Uncertainty propagation in complex networks: from noisy links to critical properties

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Many complex networks are built up from empirical data prone to experimental error. Thus, the determination of the specific weights of the links is a noisy measure. Noise propagates to those macroscopic variables researchers are interested in, such as the critical threshold for synchronization of coupled oscillators or for the spreading of a disease. Here, we apply error propagation to estimate the macroscopic uncertainty in the critical threshold for some dynamical processes in networks with noisy links. We obtain closed form expressions for the mean and standard deviation of the critical threshold depending on the properties of the noise and the moments of the degree distribution of the network. The analysis provides confidence intervals for critical predictions when dealing with uncertain measurements or intrinsic fluctuations in empirical networked systems. Furthermore, our results unveil a non-monotonous behavior of the uncertainty of the critical threshold that depends on the specific network structure.

The critical point (or threshold) refers to the minimum value of a tuning parameter that triggers a phase transition in a dynamical process. This value becomes an uncertain quantity when noise is present in the underlying network of interactions. For the class of processes where the threshold is determined by the inverse of the largest eigenvalue of the adjacency matrix, we apply error propagation to estimate the critical uncertainty induced by uncorrelated noise on the network weights. We obtain closed form expressions for the mean and standard deviation of the threshold depending on the properties of the noise and the moments of the network degree distribution. Our results can be used in practical situations, to provide confidence intervals for the predictions of the critical threshold when dealing with uncertain measurements or intrinsic fluctuations in a networked system. Furthermore, the results unveil several noise-amplifying properties of the networks, including a non-monotonous behavior of the uncertainty of the threshold depending on the heterogeneity and density of the network. Accordingly, our analysis predicts the existence of particular structures that maximize the uncertainty of the threshold only due to small noise in the weights, without altering the underlying structure of the links.

I. INTRODUCTION

The study of critical phenomena has been, and still is, a fruitful area of research in network science[1]. Critical phenomena in networks include a wide set of aspects, from structural changes in networks, or percolation phenomena[2], to epidemic[3] or synchronization[4] thresholds and many other phase transitions in dynamical processes defined on networks[1,5]. The estimation of the critical threshold is of upmost importance to predict the onset of the phase transition, and hence a major concern in several applications, such as the containment of an infectious disease[6] or the control of synchronization in the power grid[7,8]. However, an accurate estimation of the threshold is often elusive and costly because it depends on the particular details of the whole network structure, usually through its eigenvalues.

As network science becomes more and more extended, its potential applications grow fuelled by the necessity of analyzing data produced in diverse fields of research, such as sociology, biology, experimental physics, etc. However, the data collected in any of the former fields is not free from experimental error, induced for example by sampling biases, device accuracy, or mistakes in data entry. Nevertheless, the literature on network science usually dismisses these error sources, and produces results that are only valid if data is error free. Some authors have concentrated their attention on inference of missing data in networks[9–12] , however, to the best of our knowledge, no similar attention has been paid to the propagation of uncertainty from the structure to the properties of dynamical processes running on it.

The lack of works devoted to the analysis of error propagation in networks is probably due to the fact that many studies consider networks unweighted, where a link is a binary variable denoting its existence or not. However, the vast majority of networks are weighted, i.e. the existence or not is valued by its intensity. The accurate determination of the weight is unlikely, and therefore, the error in their numerical values will influence any particular measurement of the network properties.

Here, we present a study of error propagation in networks where links are subject to uncertainty in their weights, and wonder about the effect that this uncertainty will have in the determination of the critical threshold. In particular, we focus on those dynamical processes in which the critical point is known to be inversely proportional to the largest eigenvalue

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of the connectivity matrix. In section II, we present the particularities of our analysis and derive our main results, in section III we study the range of uncertainty in the critical point for different network structures, and finally, in section IV we discuss the implications and limitations of the current study, paving the way for new analysis to come.

