1 - t_2) + (\lambda_2 d_1 - \lambda_1 d_2) Q)^2\right],$$

where $\phi = n_1 n_2 / (2(n_2\lambda_1 + n_1\lambda_2) u^2)$.

In a double auction ($Q = 0$) or when $\lambda_2 d_1 = \lambda_1 d_2$,

$$\mathbb{E}[DWL] = \phi (n_2 d_1 + n_1 d_2)^2 \mathbb{E}(t_1 - t_2)^2$$

and is due to differential expected valuations. It is the product of two factors, $\phi (n_2 d_1 + n_1 d_2)^2$ which increases in d_1 and d_2, and

$$\mathbb{E}\left[(t_1 - t_2)^2\right] = \left(\overline{\theta}_1 - \overline{\theta}_2\right)^2 + (1-\rho)^2 \sigma_\theta^2 \frac{2(1+\rho) + \widehat{\sigma}_{\varepsilon_1}^2 + \widehat{\sigma}_{\varepsilon_2}^2}{\left(1 + \widehat{\sigma}_{\varepsilon_1}^2\right)\left(1 + \widehat{\sigma}_{\varepsilon_2}^2\right) - \rho^2}, \tag{7}$$

which decreases in ρ and in $\widehat{\sigma}_{\varepsilon_i}^2$, vanishes when ρ approaches 1 or when there is no uncertainty ($\sigma_\theta^2 = 0$) provided $\overline{\theta}_1 = \overline{\theta}_2$, and increases in $\left(\overline{\theta}_1 - \overline{\theta}_2\right)^2$. It is worth noting that both ETS^o and ETS increase in $\mathbb{E}\left[(t_1 - t_2)^2\right]$, which is associated to gains from trade by the two groups. However, the weight of $\mathbb{E}\left[(t_1 - t_2)^2\right]$ in ETS^o is larger than in ETS and therefore $\mathbb{E}[DWL]$ increases in $\mathbb{E}\left[(t_1 - t_2)^2\right]$.

Since d_1 and d_2 increase in $\rho, \sigma_{\varepsilon_i}^2$ we have that $\phi (n_2 d_1 + n_1 d_2)^2$ increases in ρ and $\sigma_{\varepsilon_i}^2$. It follows that $\mathbb{E}[DWL]$ may increase or decrease in ρ and $\widehat{\sigma}_{\varepsilon_i}^2$. However, if $\sigma_{\varepsilon_i}^2 = \infty$, then d_i is independent of $\sigma_{\varepsilon_j}^2$, $i = 1, 2$, $j \neq i$, and $\mathbb{E}[DWL]$ decreases with $\sigma_{\varepsilon_j}^2$.

The term $(\lambda_2 d_1 - \lambda_1 d_2) Q$ derives from the absorption of Q by the traders, and is positive whenever $d_1/d_2 \neq \lambda_1/\lambda_2$.When $d_1/d_2 = \lambda_1/\lambda_2$, the expected deadweight loss only derives from valuation differences. That is because, in this case, the non-informational trading term corresponding to the equilibrium with imperfect competition coincides with the one corresponding to the competitive equilibrium. Note that if we interpret the traders as producers competing to supply a fixed demand Q, then the condition $d_1/d_2 = \lambda_1/\lambda_2$ means that the ratio of the production of the two types of firms is aligned with the ratio of the slopes of their respective marginal costs. This condition guarantees productive efficiency provided that $\overline{\theta}_1 = \overline{\theta}_2$ and $\rho = 1$ and, since demand is fixed, this coincides with overall efficiency.

The impact of a small amount of asymmetry may be large. Suppose, for example, that the initial situation is symmetric for the groups and that σ_θ^2 is low. Then the $\mathbb{E}[DWL]$ is close to zero since we have that $d_1/d_2 = \lambda_1/\lambda_2$. However, if λ_1 is lowered somewhat then we know from Proposition 3 (iii) that d_1/d_2 increases and therefore $d_1/d_2 > \lambda_1/\lambda_2$, in which case $\mathbb{E}[DWL]$ may be quite large if Q is large. This is consistent with the results in Hortaçsu et al. (2016) who document a significant amount of efficiency losses due to heterogeneity at long maturities in US Treasury auctions.

Furthermore, if group 1 has higher transaction costs ($\lambda_1 > \lambda_2$), is more numerous ($n_1 > n_2$), and is less informed ($\sigma_{\varepsilon_1}^2 > \sigma_{\varepsilon_2}^2$) than group 2, then $d_1/d_2 < \lambda_1/\lambda_2$ (and therefore $\lambda_2 d_1 - \lambda_1 d_2 < 0$). Then the expected deadweight loss increases with Q and $\left|\overline{\theta}_1 - \overline{\theta}_2\right|$ when the stronger group values the asset no less than does the weaker group ($\overline{\theta}_1 \leq \overline{\theta}_2$). This is so since the second term in the quadratic expansion of $\mathbb{E}[DWL]$ is an interaction term that is positive for $Q > 0$ iff $(\lambda_2 d_1 - \lambda_1 d_2)\left(\overline{\theta}_1 - \overline{\theta}_2\right) > 0$, that is, when the relative distortion between groups (d_i/d_j) is large whenever $\overline{\theta}_i > \overline{\theta}_j$. For example, primary dealers in a Treasury auction may value more the bonds than other direct bidders because they have more clout in reselling them. In Hortaçsu et al. (2016) it is found that the willingness to pay of primary dealers is no lower than those of other direct bidders (as well as of indirect bidders). However, group strength and preference strength need not be aligned always. This is the case for example when a weaker group of banks is hit by a crisis, then its valuation for liquidity may increase more than for a strong group of banks.

Under full information (i.e., $\sigma_{\varepsilon_1}^2 = \sigma_{\varepsilon_2}^2 = 0$), both d_1 and d_2 are independent of ρ; in this case, then $\mathbb{E}[DWL]$ decreases with ρ. Similarly, if $\rho = 0$, then d_1 and d_2 are independent of $\sigma_{\varepsilon_1}^2$ and $\sigma_{\varepsilon_2}^2$, from which it follows that $\mathbb{E}[DWL]$ decreases with $\sigma_{\varepsilon_1}^2$ and $\sigma_{\varepsilon_2}^2$. Some of these results are summarized in our last proposition.

Proposition 9.

(i) *The expected deadweight loss may be increasing or decreasing in the information parameters (ρ, $\widehat{\sigma}_{\varepsilon_1}^2$, and $\widehat{\sigma}_{\varepsilon_2}^2$) and, therefore, price impact (d_1, d_2) and the $\mathbb{E}[DWL]$ may be negatively associated.*

(ii) *The $\mathbb{E}[DWL]$ increases with payoff asymmetry and with Q whenever group strength and preference strength are aligned (i.e., when the stronger group 2, with $\lambda_1 > \lambda_2$, $n_1 > n_2$, and $\sigma_{\varepsilon_1}^2 > \sigma_{\varepsilon_2}^2$, values the asset no less than does the weaker group, $\overline{\theta}_1 \leq \overline{\theta}_2$).*

(iii) *When groups are symmetric, the expected deadweight loss is independent of Q*, and price impact d and the $\mathbb{E}[DWL|t]$ *are positively associated, given predicted values* t, *for changes in information parameters. This need not be the case with asymmetric groups* (*e.g., for* Q *large,* $d_i/d_j > \lambda_i/\lambda_j$ *implies that* $\mathbb{E}[DWL|t]$ *increases in* d_i *and decreases in* d_j).

Market integration Our analysis can shed light also on the effects of integrating separated markets.[30] Suppose that groups 1 and 2 operate in separate markets (auctions), that is, in market i all the buyers (n_i) are of type i and supply is $n_i Q/(n_1+n_2)$. In this framework, given that all the individuals are identical in market i, the market clearing condition implies that the equilibrium quantities are given by $Q/(n_1+n_2)$. Let $(1-\rho)\,\sigma_\theta^2 > 0$, then market integration (a unified auction):

1. Increases ETS if bidders behave competitively (strictly except if $\overline{\theta}_1 = \overline{\theta}_2$, $\sigma_{\varepsilon_1}^2 = \sigma_{\varepsilon_2}^2 = \infty$, and $\lambda_1 = \lambda_2$). In the latter case payoffs are symmetric among bidders of the two groups and there is no information on values. Therefore, there are no gains from trade among the groups.

2. Increases ETS, because of gains from trade, provided $\sigma_{\varepsilon_i}^2 < \infty$ for some i, if $\overline{\theta}_1 \neq \overline{\theta}_2$ and $Q = 0$, or if groups are symmetric.

3. May decrease ETS, if Q is large and the groups asymmetric. In this case gains from trade of integration are overpowered by the inefficiency generated by group asymmetries and price impact. This may happen with no asymmetric information but asymmetric payoffs. For example, if $\overline{\theta}_1 = \overline{\theta}_2$, $\sigma_{\varepsilon_1}^2 = \sigma_{\varepsilon_2}^2 = \infty$, $\lambda_1 > \lambda_2$, and $d_1 + \lambda_1 < d_2 + \lambda_2$. In this case group 2 (with lower transaction costs) gets less in the integrated market because of higher price impact (see (4)) and diminishes total surplus. Note that this could not happen if traders behaved competitively. With asymmetric precisions of information ($\sigma_{\varepsilon_1}^2 \neq \sigma_{\varepsilon_2}^2$) and incomplete information ($\rho > 0$) but symmetric payoffs ($\overline{\theta}_1 = \overline{\theta}_2$, $\lambda_1 = \lambda_2$) integration may be also welfare decreasing. Note that this would not happen with complete information, $\rho = 0$.

6 Concluding remarks

We analyze a divisible good uniform-price auction, where two types of bidders compete. Each of these two groups contains a finite number of identical bidders. At the unique equilibrium, a group's relative price impact increases with the precision of private information and decreases

[30] See Appendix B for a more developed analysis and proof of the results.

with the group's transaction costs. Consistently with the empirical evidence for liquidity auctions (Cassola et al. 2013), we find that an increase in the transaction costs of a group of bidders induces a strategic response from the other group, whose members then submit steeper schedules. The effect is reinforced with the impact of changes in the information parameters in a crisis situation. The group that is stronger (because it has more precise private information, faces lower transaction costs, and is more oligopsonistic) has more price impact and must therefore receive a higher subsidy to behave competitively. This result is consistent with the evidence of US Treasury auctions (Hortaçsu et al. 2016) where primary dealers (strong group) exercise market power and earn significant surplus on top of having privileges in exchange for bidding minimum amounts in the auctions. The expected deadweight loss increases with the quantity auctioned and with the degree of payoff asymmetry provided the stronger group values the asset no less than does the weaker group. A small amount of asymmetry may generate large deadweight losses. The link of heterogeneity and efficiency losses is corroborated empirically for Treasury auctions by Hortaçsu et al. (2016).

Our findings have policy implications. Consider a regulator who wants to reduce inefficiency in an industry with two groups of firms (e.g., a small oligopolistic group and a competitive fringe). This regulator must bear in mind that any intervention directed toward one group will also affect the other's behavior. In addition, the regulator should set a higher subsidy rate for the group that has better information, is more oligopsonistic, and has lower transaction costs. The framework developed here can be adapted to study competition policy analyzing the effects of merger and industry capacity redistribution.

Appendix A

Proposition 1 follows from Lemmata A1 and A2.

Lemma A1. *Let $\rho < 1$. In equilibrium, the demand function for a trader of type i, $i = 1, 2$, is given by $X_i(s_i, p) = (\mathbb{E}[\theta_i | s_i, p] - p) / (d_i + \lambda_i)$, with $d_i + \lambda_i > 0$. The equilibrium coefficients satisfy the following system of equations:*

$$b_i = \left((1 - \Xi_i) \overline{\theta}_i - \Psi_i \overline{\theta}_j - \frac{\Psi_i (n_i b_i + n_j b_j - Q)}{n_j a_j} \right) \Big/ (d_i + \lambda_i), \tag{8}$$

$$a_i = \left(\Xi_i - \frac{n_i a_i}{n_j a_j} \Psi_i \right) \Big/ (d_i + \lambda_i), \text{ and} \tag{9}$$

$$c_i = \left(1 - \frac{\Psi_i (n_i c_i + n_j c_j)}{n_j a_j} \right) \Big/ (d_i + \lambda_i), \tag{10}$$

where $i, j = 1, 2$, $j \neq i$. Moreover, in equilibrium, $a_i > 0$, $i = 1, 2$.

Proof: Consider a trader of type i. Recall that at the beginning of Subsection 3.1 we obtain $X_i(s_i,p) = (\mathbb{E}[\theta_i|s_i,p]-p)/(d_i+\lambda_i)$ and $\mathbb{E}[\theta_i|s_i,p] = \mathbb{E}[\theta_i|s_i,s_j]$. Since we are looking for strategies of the form $X_i(s_i,p) = b_i + a_i s_i - c_i p$, from the market clearing condition we have that $s_j = ((n_i c_i + n_j c_j)p + Q - n_i(b_i + a_i s_i) - n_j b_j)/(n_j a_j)$. Thus, from (3), it follows that

$$\mathbb{E}[\theta_i|s_i,s_j] = (1-\Xi_i)\bar{\theta}_i - \Psi_i\bar{\theta}_j + \Psi_i\left(\frac{Q-n_ib_i-n_jb_j}{n_ja_j}\right) + \left(\Xi_i - \frac{n_ia_i}{n_ja_j}\Psi_i\right)s_i + \Psi_i\left(\frac{n_ic_i+n_jc_j}{n_ja_j}\right)p.$$

Substituting the foregoing expression in (1), and then identifying coefficients, we obtain the expressions for the demand coefficients given in (8)-(10).

Finally, we show the positiveness of the coefficients a_i, $i=1,2$. From (9), we get $a_i = \Xi_i/(d_i+\lambda_i+n_i\Psi_i/(n_ja_j))$, $i,j=1,2$, $j\neq i$. Combining the previous expressions, we have

$$a_i = \frac{n_j(\Xi_i\Xi_j - \Psi_i\Psi_j)}{n_i\Psi_i(d_j+\lambda_j)+\Xi_jn_j(d_i+\lambda_i)},\ i,j=1,2,\ j\neq i. \tag{11}$$

Direct computation yields $\Xi_i\Xi_j - \Psi_i\Psi_j = (1-\rho^2)\big/\left(\left(1+\widehat{\sigma}^2_{\varepsilon_1}\right)\left(1+\widehat{\sigma}^2_{\varepsilon_2}\right)-\rho^2\right) > 0$, whenever $\rho < 1$. Moreover, using the positiveness of $d_i+\lambda_i$, Ξ_i, and Ψ_i, $i=1,2$, we conclude that, in equilibrium, the coefficients a_i, $i=1,2$, are strictly positive. ■

Lemma A2. *In equilibrium,*

$$\begin{aligned}
b_i &= \frac{\Psi_i}{n_in_j}\frac{n_i\Xi_j\frac{a_i}{a_j}-n_j\Psi_j}{\Xi_i\Xi_j-\Psi_i\Psi_j}Q + a_i\left(\frac{\Xi_j\bar{\theta}_i-\Psi_i\bar{\theta}_j}{\Xi_i\Xi_j-\Psi_i\Psi_j}-\bar{\theta}_i\right), &(12)\\
a_i &= \Delta_ic_i, &(13)\\
c_1 &= \left(\Xi_1\Delta_1^{-1} - \frac{n_1}{n_2}\left(1-\Xi_1\Delta_1^{-1}\right)z - \frac{z}{(n_1-1)z+n_2}\right)\Big/\lambda_1,\ \textit{and} &(14)\\
c_2 &= \left(\Xi_2\Delta_2^{-1} - \frac{n_2}{n_1}\left(1-\Xi_2\Delta_2^{-1}\right)\frac{1}{z} - \frac{1}{n_1z+n_2-1}\right)\Big/\lambda_2, &(15)
\end{aligned}$$

where $\Delta_i = 1\big/\left(1+(1+\rho)^{-1}\widehat{\sigma}^2_{\varepsilon_i}\right)$, $i,j=1,2$, $j\neq i$. *Moreover,* $z\equiv c_1/c_2$ *is the unique positive solution to the cubic polynomial* $g(z) = g_3z^3+g_2z^2+g_1z+g_0$, *with*

$$\begin{aligned}
g_3 &= n_1^2(n_1-1)\left(n_2\Xi_2\Delta_2^{-1}\lambda_1 + n_1\left(1-\Xi_1\Delta_1^{-1}\right)\lambda_2\right),\\
g_2 &= n_1\left((3n_2n_1-n_1-2n_2+1)\left(n_2\Xi_2\Delta_2^{-1}\lambda_1 - n_1\Xi_1\Delta_1^{-1}\lambda_2\right)+\right.\\
&\quad \left.+\lambda_2n_1(2n_2n_1-n_1+1)-(n_1-1)(n_2+1)n_2\lambda_1\right),\\
g_1 &= n_2\left((3n_2n_1-2n_1-n_2+1)\left(n_2\Xi_2\Delta_2^{-1}\lambda_1 - n_1\Xi_1\Delta_1^{-1}\lambda_2\right)+\right.\\
&\quad \left.+\lambda_2n_1(n_2-1)(n_1+1)-(2n_2n_1-n_2+1)n_2\lambda_1\right),\ \textit{and}\\
g_0 &= -n_2^2(n_2-1)\left(n_2\left(1-\Xi_2\Delta_2^{-1}\right)\lambda_1 + n_1\Xi_1\Delta_1^{-1}\lambda_2\right).
\end{aligned}$$

