

Inference on the parameters of the Weibull distribution using records

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Abstract

The Weibull distribution is a very applicable model for lifetime data. In this paper, we have investigated inference on the parameters of Weibull distribution based on record values. We first propose a simple and exact test and a confidence interval for the shape parameter. Then, in addition to a generalized confidence interval, a generalized test variable is derived for the scale parameter when the shape parameter is unknown. The paper presents a simple and exact joint confidence region as well. In all cases, simulation studies show that the proposed approaches are more satisfactory and reliable than previous methods. All proposed approaches are illustrated using a real example.

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1. Introduction

The Weibull distribution is a well-known distribution that is widely used for lifetime models. It has numerous varieties of shapes and demonstrates considerable flexibility that enables it to have increasing and decreasing failure rates. Therefore, it is used for many applications, for example in hydrology, industrial engineering, weather forecasting and insurance. The Weibull distribution with parameters α and β , denoted by $W(\alpha, \beta)$, has a cumulative distribution function (cdf)

$$F(x) = 1 - e^{-\left(\frac{x}{\alpha}\right)^\beta}, \quad x > 0, \quad \alpha > 0, \quad \beta > 0,$$

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and probability density function (pdf)

$$f(x) = \frac{\beta}{\alpha^\beta} x^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^\beta}, \quad x > 0.$$

The Weibull distribution is a generalization of the exponential distribution and Rayleigh distribution. Also, $Y = \log(X)$ has the Gumbel distribution with parameters $b = \frac{1}{\beta}$ and $a = \log(\alpha)$, when X has a Weibull distribution with parameters α and β .

Let X_1, X_2, \dots be an infinite sequence of independent identically distributed random variables from a same population with the cdf F_θ , where θ is a parameter. An observation X_j will be called an upper record value (or simply a record) if its value exceeds that of all previous observations. Thus, X_j is a record if $X_j > X_i$ for every $i < j$. An analogous definition deals with lower record values. The record value sequence $\{R_n\}$ is defined by

$$R_n = X_{T_n}, \quad n = 0, 1, 2, \dots$$

where T_n is called the record time of n th record and is defined as $T_n = \min\{j : X_j > X_{T_{n-1}}\}$ with $T_0 = 1$.

Let R_0, \dots, R_n be the first $n+1$ upper record values from the cdf F_θ and the pdf f_θ . Then, the joint distribution of the first $n+1$ record values is given by

$$f_{\mathbf{R}}(\mathbf{r}) = f_\theta(r_n) \prod_{i=0}^{n-1} \frac{f_\theta(r_i)}{1 - F_\theta(r_i)}, \quad r_0 < r_1 < \dots < r_n, \quad (1.1)$$

where $\mathbf{r} = (r_0, r_1, \dots, r_n)$ and $\mathbf{R} = (R_0, R_1, \dots, R_n)$ (for more details see Arnold et al., 1998).

Chandler (1952) launched a statistical study of the record values, record times and inter-record times. Record values and the associated statistics are of interest and importance in the areas of meteorology, sports and economics. Ahsanullah (1995) and Arnold et al. (1998) are two good references about records and their properties.

Some papers considered inference on the Weibull distribution based on record values: Dallas (1982) discussed some distributional results based on upper record values. Balakrishnan and Chan (1994) established some simple recurrence relations satisfied by the single and the product moments, and derived the BLUE of the scale parameter when the shape parameter is known. Chan (1998) provided a conditional method to derive exact intervals for location and scale parameters of location-scale family that can be used to derive exact intervals for the shape parameter. Wu and Tseng (2006) provided some pivotal quantities to test and establish confidence interval of the shape parameter based on the first $n+1$ observed upper record values. Soliman et al. (2006) derived the Bayes estimates based on record values for the parameters with respect to squared error loss function and LINEX loss function. Asgharzadeh and Abdi (2011b) proposed joint confidence regions for the parameters. Teimouri and Gupta (2012) computed

the coefficient of skewness of upper/lower record statistics. Teimouri and Nadarajah (2013) derived exact expressions for constructing bias corrected maximum likelihood estimators (MLEs) of the parameters for the Weibull distribution based on upper records. Gouet et al. (2012) obtained the asymptotic properties for the counting process of δ -records among the first n observations.