II. ERROR PROPAGATION OF UNCERTAINTY IN THE CRITICAL THRESHOLD

We consider a dynamical process running on top of a complex network with N units. We restrict the study to the class of dynamical models in which a phase transition occurs at a critical value of the coupling intensity (the threshold), and where this value is given in terms of the largest eigenvalue λ_{max} of the network connectivity matrix $\mathbf{C}$ whose values represent the weighted structure of the network[13]

$$K_c = \frac{K_0}{\lambda_{max}(\mathbf{C})}, \tag{1}$$

where K_0 is a constant that depends on the specific details of the particular process. Without loss of generality, we fix $K_0 = 1$. Eq.(1) estimates the threshold for a wide variety of dynamical processes, including the synchronization of heterogeneous phase-oscillators[4], the onset of endemicity of a disease in epidemic models[3,14], and the phase transition in the Ising model in networks, to name a few[1,2,5]. The aim of this work is to understand how small noise in the entries of $\mathbf{C}$ affects the statistical properties of the macroscopic threshold given by Eq.(1), without looking into the details of a specific dynamical model. For the sake of simplicity, we assume that the noise in the entries is gaussian and uncorrelated (white gaussian noise) where each weight is drawn from a normal distribution $N(\mu, \sigma^2)$, being $\mu > 0$ the average weight and σ^2 its variance. Nevertheless, the proposed analysis can be extended other distributions of noise, either theoretical or obtained through empirical measurements.

To study the exact statistics of K_c in Eq.(1) induced by the presence of noise, one could use in principle the available tools from Random Matrix Theory[15,16] and Spectral Graph Theory[17,18]. However, it becomes very challenging to study noisy sparse networks with arbitrary degree distributions in these frameworks. Here, we use an alternative approach, based on applying error propagation to the mean-field approximation of Eq.(1). This approximation obviously restricts the validity range of the analysis, however, the results are found to be very accurate in some scenarios and, more importantly, they provide clear analytical insight on how the uncertainty in the structure affects the determination of the critical threshold.

Our derivation starts assuming a mean–field approach. For simplicity, we restrict to the case of undirected (symmetric) networks. Under the aforementioned conditions, the critical threshold in Eq.(1) can be approximated[19–21] by

$$K_c = \frac{\langle s \rangle}{\langle s^2 \rangle}, \tag{2}$$

where $\langle s^n \rangle$ is the n-moment of the strength distribution (the strength of a node is the sum of in-coming/out-going weights). Eq.(2) can also be obtained directly from the equations of motion of the dynamical process (for instance in the Kuramoto Model[20]) by assuming that the local field in a node is proportional to the global field weighted by the in-strength of the node[4]. For the rest of the article, we will refer to Eq.(2) as the Mean-Field approximation (MFA).

First we test the accuracy of the critical threshold in the MFA, Eq.(2), compared to the exact result, Eq.(1), in Erdös-Rényi networks with uncertainty in the weights. In Fig.(1) we plot the threshold distribution for two different values of the intensity of the uncertainty σ. We observe that the MFA accurately determines the distribution, and that the values of the expected critical threshold K_c and its variance are clearly dependent on σ. In general, we expect our results to be accurate in the cases in which the approximation of Eq.(2) remains valid.

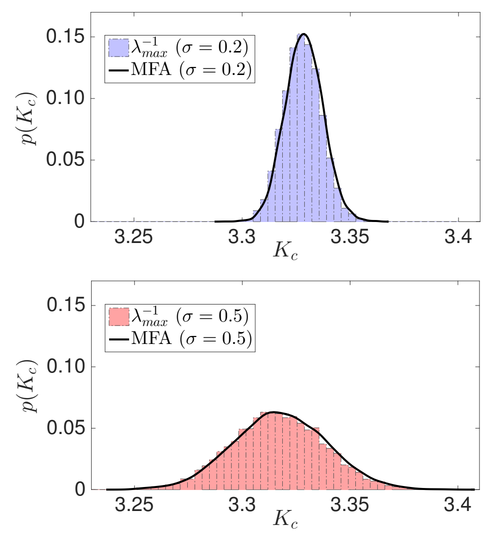

FIG. 1. Empirical distribution of the critical point K_c governed by Eq.(1) (boxes) and MFA (solid lines) in an Erdös-Rényi network with $N = 200$, $p = 0.3$, $K_0 = 1$, $\mu = 1$ for two different noise intensities ($\sigma = 0.2$ grey and $\sigma = 0.5$ red). The distribution corresponds to 10^4 independent realizations of the noise.