Proof: In relation to the expression for b_i, notice that (9) implies

$$d_i+\lambda_i = \left(\Xi_i - \frac{n_ia_i}{n_ja_j}\Psi_i\right)\Big/a_i,\ i,j=1,2,\ j\neq i. \tag{16}$$

Substituting these expressions in (8), it follows that

$$b_i = a_i \frac{\left(1-\Xi_i\right)\overline{\theta}_i - \Psi_i \overline{\theta}_j - \frac{\Psi_i\left(n_i b_i + n_j b_j - Q\right)}{n_j a_j}}{\Xi_i - \frac{n_i a_i}{n_j a_j}\Psi_i},\ i,j = 1,2,\ j \neq i. \tag{17}$$

Thus,

$$n_i b_i + n_j b_j = n_i a_i \frac{\left(1-\Xi_i\right)\overline{\theta}_i - \Psi_i \overline{\theta}_j - \frac{\Psi_i\left(n_i b_i + n_j b_j - Q\right)}{n_j a_j}}{\Xi_i - \frac{n_i a_i}{n_j a_j}\Psi_i} + n_j a_j \frac{\left(1-\Xi_j\right)\overline{\theta}_j - \Psi_j \overline{\theta}_i - \frac{\Psi_j\left(n_i b_i + n_j b_j - Q\right)}{n_i a_i}}{\Xi_j - \frac{n_j a_j}{n_i a_i}\Psi_j},$$

Isolating $n_i b_i + n_j b_j$ in the previous formula and substituting the resulting expression in (17), Expression (12) is obtained.

Concerning the expression for a_i, substituting (16) in (10), it follows that

$$c_i = a_i \left(1 - \frac{\Psi_i\left(n_i c_i + n_j c_j\right)}{n_j a_j}\right) \bigg/ \left(\Xi_i - \frac{n_i a_i}{n_j a_j}\Psi_i\right),\ i,j = 1,2,\ j \neq i. \tag{18}$$

Hence, $n_i c_i + n_j c_j = n_i a_i \frac{n_j a_j - \Psi_i(n_i c_i + n_j c_j)}{n_j a_j \Xi_i - n_i a_i \Psi_i} + n_j a_j \frac{n_i a_i - \Psi_j(n_i c_i + n_j c_j)}{n_i a_i \Xi_j - n_j a_j \Psi_j}$. Isolating $n_i c_i + n_j c_j$ in the previous formula and substituting the resulting expression in (18), we obtain a formula which is equivalent to (13). Using (13) in (10), we get the expression for c_i given in the statement of Proposition 1.

In relation to c_1 and c_2, using the expression for d_i and (13), (16) implies that

$$\lambda_i = \left(\frac{\Xi_i}{\Delta_i} - \frac{n_i \Psi_i c_i}{n_j \Delta_j c_j}\right) c_i^{-1} - \left(\left(n_i - 1\right) c_i + n_j c_j\right)^{-1},\ i,j = 1,2,\ j \neq i.$$

or, since

$$\Psi_i \Delta_j^{-1} = 1 - \Xi_i \Delta_i^{-1}, \tag{19}$$

$\lambda_i = \left(\Xi_i \Delta_i^{-1} - \frac{n_i}{n_j}\left(1 - \Xi_i \Delta_i^{-1}\right)\frac{c_i}{c_j}\right) c_i^{-1} - \left(\left(n_i - 1\right) c_i + n_j c_j\right)^{-1}$, $i,j = 1,2,\ j \neq i$, which imply (14) and (15) since $z = c_1/c_2$. Moreover, dividing the previous two equalities, it follows that

$$\frac{\lambda_1}{\lambda_2} = \frac{\Xi_1 \Delta_1^{-1} - \frac{n_1}{n_2}\left(1 - \Xi_1 \Delta_1^{-1}\right) z - z\left(\left(n_1 - 1\right) z + n_2\right)^{-1}}{\Xi_2 \Delta_2^{-1} z - \frac{n_2}{n_1}\left(1 - \Xi_2 \Delta_2^{-1}\right) - z\left(n_1 z + n_2 - 1\right)^{-1}}. \tag{20}$$

After some algebra, (20) is equivalent to $g(z) = 0$, where $g(z) = g_3 z^3 + g_2 z^2 + g_1 z + g_0$, as stated in this lemma. Notice that $g(0) < 0$ and $\lim\limits_{z \to \infty} g(z) = \infty$. Consequently, there exists $z \in (0, \infty)$ such that $g(z) = 0$. Furthermore, we have that $g_2/n_1 > g_1/n_2$. Combining this inequality with the fact that $g_3 > 0$ and $g_0 < 0$ allows us to conclude that at least there is only one sign change of the coefficients of $g(z)$. To show that, we distinguish 3 cases:

(1) $0 \geq \frac{g_2}{n_1} > \frac{g_1}{n_2}$. This implies that $0 \geq g_2$ and $0 > g_1$. As $g_3 > 0$ and $g_0 < 0$, we conclude that there is only one sign change of the coefficients of $g(z)$.

(2) $\frac{g_2}{n_1} > 0 \geq \frac{g_1}{n_2}$. This implies that $g_2 > 0 \geq g_1$. As $g_3 > 0$ and $g_0 < 0$, we conclude that there is only one sign change of the coefficients of $g(z)$.

(3) $\frac{g_2}{n_1} > \frac{g_1}{n_2} > 0$. This implies that $g_2 > 0$ and $g_1 > 0$. As $g_3 > 0$ and $g_0 < 0$, we conclude that there is only one sign change of the coefficients of $g(z)$.

This property implies that there exists only one sign change in the coefficients of $g(z)$. Applying the Descartes' rule, we conclude that there exists a unique positive root of $g(z)$. ■

Proposition 2A. *Let* $\rho < 1$.

(a) *There exists an equilibrium if and only if* $c_i > 0$, $i = 1, 2$, *where*

$$c_1 = \frac{q_N(z)}{((n_1 - 1) z + n_2) n_2 \lambda_1} \text{ and } c_2 = \frac{q_D(z)}{(n_1 z + n_2 - 1) n_1 z \lambda_2}, \tag{21}$$

where $z = c_1/c_2$ *and the expressions of* $q_N(Z)$ *and* $q_D(Z)$ *are given by*

$$\begin{aligned} q_N(Z) &= n_2^2 \Xi_1 \Delta_1^{-1} + n_2 \left(\Xi_1 \Delta_1^{-1} (2n_1 - 1) - (n_1 + 1)\right) Z - (n_1 - 1)\left(1 - \Xi_1 \Delta_1^{-1}\right) n_1 Z^2 \text{ and} \\ q_D(Z) &= -n_2 (n_2 - 1)\left(1 - \Xi_2 \Delta_2^{-1}\right) + n_1 \left(\Xi_2 \Delta_2^{-1} (2n_2 - 1) - (n_2 + 1)\right) Z + n_1^2 \Xi_2 \Delta_2^{-1} Z^2. \end{aligned}$$

(b) *Complete information. When* $\rho = 0$, *or* $\sigma^2_{\varepsilon_i} = 0$, *or* $\sigma^2_{\varepsilon_i} = \infty$, $i = 1, 2$, *equilibrium exists iff* $n_1 + n_2 \geq 3$.

(c) *Incomplete information.*

(c.1) *Let* $\rho \sigma^2_{\varepsilon_1} \sigma^2_{\varepsilon_2} > 0$. *Then* $c_i > 0$ *(*$i = 1, 2$*) if and only if* $\overline{z}_N > \overline{z}_D$, *where* $\overline{z}_N$ *and* $\overline{z}_D$ *denote the highest root of* $q_N(Z)$ *and* $q_D(Z)$, *respectively.*

(c.2) *Let* $\rho \sigma^2_{\varepsilon_i} > 0$ *and* $\sigma^2_{\varepsilon_j} = 0$, $i \neq j$. *Then* $c_i > 0$ $(i = 1, 2)$ *if* $n_j \geq 2$, *or if* $n_j = 1$, n_i *large enough and* ρ *low enough.*

Remark 8. For an equilibrium to exist we must have $c_i > 0$ $(i = 1, 2)$ and these inequalities hold if and only if $\overline{z}_D < z < \overline{z}_N$. If $n_1 = 1$ and $n_2 = 1$, then $\overline{z}_N = 1/\left(2\Delta_1 \Xi_1^{-1} - 1\right)$ and $\overline{z}_D = 2\Delta_2 \Xi_2^{-1} - 1$. Since $\Delta_i \Xi_i^{-1} \geq 1$, $i = 1, 2$, and $\Delta_1 \Xi_1^{-1} = \Delta_2 \Xi_2^{-1} = 1$ do not hold, we can use direct computation to obtain $\overline{z}_N < \overline{z}_D$. Applying Proposition 2A, we conclude that no equilibrium exists in this case. Therefore, $n_1 + n_2 \geq 3$ is a necessary condition for the existence of an equilibrium.

Remark 9. In c.1, we obtain that $\lim_{\lambda_1 \to 0} z = \overline{z}_N$ and $\lim_{\lambda_2 \to 0} z = \overline{z}_D$.

Remark 10. In c.2, when $\sigma^2_{\varepsilon_2} = 0$, $\overline{z}_D = 1/n_1$ if $n_2 = 1$, whereas $\overline{z}_D = 0$ if $n_2 \geq 2$.

Proof: **(a)** (Necessity). From Proposition 1 we know that $a_i > 0$, $i = 1, 2$, whenever $\rho < 1$. Combining this property with expressions given in (13), we have that, in equilibrium, the coefficients c_i, $i = 1, 2$, are strictly positive. Moreover, (14) and (15) can be rewritten as the expressions given in (21).

(Sufficiency). Suppose that the candidates equilibrium coefficients c_1 and c_2 are positive and satisfy (21). Then, the ratio $z = c_1/c_2 > 0$ and satisfies (20). Then, we conclude that an equilibrium exists and it is unique since we know that (20) has a unique positive solution. Finally, substituting this value of z in the expressions stated in Lemma A2, we obtain the equilibrium coefficients of the demand functions.

(b) When $\rho = 0$ or $\sigma^2_{\varepsilon_i} = 0$, $i = 1, 2$, the demand functions are given by

$$X_i(s_i, p) = (\mathbb{E}[\theta_i|s_i] - p) / (d_i + \lambda_i), \ i = 1, 2,$$

while when $\sigma^2_{\varepsilon_i} = \infty$, $i = 1, 2$, the demand functions hold

$$X_i(s_i, p) = (\overline{\theta}_i - p) / (d_i + \lambda_i), \ i = 1, 2.$$

Moreover recall that the SOCs imply $d_i + \lambda_i > 0$. Moreover, in all these cases we can express the coefficients of the demand functions in terms of d_i, $i = 1, 2$. In particular, $c_i = 1/(d_i + \lambda_i) > 0$, $i = 1, 2$. Given our expression for d_i, we characterize d_i, $i = 1, 2$, as the positive solutions of the following system of equations:

$$d_i = \left(\frac{n_i - 1}{d_i + \lambda_i} + \frac{n_j}{d_j + \lambda_j}\right)^{-1} \text{ for } i, j = 1, 2 \text{ and } j \neq i.$$

After some algebra, we conclude that this system has positive solutions if and only if $n_1 + n_2 \geq 3$.

(c.1) (Necessity). Let $\overline{z}_N$ and $\overline{z}_D$ denote the highest root of $q_N(Z)$ and $q_D(Z)$, respectively. Notice that the positiveness of c_i, $i = 1, 2$, is equivalent to $\overline{z}_N > z > \overline{z}_D$. Therefore, $\overline{z}_N > \overline{z}_D$.

(Sufficiency). Suppose that $\overline{z}_N > \overline{z}_D$. Recall that Lemma A2 shows that there exists a unique positive value of z that solves (20), which can be rewritten as

$$\frac{\lambda_1}{\lambda_2} = \frac{n_1 (n_2 - 1 + n_1 z) q_N(z)}{(n_2 + (n_1 - 1) z) n_2 q_D(z)}. \tag{22}$$

This implies that $\overline{z}_N > z > \overline{z}_D$. Notice that these inequalities guarantee the positiveness of c_i, $i = 1, 2$.

(c.2) Suppose that $\rho\sigma^2_{\varepsilon_1} > 0$ and $\sigma^2_{\varepsilon_2} = 0$. In this case $\Xi_2\Delta_2^{-1} = 1$ and, hence, $q_D(Z) = Zn_1(n_2 + Zn_1 - 2)$. On the one hand, if $n_2 = 1$, then $\overline{z}_D = 1/n_1$. As in c.1) the condition that guarantees the existence of equilibrium is $\overline{z}_N > \overline{z}_D$, which is equivalent to $n_1\left(2\Xi_1\Delta_1^{-1} - 1\right) > \Xi_1\Delta_1^{-1}$, i.e., $\Xi_1\Delta_1^{-1} > 1/2$ and $n_1 > \Xi_1\Delta_1^{-1}/\left(2\Xi_1\Delta_1^{-1} - 1\right)$ or, using the expressions of Ξ_1 and Δ_1, $1 - \rho^2 + (1 - 2\rho)\widehat{\sigma}^2_{\varepsilon_1} > 0$ and $n_1 > 1 + \widehat{\sigma}^2_{\varepsilon_1}\rho/(1 - \rho^2 + (1 - 2\rho)\widehat{\sigma}^2_{\varepsilon_1})$, which implies when ρ is low enough and n_1 is large enough.

On the other hand, if $n_2 \geq 2$, $q_D(Z) > 0$ for all $Z > 0$ and, therefore, we have that $c_2 > 0$ is satisfied. The positiveness of c_1 requires that $\overline{z}_N > z$. But, this inequality holds since z solves

Equation (22). To sum up, when $\sigma_{\varepsilon_2}^2 = 0$, an equilibrium exists if $n_2 = 1$, n_1 large enough and ρ low enough, or if $n_2 \geq 2$.

Now, suppose that $\rho\sigma_{\varepsilon_2}^2 > 0$ and $\sigma_{\varepsilon_1}^2 = 0$. In this case $\Xi_1\Delta_1^{-1} = 1$ and, hence, $q_N(Z) = n_2^2 + n_2(n_1 - 2)Z$. On the one hand, if $n_1 = 1$, then $\overline{z}_N = n_2$. As in c.1) the condition that guarantees the existence of equilibrium is $\overline{z}_N > \overline{z}_D$, which is equivalent to $n_2\left(2\Xi_2\Delta_2^{-1} - 1\right) > \Xi_2\Delta_2^{-1}$, i.e., $\Xi_2\Delta_2^{-1} > 1/2$ and $n_2 > \Xi_2\Delta_2^{-1}/\left(2\Xi_2\Delta_2^{-1} - 1\right)$ or, using the expressions of Ξ_2 and Δ_2, $1 - \rho^2 + (1 - 2\rho)\widehat{\sigma}_{\varepsilon_2}^2 > 0$ and $n_2 > 1 + \widehat{\sigma}_{\varepsilon_2}^2\rho/(1 - \rho^2 + (1 - 2\rho)\widehat{\sigma}_{\varepsilon_2}^2)$, which implies when ρ is low enough and n_2 is large enough.

On the other hand, if $n_1 \geq 2$, $q_N(Z) > 0$ for all $Z > 0$ and, therefore, we have that $c_1 > 0$ is satisfied. The positiveness of c_2 requires that $z > \overline{z}_D$. But, this inequality holds since the equilibrium value, z, solves Equation (22). To sum up, when $\sigma_{\varepsilon_1}^2 = 0$, an equilibrium exists if $n_1 = 1$, n_2 large enough and ρ low enough, or if $n_1 \geq 2$. ■

Lemma 2A. *The condition $\overline{z}_N > \overline{z}_D$ given in the statement of Proposition 2A is satisfied in the following cases:*

(i) if $\rho < 1$ and n_1, n_2 are large enough;

(ii) given n_i, n_j is large enough and ρ low enough, for $i, j = 1, 2$ and $j \neq i$.