In this paper, we consider inference about the parameters of Weibull distribution based on record values. First, we will propose a simple and exact method for constructing confidence interval and testing the hypotheses about the shape parameter β . Then using the concepts of generalized p-value and generalized confidence interval, a generalized approach for inference about the scale parameter α will be derived. Tsui and Weerahandi (1989) introduced the concept of generalized p-value, and Weerahandi (1993) introduced the concept of generalized confidence interval. These approaches have been used successfully to address several complex problems (see Weerahandi, 1995) such as confidence interval for the common mean of several log-normal distributions (Behboodian and Jafari, 2006), confidence interval for the mean of Weibull distribution (Krishnamoorthy et al., 2009), inference about the stress-strength reliability involving two independent Weibull distributions (Krishnamoorthy and Lin, 2010), and comparing two dependent generalized variances (Jafari, 2012).

We also present an exact joint confidence region for the parameters. Our simulation studies show that the area of our joint confidence region is smaller than those provided by other existing methods.

The rest of this article is organized as follows: A simple method for inference about shape parameter and a generalized approach for inference about the scale parameter are proposed in Section 2. Furthermore, a simulation study is performed and a real example is proposed in this Section. We also present a joint confidence region for the parameters α and β in Section 3.

2. Inference on the parameters

Suppose R_0, R_1, \dots, R_n are the first $n+1$ upper record values from a Weibull distribution with parameters α and β . In this section, we consider inference on the parameters α and β . From (1.1), the joint distribution of these record values can be written as

$$f_{\mathbf{R}}(\mathbf{r}) = \frac{\beta^{n+1}}{\alpha^{\beta(n+1)}} e^{-\left(\frac{r_n}{\alpha}\right)^\beta} \prod_{i=0}^n r_i^{\beta-1} \quad 0 < r_0 < r_1 < \dots < r_n. \quad (2.1)$$

Therefore, $(R_n, \sum_{i=0}^n \log(R_i))$ are sufficient statistics for (α, β) . Moreover, it can be easily shown that the MLE's of the parameters α and β are

$$\hat{\beta} = \frac{n+1}{\sum_{i=0}^n \log\left(\frac{R_n}{R_i}\right)}, \quad \hat{\alpha} = \frac{R_n}{(n+1)^{\frac{1}{\hat{\beta}}}}. \quad (2.2)$$

Theorem 2.1 Let R_0, R_1, \dots, R_n be the first $n + 1$ upper record values from a Weibull distribution. Then

- i. $U = 2\beta \sum_{i=0}^n \log\left(\frac{R_n}{R_i}\right)$ has a chi-square distribution with $2n$ degrees of freedom.
- ii. $V = 2\left(\frac{R_n}{\alpha}\right)^\beta$ has a chi-square distribution with $2n + 2$ degrees of freedom.
- iii. U and V are independent.

Proof. i. Define

$$Q_m = \frac{R_m}{R_{m-1}}, \quad m = 1, 2, \dots, n. \quad (2.3)$$

From Arnold et al. (1998) page 20, Q_m 's are independent random variables with

$$P(Q_m > q) = q^{-\beta m}, \quad q > 1,$$

and

$$2\beta m \log(Q_m) = 2\beta m \log\left(\frac{R_m}{R_{m-1}}\right) \sim \chi_{(2)}^2.$$

Therefore,

$$\begin{aligned} U &= 2\beta \sum_{i=0}^n \log\left(\frac{R_n}{R_i}\right) = 2\beta \sum_{i=0}^{n-1} \log\left(\frac{R_n}{R_{n-1}} \cdot \frac{R_{n-1}}{R_{n-2}} \cdots \frac{R_{i+1}}{R_i}\right) \\ &= 2\beta \sum_{i=0}^{n-1} \sum_{m=i+1}^n \log\left(\frac{R_m}{R_{m-1}}\right) = 2\beta \sum_{m=1}^n \sum_{i=0}^{m-1} \log(Q_m) = \sum_{m=1}^n 2\beta m \log(Q_m), \end{aligned}$$

has a chi-square distribution with $2n$ degrees of freedom.

ii. Define

$$Y = \left(\frac{X}{\alpha}\right)^\beta,$$

where X has a Weibull distribution with parameters α and β . Then, Y has an exponential distribution with parameter one. Therefore, we can conclude that V has a chi-square distribution with $2n + 2$ degrees of freedom (see Arnold et al., 1998, page 9).

iii. Let β be known. Then, it can be concluded from (2.1) that R_n is a complete sufficient statistic for α . Also, Q_m 's in (2.3) are ancillary statistics. Therefore, R_n and Q_m 's are independent, and the proof is completed. ■

2.1. Inference on the shape parameter

Here, we consider inference on the shape parameter, β from a Weibull distribution based on record values, and propose a simple and an exact method for constructing a confidence interval and testing the one-sided hypotheses

$$H_0 : \beta \leq \beta_0 \quad \text{vs.} \quad H_1 : \beta > \beta_0, \quad (2.4)$$

and the two-sided hypotheses

$$H_0 : \beta = \beta_0 \quad \text{vs.} \quad H_1 : \beta \neq \beta_0, \quad (2.5)$$

where β_0 is a specified value.