Using Eq.(2), we can express K_c in terms of the moments of the degree distribution and noise parameters. The detailed calculations are shown in Appendix A. We obtain

$$K_c = \frac{\sum_{i=1}^{N} \mu_i k_i}{\sum_{i=1}^{N} \mu_i^2 (k_i^2 - k_i) + \sum_{i=1}^{N} \langle w^2 \rangle_i k_i}, \tag{3}$$

where μ_i is the average weight of node i, and $\langle w^2 \rangle_i$ the average second moment of the weight distribution for node i. In random homogeneous networks, for sufficiently large degree ($k_i \gg 1$), we can approximate $\mu_i = \mu$, and $\langle w^2 \rangle_i = \sigma^2 + \mu^2$ in Eq.(3). This approximation allows to write down a simple relation between the mean of the critical threshold and the uncertainty of the network as

$$\langle K_c \rangle \approx \frac{\mu \langle k \rangle}{\mu^2 \langle k^2 \rangle + \sigma^2 \langle k \rangle}. \quad (4)$$

Interestingly, the naïve approximation in Eq.(4) already informs that the critical threshold decreases as the noise intensity σ increases. This can be understood because the noise increases the structural heterogeneity of the network, and heterogeneity tends to make the epidemic threshold to vanish. Note that for $\mu = 1$ and $\sigma = 0$, we recover the usual threshold for unweighted, undirected networks[21] and for $\sigma \ll 1$, $\langle K_c \rangle \approx \langle k \rangle / \mu \langle k^2 \rangle$.

Now, we estimate confidence intervals for the uncertainty of K_c, that is the standard deviation named here δK_c (or the variance $(\delta K_c)^2$). For this purpose, we use the method of error propagation[22,23], that quantifies how the error in the microscopic variables of a system (the $2N$ random variables in our nodal description) propagate through a macroscopic quantity (the critical threshold K_c). In a first-order expansion, we have

$$(\delta K_c)^2 \approx J_0^T \mathbf{V} J_0, \quad (5)$$

with $J \in R^{2N}$ the Jacobian of the system evaluated at the mean values of the random variables $\vec{\mu}$ and $\langle \vec{w^2} \rangle$ and $\mathbf{V} \in R^{2N \times 2N}$ the covariance matrix, which depends on the full connectivity matrix $\mathbf{C}$. The details of these calculations (for white gaussian noise and fixing $K_0 = 1$) are shown in Appendix B. Finally, we obtain the following closed form expression

$$(\delta K_c)^2 \approx a[\mu^4(2\langle k \rangle \langle k^3 \rangle - \langle k^2 \rangle^2) - 2\mu^2\sigma^2(\langle k \rangle \langle k^2 \rangle - \langle k \rangle^2) + \sigma^4 \langle k \rangle^2] \quad (6)$$

with $a = 2\sigma^2 \langle k \rangle / [N(\mu^2 \langle k^2 \rangle + \sigma^2 \langle k \rangle)^4]$.

Eq.(6) shows that, beyond the non-linear dependence on the network and noise parameters, the uncertainty in the threshold is a finite-size effect, and decays with $N^{-1/2}$. To compare networks of different sizes, we will scale the threshold by the size N in the current analysis.

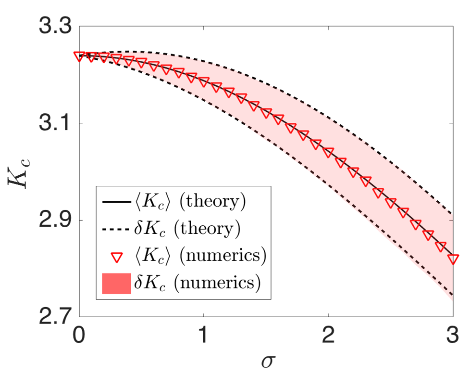

FIG. 2. Numerics (Eq.(1)) vs theory (Eqs.(4,6)): mean and standard deviation of the threshold K_c depending on the noise intensity σ for an Erdös-Rényi network with $N = 200$, $p = 0.3$, $\mu = 1$, and 5000 independent realizations for each value of the noise intensity σ.