Proof: We distinguish two cases: $n_1 > 1$ and $n_1 = 1$.

Case 1: $n_1 > 1$. In this case

$$\overline{z}_N = \frac{n_2\left(\left(n_1 - 1\right)\left(2\Xi_1\Delta_1^{-1} - 1\right) - \left(2 - \Xi_1\Delta_1^{-1}\right) + \sqrt{\left(2 - \Xi_1\Delta_1^{-1}\right)^2 + \left(n_1 - 1\right)\left(n_1 + 3 - 6\Xi_1\Delta_1^{-1}\right)}\right)}{2n_1\left(n_1 - 1\right)\left(1 - \Xi_1\Delta_1^{-1}\right)} \tag{23}$$

and

$$\overline{z}_D = \frac{n_2 + 1 - \Xi_2\Delta_2^{-1}\left(2n_2 - 1\right) + \sqrt{\left(2 - \Xi_2\Delta_2^{-1}\right)^2 + \left(n_2 - 1\right)\left(n_2 + 3 - 6\Xi_2\Delta_2^{-1}\right)}}{2\Xi_2\Delta_2^{-1}n_1}. \tag{24}$$

Proposition 2A indicates that an equilibrium exists if and only if $\overline{z}_N > \overline{z}_D$, or equivalently, $n_1\overline{z}_N/n_2 > n_1\overline{z}_D/n_2$. Using the expressions of $\overline{z}_N$ and $\overline{z}_D$, we have that $n_1\overline{z}_N/n_2$ is increasing in n_1 and $n_1\overline{z}_D/n_2$ is decreasing in n_2. Taking limits, it follows that

$$\lim_{n_1\to\infty} n_1\overline{z}_N/n_2 = \Xi_1\Delta_1^{-1}/\left(1 - \Xi_1\Delta_1^{-1}\right) \text{ and } \lim_{n_2\to\infty} n_1\overline{z}_D/n_2 = \left(1 - \Xi_2\Delta_2^{-1}\right)/\left(\Xi_2\Delta_2^{-1}\right).$$

Moreover, using the expressions of Ξ_i and Δ_i, $i = 1, 2$, we have that

$$\frac{\Xi_1\Delta_1^{-1}}{1 - \Xi_1\Delta_1^{-1}} - \frac{1 - \Xi_2\Delta_2^{-1}}{\Xi_2\Delta_2^{-1}} = \frac{\left(1 - \rho^2\right)\left(1 + \rho + \widehat{\sigma}_{\varepsilon_1}^2\right)\left(\left(1 + \widehat{\sigma}_{\varepsilon_1}^2\right)\left(1 + \widehat{\sigma}_{\varepsilon_2}^2\right) - \rho^2\right)}{\rho\widehat{\sigma}_{\varepsilon_1}^2\left(1 + \rho + \widehat{\sigma}_{\varepsilon_2}^2\right)\left(1 - \rho^2 + \widehat{\sigma}_{\varepsilon_1}^2\right)} > 0.$$

Hence, we get that, as $\rho < 1$, $\lim_{n_1 \to \infty} n_1 \overline{z}_N / n_2 > \lim_{n_2 \to \infty} n_1 \overline{z}_D / n_2$. This implies that whenever $\rho < 1$ and n_1 and n_2 large enough, the existence of the equilibrium is guaranteed.

Consider now a fixed positive integer n_1, such that $n_1 > 1$. Using the fact that $\overline{z}_N$ is the positive root of $q_N(Z)$, it follow that $n_1 \overline{z}_N / n_2 > \Xi_1 \Delta_1^{-1} / \left(2 - \Xi_1 \Delta_1^{-1}\right)$. Moreover,

$$\Xi_1 \Delta_1^{-1} / \left(2 - \Xi_1 \Delta_1^{-1}\right) > \left(1 - \Xi_2 \Delta_2^{-1}\right) / \left(\Xi_2 \Delta_2^{-1}\right), \tag{25}$$

whenever ρ is low enough. Therefore,

$$n_1 \overline{z}_N / n_2 > \Xi_1 \Delta_1^{-1} / \left(2 - \Xi_1 \Delta_1^{-1}\right) > \left(1 - \Xi_2 \Delta_2^{-1}\right) / \left(\Xi_2 \Delta_2^{-1}\right) = \lim_{n_2 \to \infty} n_1 \overline{z}_D / n_2.$$

Hence, we conclude that if n_2 is large enough, as $n_1 \overline{z}_D / n_2$ is decreasing in n_2, the previous inequalities imply that $n_1 \overline{z}_N / n_2 > n_1 \overline{z}_D / n_2$ or, equivalently, $\overline{z}_N > \overline{z}_D$. Applying Proposition 2A, it follows that in this case there exists an equilibrium provided that n_2 is high enough and ρ low enough.

Consider now a fixed positive integer n_2, such that $n_2 \geq 1$, and assume again that $\rho < 1$. Using the fact that $\overline{z}_D$ is the positive root of $q_D(Z)$, it follow that $n_1 \overline{z}_D / n_2 \leq \left(2 - \Xi_2 \Delta_2^{-1}\right) / \left(\Xi_2 \Delta_2^{-1}\right)$. In addition, when ρ is low enough, then we have that

$$\left(2 - \Xi_2 \Delta_2^{-1}\right) / \left(\Xi_2 \Delta_2^{-1}\right) < \Xi_1 \Delta_1^{-1} / \left(1 - \Xi_1 \Delta_1^{-1}\right) = \lim_{n_1 \to \infty} n_1 \overline{z}_N / n_2.$$

Thus, we have that $n_1 \overline{z}_D / n_2 < \lim_{n_1 \to \infty} n_1 \overline{z}_N / n_2$. Using the fact that $n_1 \overline{z}_N / n_2$ increases with n_1, we have that, when n_1 is high enough, $n_1 \overline{z}_D / n_2 < n_1 \overline{z}_N / n_2$ or, equivalently, $\overline{z}_D < \overline{z}_N$, which guarantees the existence of equilibrium. To sum up, we have that given n_2, there exists an equilibrium provided that n_1 is high enough and ρ low enough.

Case 2: $n_1 = 1$. In this case, we have that $\overline{z}_N = n_2 \Xi_1 \Delta_1^{-1} / \left(2 - \Xi_1 \Delta_1^{-1}\right)$ and

$$\overline{z}_D = \frac{n_2 + 1 - \Xi_2 \Delta_2^{-1}(2n_2 - 1) + \sqrt{\left(2 - \Xi_2 \Delta_2^{-1}\right)^2 + (n_2 - 1)\left(n_2 + 3 - 6\Xi_2 \Delta_2^{-1}\right)}}{2\Xi_2 \Delta_2^{-1}}.$$

Furthermore, whenever ρ is low enough, (25) holds. Therefore, it follows that

$$\overline{z}_N / n_2 = \Xi_1 \Delta_1^{-1} / \left(2 - \Xi_1 \Delta_1^{-1}\right) > \left(1 - \Xi_2 \Delta_2^{-1}\right) / \left(\Xi_2 \Delta_2^{-1}\right) = \lim_{n_2 \to \infty} \overline{z}_D / n_2.$$

Using the fact that $\overline{z}_D / n_2$ decreases with n_2, the previous inequality implies that $\overline{z}_N / n_2 > \overline{z}_D / n_2$ whenever n_2 is high enough, i.e., $\overline{z}_N > \overline{z}_D$, which guarantees the existence of equilibrium. To sum up, we have that when $n_1 = 1$, there exists an equilibrium provided that n_2 is high enough and ρ low enough. ■

Proof of Proposition 2: This proposition directly follows from Proposition 2A and Lemma 2A. ■

Remark (symmetric groups). Let $n_i = n$, $\lambda_i = \lambda$, and $\sigma^2_{\varepsilon_i} = \sigma^2_\varepsilon$, $i = 1, 2$. Here $z = 1$ in equilibrium. From Proposition 2A we know that, if an equilibrium exists, then the value of z is in the interval $(\overline{z}_D, \overline{z}_N)$. It follows that $\overline{z}_N > 1 > \overline{z}_D$ or, equivalently, that $q_N(1) > 0$ and $q_D(1) > 0$. After performing some algebra, we find that the foregoing inequalities are satisfied iff $n > 1 + \rho\widehat{\sigma}^2_\varepsilon / \left((1-\rho)\left(1+\rho+\widehat{\sigma}^2_\varepsilon\right)\right)$, where $\widehat{\sigma}^2_\varepsilon = \sigma^2_\varepsilon/\sigma^2_\theta$.

Proof of Proposition 3: In what follows we prove the following comparative statics results for $i, j = 1, 2$, $i \neq j$:

(a) $\partial b_i/\partial\overline{\theta}_i > 0$, $\partial a_i/\partial\overline{\theta}_i = 0$, and $\partial c_i/\partial\overline{\theta}_i = 0$,

(b) $\partial b_i/\partial\overline{\theta}_j < 0$, $\partial a_i/\partial\overline{\theta}_j = 0$, and $\partial c_i/\partial\overline{\theta}_j = 0$,

(c) $\partial b_i/\partial Q > 0$,$\partial a_i/\partial Q = 0$, and $\partial c_i/\partial Q = 0$,

(d) $\partial a_i/\partial\lambda_i < 0$ and $\partial c_i/\partial\lambda_i < 0$,

(e) $\partial a_i/\partial\lambda_j < 0$ and $\partial c_i/\partial\lambda_j < 0$,

(f) $\partial a_i/\partial\rho < 0$ and $\partial c_i/\partial\rho < 0$,

(g) $\partial\left(d_i/d_j\right)/\partial\sigma^2_{\varepsilon_i} < 0$, $\partial\left(d_i/d_j\right)/\partial\sigma^2_{\varepsilon_j} > 0$, $\partial\left(d_i/d_j\right)/\partial\lambda_i < 0$, and $\partial\left(d_i/d_j\right)/\partial\lambda_j > 0$,

(h) $\partial a_i/\partial\sigma^2_{\varepsilon_i} < 0$ and $\partial c_i/\partial\sigma^2_{\varepsilon_i} < 0$,

(i) $\partial a_i/\partial\sigma^2_{\varepsilon_j} < 0$ and $\partial c_i/\partial\sigma^2_{\varepsilon_j} < 0$, and

(j) $\partial d_i/\partial n_i < 0$ and $\partial d_j/\partial n_i < 0$.

From Lemma A1, we know that the equilibrium coefficients that depend on $\overline{\theta}_i$, $\overline{\theta}_j$ and Q are b_1 and b_2. Using Lemma A2 and after some algebra, the results given in (a), (b) and (c) are obtained. In what follows, without any loss of generality, let $i = 1$. First, we prove that $\partial z/\partial\lambda_1 < 0$. From Lemma A2, we know that z is the unique positive solution of the following equation:

$$\frac{\lambda_1}{\lambda_2} - \frac{N(z)}{D(z)} = 0, \tag{26}$$

where

$$\begin{aligned} N(z) &= \Xi_1\Delta_1^{-1} - n_1\left(1 - \Xi_1\Delta_1^{-1}\right)z/n_2 - z\left((n_1-1)z + n_2\right)^{-1} \text{ and} \\ D(z) &= \Xi_2\Delta_2^{-1}z - n_2\left(1 - \Xi_2\Delta_2^{-1}\right)/n_1 - z\left(n_1 z + n_2 - 1\right)^{-1}, \end{aligned}$$

with $\Xi_i\Delta_i^{-1} = \left(1 - \rho^2 + \widehat{\sigma}^2_{\varepsilon_j}\right)\left(1 + \rho + \widehat{\sigma}^2_{\varepsilon_i}\right)\left(\left(\left(1+\widehat{\sigma}^2_{\varepsilon_i}\right)\left(1+\widehat{\sigma}^2_{\varepsilon_j}\right) - \rho^2\right)(1+\rho)\right)^{-1}$, $i, j = 1, 2, i \neq j$. Applying the Implicit Function Theorem,

$$\frac{\partial z}{\partial\lambda_i} = -\frac{\partial\left(\lambda_1/\lambda_2 - N(z)/D(z)\right)/\partial\lambda_i}{\partial\left(\lambda_1/\lambda_2 - N(z)/D(z)\right)/\partial z}, \; i = 1, 2.$$

As $\partial\left(\lambda_1/\lambda_2 - N(z)/D(z)\right)/\partial\lambda_1 > 0$, $\partial\left(\lambda_1/\lambda_2 - N(z)/D(z)\right)/\partial\lambda_2 < 0$, and

$$\frac{\partial\left(\lambda_1/\lambda_2 - N(z)/D(z)\right)}{\partial z} > 0, \tag{27}$$

because of $z \in (\overline{z}_D, \overline{z}_N)$, we conclude that $\partial z/\partial\lambda_1 < 0$ and $\partial z/\partial\lambda_2 > 0$.

Next, we study the relationship between $c's$ and λ_1. Differentiating (15), we have

$$\frac{\partial c_2}{\partial\lambda_1} = \frac{\partial c_2}{\partial z}\frac{\partial z}{\partial\lambda_1} = \frac{1}{\lambda_2}\left(\frac{n_2\left(1-\Xi_2\Delta_2^{-1}\right)}{n_1 z^2} + \frac{n_1}{\left(n_1 z + n_2 - 1\right)^2}\right)\frac{\partial z}{\partial\lambda_1} < 0,$$

since $\partial z/\partial\lambda_1 < 0$. Moreover, as $c_1 = zc_2$, it follows that $\partial c_1/\partial\lambda_1 = \left(\partial z/\partial\lambda_1\right)c_2 + z\left(\partial c_2/\partial\lambda_1\right) < 0$, because of the positiveness of c_2 and z, and the negativeness of $\partial z/\partial\lambda_1$ and $\partial c_2/\partial\lambda_1$. In relation to a_1 and a_2, from (13), direct computation yields $\partial a_1/\partial\lambda_1 < 0$ and $\partial a_2/\partial\lambda_1 < 0$, since $\partial c_1/\partial\lambda_1 < 0$ and $\partial c_2/\partial\lambda_1 < 0$.