Based on Theorem 2.1, $U = 2\beta \sum_{i=0}^n \log \left(\frac{R_n}{R_i} \right)$ has a chi-square distribution with $2n$ degrees of freedom. Therefore, a $100(1 - \gamma)\%$ confidence interval for β can be obtained as

$$\left(\frac{\chi_{(2n), \gamma/2}^2}{2 \sum_{i=0}^n \log \left(\frac{R_n}{R_i} \right)}, \frac{\chi_{(2n), 1-\gamma/2}^2}{2 \sum_{i=0}^n \log \left(\frac{R_n}{R_i} \right)} \right), \quad (2.6)$$

where $\chi_{(k), \gamma}^2$ is the γ th percentile of the chi-square distribution with k degrees of freedom. Also, for testing the hypotheses in (2.4) and (2.5), we can define the test statistic

$$U_0 = 2\beta_0 \sum_{i=0}^n \log \left(\frac{R_n}{R_i} \right),$$

and the null hypothesis in (2.4) is rejected at nominal level γ if

$$U_0 > \chi_{(2n), 1-\gamma}^2,$$

and the null hypothesis in (2.5) is rejected if

$$U_0 < \chi_{(2n), \gamma/2}^2 \quad \text{or} \quad U_0 > \chi_{(2n), 1-\gamma/2}^2.$$

Wu and Tseng (2006) proposed the random variable

$$W(\beta) = \frac{\sum_{i=0}^n R_i^\beta}{(n+1) \left(\prod_{i=0}^n R_i \right)^{\frac{\beta}{n+1}}},$$

for inference about the shape parameter, and showed that $W(\beta)$ is an increasing function with respect to β . Also, its distribution does not depend on the parameters α and β . In fact, $W(\beta)$ is distributed as

$$W^* = \frac{\sum_{i=0}^n R_i^*}{(n+1)(\prod_{i=0}^n R_i^*)^{\frac{1}{n+1}}},$$

where R_i^* is the i th record from the exponential distribution with parameter one. However, its exact distribution is very complicated, and Wu and Tseng (2006) obtained the percentiles of $W(\beta)$ using Monte Carlo simulation. The confidence limits for β are obtained by solving the following equations numerically as

$$W(\beta) = W_{1-\gamma/2}^*, \quad W(\beta) = W_{\gamma/2}^*, \quad (2.7)$$

where W_{δ}^* is the δ th percentile of the distribution of W^* .

2.2. Inference on the scale parameter

Here, we consider inference about the scale parameter, α for a Weibull distribution based on record values, and propose an approach for constructing a confidence interval and testing the one-sided hypotheses

$$H_0 : \alpha \leq \alpha_0 \quad \text{vs.} \quad H_1 : \alpha > \alpha_0, \quad (2.8)$$

and the two-sided hypotheses

$$H_0 : \alpha = \alpha_0 \quad \text{vs.} \quad H_1 : \alpha \neq \alpha_0, \quad (2.9)$$

where α_0 is a specified value.

We did not find any approach in literature for inference about α based on record values when the shape parameter is unknown. Here, we use the concepts of generalized p-value and generalized confidence interval introduced by Tsui and Weerahandi (1989), and Weerahandi (1993), respectively. In the Appendix, we briefly review these concepts, and refer readers to Weerahandi (1995) for more details.

Let

$$T = r_n \left(\frac{2}{V} \right)^{\frac{2C_r}{U}} = r_n \left(\frac{\alpha}{R_n} \right)^{\frac{C_r}{\sum_{i=0}^n \log\left(\frac{R_n}{R_i}\right)}}, \quad (2.10)$$

where $C_r = \sum_{i=0}^n \log\left(\frac{r_n}{r_i}\right)$, and $r_i, i = 0, 1, \dots, n$ is the observed value of $R_i, i = 0, 1, \dots, n$, and U and V are independent random variables that are defined in Theorem 2.1. The

observed value of T is α , and distribution of T does not depend on unknown parameters α and β . Therefore, T is a generalized pivotal variable for α , and can be used to construct a generalized confidence interval for α .