In Fig.(2), we show the accuracy of the theoretical expressions for an Erdös-Rényi network, confirming the validity of the approach, at least for small noise and homogeneous structures. Note that the linear approximation used in Eq.(5) is valid as fas as[23]

$$J_0^T \mathbf{V} J_0 \gg \frac{1}{2} \mathrm{Tr}[(\mathbf{H_0} \mathbf{V})^2] \quad (7)$$

where $\mathbf{H_0} \in R^{2N \times 2N}$ is the Hessian matrix of the system evaluated at the mean values of the random variables. The detailed calculations of $\mathbf{H_0}$ are shown in Appendix B. Both terms in Eq.(7) depend implicitly on the value of the noise, so their scaling with σ will determine the range of validity of Eq.(6). We numerically examine the goodness of both the linear, Eq.(5) and the second-order approximation for the uncertainty δK_c

$$(\delta K_c)^2 \approx J_0^T \mathbf{V} J_0 + \frac{1}{2} \mathrm{Tr}[(\mathbf{H_0} \mathbf{V})^2] \quad (8)$$

against the numerical results obtained for the Erdös-Rényi network analyzed so far, and also for a real world network with large size and heterogeneous connectivity patterns (the worldwide air transportation network). The air transportation network was constructed using data from the website openflights.org, which has information about the traffic between airports updated to 2012, data available from[6]. This network accounts for the largest connected component, with 3154 nodes and 18,592 edges.

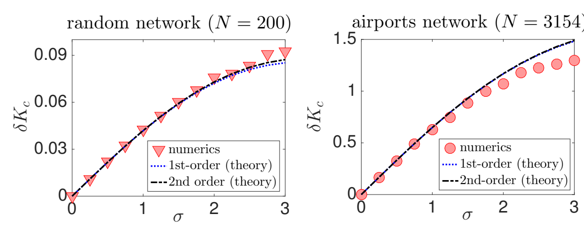

FIG. 3. Numerics vs theory: standard deviation of the critical threshold δK_c depending on the noise intensity σ with $\mu = 1$ for a (left) fixed Erdös-Rényi network ($N = 200$, $\langle k \rangle = 60$, $p = 0.3$) and (right) the empirical network of airports ($N = 3154$, $\langle k \rangle \approx 6$) for 2000 independent realizations for each value of the noise. Results have been rescaled by N.

Fig.(3) shows that the first and second order solutions are practically indistinguishable for small noise, therefore validating the result in Eq.(6) in this regime. The deviation of the theory from the actual values in the empirical network (right plot in Fig.(3)) points towards another direction: the goodness of the MFA itself. Basically, the theory is expected to be accurate for networks that deviate from a random structure as long as the MFA in Eq.(2) holds. It is not the main goal of this article to convey an exhaustive verification of the theory for particular networks, and we refer the reader to the literature[19,21,24] for details on the validity of the MFA. Moreover, it is important to remark that even if the MFA holds, the method of error propagation (at any order) can only be applied in our problem when the mean of the signal μ is sufficiently large compared to the noise.

III. THE ROLE OF THE STRUCTURE IN THE ERROR PROPAGATION

The network structure plays an important role in the uncertainty range of K_c. After the finding of Eq.(6), some interesting questions arise: does the heterogeneity induce an increase of the critical fluctuations with respect to a homogeneous network? Is the behavior of (δK_c) monotonous with the moments of the degree distribution of the network? If not, is there any particular structure that maximizes the uncertainty of the critical point induced by noise in the weights?

To answer these questions, we consider the regime where networks are sufficiently large and $\sigma \ll \mu$. Then, we can approximate Eq.(6) by its leading term, neglecting terms in σ larger than $\mathscr{O}(\sigma^2)$

$$(\delta K_c)^2 \approx 2\sigma^2 \frac{2\langle k\rangle\langle k^3\rangle - \langle k^2\rangle^2}{N\langle k\rangle^3}\langle K_c\rangle^4. \tag{9}$$

Note that δK_c increases linearly with the noise intensity and scales with $\langle K_c\rangle^2$. We know that $\langle K_c\rangle^2$ is reduced by the heterogeneity of the degree distribution, and therefore one would expect δK_c to follow the same trend. However, the nonlinear dependence on the moments of the degree distribution could change this intuition.