Now, we study how the correlation coefficient ρ affects a_1. Let $y = a_1/a_2$. As $a_1 = \Delta_1 c_1$ and $a_2 = \Delta_2 c_2$, then $z = \Delta_2 y/\Delta_1$. Substituting this expression in (20), and after some algebra, we have that

$$\frac{\lambda_1}{\lambda_2}y = \frac{\widetilde{N}(y,\rho)}{\widetilde{D}(y,\rho)}, \tag{28}$$

where $\widetilde{N}(y,\rho) = \frac{1-\rho^2+\widehat{\sigma}^2_{\varepsilon_2}-\frac{n_1}{n_2}\widehat{\sigma}^2_{\varepsilon_1}\rho y}{\left(1+\widehat{\sigma}^2_{\varepsilon_1}\right)\left(1+\widehat{\sigma}^2_{\varepsilon_2}\right)-\rho^2} - \left(\left(n_1-1\right)\frac{1+\rho+\widehat{\sigma}^2_{\varepsilon_1}}{1+\rho} + n_2\frac{1+\rho+\widehat{\sigma}^2_{\varepsilon_2}}{1+\rho}\frac{1}{y}\right)^{-1}$ and

$$\widetilde{D}(y,\rho) = \frac{1-\rho^2+\widehat{\sigma}^2_{\varepsilon_1}-\frac{n_2}{n_1}\widehat{\sigma}^2_{\varepsilon_2}\rho\frac{1}{y}}{\left(1+\widehat{\sigma}^2_{\varepsilon_1}\right)\left(1+\widehat{\sigma}^2_{\varepsilon_2}\right)-\rho^2} - \left(n_1\frac{1+\rho+\widehat{\sigma}^2_{\varepsilon_1}}{1+\rho}y + \left(n_2-1\right)\frac{1+\rho+\widehat{\sigma}^2_{\varepsilon_2}}{1+\rho}\right)^{-1}.$$

Moreover, $a_1 = \widetilde{N}(y,\rho)/\lambda_1$ and $a_2 = \widetilde{D}(y,\rho)/\lambda_2$. Hence,

$$\frac{\partial a_1}{\partial\rho} = \frac{\left(\partial\widetilde{N}(y,\rho)/\partial y\right)\left(\partial y/\partial\rho\right) + \partial\widetilde{N}(y,\rho)/\partial\rho}{\lambda_1}.$$

Thus, in order to show $\partial a_1/\partial\rho < 0$, it suffices to prove that

$$\frac{\partial\widetilde{N}(y,\rho)}{\partial y}\frac{\partial y}{\partial\rho} + \frac{\partial\widetilde{N}(y,\rho)}{\partial\rho} < 0. \tag{29}$$

Direct computation yields $\partial\widetilde{N}(y,\rho)/\partial y < 0$. Then, (29) is equivalent to

$$\frac{\partial y}{\partial\rho} > -\frac{\partial\widetilde{N}(y,\rho)/\partial\rho}{\partial\widetilde{N}(y,\rho)/\partial y}. \tag{30}$$

Moreover, recall that y in equilibrium is the unique positive value that satisfies (28). Thus, applying the Implicit Function Theorem, it follows that

$$\frac{\partial y}{\partial \rho} = -\frac{\partial\left(\lambda_1 y/\lambda_2 - \widetilde{N}(y,\rho)/\widetilde{D}(y,\rho)\right)\Big/\partial\rho}{\partial\left(\lambda_1 y/\lambda_2 - \widetilde{N}(y,\rho)/\widetilde{D}(y,\rho)\right)\Big/\partial y}.$$

Then, (30) can be rewritten as

$$-\frac{\partial\left(\lambda_1 y/\lambda_2 - \widetilde{N}(y,\rho)/\widetilde{D}(y,\rho)\right)\Big/\partial\rho}{\partial\left(\lambda_1 y/\lambda_2 - \widetilde{N}(y,\rho)/\widetilde{D}(y,\rho)\right)\Big/\partial y} > -\frac{\partial\widetilde{N}(y,\rho)/\partial\rho}{\partial\widetilde{N}(y,\rho)/\partial y},$$

or using the fact that in equilibrium $\partial\left(\lambda_1 y/\lambda_2 - \widetilde{N}(y,\rho)/\widetilde{D}(y,\rho)\right)\Big/\partial y > 0$, (30) is satisfied iff

$$-\frac{\partial\left(\lambda_1 y/\lambda_2 - \widetilde{N}(y,\rho)/\widetilde{D}(y,\rho)\right)}{\partial\rho} > -\left(\frac{\partial\widetilde{N}(y,\rho)/\partial\rho}{\partial\widetilde{N}(y,\rho)/\partial y}\right)\frac{\partial\left(\lambda_1 y/\lambda_2 - \widetilde{N}(y,\rho)/\widetilde{D}(y,\rho)\right)}{\partial y}. \tag{31}$$

Notice that

$$-\frac{\partial\left(\lambda_1 y/\lambda_2 - \widetilde{N}(y,\rho)/\widetilde{D}(y,\rho)\right)}{\partial\rho} = -\frac{\left(\partial\widetilde{N}(y,\rho)/\partial\rho\right)\widetilde{D}(y,\rho) - \widetilde{N}(y,\rho)\left(\partial\widetilde{D}(y,\rho)/\partial\rho\right)}{\widetilde{D}^2(y,\rho)},$$

or using (26),

$$-\frac{\partial\left(\lambda_1 y/\lambda_2 - \widetilde{N}(y,\rho)/\widetilde{D}(y,\rho)\right)}{\partial\rho} = -\frac{\partial\widetilde{N}(y,\rho)/\partial\rho - \lambda_1 y\left(\partial\widetilde{D}(y,\rho)/\partial\rho\right)\Big/\lambda_2}{\widetilde{D}(y,\rho)}.$$

Analogously,

$$\frac{\partial\left(\lambda_1 y/\lambda_2 - \widetilde{N}(y,\rho)/\widetilde{D}(y,\rho)\right)}{\partial y} = \frac{\lambda_1}{\lambda_2} - \frac{\partial\widetilde{N}(y,\rho)/\partial y - \lambda_1 y\left(\partial\widetilde{D}(y,\rho)/\partial y\right)\Big/\lambda_2}{\widetilde{D}(y,\rho)}.$$

Therefore, (31) is equivalent to

$$\begin{aligned}\frac{\partial\widetilde{N}(y,\rho)/\partial\rho - \lambda_1 y\left(\partial\widetilde{D}(y,\rho)/\partial\rho\right)\Big/\lambda_2}{\widetilde{D}(y,\rho)} \quad &> \quad -\frac{\partial\widetilde{N}(y,\rho)/\partial\rho}{\partial\widetilde{N}(y,\rho)/\partial y}\times \\ &\left(\frac{\lambda_1}{\lambda_2} - \frac{\partial\widetilde{N}(y,\rho)/\partial y - \lambda_1 y\left(\partial\widetilde{D}(y,\rho)/\partial y\right)\Big/\lambda_2}{\widetilde{D}(y,\rho)}\right),\end{aligned}$$

or,

$$-\frac{y\left(\partial\widetilde{D}(y,\rho)/\partial\rho\right)}{\widetilde{D}(y,\rho)} > -\frac{\partial\widetilde{N}(y,\rho)/\partial\rho}{\partial\widetilde{N}(y,\rho)/\partial y}\left(1 + \frac{y\left(\partial\widetilde{D}(y,\rho)/\partial y\right)}{\widetilde{D}(y,\rho)}\right). \tag{32}$$

Moreover, recall that $a_2 = \widetilde{D}(y,\rho)/\lambda_2$. The positiveness of a_2 tells us that $\widetilde{D}(y,\rho) > 0$. After some algebra, we have that $\partial\widetilde{D}(y,\rho)/\partial\rho < 0$, $\partial\widetilde{N}(y,\rho)/\partial\rho < 0$ and $\partial\widetilde{D}(y,\rho)/\partial y > 0$. Hence, we conclude that the left-hand side (LHS) of (32) is positive, whereas the right-hand side (RHS) of (32) is negative since $\partial\widetilde{N}(y,\rho)/\partial y < 0$. Consequently, the fact that (32) is satisfied allows us to conclude that $\partial a_1/\partial\rho < 0$.

Concerning the effect of ρ on c_1, recall that $c_1 = a_1/\Delta_1 = \left(1+\rho+\widehat{\sigma}^2_{\varepsilon_1}\right) a_1/\left(1+\rho\right)$. This expression tells us that c_1 is the product of two decreasing positive functions in ρ. Therefore, $\partial c_1/\partial\rho < 0$.

Next, we prove that $\partial\left(d_1/d_2\right)/\partial\sigma^2_{\varepsilon_1} < 0$ and $\partial\left(d_1/d_2\right)/\partial\sigma^2_{\varepsilon_2} > 0$. From the expressions of d_1, d_2, and z, it follows that $d_1/d_2 = \left(n_1 z + (n_2-1)\right)/\left((n_1-1)z+n_2\right)$. Applying the chain rule, we get $\partial\left(d_1/d_2\right)/\partial\sigma^2_{\varepsilon_i} = \left(\partial\left(d_1/d_2\right)/\partial z\right)\left(\partial z/\partial\sigma^2_{\varepsilon_i}\right)$. As $\partial\left(d_1/d_2\right)/\partial z > 0$, we know that the sign of $\partial\left(d_1/d_2\right)/\partial\sigma^2_{\varepsilon_i}$ is the same as the sign of $\partial z/\partial\sigma^2_{\varepsilon_i}$. Applying the Implicit Function Theorem,

$$\frac{\partial z}{\partial\sigma^2_{\varepsilon_i}} = -\frac{\partial\left(\lambda_1/\lambda_2 - N(z)/D(z)\right)/\partial\sigma^2_{\varepsilon_i}}{\partial\left(\lambda_1/\lambda_2 - N(z)/D(z)\right)/\partial z}.$$

Using $\partial\left(\lambda_1/\lambda_2 - N(z)/D(z)\right)/\partial\sigma^2_{\varepsilon_1} > 0$, $\partial\left(\lambda_1/\lambda_2 - N(z)/D(z)\right)/\partial\sigma^2_{\varepsilon_2} < 0$, and the inequality given in (27), we obtain $\partial z/\partial\sigma^2_{\varepsilon_1} < 0$ and $\partial z/\partial\sigma^2_{\varepsilon_2} > 0$, and hence, we conclude that $\partial\left(d_1/d_2\right)/\partial\sigma^2_{\varepsilon_1} < 0$ and $\partial\left(d_1/d_2\right)/\partial\sigma^2_{\varepsilon_2} > 0$. Analogously, the negativeness of $\partial z/\partial\lambda_1$ and the positiveness of $\partial z/\partial\lambda_2$ allows us to conclude that $\partial\left(d_1/d_2\right)/\partial\lambda_1 < 0$ and $\partial\left(d_1/d_2\right)/\partial\lambda_2 > 0$.

Now, we study how a_1 and c_1 vary with a change in $\sigma^2_{\varepsilon_i}$, $i=1,2$. In order to do that first we analyze the effect of $\sigma^2_{\varepsilon_i}$ on d_1 and d_2. From Proposition 1, we know that $d_i = \left(\left(n_i-1\right)c_i + n_j c_j\right)^{-1}$ and $a_i = \Delta_i c_i > 0$, $i=1,2$. Therefore, $d_i = \left(\left(n_i-1\right)\Delta_i^{-1}a_i + n_j\Delta_j^{-1}a_j\right)^{-1}$, $i,j=1,2$, $j\neq i$. Substituting the expressions of (11) and the expression for Δ_i given in Lemma A2, it follows that

$$d_i = \left(\frac{\left(n_i-1\right)n_j}{\Omega_i} + \frac{n_j n_i}{\Omega_j}\right)^{-1},$$

where $\Omega_i = n_j\Upsilon_i\left(d_i+\lambda_i\right) + n_i\left(\Upsilon_i - 1\right)\left(d_j+\lambda_j\right)$ and $\Omega_j = n_i\Upsilon_j\left(d_j+\lambda_j\right) + n_j\left(\Upsilon_j-1\right)\left(d_i+\lambda_i\right)$, with $\Upsilon_i = \Xi_j/\left(\Xi_j - \Psi_i\right) = \left(1-\rho^2+\widehat{\sigma}^2_{\varepsilon_i}\right)/\left(\left(1-\rho\right)\left(1+\rho+\widehat{\sigma}^2_{\varepsilon_i}\right)\right) > 1$, $i,j=1,2$, $j\neq i$. Therefore, we derive the following equations that are satisfied in equilibrium: $F_i\left(\sigma^2_{\varepsilon_1},\sigma^2_{\varepsilon_2},d_1,d_2\right) = 0$, $i=1,2$, where

$$F_i\left(\sigma^2_{\varepsilon_1},\sigma^2_{\varepsilon_2},d_1,d_2\right) = \frac{\left(n_i-1\right)n_j d_i}{\Omega_i} + \frac{n_i n_j d_i}{\Omega_j} - 1,$$

$i,j=1,2$, $j\neq i$. Let $DF_{d_1,d_2}\left(\sigma^2_{\varepsilon_1},\sigma^2_{\varepsilon_2},d_1,d_2\right)$ denote the following matrix:

$$\begin{pmatrix} \partial F_1\left(\sigma^2_{\varepsilon_1},\sigma^2_{\varepsilon_2},d_1,d_2\right)/\partial d_1 & \partial F_1\left(\sigma^2_{\varepsilon_1},\sigma^2_{\varepsilon_2},d_1,d_2\right)/\partial d_2 \\ \partial F_2\left(\sigma^2_{\varepsilon_1},\sigma^2_{\varepsilon_2},d_1,d_2\right)/\partial d_1 & \partial F_2\left(\sigma^2_{\varepsilon_1},\sigma^2_{\varepsilon_2},d_1,d_2\right)/\partial d_2 \end{pmatrix}.$$

After some tedious algebra, it can be shown that the determinant of $DF_{d_1,d_2}\left(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2\right)$ is strictly positive. In particular, it is not null and, therefore, this matrix is invertible. Hence, we can apply the Implicit Function Theorem, we have

$$\begin{pmatrix} \partial d_1/\partial\sigma^2_{\varepsilon_1} & \partial d_1/\partial\sigma^2_{\varepsilon_2} \\ \partial d_2/\partial\sigma^2_{\varepsilon_1} & \partial d_2/\partial\sigma^2_{\varepsilon_1} \end{pmatrix} =$$

$$-\left(DF_{d_1,d_2}\left(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2\right)\right)^{-1} \begin{pmatrix} \partial F_1\left(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2\right)/\partial\sigma^2_{\varepsilon_1} & \partial F_1\left(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2\right)/\partial\sigma^2_{\varepsilon_2} \\ \partial F_2\left(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2\right)/\partial\sigma^2_{\varepsilon_1} & \partial F_2\left(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2\right)/\partial\sigma^2_{\varepsilon_2} \end{pmatrix}. \tag{33}$$

It is easy to see that all the elements of $\left(DF_{d_1,d_2}\left(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2\right)\right)^{-1}$ are positive. Moreover, $\partial F_i\left(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2\right)/\partial\sigma^2_{\varepsilon_i} < 0$ and $\partial F_i\left(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2\right)/\partial\sigma^2_{\varepsilon_j} < 0$, $i, j = 1, 2$, $j \neq i$. Hence, (33) implies that $\partial d_i/\partial\sigma^2_{\varepsilon_i} > 0$ and $\partial d_i/\partial\sigma^2_{\varepsilon_j} > 0$.

Next, we study the comparative statics of c_1 and c_2 with respect to $\sigma^2_{\varepsilon_1}$. Recall that $c_i = n_j/\Omega_i$, $i, j = 1, 2$, $j \neq i$. Using the fact that Υ_1, d_1, and d_2 are increasing in $\sigma^2_{\varepsilon_1}$ and that Υ_2 is independent of $\sigma^2_{\varepsilon_1}$, we have that Ω_1 and Ω_2 are increasing in $\sigma^2_{\varepsilon_1}$, which allows us to conclude that c_1 and c_2 are decreasing in $\sigma^2_{\varepsilon_1}$. Combining these results with the fact that Δ_1 is decreasing in $\sigma^2_{\varepsilon_1}$ and Δ_2 is independent of $\sigma^2_{\varepsilon_1}$, it follows that a_1 and a_2 are decreasing in $\sigma^2_{\varepsilon_1}$, since $a_1 = \Delta_1 c_1$ and $a_2 = \Delta_2 c_2$.

Finally, concerning h), notice that doing a similar reasoning as before we derive the following equations that are satisfied in equilibrium: $F_i\left(n_1, n_2, d_1, d_2\right) = 0$, $i = 1, 2$, where $F_i\left(n_1, n_2, d_1, d_2\right) = F_i\left(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2\right)$, $i, j = 1, 2$, $j \neq i$. Hence,

$$\begin{pmatrix} \partial d_1/\partial n_1 & \partial d_1/\partial n_2 \\ \partial d_2/\partial n_1 & \partial d_2/\partial n_2 \end{pmatrix} =$$

$$-\left(DF_{d_1,d_2}\left(n_1, n_2, d_1, d_2\right)\right)^{-1} \begin{pmatrix} \partial F_1\left(n_1, n_2, d_1, d_2\right)/\partial n_1 & \partial F_1\left(n_1, n_2, d_1, d_2\right)/\partial n_2 \\ \partial F_2\left(n_1, n_2, d_1, d_2\right)/\partial n_1 & \partial F_2\left(n_1, n_2, d_1, d_2\right)/\partial n_2 \end{pmatrix}.$$

Taking into account that all the elements of the previous two matrices are positive, we conclude that $\partial d_i/\partial n_i < 0$ and $\partial d_i/\partial n_j < 0$, $i, j = 1, 2$, $j \neq i$. ■

Proof of Corollary 2(i): Suppose that $\sigma^2_{\varepsilon_1} \geq \sigma^2_{\varepsilon_2}$, $\lambda_1 \geq \lambda_2$, and $n_1 \geq n_2$. Using the expressions of Ξ_i and Δ_i, $i = 1, 2$, it is easy to see that in this case $\Xi_2\Delta_2^{-1} > \Xi_1\Delta_1^{-1}$. Next, we distinguish two cases:

Case 1: $\left(n_1 + n_2 - 2\right) n_1/\left(\left(n_1 + n_2\right)\left(n_1 + n_2 - 1\right)\right) \geq 1 - \Xi_2\Delta_2^{-1}$. Evaluating the polynomial $g(z)$, stated in the proof of Lemma A2, at $z = 1$, we have that in this case $g(1) \geq 0$. This implies that $z \leq 1$, and therefore, $c_1 \leq c_2$. In addition, using the expressions of d_1 and d_2, we get sgn$\{d_1 - d_2\}$ =sgn$\{c_1 - c_2\}$, which implies $d_1 \leq d_2$. Finally, notice that $\Delta_1 \leq \Delta_2$ whenever $\sigma^2_{\varepsilon_1} \geq \sigma^2_{\varepsilon_2}$. Hence, $a_1/a_2 = z\Delta_1/\Delta_2 \leq 1$.