Let

$$T^* = T - \alpha = r_n \left(\frac{2}{V} \right)^{\frac{2C_T}{U}} - \alpha.$$

Then, T^* is a generalized test variable for α , because i) the observed value of T^* does not depend on any parameters, ii) the distribution function of T^* is free from nuisance parameters and only depends on the parameter α , and iii) the distribution function of T^* is an increasing function with respect to the parameter α , and so, the distribution of T^* is stochastically decreasing in α . Therefore, the generalized p-value for testing the hypotheses in (2.8) is given as

$$p = P(T^* < 0 | H_0) = P(T < \alpha_0), \quad (2.11)$$

and the generalized p-value for testing the hypotheses in (2.9) is given as

$$p = 2 \min \{P(T > \alpha_0), P(T < \alpha_0)\}. \quad (2.12)$$

The generalized confidence interval for α based on T , and the generalized p-values in (2.11) and (2.12) can be computed using Monte Carlo simulation (Weerahandi, 1995; Behboodian and Jafari, 2006) based on the following algorithm:

Algorithm 2.1 For given r_0, r_1, \dots, r_n ,

1. Generate $U \sim \chi_{(2n)}^2$ and $V \sim \chi_{(2n+2)}^2$.
2. Compute T in (2.10).
3. Repeat steps 1 and 2 for a large number times, (say $M = 10000$), and obtain the values T_1, \dots, T_M .
4. Set $D_l = 1$ if $T_l < \alpha_0$ else $D_l = 0$, $l = 1, \dots, M$.

The $100(1 - \gamma)\%$ generalized confidence interval for α is $[T_{(\gamma/2)}, T_{(1-\gamma/2)}]$, where $T_{(\delta)}$ is the δ th percentile of T_l 's. Also, the generalized p-value for testing the one-sided hypotheses in (2.11) is obtained by $\frac{1}{M} \sum_{l=1}^M D_l$.

2.3. Real example

Roberts (1979) gave monthly and annual maximal of one-hour mean concentration of sulfur dioxide (in pphm, parts per hundred million) from Long Beach, California, for 1956 to 1974. Chan (1998) showed that the Weibull distribution is a reasonable model

for this data set. Wu and Tseng (2006) also study this data set. The upper record values for the month of October from the data are

$$26, 27, 40, 41.$$

The 95% confidence interval for the scale parameter α based on our generalized confidence interval with $M = 10000$ is obtained as (5.4869, 39.9734). The 95% confidence interval for the shape parameter β in (2.6) is obtained as (0.6890, 8.0462), and based on Wu and Tseng's method in (2.7) is obtained as (0.6352, 7.7423). Also, the generalized p-value is equal to 0.0227 for testing the hypotheses in (2.8) with $\alpha_0 = 5$. Therefore, the null hypothesis is rejected.

2.4. Simulation study

We performed a simulation study in order to evaluate the accuracy of the proposed methods for constructing confidence interval for the parameters of Weibull distribution. For this purpose, we generated $n + 1$ record values from a Weibull distribution, and considered $\alpha = 1, 2$. For the simulation with 10000 runs and different values of the shape parameter β , the empirical coverage probabilities and expected lengths of the methods with the confidence coefficient 0.95 were obtained. The results of our generalized confidence interval for inference on α using the algorithm 2.1 with $M = 10000$ are presented in Table 1, and the results of our exact method (E) and the Wu method (W) for inference on β are given in Table 2. We can conclude that

Table 1: Empirical coverage probabilities and expected lengths of the generalized confidence interval for the parameter α with confidence level 0.95.

	α	n	β						
			0.5	1.0	1.2	1.5	2.0	3.0	5.0
Empirical Coverage	1.0	3	0.951	0.949	0.953	0.947	0.947	0.948	0.948
		7	0.952	0.949	0.950	0.950	0.951	0.953	0.952
		9	0.951	0.948	0.953	0.951	0.948	0.949	0.950
		14	0.945	0.949	0.950	0.950	0.954	0.952	0.952
	2.0	3	0.949	0.952	0.947	0.949	0.951	0.950	0.953
		7	0.948	0.953	0.950	0.946	0.954	0.948	0.951
		9	0.952	0.948	0.953	0.950	0.952	0.953	0.954
		14	0.950	0.946	0.949	0.951	0.951	0.952	0.955
Expected Length	1.0	3	16.740	3.581	2.804	2.155	1.653	1.211	0.847
		7	13.575	3.198	2.477	1.942	1.475	1.041	0.686
		9	13.505	3.138	2.469	1.918	1.446	1.008	0.651
		14	13.122	3.082	2.403	1.854	1.376	0.943	0.596
	2.0	3	33.516	7.187	5.579	4.341	3.323	2.427	1.704
		7	27.960	6.344	4.999	3.899	2.960	2.080	1.364
		9	27.342	6.304	4.935	3.831	2.890	2.016	1.302
		14	26.626	6.129	4.779	3.705	2.757	1.886	1.191

Table 2: Empirical coverage probabilities and expected lengths of the methods for constructing confidence interval for the parameter β with confidence level 0.95.