To understand this effect, we choose first as a reference the most homogeneous network we can consider, a regular network, i.e. $k_i = k, \forall i$. We compute K_c and δK_c for a regular network, obtaining

$$\langle K_c\rangle_{\text{reg}} \approx \frac{1}{\mu k},$$
$$(\delta K_c)^2_{\text{reg}} \approx \frac{2\sigma^2}{N\mu^4 k^3}. \tag{10}$$

The role of the heterogeneity will be detected by comparing $(\delta K_c)^2$ with $(\delta K_c)^2_{\text{reg}}$ for networks with the same size and average degree, and for the same noise parameters μ and σ. After some algebra, the condition for a given network to display higher uncertainty in K_c than a random regular network reads

$$\langle k^3\rangle > \frac{\langle k^2\rangle^2}{2\langle k\rangle}\left(1 + \frac{\langle k^2\rangle^2}{\langle k\rangle^4}\right). \tag{11}$$

Now, we can use Eq.(11) to evaluate the role of heterogeneity. Let us consider a power-law distribution $p(k) \approx k^{-\gamma}$, where the exponent γ controls the tail of the distribution. For the value $\gamma = 3$, one recovers the well-know scale-free network that emerges from preferential attachment[25]. For lower (higher) values of γ, the network becomes more (less) heterogeneous. For a finite power-law network, the moments of the degree distribution are given by

$$\langle k^n\rangle = \frac{(-\gamma+1)(k_{\text{max}}^{n-\gamma+1} - k_{\text{min}}^{n-\gamma+1})}{(n-\gamma+1)(k_{\text{max}}^{\gamma+1} - k_{\text{min}}^{\gamma+1})}. \tag{12}$$

By fixing the value of k_{min}, we can explore the space of networks with a given (γ, k_{max}), thus revealing the effect of heterogeneity and size. To simplify the visualization, we define

$$q = \log\left[\frac{2\langle k\rangle\langle k^3\rangle}{\langle k^2\rangle^2\left(1 + \frac{\langle k^2\rangle^2}{\langle k\rangle^4}\right)}\right]. \tag{13}$$

This way, when $q = 0$, the uncertainty of the critical threshold of a network is the same than that of the regular one, and for positive (negative) values of q, we are measuring an increase (decrease) of δK with respect to the homogeneous network. In Fig.(4) we show the theoretical results obtained for the q value of networks in the space (γ, k_{max}). We note that the three

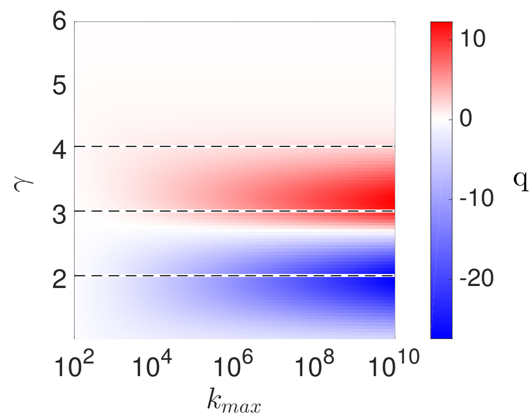

FIG. 4. Colormap showing the theoretical dependence of q on the exponent γ and the maximum degree of the network k_{max}. The value of k_{min} is fixed to $k_{\text{min}} = 5$ and the resolution of the map is 100x100.

horizontal lines correspond to the cases where the network has an integer exponent of 2, 3 or 4. In these cases, the first, second or third moments diverge. It is also important to remark that below $\gamma = 2$, it is not feasible to generate networks with a pure power-law distribution[2]. Besides these considerations, we observe an interesting result. As expected, for large values of the exponent γ, the networks show similar uncertainty to that of a regular network. However, for $\gamma < 4$, uncertainty significantly increases, reaching a maximum as the exponent approaches $\gamma = 3$, before decreasing again. When approaching the value of $\gamma = 3$, the network maximizes the third moment of the degree distribution, while minimizing its second moment, and therefore emerges as the optimal uncorrelated structure amplifying the uncertainty in the threshold. Conversely, uncertainty is minimal for maximally heterogeneous networks, corresponding to an exponent $\gamma \approx 2$. Interestingly, the non monotonous dependence on γ is amplified as we increase the size of the system (in terms of its maximum degree).