Case 2: $(n_1 + n_2 - 2)\, n_1/\left((n_1 + n_2)(n_1 + n_2 - 1)\right) < 1 - \Xi_2\Delta_2^{-1}$. Notice that

$$\frac{(n_1 + n_2 - 2)\, n_2}{(n_1 + n_2)(n_1 + n_2 - 1)} - \left(1 - \Xi_1\Delta_1^{-1}\right) \leq \frac{(n_1 + n_2 - 2)\, n_1}{(n_1 + n_2)(n_1 + n_2 - 1)} - \left(1 - \Xi_2\Delta_2^{-1}\right),$$

since $\Xi_2\Delta_2^{-1} > \Xi_1\Delta_1^{-1}$ and $n_1 \geq n_2$. Thus, in this case we have that $q_N(1) < 0$ and $q_D(1) < 0$. Taking into account the shape of these polynomials, the previous two inequalities imply that $\overline{z}_D > 1 > \overline{z}_N$. However, Proposition 2A indicates that in this case there is no equilibrium. ■

Corollary 2A(ii). *Suppose that $\sigma^2_{\varepsilon_1} \geq \sigma^2_{\varepsilon_2}$, $\lambda_1 \geq \lambda_2$, and $n_1 \geq n_2$, and suppose that at least one of these inequalities is strict. If*

$$(1-\rho)\, n_1 \left(\frac{n_2\left(1 + \rho + \widehat{\sigma}^2_{\varepsilon_1}\right)}{n_2\left(1 - \rho^2 + \widehat{\sigma}^2_{\varepsilon_1}\right) + n_1\rho\widehat{\sigma}^2_{\varepsilon_1}} + \frac{(n_2 - 1)\left(1 + \rho + \widehat{\sigma}^2_{\varepsilon_2}\right)}{n_1\left(1 - \rho^2 + \widehat{\sigma}^2_{\varepsilon_2}\right) + n_2\rho\widehat{\sigma}^2_{\varepsilon_2}} \right) \leq 1, \tag{34}$$

then $d_1 + \lambda_1 < d_2 + \lambda_2$ always holds. Otherwise, $d_1 + \lambda_1 > d_2 + \lambda_2$ iff λ_1/λ_2 is high enough.

Proof: By virtue of (16), the inequality $d_1 + \lambda_1 > d_2 + \lambda_2$ is equivalent to

$$\left(\Xi_1 - \frac{n_1}{n_2}\frac{a_1}{a_2}\Psi_1 \right) \Big/ a_1 > \left(\Xi_2 - \frac{n_2}{n_1}\frac{a_2}{a_1}\Psi_2 \right) \Big/ a_2.$$

Using (13) and (19), and after some algebra, the previous inequality is equivalent to

$$z < \widehat{z}, \tag{35}$$

where $\widehat{z} = \left(\Xi_1\Delta_1^{-1} + n_2 n_1^{-1}\left(1 - \Xi_2\Delta_2^{-1}\right)\right) / \left(\Xi_2\Delta_2^{-1} + n_1 n_2^{-1}\left(1 - \Xi_1\Delta_1^{-1}\right)\right)$. We distinguish two cases:

Case I: $\frac{\Xi_2\Delta_2^{-1} + \Xi_1\Delta_1^{-1} - 1}{\left(1 - \Xi_1\Delta_1^{-1}\right) n_1 n_2^{-1} + \Xi_2\Delta_2^{-1}} \leq \frac{\Xi_1\Delta_1^{-1} + n_2 n_1^{-1}\left(1 - \Xi_2\Delta_2^{-1}\right)}{n_1\left(\Xi_1\Delta_1^{-1} + n_2 n_1^{-1}\left(1 - \Xi_2\Delta_2^{-1}\right)\right) + (n_2 - 1)\left(\left(1 - \Xi_1\Delta_1^{-1}\right) n_1 n_2^{-1} + \Xi_2\Delta_2^{-1}\right)}$. Using the expressions of Ξ_i and Δ_i, we get that the previous inequality is equivalent to (34). Moreover, after some algebra, we have that $q_D(\widehat{z}) \leq 0$. Consequently, $\widehat{z} \leq \overline{z}_D < z$, which implies that in this case $d_1 + \lambda_1 < d_2 + \lambda_2$ holds. Note that inequality holds for ρ close to 1.

Case II: $\frac{\Xi_2\Delta_2^{-1} + \Xi_1\Delta_1^{-1} - 1}{\left(1 - \Xi_1\Delta_1^{-1}\right) n_1 n_2^{-1} + \Xi_2\Delta_2^{-1}} > \frac{\Xi_1\Delta_1^{-1} + n_2 n_1^{-1}\left(1 - \Xi_2\Delta_2^{-1}\right)}{n_1\left(\Xi_1\Delta_1^{-1} + n_2 n_1^{-1}\left(1 - \Xi_2\Delta_2^{-1}\right)\right) + (n_2 - 1)\left(\left(1 - \Xi_1\Delta_1^{-1}\right) n_1 n_2^{-1} + \Xi_2\Delta_2^{-1}\right)}$. In this case, taking into account that z is the unique positive solution of (20), (35) is equivalent to

$$\frac{\lambda_1}{\lambda_2} > \frac{\frac{\Xi_2\Delta_2^{-1} + \Xi_1\Delta_1^{-1} - 1}{\left(1 - \Xi_1\Delta_1^{-1}\right) n_1 n_2^{-1} + \Xi_2\Delta_2^{-1}} - \frac{\left(\Xi_1\Delta_1^{-1} + n_2 n_1^{-1}\left(1 - \Xi_2\Delta_2^{-1}\right)\right)}{(n_1 - 1)\left(\Xi_1\Delta_1^{-1} + n_2 n_1^{-1}\left(1 - \Xi_2\Delta_2^{-1}\right)\right) + n_2\left(\left(1 - \Xi_1\Delta_1^{-1}\right) n_1 n_2^{-1} + \Xi_2\Delta_2^{-1}\right)}}{\frac{\Xi_2\Delta_2^{-1} + \Xi_1\Delta_1^{-1} - 1}{\left(1 - \Xi_1\Delta_1^{-1}\right) n_1 n_2^{-1} + \Xi_2\Delta_2^{-1}} - \frac{\Xi_1\Delta_1^{-1} + n_2 n_1^{-1}\left(1 - \Xi_2\Delta_2^{-1}\right)}{n_1\left(\Xi_1\Delta_1^{-1} + n_2 n_1^{-1}\left(1 - \Xi_2\Delta_2^{-1}\right)\right) + (n_2 - 1)\left(\left(1 - \Xi_1\Delta_1^{-1}\right) n_1 n_2^{-1} + \Xi_2\Delta_2^{-1}\right)}}. \tag{36}$$

Taking into account that $\sigma_{\varepsilon_1}^2 \geq \sigma_{\varepsilon_2}^2$, $\lambda_1 \geq \lambda_2$, and $n_1 \geq n_2$, and after some algebra, we get that both sides of the inequality are higher than (or equal to) 1. Therefore, if the value of λ_1/λ_2 is high enough, then we obtain $d_1 + \lambda_1 > d_2 + \lambda_2$. Otherwise, the opposite inequality holds. In short, the result indicates that the inequality $d_1+\lambda_1 > d_2+\lambda_2$ holds whenever (a) the differences between groups stem mainly from transaction costs;[31] and (b) λ_1/λ_2 is high enough. Finally, under complete information, the inequality (36) becomes $\frac{\lambda_1}{\lambda_2} > 1$ and, therefore, we have that in this case $d_1 + \lambda_1 > d_2 + \lambda_2$ iff $\lambda_1 > \lambda_2$. ■

Proof of Proposition 4: **(i)** First, suppose that $\overline{\theta}_1 = \overline{\theta}_2$. Then $\mathbb{E}[p] = \overline{\theta}_1 - Q \Big/ \left(\frac{n_1}{d_1+\lambda_1} + \frac{n_2}{d_2+\lambda_2} \right)$. From Proposition 3 we know that d_i and d_j decrease with n_i, and increase with $\sigma_{\varepsilon_i}^2$, λ_i, and ρ. Using these results in the previous expression, we conclude that the expected price is increasing in n_i and decreasing in λ_i, $\sigma_{\varepsilon_i}^2$, and ρ.

Now, suppose that $\overline{\theta}_1 \neq \overline{\theta}_2$. The results we have just derived may not hold if $\left|\overline{\theta}_1 - \overline{\theta}_2\right|$ is large enough. For example, let us focus on the relationship between the expected price and n_1. To study this relationship, we first show that $n_2 (d_1 + \lambda_1) / (n_1 (d_2 + \lambda_2))$ decreases with n_1. Recall that $d_2 = ((n_2 - 1) n_1/\Omega_2 + n_1 n_2/\Omega_1)^{-1}$. Using the expressions of Ω_i, $i = 1, 2$, we have

$$1 = \left(\frac{n_2 - 1}{\Upsilon_2 + (\Upsilon_2 - 1) \frac{n_2(d_1+\lambda_1)}{n_1(d_2+\lambda_2)}} + \frac{n_2}{\Upsilon_1 \frac{n_2(d_1+\lambda_1)}{n_1(d_2+\lambda_2)} + \Upsilon_1 - 1} \right)^{-1} + \frac{\lambda_2}{d_2 + \lambda_2}.$$

The fact that d_2 decreases with n_1 implies that $\lambda_2/(d_2 + \lambda_2)$ increases with n_1. Then, the previous inequality tells us that $\frac{n_2-1}{\Upsilon_2+(\Upsilon_2-1)\frac{n_2(d_1+\lambda_1)}{n_1(d_2+\lambda_2)}} + \frac{n_2}{\Upsilon_1 \frac{n_2(d_1+\lambda_1)}{n_1(d_2+\lambda_2)} + \Upsilon_1 - 1}$ increases with n_1. For this to be possible, $\frac{n_2(d_1+\lambda_1)}{n_1(d_2+\lambda_2)}$ needs to be decreasing in n_1. Given that the expected price satisfies

$$\mathbb{E}[p] = \left(1 + \frac{n_2 (d_1 + \lambda_1)}{n_1 (d_2 + \lambda_2)}\right)^{-1} \overline{\theta}_1 + \left(1 - \left(1 + \frac{n_2 (d_1 + \lambda_1)}{n_1 (d_2 + \lambda_2)}\right)^{-1}\right) \overline{\theta}_2 - \left(\frac{n_1}{d_1 + \lambda_1} + \frac{n_2}{d_2 + \lambda_2}\right)^{-1} Q,$$

we have that the relationship between the expected price and n_1 is ambiguous. For instance, if $\overline{\theta}_2$ is low enough, then the fact that d_1, d_2, and $n_2 (d_1 + \lambda_1) / (n_1 (d_2 + \lambda_2))$ are decreasing in n_1 allows us to conclude that the expected price increases with n_1. However, if $\overline{\theta}_2$ is large and $\overline{\theta}_1$ and Q are low enough, then the expected price decreases with n_1.

(ii) From the expression for the expected revenue it follows that $QE[p]$ increases with $\overline{\theta}_i$, $i = 1, 2$. In addition, direct computation yields that $Q = \left(\frac{n_1}{d_1+\lambda_1} \overline{\theta}_1 + \frac{n_2}{d_2+\lambda_2} \overline{\theta}_2 \right) \Big/ 2$. Using Proposition 3, it follows that this quantity increases with n_i and $\overline{\theta}_i$ and decreases with ρ, λ_i and $\sigma_{\varepsilon_i}^2$, $i = 1, 2$. ■

[31] This claim follows because if $n_1 = n_2$ and $\widehat{\sigma}_{\varepsilon_1}^2 = \widehat{\sigma}_{\varepsilon_2}^2$, then the inequality given in (34) does not hold.

Proof of Corollary 3: Using the expression for the expected price, it follows that

$$\left(\min\left\{\overline{\theta}_1,\overline{\theta}_2\right\}-\frac{Q}{\frac{n_1}{d_1+\lambda_1}+\frac{n_2}{d_2+\lambda_2}}\right)Q\leq\mathbb{E}\left[p\right]Q\leq\left(\max\left\{\overline{\theta}_1,\overline{\theta}_2\right\}-\frac{Q}{\frac{n_1}{d_1+\lambda_1}+\frac{n_2}{d_2+\lambda_2}}\right)Q.$$

Notice both the left-hand side (LHS) and the right-hand side (RHS) of this expression correspond to the expected revenue in an auction where all participants have an expected valuation of $\min\left\{\overline{\theta}_1,\overline{\theta}_2\right\}$ and of $\max\left\{\overline{\theta}_1,\overline{\theta}_2\right\}$, respectively. Using Proposition 4(i), we know that both LHS and RHS increase with n_i but decrease with λ_i and $\sigma^2_{\varepsilon_i}$. Hence, we obtain that $QE\left[p\right]$ is lower than the expected revenue of the symmetric auction in which both groups are ex-ante identical, with large size (each group with $\max\left\{n_1,n_2\right\}$ bidders), with high expected valuation ($\max\left\{\overline{\theta}_1,\overline{\theta}_2\right\}$), low transaction costs ($\min\{\lambda_1,\lambda_2\}$), and precise signals ($\min\{\sigma^2_{\varepsilon_1},\sigma^2_{\varepsilon_2}\}$), and larger than the expected revenue of the symmetric auction in which both groups are ex-ante identical but with the opposite characteristics (i.e., $\min\left\{n_1,n_2\right\}$, $\min\left\{\overline{\theta}_1,\overline{\theta}_2\right\}$, $\max\{\lambda_1,\lambda_2\}$, and $\max\{\sigma^2_{\varepsilon_1},\sigma^2_{\varepsilon_2}\}$). ■

Proof of Proposition 5: Using (23) and (24), it follows that $\lim_{n_1\to\infty}\overline{z}_N=\lim_{n_1\to\infty}\overline{z}_D=0$. Furthermore, after some algebra, we have that the necessary and sufficient condition for the existence of an equilibrium (i.e., $\lim_{n_1\to\infty}\overline{z}_N/\overline{z}_D>1$) is equivalent to $n_2>\bar{n}_2\left(\rho,\widehat{\sigma}^2_{\varepsilon_1},\widehat{\sigma}^2_{\varepsilon_2}\right)$, where

$$\bar{n}_2\left(\rho,\widehat{\sigma}^2_{\varepsilon_1},\widehat{\sigma}^2_{\varepsilon_2}\right)=\frac{\rho\left(\left(2-\rho\right)\widehat{\sigma}^2_{\varepsilon_2}+2\left(1-\rho^2\right)\right)\widehat{\sigma}^2_{\varepsilon_1}}{\left(1-\rho^2\right)\left(\left(1+\widehat{\sigma}^2_{\varepsilon_1}\right)\left(1+\widehat{\sigma}^2_{\varepsilon_2}\right)-\rho^2\right)}.$$

Moreover, taking the limit in (20), it follows that $\lim_{n_1\to\infty}z=0$ and

$$\lim_{n_1\to\infty}n_1z=n_2\Xi_1\Delta_1^{-1}/\left(1-\Xi_1\Delta_1^{-1}\right). \tag{37}$$