	α	n	Method	β						
				0.5	1.0	1.2	1.5	2.0	3.0	5.0
Empirical Coverage	1.0	3	W	0.950	0.952	0.952	0.949	0.949	0.953	0.947
			E	0.949	0.953	0.953	0.950	0.948	0.953	0.946
		7	W	0.951	0.949	0.950	0.948	0.949	0.949	0.953
			E	0.951	0.950	0.948	0.950	0.953	0.953	0.950
		9	W	0.949	0.949	0.945	0.949	0.947	0.949	0.950
			E	0.948	0.950	0.948	0.948	0.949	0.952	0.949
		14	W	0.946	0.949	0.951	0.951	0.953	0.949	0.951
			E	0.947	0.948	0.950	0.951	0.953	0.952	0.952
	2.0	3	W	0.954	0.955	0.948	0.950	0.950	0.952	0.949
			E	0.953	0.953	0.947	0.949	0.949	0.952	0.950
		7	W	0.950	0.955	0.947	0.948	0.952	0.947	0.948
			E	0.949	0.952	0.951	0.948	0.952	0.947	0.950
		9	W	0.953	0.948	0.956	0.950	0.953	0.951	0.952
			E	0.952	0.948	0.951	0.951	0.953	0.951	0.953
		14	W	0.948	0.947	0.949	0.950	0.951	0.951	0.952
			E	0.950	0.947	0.949	0.949	0.953	0.951	0.955
Expected Length	1.0	3	W	1.704	3.431	4.194	5.279	6.913	10.396	17.479
			E	1.630	3.285	4.013	5.041	6.611	9.937	16.722
		7	W	0.932	1.879	2.224	2.808	3.752	5.578	9.276
			E	0.853	1.716	2.038	2.574	3.437	5.115	8.499
		9	W	0.806	1.603	1.928	2.412	3.222	4.797	8.024
			E	0.730	1.450	1.748	2.185	2.928	4.352	7.267
		14	W	0.625	1.262	1.509	1.888	2.505	3.766	6.263
			E	0.558	1.125	1.343	1.685	2.236	3.353	5.590
	2.0	3	W	1.713	3.458	4.156	5.277	6.856	10.307	16.998
			E	1.638	3.306	3.967	5.053	6.560	9.859	16.266
		7	W	0.934	1.866	2.208	2.822	3.738	5.629	9.392
			E	0.854	1.710	2.026	2.589	3.419	5.151	8.581
		9	W	0.808	1.600	1.923	2.418	3.189	4.798	8.032
			E	0.733	1.451	1.743	2.193	2.890	4.347	7.276
		14	W	0.628	1.260	1.496	1.888	2.523	3.768	6.302
			E	0.560	1.124	1.338	1.688	2.250	3.366	5.624

- i. The empirical coverage probabilities of all methods are close to the confidence level 0.95.
- ii. The expected lengths of E and W increase when the parameter β increases. Additionally, the expected length of E is smaller than W especially when β is large.

- iii. The expected length of our generalized confidence interval for α decreases when the parameter β increases. Moreover, it is very large when β is small.
- iv. The expected lengths of all methods decrease when the number of records increases.
- v. The empirical coverage probabilities and expected lengths of W and E do not change when the parameter α changes.

3. Joint confidence regions for the parameters

Suppose R_0, R_1, \dots, R_n are the first $n+1$ upper record values from a Weibull distribution with parameters α and β . In this section, we presented a joint confidence region for the parameters α and β . This is important because it can be used to find confidence bounds for any function of the parameters such as the reliability function $R(t) = \exp(-(\frac{t}{\alpha})^\beta)$. For more references about the joint confidence region based on records, see Asgharzadeh and Abdi (2011a,b) and Asgharzadeh et al. (2011).