Using the expressions included in the statement of Lemma A2, and after some tedious algebra, we get $\lim_{n_1\to\infty}b_1=q$, $\lim_{n_1\to\infty}a_1=0$, $\lim_{n_1\to\infty}c_1=0$, $\lim_{n_1\to\infty}a_2=\Delta_2\lim_{n_1\to\infty}c_2$,

$$\begin{aligned}\lim_{n_1\to\infty}b_2 &= \frac{\widehat{\sigma}^2_{\varepsilon_2}\left(\frac{(n_2-1)\left(1-\rho^2\right)}{\left(1-\rho^2+\widehat{\sigma}^2_{\varepsilon_2}\right)}+\frac{\left(1-\rho^2+\widehat{\sigma}^2_{\varepsilon_1}(1-2\rho)\right)}{\left(1+\widehat{\sigma}^2_{\varepsilon_1}\right)\left(1+\widehat{\sigma}^2_{\varepsilon_2}\right)-\rho^2}\right)\left(\overline{\theta}_2-\rho\overline{\theta}_1+q\rho\lambda_1\right)}{\left(1-\rho\right)\lambda_2\left(n_2\left(1+\rho\right)-\frac{\rho\widehat{\sigma}^2_{\varepsilon_1}\left(1+\rho+\widehat{\sigma}^2_{\varepsilon_2}\right)}{\left(1+\widehat{\sigma}^2_{\varepsilon_1}\right)\left(1+\widehat{\sigma}^2_{\varepsilon_2}\right)-\rho^2}\right)}+\\ &\quad +q\frac{\rho^2\widehat{\sigma}^2_{\varepsilon_2}\widehat{\sigma}^2_{\varepsilon_1}}{n_2\left(1-\rho^2\right)\left(\left(1+\widehat{\sigma}^2_{\varepsilon_1}\right)\left(1+\widehat{\sigma}^2_{\varepsilon_2}\right)-\rho^2\right)},\text{ and}\\ \lim_{n_1\to\infty}c_2 &= \frac{n_2-\bar{n}_2\left(\rho,\widehat{\sigma}^2_{\varepsilon_1},\widehat{\sigma}^2_{\varepsilon_2}\right)}{\lambda_2\frac{1-\rho^2+\widehat{\sigma}^2_{\varepsilon_2}}{1-\rho}\left(\frac{n_2}{\left(1+\rho+\widehat{\sigma}^2_{\varepsilon_2}\right)}-\frac{\rho\widehat{\sigma}^2_{\varepsilon_1}}{(1+\rho)\left(\left(1+\widehat{\sigma}^2_{\varepsilon_1}\right)\left(1+\widehat{\sigma}^2_{\varepsilon_2}\right)-\rho^2\right)}\right)}.\end{aligned} \tag{38}$$

Next, in relation to the expressions for d_1 and d_2, we have that

$$\lim_{n_1\to\infty} d_1 = \lim_{n_1\to\infty} \left((n_1 - 1)\, c_1 + n_2 c_2 \right)^{-1} = \left(\lim_{n_1\to\infty} \left(\frac{(n_1-1)}{n_1} n_1 z + n_2 \right) \lim_{n_1\to\infty} c_2 \right)^{-1} > 0.$$

The fact that $n_1 z$ and c_2 converge to a positive finite number (see (37) and (38)) implies that d_1 does not converge to zero (provided that $\rho\widehat{\sigma}^2_{\varepsilon_1} > 0$; if $\rho\widehat{\sigma}^2_{\varepsilon_1} = 0$, then it is easy to see that $\lim_{n_1\to\infty} n_1 z = \infty$). A similar result is obtained with the limit of d_2. In particular,

$$\lim_{n_1\to\infty} d_2 = \left(\left(\lim_{n_1\to\infty} n_1 z + n_2 - 1 \right) \lim_{n_1\to\infty} c_2 \right)^{-1} > \lim_{n_1\to\infty} d_1 > 0. \blacksquare$$

Proof of Proposition 6: Suppose that n_1 and n_2 go to infinity and that $n_1/(n_1+n_2)$ converges to μ_1. Taking limits in the equation that characterizes z (i.e., Equation (20)) and operating, we have

$$z = \frac{\Xi_1 \Delta_1^{-1} + \mu_2 \mu_1^{-1} \left(1 - \Xi_2 \Delta_2^{-1}\right) \lambda_1 \lambda_2^{-1}}{\mu_1 \mu_2^{-1} \left(1 - \Xi_1 \Delta_1^{-1}\right) + \Xi_2 \Delta_2^{-1} \lambda_1 \lambda_2^{-1}}. \tag{39}$$

Moreover, taking the limit in the expressions of the equilibrium coefficients given in Lemma A2, it follows that

$$\begin{aligned} b_i &= a_i \left(\frac{\Xi_j \overline{\theta}_i - \Psi_i \overline{\theta}_j}{\Xi_i \Xi_j - \Psi_i \Psi_j} - \overline{\theta}_i \right) + \frac{\Psi_i}{\mu_i \mu_j} \frac{\mu_i \Xi_j \frac{a_i}{a_j} - \mu_j \Psi_j}{\left(\Xi_i \Xi_j - \Psi_i \Psi_j\right)} q,\ a_i = \Delta_i c_i,\ i, j = 1, 2,\ j \neq i, \\ c_1 &= \frac{\Xi_1 \Delta_1^{-1} - \mu_1 \mu_2^{-1} \left(1 - \Xi_1 \Delta_1^{-1}\right) z}{\lambda_1}, \text{ and } c_2 = \frac{\Xi_2 \Delta_2^{-1} - \mu_2 \mu_1^{-1} \left(1 - \Xi_2 \Delta_2^{-1}\right) z^{-1}}{\lambda_2}. \end{aligned}$$

Substituting (39) in the previous expressions and after some algebra, we get

$$b_i = a_i \left(\frac{\Xi_j \overline{\theta}_i - \Psi_i \overline{\theta}_j}{\Xi_i \Xi_j - \Psi_i \Psi_j} - \overline{\theta}_i \right) + \frac{\lambda_j \Psi_i}{\mu_j \Xi_j \lambda_i + \mu_i \Psi_i \lambda_j} q, \tag{40}$$

$$a_i = \frac{\mu_j \left(\Xi_i \Xi_j - \Psi_i \Psi_j\right)}{\mu_j \Xi_j \lambda_i + \mu_i \Psi_i \lambda_j}, \text{ and} \tag{41}$$

$$c_i = \frac{\mu_j \left(\Xi_j - \Psi_i\right)}{\mu_j \Xi_j \lambda_i + \mu_i \Psi_i \lambda_j},\ i, j = 1, 2,\ j \neq i. \tag{42}$$

Next, we derive the equilibrium in the following continuous setup: Consider now that there is a continuum of bidders $[0, 1]$. Let q denote the aggregate quantity supplied in the market. Suppose that a fraction μ_1 of these bidders are traders of type 1 and the remainder fraction, μ_2, are bidders of type 2.

Consider a trader of type i. This bidder chooses to maximize

$$\mathbb{E}\left[\pi_i | s_i, p\right] = \left(\mathbb{E}\left[\theta_i | s_i, p\right] - p\right) x_i - \lambda_i x_i^2 / 2.$$

The FOC is given by $\mathbb{E}[\theta_i|s_i,p]-p-\lambda_i x_i=0$, or equivalently,

$$X_i(s_i,p)=\left(\mathbb{E}[\theta_i|s_i,p]-p\right)/\lambda_i. \tag{43}$$

Positing linear strategies, the market clearing condition implies that

$$p=\left(\mu_i(b_i+a_is_i)+\mu_j(b_j+a_js_j)-q\right)/\left(\mu_ic_i+\mu_jc_j\right), \tag{44}$$

provided that $\mu_ic_i+\mu_jc_j\neq 0$. Using the expression for p and assuming that $a_i\neq 0$, $i=1,2$, it follows that $\mathbb{E}[\theta_i|s_i,p]=\mathbb{E}[\theta_i|s_i,s_j]$. Hence, $\mathbb{E}[\theta_i|s_i,p]=\overline{\theta}_i+\Xi_i\left(s_i-\overline{\theta}_i\right)+\Psi_i\left(s_j-\overline{\theta}_j\right)$. Using (44), $s_j=\left(q-\mu_ib_i-\mu_jb_j-\mu_ia_is_i+p\left(\mu_ic_i+\mu_jc_j\right)\right)/\left(\mu_ja_j\right)$. Therefore,

$$\mathbb{E}[\theta_i|s_i,p]=\overline{\theta}_i+\Xi_i\left(s_i-\overline{\theta}_i\right)+\Psi_i\left(\left(q-\mu_ib_i-\mu_jb_j-\mu_ia_is_i+p\left(\mu_ic_i+\mu_jc_j\right)\right)/\left(\mu_ja_j\right)-\overline{\theta}_j\right).$$

Substituting this expression in (43), and identifying coefficients, it follows that

$$\begin{aligned}
b_i &= \left((1-\Xi_i)\overline{\theta}_i-\Psi_i\overline{\theta}_j+\Psi_i\left(q-\left(\mu_ib_i+\mu_jb_j\right)\right)/\left(\mu_ja_j\right)\right)/\lambda_i, && (45)\\
a_i &= \left(\Xi_i-\Psi_i\mu_ia_i/\left(\mu_ja_j\right)\right)/\lambda_i, \text{ and} && (46)\\
c_i &= \left(1-\Psi_i\left(\mu_ic_i+c_j\mu_j\right)/\left(\mu_ja_j\right)\right)/\lambda_i,\ i,j=1,2,\ j\neq i. && (47)
\end{aligned}$$

Note that (46) implies that $a_i/a_j=\lambda_j\left(\Xi_i-\Psi_i\mu_ia_i/\left(\mu_ja_j\right)\right)/\left(\lambda_i\left(\Xi_j-\Psi_j\mu_ja_j/\left(\mu_ia_i\right)\right)\right)$. Hence, $a_i/a_j=\mu_j\left(\Psi_j\lambda_i\mu_j+\Xi_i\lambda_j\mu_i\right)/\left(\mu_i\left(\Psi_i\lambda_j\mu_i+\lambda_i\Xi_j\mu_j\right)\right)$. Then, plugging the previous expression into (46), we get (41).

Furthermore, using (45), and after some algebra, we get

$$\mu_ib_i+\mu_jb_j=\frac{\frac{\mu_i}{\lambda_i}\left((1-\Xi_i)\overline{\theta}_i-\Psi_i\overline{\theta}_j+\frac{\Psi_i}{\mu_ja_j}q\right)+\frac{\mu_j}{\lambda_j}\left((1-\Xi_j)\overline{\theta}_j-\Psi_j\overline{\theta}_i+\frac{\Psi_j}{\mu_ia_i}q\right)}{\frac{\Psi_i}{\lambda_i}\frac{\mu_i}{\mu_ja_j}+\frac{\Psi_j}{\lambda_j}\frac{\mu_j}{\mu_ia_i}+1}.$$

Substituting (41) and the last expression in (45), we get (40). Moreover, from (47), and after some algebra, we get $\mu_ic_i+c_j\mu_j=\left(\frac{\mu_i}{\lambda_i}+\frac{\mu_j}{\lambda_j}\right)\Big/\left(\frac{\Psi_i}{\lambda_i}\frac{\mu_i}{\mu_ja_j}+\frac{\Psi_j}{\lambda_j}\frac{\mu_j}{\mu_ia_i}+1\right)$. Using (41) and the last expression in (47), we obtain (42). Consequently, we conclude that the equilibrium coefficients of the limiting case converge to the equilibrium coefficients of the continuous setup. Finally, taking into account the expressions for Ξ_i, Ξ_j, Ψ_i, and Ψ_j, we obtain the expressions stated in Proposition 6. ■

Lemma A3. *The equilibrium quantities solve the following distorted benefit maximization program:*

$$\max_{x_1,x_2}\ \mathbb{E}\left[n_1\left(\theta_1x_1-(d_1+\lambda_1)x_1^2/2\right)+n_2\left(\theta_2x_2-(d_2+\lambda_2)x_2^2/2\right)\middle|\, t\right]$$
$$s.t.\ n_1x_1+n_2x_2=Q,$$

taking as given the equilibrium parameters d_1 and d_2.

Proof: The Lagrangian function of the maximization program is given by

$$\mathcal{L}(x_1, x_2, \mu) = n_1\left(t_1 x_1 - (d_1+\lambda_1)\, x_1^2/2\right) + n_2\left(t_2 x_2 - (d_2+\lambda_2)\, x_2^2/2\right) - \mu\left(n_1 x_1 + n_2 x_2 - Q\right),$$

where μ denotes the Lagrange multiplier. Differentiating, we obtain the following FOCs:

$$n_1\left(t_1 - (d_1+\lambda_1)\, x_1\right) - \mu n_1 = 0, \tag{48}$$

$$n_2\left(t_2 - (d_2+\lambda_2)\, x_2\right) - \mu n_2 = 0, \text{ and} \tag{49}$$

$$n_1 x_1 + n_2 x_2 = Q. \tag{50}$$

From (48) and (49), it follows that $x_i = (t_i - \mu)/(d_i+\lambda_i)$, $i = 1, 2$. Substituting these expressions in (50) and operating, we have $\mu = \left(\frac{n_1 t_1}{d_1+\lambda_1} + \frac{n_2 t_2}{d_2+\lambda_2} - Q\right)\left(\frac{n_1}{d_1+\lambda_1} + \frac{n_2}{d_2+\lambda_2}\right)^{-1}$. Then, plugging this expression into (48) and (49), we get the expressions of the equilibrium quantities given in (4). In addition, since the objective function is concave and the constraint is a linear equation, we conclude that the critical point is a global maximum. Hence, the equilibrium quantities are the solutions of the optimization problem stated in Lemma A3. ■

Proof of Proposition 7: Recall that in the competitive setup, the FOC of the two optimization problems are given by $\mathbb{E}\left[\theta_i | s_i, p\right] - p - \lambda_i x_i = 0$, $i = 1, 2$. Doing similar computations as in the proof of Lemma A1, we derive the following system of equations:[32]

$$b_i = \left((1-\Xi_i)\,\overline{\theta}_i - \Psi_i \overline{\theta}_j + \Psi_i\left(\frac{Q - n_i b_i - n_j b_j}{n_j a_j}\right)\right)\Big/\lambda_i,\ a_i = \left(\Xi_i - \frac{n_i}{n_j}\frac{a_i}{a_j}\Psi_i\right)\Big/\lambda_i, \text{ and}$$

$$c_i = \left(1 - \Psi_i\left(\frac{n_i c_i + n_j c_j}{n_j a_j}\right)\right)\Big/\lambda_i,\ i, j = 1, 2,\ j \neq i.$$

Taking into account that $Q = (n_i + n_j)q$ and $\mu_i = n_i/(n_i + n_j)$, we have that the previous system is identical to the system of equations given in (45)-(47). As a consequence, the equilibrium coefficients given in the statement of Proposition 6 coincide with the equilibrium coefficients in the competitive setup. ■

Proof of Proposition 8: Performing similar computations as in the proof of Lemma A1, we obtain that the equilibrium coefficients with subsidies $\kappa_i = d_i(c_1^o, c_2^o)$ satisfy

$$b_i = \frac{(1-\Xi_i)\,\overline{\theta}_i - \Psi_i\overline{\theta}_j - \frac{\Psi_i(n_i b_i + n_j b_j - Q)}{n_j a_j}}{d_i + \lambda_i - d_i(c_1^o, c_2^o)},\ a_i = \frac{\Xi_i - \frac{n_i a_i}{n_j a_j}\Psi_i}{d_i + \lambda_i - d_i(c_1^o, c_2^o)} > 0, \text{ and}$$

$$c_i = \frac{1 - \frac{\Psi_i(n_i c_i + n_j c_j)}{n_j a_j}}{d_i + \lambda_i - d_i(c_1^o, c_2^o)},\ i, j = 1, 2,\ j \neq i.$$

[32] To ease the notation the superscript o is omitted in this proof.