3.1. Asgharzadeh and Abdi method

Asgharzadeh and Abdi (2011b) present exact joint confidence regions for the parameters of Weibull distribution based on the record values using the idea presented by Wu and Tseng (2006). The following inequalities determine $100(1-\gamma)\%$ joint confidence regions for α and β :

$$A_j = \begin{cases} \frac{\log\left(\left(\frac{n-j+1}{j}\right)k_1 + 1\right)}{\log\left(\frac{R_n}{R_{j-1}}\right)} < \beta < \frac{\log\left(\left(\frac{n-j+1}{j}\right)k_2 + 1\right)}{\log\left(\frac{R_n}{R_{j-1}}\right)} \\ R_n \left(\frac{2}{\chi_{(2n+2), (1+\sqrt{1-\gamma})/2}^2}\right)^{\frac{1}{\beta}} < \alpha < R_n \left(\frac{2}{\chi_{(2n+2), (1-\sqrt{1-\gamma})/2}^2}\right)^{\frac{1}{\beta}} \end{cases}, \quad (3.1)$$

for $j = 1, \dots, n$, where

$$k_1 = F_{(2n-2j+2, 2j), (1-\sqrt{1-\gamma})/2} \quad k_2 = F_{(2n-2j+2, 2j), (1+\sqrt{1-\gamma})/2},$$

and $F_{(a,b),\gamma}$ is the γ th percentile of the F distribution with a and b degrees of freedom. Note that for each j , we have a joint confidence region for α and β . Asgharzadeh and Abdi (2011b) found that in most cases $A_{\lfloor \frac{n+1}{5} \rfloor}$ and $A_{\lfloor \frac{n+1}{5} \rfloor + 1}$ provide the smallest confidence areas, where $\lfloor x \rfloor$ is the largest integer value smaller than x .

3.2. A new joint confidence region

From Theorem 2.1, $U = 2\beta \sum_{i=0}^n \log \left(\frac{R_n}{R_i} \right)$ has a chi-square distribution with $2n$ degrees of freedom and $V = 2 \left(\frac{R_n}{\alpha} \right)^\beta$ has a chi-square distribution with $2n+2$ degrees of freedom, and U and V are independent. Therefore, an exact joint confidence region for the parameters α and β of Weibull distribution based on the record values can be given as

$$B = \left\{ \begin{array}{l} \frac{\chi_{(2n), (1-\sqrt{1-\gamma})/2}^2}{2 \sum_{i=0}^n \log \left(\frac{R_n}{R_i} \right)} < \beta < \frac{\chi_{(2n), (1+\sqrt{1-\gamma})/2}^2}{2 \sum_{i=0}^n \log \left(\frac{R_n}{R_i} \right)} \\ R_n \left(\frac{2}{\chi_{(2n+2), (1+\sqrt{1-\gamma})/2}^2} \right)^{\frac{1}{\beta}} < \alpha < R_n \left(\frac{2}{\chi_{(2n+2), (1-\sqrt{1-\gamma})/2}^2} \right)^{\frac{1}{\beta}} \end{array} \right. \quad (3.2)$$

Remark 3.1 All record values are used in the proposed joint confidence region in (3.2) but not in the proposed joint confidence regions in (3.1).

3.3. Real example

Here, we consider the upper record values in the example given in Section 2.3. Therefore, the 95% joint confidence regions for α and β based on Asgharzadeh and Abdi (2011b) in (3.1) are

$$\begin{aligned} A_1 &= \left\{ (\alpha, \beta) : 0.5826 < \beta < 11.9955, 41(0.1029)^{\frac{1}{\beta}} < \alpha < 41(1.1318)^{\frac{1}{\beta}} \right\} \\ A_2 &= \left\{ (\alpha, \beta) : 0.1646 < \beta < 6.4905, 41(0.1029)^{\frac{1}{\beta}} < \alpha < 41(1.1318)^{\frac{1}{\beta}} \right\} \\ A_3 &= \left\{ (\alpha, \beta) : 0.1720 < \beta < 58.9824, 41(0.1029)^{\frac{1}{\beta}} < \alpha < 41(1.1318)^{\frac{1}{\beta}} \right\} \end{aligned}$$

and the 95% joint confidence region for α and β in (3.2) is

$$B = \left\{ (\alpha, \beta) : 0.5305 < \beta < 9.0277, 41(0.1029)^{\frac{1}{\beta}} < \alpha < 41(1.1318)^{\frac{1}{\beta}} \right\}.$$

The plot of all joint confidence regions are given in Figure 1. Also, the area of the joint confidence regions A_1 , A_2 , A_3 , and B are 194.9723, 166.7113, 369.7654, and 172.5757, respectively.