Comparing this system of equations and the one derived in the proof of Proposition 7, we obtain that the equilibrium coefficients of the price-taking equilibrium solves this system. Therefore, we conclude that the quadratic subsidies $\kappa_i x_i^2/2$, with $\kappa_i = d_i(c_1^o, c_2^o)$, $i = 1, 2$, induce an efficient allocation. ■

Lemma A4. *The expected deadweight loss at an anonymous allocation* $(x_1(t), x_2(t))$ *satisfies*

$$\mathbb{E}[DWL] = \frac{1}{2}\lambda_1 n_1 \mathbb{E}\left[(x_1(t) - x_1^o(t))^2\right] + \frac{1}{2}\lambda_2 n_2 \mathbb{E}\left[(x_2(t) - x_2^o(t))^2\right]. \tag{51}$$

Proof: Notice that $ETS = \mathbb{E}[\mathbb{E}[TS|t]]$, where

$$\begin{aligned}\mathbb{E}[TS|t] &= \mathbb{E}\left[n_1\left(\theta_1 x_1(t) - \lambda_1 (x_1(t))^2/2\right) + n_2\left(\theta_2 x_2(t) - \lambda_2 (x_2(t))^2/2\right)\middle| t\right] = \\ & \quad n_1\left(t_1 x_1(t) - \lambda_1 (x_1(t))^2/2\right) + n_2\left(t_2 x_2(t) - \lambda_2 (x_2(t))^2/2\right).\end{aligned}$$

A Taylor series expansion of $\mathbb{E}[TS|t]$ around the price-taking equilibrium $(x_1^o(t), x_2^o(t))$, stopping at the second term due to the fact that $\mathbb{E}[TS|t]$ is quadratic, yields

$$\begin{aligned}\mathbb{E}[TS|t](x(t)) &= \mathbb{E}[TS|t](x^o(t)) + \nabla\mathbb{E}[TS|t](x^o(t))(x(t) - x^o(t)) + \\ & \quad + \frac{1}{2}(x(t) - x^o(t))' D^2\mathbb{E}[TS|t](x^o(t))(x(t) - x^o(t)),\end{aligned}$$

where $\nabla\mathbb{E}[TS|t](x^o(t))$ and $D^2\mathbb{E}[TS|t](x^o(t))$ are, respectively, the gradient and the Hessian matrix of $\mathbb{E}[TS|t]$ evaluated at $x^o(t)$. Notice that we know

$$\nabla\mathbb{E}[TS|t](x^o(t)) = (n_1(t_1 - \lambda_1 x_1^o(t)), n_2(t_2 - \lambda_2 x_2^o(t))).$$

Using the expressions of $x_1^o(t)$ and $x_2^o(t)$, and after some algebra it follows that

$$\nabla\mathbb{E}[TS|t](x^o(t))(x(t) - x^o(t)) = 0.$$

In addition, $D^2\mathbb{E}[TS|t](x^o(t)) = \begin{pmatrix} -\lambda_1 n_1 & 0 \\ 0 & -\lambda_2 n_2 \end{pmatrix}$. Hence,

$$\mathbb{E}[TS|t](x(t)) - \mathbb{E}[TS|t](x^o(t)) = -\frac{1}{2}\lambda_1 n_1 (x_1(t) - x_1^o(t))^2 - \frac{1}{2}\lambda_2 n_2 (x_2(t) - x_2^o(t))^2$$

and, therefore, (51) is satisfied. ■

Proof of Proposition 9: **(i)** Suppose that $Q = 0$. Then, $\mathbb{E}[DWL]$ is given by

$$\mathbb{E}[DWL] = \frac{n_1 n_2 (n_2 d_1 + n_1 d_2)^2}{2(n_2\lambda_1 + n_1\lambda_2) u^2}\mathbb{E}(t_1 - t_2)^2.$$

Hence,

$$\frac{d\mathbb{E}[DWL]}{d\sigma_{\varepsilon_1}^2} = \frac{\partial\mathbb{E}[DWL]}{\partial d_1}\frac{\partial d_1}{\partial\sigma_{\varepsilon_1}^2} + \frac{\partial\mathbb{E}[DWL]}{\partial d_2}\frac{\partial d_2}{\partial\sigma_{\varepsilon_1}^2} + \frac{\partial\mathbb{E}[DWL]}{\partial\sigma_{\varepsilon_1}^2}.$$

It is easy to see that in this case $\partial\mathbb{E}\left[DWL\right]/\partial d_i > 0$, $i = 1,2$, and $\partial\mathbb{E}\left[DWL\right]/\partial\sigma^2_{\varepsilon_1} < 0$. Combining these results with Proposition 3, we have that the first two terms of $d\mathbb{E}\left[DWL\right]/d\sigma^2_{\varepsilon_1}$ are positive, while the last one is negative.

We know that d_1 and d_2 are independent of $\sigma^2_{\varepsilon_1}$ when $\rho = 0$. By continuity, we know that for very low values of ρ is $\partial d_1/\partial\sigma^2_{\varepsilon_1}$ and $\partial d_2/\partial\sigma^2_{\varepsilon_1}$ are positive, but very low. Hence, we conclude that the last term in $d\mathbb{E}\left[DWL\right]/\,d\sigma^2_{\varepsilon_1}$ dominates and, hence, in this case we have that $d\mathbb{E}\left[DWL\right]/\,d\sigma^2_{\varepsilon_1} < 0$ although $\partial d_1/\partial\sigma^2_{\varepsilon_1} > 0$, $i = 1,2$. By contrast, if we consider the case in which ρ is not low and $\left(\overline{\theta}_1 - \overline{\theta}_2\right)^2$ is high enough, then the first two terms in $d\mathbb{E}\left[DWL\right]/\,d\sigma^2_{\varepsilon_1}$ dominate, which implies that $d\mathbb{E}\left[DWL\right]/\,d\sigma^2_{\varepsilon_1} > 0$, $i = 1,2$.

(ii) Omitted since it is trivial.

(iii) When groups are symmetric ($n_2 = n_1 = n$, $\lambda_1 = \lambda_2 = \lambda$, and $\sigma^2_{\varepsilon_1} = \sigma^2_{\varepsilon_2} = \sigma^2_{\varepsilon}$) $d_1 = d_2 = d$ and $\lambda_2 d_1 - \lambda_1 d_2 = 0$. Therefore, the expected deadweight loss consists of only one term, the first one, that is independent of Q. Moreover, after some computations, we get that $\mathbb{E}\left[DWL|t\right] = nd^2\left(t_2 - t_1\right)^2/\left(4\left(d+\lambda\right)^2\lambda\right)$. Using this expression and the fact that $\partial\mathbb{E}\left[DWL|t\right]/\partial d > 0$ allows us to conclude that an increase in an information parameter (ρ or $\widehat{\sigma}^2_{\varepsilon}$) raises both d and $\mathbb{E}\left[DWL|t\right]$.

However, with asymmetric groups the previous results may not hold. In this case, suppose that Q is large enough. Then, for $i = 1,2$, $j \neq i$,

$$\operatorname{sgn}\left(\frac{\partial E\left[DWL|t\right]}{\partial d_i}\right) = \operatorname{sgn}\left(\partial\left(\frac{n_1 n_2\left(\lambda_1 d_2 - \lambda_2 d_1\right)^2}{2\left(\lambda_1 n_2 + \lambda_2 n_1\right)u^2}Q^2\right)\Big/\,\partial d_i\right) = \operatorname{sgn}\left(\lambda_j d_i - \lambda_i d_j\right).$$

When Q is large enough we have that if $d_i/d_j > \lambda_i/\lambda_j$, then $\partial E\left[DWL|t\right]/\partial d_i > 0$ and $\partial E\left[DWL|t\right]/\partial d_j < 0$. Thus, with asymmetric groups price impact (d_1, d_2) and the $E\left[DWL|t\right]$ are not always positively associated, given predicted values t, for changes in information parameters. ■

Appendix B

Bid shading and expected discount

Recall that $\tilde{t} = \left(n_1 t_1 + n_2 t_2\right)/\left(n_1 + n_2\right)$. From the demand of bidders it follows that $p\left(t\right) = t_i - (d_i + \lambda_i)x_i\left(t\right)$, $i = 1,2$. For a trader of type i, the expected marginal benefit of buying x_i units of the asset is $t_i - \lambda_i x_i$. Hence, the average marginal benefit is given by $\tilde{t} - \left(\lambda_1 n_1 x_1 + \lambda_2 n_2 x_2\right)/\left(n_1 + n_2\right)$. The magnitude of (average) *bid shading* is the difference between the average marginal valuation and the auction price, that is, $(d_1 n_1 x_1 + d_2 n_2 x_2) / (n_1 + n_2)$.[33] We can use Equation (4) to write bid shading as

$$\frac{(n_2 d_2 (d_1 + \lambda_1) + n_1 d_1 (d_2 + \lambda_2)) Q}{(n_1 + n_2) (n_1 (d_2 + \lambda_2) + n_2 (d_1 + \lambda_1))} + \frac{(t_2 - t_1) (d_2 - d_1) n_2 n_1}{(n_1 + n_2) (n_1 (d_2 + \lambda_2) + n_2 (d_1 + \lambda_1))}. \tag{52}$$

At this juncture, some additional remarks are in order.

- Bid shading increases with Q and decreases when the group that values more the asset, say group 1, ($t_1 > t_2$) has less price impact ($d_1 < d_2$).
- When $d_1 = d_2 = d$ as in the symmetric case, for instance, bid shading consists of only one term (the first one) and it is equal to $dQ/ (n_1 + n_2)$.
- When $d_1 \neq d_2$, the second term of (52) is negative and bid shading decreases whenever the group that values the asset more highly ($t_i > t_j$) has less price impact ($d_i < d_j$).
- If group 1 has higher transaction costs ($\lambda_1 > \lambda_2$), is more numerous ($n_1 > n_2$), and is less informed ($\sigma^2_{\varepsilon_1} > \sigma^2_{\varepsilon_2}$) than group 2, then $c_1 < c_2$, and so $d_1 < d_2$. If $t_1 > t_2$, then the second term of (52) is negative and the two terms have opposite signs. Therefore, if Q is low (e.g., $Q = 0$) or if the difference in predicted values of the asset is high, then *negative* bid shading obtains.

The *expected discount* is defined as $\mathbb{E}\left[\tilde{t}\right] - \mathbb{E}\left[p(t)\right]$. Using (6), some algebra yields the following expression for the expected discount:

$$\frac{(d_1 + \lambda_1)(d_2 + \lambda_2)}{n_1 (d_2 + \lambda_2) + n_2 (d_1 + \lambda_1)} Q + \frac{n_1 n_2 (d_2 + \lambda_2 - d_1 - \lambda_1) \left(\overline{\theta}_2 - \overline{\theta}_1\right)}{(n_1 + n_2)(n_1 (d_2 + \lambda_2) + n_2 (d_1 + \lambda_1))}. \tag{53}$$

Here our related comments are as follows.

- When $d_1 + \lambda_1 = d_2 + \lambda_2 = d + \lambda$ (as in the symmetric case), the expected discount is $(d + \lambda)Q/ (n_1 + n_2)$.
- The first term is always positive provided $Q > 0$, whereas the second term is positive whenever $(d_2 + \lambda_2 - d_1 - \lambda_1)\left(\overline{\theta}_2 - \overline{\theta}_1\right) > 0$. Therefore, the expected discount is lower whenever the group that values the asset more highly ($\overline{\theta}_2 > \overline{\theta}_1$) has a lower "total transaction cost" ($d_2 + \lambda_2 < d_1 + \lambda_1$).

[33] According to Cassola et al. (2013) average bid shading almost tripled after the turmoil in August 2007 in the ECB liquidity auctions.

- If group 1 ex ante values the asset more ($\overline{\theta}_1 > \overline{\theta}_2$), has higher transaction costs ($\lambda_1 > \lambda_2$), is more numerous ($n_1 > n_2$), and is less informed ($\sigma^2_{\varepsilon_1} > \sigma^2_{\varepsilon_2}$), then Corollary 2 shows that $d_1 + \lambda_1 > d_2 + \lambda_2$ whenever (a) the differences between groups are due mostly to transaction costs and (b) λ_1/λ_2 is high enough. In this case, both terms are positive and so the expected discount is positive. Yet, if both groups have similar transactions costs, then the two terms in (53) have opposite signs. In particular, we expect a negative discount when Q is low.

Effects of Integrating Separated Markets

We compare the expected total surplus of our setup with a framework with two separated markets, the market 1 and the market 2. In market i all the buyers (n_i) are of type i and the quantity supplied is $n_i Q/(n_1 + n_2)$. In this last framework, given that all the individuals are identical, the market clearing condition implies that the equilibrium quantities in market i are given by $Q/(n_i + n_j)$. Hence, the expected total surplus in market i, denoted by $\mathbb{E}\left[TS\right]_{\text{Market } i}$, satisfies

$$\mathbb{E}\left[TS\right]_{\text{Market } i} = \frac{n_i \overline{\theta}_i}{n_1 + n_2} Q - \frac{\lambda_i n_i}{\left(n_1 + n_2\right)^2} \frac{Q^2}{2}$$

and, consequently, the sum of expected total surplus in this setting

$$\mathbb{E}\left[TS\right]_{\text{Market 1}} + \mathbb{E}\left[TS\right]_{\text{Market 2}} = \frac{n_1 \overline{\theta}_1 + n_2 \overline{\theta}_2}{n_1 + n_2} Q - \frac{n_1 \lambda_1 + n_2 \lambda_2}{\left(n_1 + n_2\right)^2} \frac{Q^2}{2}.$$

On the other hand, in our setup (the integrated market) the expected total surplus is given by

$$\mathbb{E}\left[TS\right]_{\text{Integrated Market}} = \left(\frac{n_1(d_2+\lambda_2)\overline{\theta}_1 + n_2(d_1+\lambda_1)\overline{\theta}_2}{u} - \frac{n_1 n_2(\lambda_1 d_2 - \lambda_2 d_1)\left(\overline{\theta}_1 - \overline{\theta}_2\right)}{u^2} \right) Q$$
$$- \frac{n_1\lambda_1(d_2+\lambda_2)^2 + n_2\lambda_2(d_1+\lambda_1)^2}{2u^2} Q^2 + \frac{n_1 n_2(n_2(2d_1+\lambda_1) + n_1(2d_2+\lambda_2))}{2u^2} \mathbb{E}\left[(t_1 - t_2)^2\right],$$

where u and $\mathbb{E}\left[(t_1 - t_2)^2\right]$ are given by (5) and (7), respectively.

At first glance, we can observe that when $Q = 0$, it follows that

$$\mathbb{E}\left[TS\right]_{\text{Integrated Market}} > \mathbb{E}\left[TS\right]_{\text{Market 1}} + \mathbb{E}\left[TS\right]_{\text{Market 2}},$$

whenever $\overline{\theta}_1 \neq \overline{\theta}_2$, or when there is an informative private signal (i.e., when $\sigma^2_{\varepsilon_1} = \sigma^2_{\varepsilon_2} = \infty$ does not hold). Under this parameter configuration, this inequality also holds if the groups are symmetric (i.e., $\lambda_1 = \lambda_2 = \lambda$, $n_1 = n_2 = n$, $\sigma^2_{\varepsilon_1} = \sigma^2_{\varepsilon_2} = \sigma^2_{\varepsilon}$, and $\overline{\theta}_1 = \overline{\theta}_2 = \overline{\theta}$). To see this, note that in this case

$$\mathbb{E}\left[TS\right]_{\text{Integrated Market}} = \overline{\theta} Q - \frac{\lambda}{4n} Q^2 + \frac{n\left(2d + \lambda\right)}{4\left(d + \lambda\right)^2} \mathbb{E}\left[(t_1 - t_2)^2\right] \text{ and}$$

$$\mathbb{E}\left[TS\right]_{\text{Market 1}} + \mathbb{E}\left[TS\right]_{\text{Market 2}} = \overline{\theta}Q - \frac{\lambda}{4n}Q^2.$$

Therefore, we can conclude that under symmetry, market integration increases the expected total surplus.

In what follows we show that market integration is also good in terms of $\mathbb{E}\left[TS\right]$ under perfect competition, even though the groups are asymmetric. Notice that under perfect competition $(d_1 = d_2 = 0)$, we have that

$$\mathbb{E}\left[TS\right]_{\text{Integrated Market}} = \frac{1}{n_1\lambda_2 + n_2\lambda_1}\left(\left(n_1\lambda_2\overline{\theta}_1 + n_2\lambda_1\overline{\theta}_2\right)Q - \frac{\lambda_1\lambda_2}{2}Q^2 + \frac{n_1 n_2}{2}\mathbb{E}\left[\left(t_1 - t_2\right)^2\right]\right).$$

Hence,

$$\mathbb{E}\left[TS\right]_{\text{Integrated Market}} - \left(\mathbb{E}\left[TS\right]_{\text{Market 1}} + \mathbb{E}\left[TS\right]_{\text{Market 2}}\right) =$$
$$\frac{n_1 n_2}{n_1\lambda_2 + n_2\lambda_1}\left(\frac{\left(\overline{\theta}_1 - \overline{\theta}_2\right)\left(\lambda_2 - \lambda_1\right)}{n_1 + n_2}Q + \frac{\left(\lambda_2 - \lambda_1\right)^2}{2\left(n_1 + n_2\right)^2}Q^2 + \frac{\mathbb{E}\left[\left(t_1 - t_2\right)^2\right]}{2}\right),$$

which is a convex function in Q. Notice that, if $\left(\overline{\theta}_1 - \overline{\theta}_2\right)\left(\lambda_2 - \lambda_1\right) \geq 0$, it follows that

$$\mathbb{E}\left[TS\right]_{\text{Integrated Market}} \geq \mathbb{E}\left[TS\right]_{\text{Market 1}} + \mathbb{E}\left[TS\right]_{\text{Market 2}},$$

with an equality only holds when $\overline{\theta}_1 = \overline{\theta}_2$ $\lambda_1 = \lambda_2$ and $\sigma^2_{\varepsilon_1} = \sigma^2_{\varepsilon_2} = \infty$. Otherwise, if $\left(\overline{\theta}_1 - \overline{\theta}_2\right)\left(\lambda_2 - \lambda_1\right) < 0$, the expected total surplus is also larger in the integrated market. In order to show this, we examine the value of $\mathbb{E}\left[TS\right]_{\text{Integrated Market}} - \left(\mathbb{E}\left[TS\right]_{\text{Market 1}} + \mathbb{E}\left[TS\right]_{\text{Market 2}}\right)$ at the minimum, which is $Q = \frac{n_1+n_2}{\lambda_1-\lambda_2}\left(\overline{\theta}_1 - \overline{\theta}_2\right)$. Direct computation yields that this function at its minimum is strictly positive and, therefore, for all $Q \geq 0$, this difference is also strictly positive. Therefore, we can conclude that, under perfect competition the following inequality holds

$$\mathbb{E}\left[TS\right]_{\text{Integrated Market}} \geq \mathbb{E}\left[TS\right]_{\text{Market 1}} + \mathbb{E}\left[TS\right]_{\text{Market 2}},$$

and the equality is satisfied when $\overline{\theta}_1 = \overline{\theta}_2$, $\lambda_1 = \lambda_2$ and $\sigma^2_{\varepsilon_1} = \sigma^2_{\varepsilon_2} = \infty$.

To find the reverse result, we have to restrict our attention to a setup with strategic behavior. Moreover, we assume that $\overline{\theta}_1 = \overline{\theta}_2 = \overline{\theta}$ and $\sigma^2_{\varepsilon_1} = \sigma^2_{\varepsilon_2} = \infty$. Then, it follows that

$$\mathbb{E}\left[TS\right]_{\text{Integrated Market}} = \overline{\theta}Q - \frac{n_1\lambda_1\left(d_2 + \lambda_2\right)^2 + n_2\lambda_2\left(d_1 + \lambda_1\right)^2}{u^2}\frac{Q^2}{2} \text{ and}$$

$$\mathbb{E}\left[TS\right]_{\text{Market 1}} + \mathbb{E}\left[TS\right]_{\text{Market 2}} = \overline{\theta}Q - \frac{n_1\lambda_1 + n_2\lambda_2}{\left(n_1 + n_2\right)^2}\frac{Q^2}{2}.$$

Therefore, in this case

$$\mathbb{E}\left[TS\right]_{\text{Integrated Market}} - \left(\mathbb{E}\left[TS\right]_{\text{Market 1}} + \mathbb{E}\left[TS\right]_{\text{Market 2}}\right) =$$

$n_1 n_2 \left(d_1 + \lambda_1 - \lambda_2 - d_2\right) \frac{2(d_1+\lambda_1)(\lambda_1-\lambda_2)(n_1+n_2)+(n_1(2\lambda_1-\lambda_2)+\lambda_1 n_2)(d_2+\lambda_2-d_1-\lambda_1)}{(n_1+n_2)^2 u^2} \frac{Q^2}{2}$.

In particular, whenever $\lambda_1 > \lambda_2$, $d_1 + \lambda_1 < d_2 + \lambda_2$, and $Q \neq 0$, it follows that

$$\mathbb{E}\left[TS\right]_{\text{Integrated Market}} < \mathbb{E}\left[TS\right]_{\text{Market 1}} + \mathbb{E}\left[TS\right]_{\text{Market 2}}.$$

To intuitively understand this result notice that, when bidders do not have private information and ex-ante identically value the asset ($\overline{\theta}_1 = \overline{\theta}_2 = \overline{\theta}$), the individual equilibrium quantity for the group i can be written as

$$\frac{Q}{n_i + n_j} - \frac{\left(d_i + \lambda_i - d_j - \lambda_j\right) Q n_j}{\left(n_i + n_j\right) u}.$$

Then, when $d_2 + \lambda_2 > d_1 + \lambda_1$ the group 2 with lower transaction costs ($\lambda_1 > \lambda_2$), in equilibrium, obtains a lower quantity in the integrated market with respect to the framework with separated markets. This result is due to the fact that in the integrated market the individuals of group 2 reduce their demand strategically and, as a result their equilibrium quantities are lower, while the individuals with higher transaction costs get a higher quantity. All this leads to a lower expected total surplus in the integrated market.

Finally, we end this proof with another particular case (now with incomplete information) in which market integration also reduces the expected total surplus. Suppose that $\overline{\theta}_1 = \overline{\theta}_2 = \overline{\theta}$, $\lambda_1 = \lambda_2 = \lambda$, $n_1 = n_2 = n$, $\rho \neq 0$, and $\widehat{\sigma}^2_{\varepsilon_1} \neq \widehat{\sigma}^2_{\varepsilon_2}$. In this case the sum of the expected total surplus with separated markets

$$\mathbb{E}\left[TS\right]_{\text{Market 1}} + \mathbb{E}\left[TS\right]_{\text{Market 2}} = \overline{\theta} Q - \frac{\lambda}{4n} Q^2.$$

On the other hand, in the integrated market

$$\mathbb{E}\left[TS\right]_{\text{Integrated Market}} =$$

$$\overline{\theta} Q - \frac{\lambda}{2n} \frac{\left(d_2 + \lambda\right)^2 + \left(d_1 + \lambda\right)^2}{\left(d_1 + d_2 + 2\lambda\right)^2} Q^2 + n \frac{d_1 + d_2 + \lambda}{\left(d_1 + d_2 + 2\lambda\right)^2} \mathbb{E}\left[\left(t_1 - t_2\right)^2\right].$$

Suppose that Q is large enough, the comparison of the previous two expressions is simply reduced to the comparison of the coefficients associated with Q^2 (in absolute values).[34] Note that

$$\frac{\lambda}{2n} \frac{\left(d_2 + \lambda\right)^2 + \left(d_1 + \lambda\right)^2}{\left(d_1 + d_2 + 2\lambda\right)^2} - \frac{\lambda}{4n} = \frac{\lambda \left(d_1 - d_2\right)^2}{4n \left(d_1 + d_2 + 2\lambda\right)^2} > 0.$$

This implies that there are values of Q such that

$$\mathbb{E}\left[TS\right]_{\text{Integrated Market}} < \mathbb{E}\left[TS\right]_{\text{Market 1}} + \mathbb{E}\left[TS\right]_{\text{Market 2}}.$$

[34] To avoid the anomaly of negative expected total surplus, we have to assume that $\overline{\theta}$ is high enough.

References

Akgün, U. (2004): "Mergers with supply functions," *Journal of Industrial Economics,* 52 (4), 535-546.

Anderson, E. AND X. Hu (2008): "Finding supply function equilibria with asymmetric firms," *Operations Research,* 56 (3), 697-711.

Armantier, O., AND E. Sbaï (2006): "Estimation and comparison of treasury auction formats when bidders are asymmetric," *Journal of Applied Econometrics,* 21, 745–779.

Arnone, M., AND G. Iden (2003): "Primary dealers in government securities: policy issues and selected countries' experience," IMF Working Paper 03/45.

Athey, S. (2001): "Single crossing properties and the existence of pure strategy equilibria in games of incomplete information," *Econometrica,* 69 (4), 861–890.

Athey, S., D. Coey AND J. Levin (2013): "Set-asides and subsidies in auctions," *American Economic Journal: Microeconomics,* 5 (1), 1-27.

Ausubel, L. M., P. Cramton, M. Pycia, M. Rostek, AND M. Weretka (2014): "Demand reduction and inefficiency in multi-unit auctions," *The Review of Economic Studies,* 81 (4), 1366-1400.

Back, K., AND J.F. Zender (1993): "Auctions of divisible goods: on the rationale for the Treasury experiment," *The Review of Financial Studies,* 6, 733–764.

Bajari, P. (1998): "A structural econometric model of the first price sealed bid auction with asymmetric bidders," Manuscript, Stanford University.

Bergemann, D., T. Heumann, AND S. Morris (2015): "Information and market power," Cowles Foundation working paper no. 2017.

Bindseil, U., K. Nyborg, AND I. Strebulaev (2009): "Bidding and performance in repo auctions: Evidence from ECB open market operations," *Journal of Money, Credit and Banking,* 41 (7), 1391-1421.

Bjonnes, G.H. (2001): "Winner's curse in discriminatory price auctions: Evidence from the Norwegian Treasury bill auctions," SIFR Research Report Series 3, Institute for Financial Research.

Borenstein, S., J. Bushnell, AND F. Wolak (2002): "Measuring market inefficiencies in California's restructured wholesale electricity market," *American Economic Review,* 92 (5), 1376-1405.

Boyarchenko, N., D.O. Lucca, AND L. Veldkamp (2015): "Intermediaries as information aggregators: an application to US Treasury Auctions," Federal Reserve Bank of New York Staff Report no. 726.

Brandts, J., Reynolds, S. AND A. Schram (2014): "Pivotal suppliers and market power in experimental supply function competition," *Economic Journal,* 124 (579), 887-916.

Bustos-Salvagno, J. (2015): "Bidding behavior in the Chilean electricity market," *Energy Economics*, 51, 288-299.

Cantillon, E. (2008): "The effect of bidders' asymmetries on expected revenue in auctions," *Games and Economic Behavior,* 62 (1), 1-25.

Cassola, N., A. Hortaçsu, AND J. Kastl (2013): "The 2007 subprime market crisis through the lens of European Central Bank auctions for short-term funds," *Econometrica*, 81 (4), 1309-1345.

Ciarreta, A., AND M.P. Espinosa (2010): "Supply function competition in the Spanish wholesale electricity market," *Energy Journal,* 31 (4), 137-157.

Cramton, P., AND S. Stoft (2006): "Uniform price auctions in electricity markets," U. Maryland Working paper.

Cramton, P., AND S. Stoft (2007): "Why we need to stick with uniform price auctions in electricity markets," *Electricity Journal,* 20 (1), 26-37.

Crawford, G., J. Crespo, AND H. Tauchen (2007): "Bidding asymmetries in multi-unit auctions: Implications of bid function equilibria in the British spot market for electricity," *International Journal of Industrial Organization,* 25 (6), 1233-1268.

Du, S. AND H. Zhu (2015): "Welfare and optimal trading frequency in dynamic double auctions," Working paper.

Du, S. AND H. Zhu (2016): "Bilateral trading in divisible double auctions," Working paper.

Engelbrecht-Wiggans, R., J.A. List, AND D. Lucking-Reiley (2006): "Demand reduction in multi-unit auctions with varying numbers of bidders: Theory and field experiments," *International Economic Review,* 47, 203–231.

European Commission (2007): "DG Competition report on energy sector inquiry," 10 January 2007, SEC (2006) 1724, Brussels, Commission of the European Communities.

Ewerhart, C., N. Cassola, AND N. Valla (2010): "Declining valuations and equilibrium bidding in central bank refinancing operations," *International Journal of Industrial Organization*, 28, 30–43.

Green, R., AND D. Newbery (1992): "Competition in the British electricity spot market," *Journal of Political Economy*, 100, 929–953.

Grossman, S., AND J. Stiglitz (1980): "On the impossibility of informationally efficient markets," *American Economic Review,* 70, 393-408.

Holmberg, P., D. Newbery, AND D. Ralph (2013): "Supply function equilibria: Step functions and continuous representations," *Journal of Economic Theory,* 148(4), 1509-1551.

Holmberg, P., AND F. Wolak (2015): "Electricity markets: Designing auctions where suppliers have uncertain costs," IFN working paper no. 1099.

Hortaçsu, A., AND J. Kastl (2012): "Valuing dealers' informational advantage: a study of Canadian Treasury auctions," *Econometrica,* 80 (6), 2511-2542.

Hortaçsu, A., J. Kastl, AND A. Zhang (2016): "Bid shading and bidder surplus in the US Treasury auction system," Working Paper.

Hortaçsu, A., AND D. McAdams (2010): "Mechanism choice and strategic bidding in divisible good auctions: An empirical analysis of the Turkish Treasury auction market," *Journal of Political Economy,* 118 (5), 833-865.

Hortaçsu, A., AND S. Puller (2008): "Understanding strategic bidding in multi-unit auctions: A case study of the Texas electricity spot market," *RAND Journal of Economics,* 39 (1), 86-114.

Joskow, P., AND E. Kahn (2002): "A quantitative analysis of pricing behavior in California's wholesale electricity market during summer 2000," *The Energy Journal,* 23 (4), 1-35.

Kagel, J.H., AND D. Levin (2001): "Behavior in multi-unit demand auctions: Experiments with uniform price and dynamic Vickrey auctions," *Econometrica,* 69, 413–454.

Kastl, J. (2011): "Discrete bids and empirical inference in divisible good auctions," *The Review of Economic Studies,* 78 (3), 974-1014.

Klemperer, P., AND M. Meyer (1989): "Supply function equilibria in oligopoly under uncertainty," *Econometrica*, 57, 1243–1277.

Kyle, A. (1985): "Continuous auctions and insider trading," *Econometrica*, 53, 1315-1335.

Kyle, A. (1989): "Informed speculation with imperfect competition," *The Review of Economic Studies,* 56, 317-356.

Lambert, N., M. Ostrovsky, AND M. Pavnov (2016): "Strategic trading in informationally complex environments," Working paper.

Lebrun, B. (1996): "Existence of an equilibrium in first price auctions," *Economic Theory,* 7 (3), 421–443.

Lebrun, B. (1999): "First price auctions in the asymmetric N bidder case," *International Economic Review,* 40 (1), 125–142.

Loertscher, S. AND L. Marx (2016): "Auctions with bid credits and resale", Working Paper.

Lopomo, G., L. Marx, D. McAdams, AND B. Murray (2011): "Carbon allowance auction design: an assessment of options for the United States," *Review of Environmental Economics and Policy,* 5 (1) 25-43.

Malvey, P.F., AND C.F. Archibald (1998): "Uniform-price auctions: Update of the Treasury experience," US Treasury.

Maskin, E., AND J.G. Riley (2000a): "Asymmetric auctions," *The Review of Economic Studies,* 67 (3), 413-438.

Maskin, E., AND J.G. Riley (2000b): "Equilibrium in sealed high bid auctions," *The Review of Economic Studies,* 67 (3), 439–454.

Maskin, E., AND J.G. Riley (2003): "Uniqueness in sealed high bid auctions," *Games and Economic Behavior* 45 (2), 395–409.

McAdams, D. (2003): "Isotone equilibrium in games of incomplete information," *Econometrica* 71 (4), 1191-1214.

McAdams, D. (2006): "Monotone equilibrium in multi-unit auctions," *Review of Economic Studies* 73, 1039-1056.

Pesendorfer, M. (2000): "A study of collusion in first-price auctions," *The Review of Economic Studies,* 67 (3), 381–411.

Pai, M. AND R. Vohra (2012): "Auction design with fairness concerns: subsidies vs. set-asides," Discussion Paper, Center for Mathematical Studies in Economics and Management Science, Northwestern University.

Porter, R.H., AND J.D. Zona (1999): "Ohio school milk markets: An analysis of bidding," *RAND Journal of Economics,* 30 (2), 263–288.

Pycia, M., AND K. Woodward (2016): "Pay-as-bid. Selling divisible goods," Working paper.

Reny, P. (2011): "On the existence of monotone pure-strategy equilibria in Bayesian games," *Econometrica* 79 (2), 499-553.

Reny, P., AND S. Zamir (2004): "On the existence of pure strategy monotone equilibria in asymmetric first-price auctions," *Econometrica,* 72 (4), 1105-1125.

Rostek, M., AND M. Weretka (2012): "Price inference in small markets," *Econometrica* 80 (2), 687-711.

Sadzik, T., AND P. Andreyanov (2016): "Robust mechanism design of exchange," Working paper.

Umlauf, S.R. (1993): "An empirical study of the Mexican Treasury bill auction," *Journal of Financial Economics,* 33, 313-340.

Vives, X. (2010): "Asset auctions, information, and liquidity," *Journal of the European Economic Association,* 8 (2-3), 467-477.

Vives, X. (2011): "Strategic supply function competition with private information," *Econometrica,* 79 (6), 1919-1966.

Vives, X. (2014): "On the possibility of informationally efficient markets," *Journal of the European Economic Association,* 12 (5), 1200-1239.

Wang, J.J.D., AND J.F. Zender (2002): "Auctioning divisible goods," *Economic Theory,* 19 (4), 673–705.

Wilson, R. (1979): "Auctions of shares," *Quarterly Journal of Economics,* 93, 675–698.

Wolfram, C. (1998): "Strategic bidding in a multiunit auction: An empirical analysis of bids to supply electricity in England and Wales," *RAND Journal of Economics*, 29, 703–725.