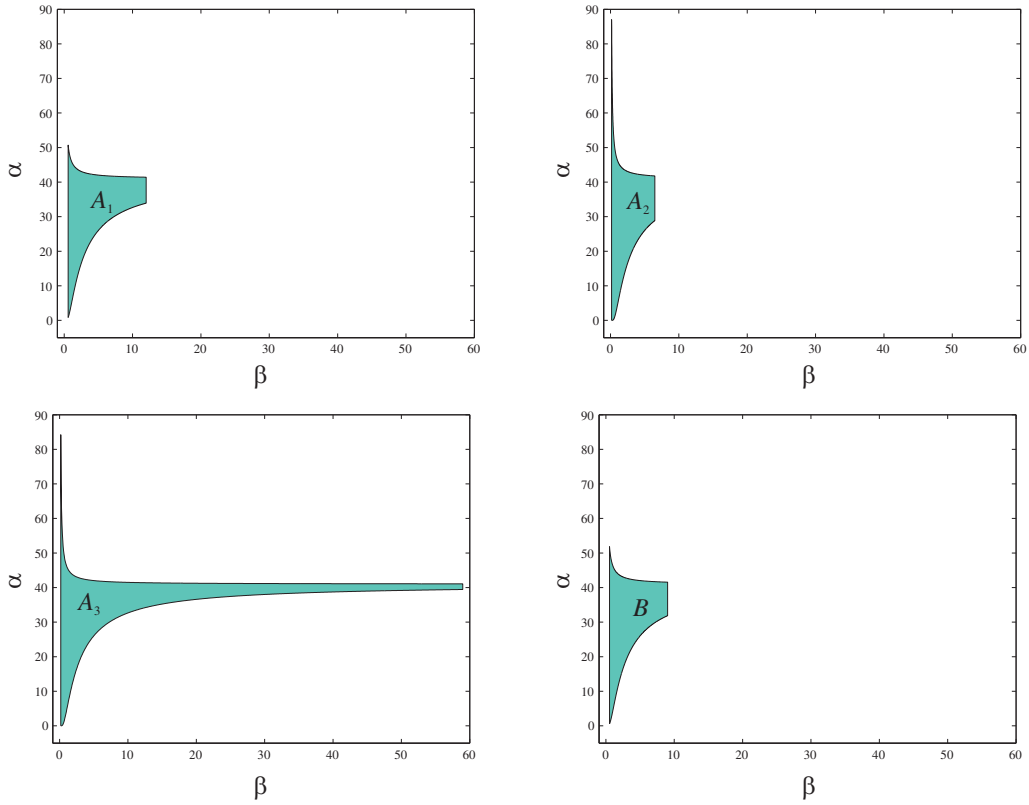


Figure 1: The plot of the joint confidence regions A_1 , A_2 , A_3 , and B .

3.4. Simulation study

We performed a similar simulation given in Section 2.4 with considering $\alpha = 1$, in order to compare the joint confidence regions proposed by Asgharzadeh and Abdi (2011b) and our joint confidence region (B) in (3.2). Here, we consider the confidence areas $A_{\lfloor \frac{n+1}{5} \rfloor}$ and $A_{\lfloor \frac{n+1}{5} \rfloor + 1}$ because the coverage probabilities of all A_i 's are close to the confidence coefficient and Asgharzadeh and Abdi (2011b) found that in most cases these two confidence areas provide the smallest confidence areas. The empirical coverage probabilities and expected areas of the methods for the confidence coefficient 95% are given in Table 3. We can conclude that

1. The coverage probabilities of the all methods are close to the confidence coefficient 0.95.
2. The expected area of our method is smaller than the expected areas of the proposed methods by Asgharzadeh and Abdi (2011b).
3. The expected areas of all methods decrease when the number of records increases.
4. The expected areas of all methods decrease when the parameter β increases.

Table 3: Empirical coverage probabilities of the methods for constructing joint confidence region for the parameters α and β with $\gamma = 0.05$.

	n	Region	β						
			0.5	1.0	1.2	1.5	2.0	3.0	5.0
Coverage Probability	4	A_1	0.950	0.949	0.949	0.951	0.954	0.946	0.950
		A_2	0.949	0.951	0.950	0.951	0.953	0.949	0.952
		B	0.949	0.950	0.949	0.952	0.954	0.949	0.950
	6	A_1	0.951	0.951	0.952	0.952	0.954	0.949	0.949
		A_2	0.952	0.948	0.950	0.948	0.953	0.948	0.952
		B	0.953	0.949	0.951	0.951	0.953	0.950	0.951
	9	A_2	0.953	0.949	0.950	0.955	0.949	0.948	0.953
		A_3	0.951	0.951	0.951	0.956	0.949	0.948	0.949
		B	0.950	0.952	0.951	0.953	0.949	0.948	0.952
	14	A_3	0.947	0.950	0.954	0.948	0.954	0.951	0.952
		A_4	0.946	0.951	0.952	0.948	0.951	0.950	0.952
		B	0.948	0.949	0.952	0.947	0.951	0.953	0.952
	29	A_6	0.951	0.953	0.950	0.950	0.948	0.948	0.950
		A_7	0.950	0.952	0.951	0.949	0.948	0.948	0.950
		B	0.953	0.955	0.951	0.953	0.950	0.948	0.952
Expected Area	4	A_1	27.787	8.548	7.339	6.331	5.725	5.330	5.203
		A_2	30.020	8.976	7.651	6.682	5.989	5.593	5.504
		B	22.985	7.371	6.388	5.596	5.099	4.792	4.713
	6	A_1	21.062	6.036	5.213	4.576	4.102	3.783	3.701
		A_2	20.035	5.824	5.046	4.436	3.985	3.701	3.648
		B	14.551	4.714	4.144	3.714	3.399	3.192	3.162
	9	A_2	14.631	4.081	3.533	3.059	2.756	2.574	2.510
		A_3	14.651	4.086	3.545	3.077	2.767	2.585	2.541
		B	9.639	3.137	2.774	2.471	2.275	2.160	2.133
	14	A_3	9.436	2.702	2.320	2.035	1.816	1.701	1.661
		A_4	9.388	2.686	2.304	2.035	1.812	1.707	1.668
		B	5.784	1.999	1.763	1.599	1.471	1.405	1.385
	29	A_6	4.244	1.298	1.124	0.988	0.905	0.849	0.828
		A_7	4.202	1.291	1.118	0.985	0.904	0.850	0.828
		B	2.380	0.932	0.838	0.761	0.719	0.689	0.678

Appendix. Generalized p-value and generalized confidence interval

Let X be a random variable whose distribution depends on a parameter of interest θ , and a nuisance parameter λ . Let x denote the observed value of X . A generalized pivotal quantity for θ is a random quantity denoted by $T(X;x;\theta)$ that satisfies the following conditions:

- (i) The distribution of $T(X;x;\theta)$ is free of any unknown parameters.

(ii) The value of $T(\mathbf{X}; \mathbf{x}; \theta)$ at $\mathbf{X} = \mathbf{x}$, i.e., $T(\mathbf{x}; \mathbf{x}; \theta)$ is free of the nuisance parameter λ .

Appropriate percentiles of $T(\mathbf{X}; \mathbf{x}; \theta)$ form a confidence interval for θ . Specifically, if $T(\mathbf{x}; \mathbf{x}; \theta) = \theta$, and T_δ denotes the 100δ percentage point of $T(\mathbf{X}; \mathbf{x}; \theta)$ then $(T_{\gamma/2}, T_{1-\gamma/2})$ is a $1 - \gamma$ generalized confidence interval for θ . The percentiles can be found because, for a given \mathbf{x} , the distribution of $T(\mathbf{X}; \mathbf{x}; \theta)$ does not depend on any unknown parameters.

In the above setup, suppose we are interested in testing the hypotheses

$$H_0 : \theta \leq \theta_0 \quad \text{vs.} \quad H_1 : \theta > \theta_0, \quad (\text{A.1})$$

for a specified θ_0 . The generalized test variable, denoted by $T^*(\mathbf{X}; \mathbf{x}; \theta)$, is defined as follows:

- (i) The value of $T^*(\mathbf{X}; \mathbf{x}; \theta)$ at $\mathbf{X} = \mathbf{x}$ is free of any unknown parameters.
- (ii) The distribution of $T^*(\mathbf{X}; \mathbf{x}; \theta)$ is stochastically monotone (i.e., stochastically increasing or stochastically decreasing) in θ for any fixed \mathbf{x} and λ .
- (iii) The distribution of $T^*(\mathbf{X}; \mathbf{x}; \theta)$ is free of any unknown parameters.

Let $t^* = T^*(\mathbf{x}; \mathbf{x}; \theta_0)$, the observed value of $T^*(\mathbf{X}; \mathbf{x}; \theta)$ at $(\mathbf{X}; \theta) = (\mathbf{x}; \theta_0)$. When the above conditions hold, the generalized p-value for testing the hypotheses in (A.1) is defined as

$$p = P(T^*(\mathbf{X}; \mathbf{x}; \theta_0) \leq t^*) \quad (\text{A.2})$$

where $T^*(\mathbf{X}; \mathbf{x}; \theta)$ is stochastically decreasing in θ . The test based on the generalized p-value rejects H_0 when the generalized p-value is smaller than a nominal level γ . However, the size and power of such a test may depend on the nuisance parameters.

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