To validate the previous theoretical prediction, we generate synthetic power-law networks using the modified preferential attachment algorithm with an attractiveness parameter that control the exponent[26]. Fixing the value of the minimum degree k_{min}, and tuning the exponent and the size of the network, we detect a maximum in the uncertainty δK_c for the exponent $\gamma = 3$, as shown in Fig.(5) thus confirming the prediction of the theory. We observe good qualitative agreement for the non monotonous dependency on the heterogeneity, and also that system size reinforces this dependency. The results point towards the difficulty of accurately determine the critical threshold of scale-free networks, with exponent $\gamma \approx 3$, because δK_c is maximized in the presence of noisy weights for these networks.

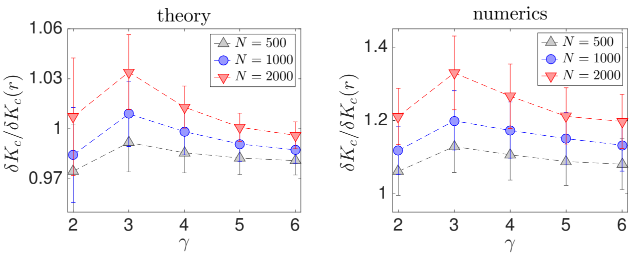

FIG. 5. Relative value of the theoretical (left) and numerical (right) uncertainty δK_c for scale-free networks in the range $\gamma \in [2,6]$ for sizes $N = 500$, 1000 and 2000, $\mu = 1$, $\sigma = 0.05$ and minimum degree fixed at $k_{\min} = 5$ compared to regular networks with the same average degree, and the same characteristics of the noise. The results are obtained with 200 realizations of the noise for each network and then averaging with 200 networks for each configuration of the modified preferential attachment algorithm. The high variance at each point shows that the results are very sensitive to the particular structure of the network, although the general trend is captured.

IV. DISCUSSION AND CONCLUSIONS

The results found in section III are of theoretical and practical relevance for the field of network science and they should be investigated further in detail. We have shown that particular network structures, as power-law degree distribution networks with exponent $\gamma \approx 3$ maximize the uncertainty of the critical threshold in the presence of noisy weights. This fact should be taken into account in the prediction of the critical threshold in empirical networks (which are usually heterogeneous) because, as proven, the accuracy in the estimation crucially depends on the underlying structure of the network. Moreover, the results might have a strong impact in the context of network optimization and adaptation[27–29], specially considering the ubiquity and theoretical relevance[2,25] of power-law networks with exponent $\gamma \approx 3$ and the well-stablished hypothesis that many biological networks are operating near the critical point[30,31]. In particular, one could wonder to which extent the existence of power-law networks with an exponent close to 3, maximizing the range of critical values has been evolutionary favourable. In this sense, the current results make a natural connection with the previous work in[32], where it was shown that scale-free networks with exponent $\gamma = 3$ are able to achieve a larger variety of macrostates with respect to homogeneous networks (specifically near the critical threshold) by deterministically tuning the weights of the links.

From the methodological side, the formalism introduced in section II represents a first step in the use of error propagation methods to the analysis of complex networks with dynamical processes on top of them. The formalism is flexible and it can be applied to other network properties and in other scenarios, being of special importance the case of colored noise obtained directly from empirical measurements. We conjecture that this line of research will receive more attention in the future due to the increasing amount of data (not free of errors), that is being collected for a large variety of systems. We remark also that the current method is based on a MFA of the largest eigenvalue of the connectivity matrix, and this approximation neglects strong correlations of the eigenvalues in the presence of noise[33,34]. While definitely more results are needed, the present formalism provides analytical insight to the studied phenomena, and turns out to give very accurate quantitative predictions if a few assumptions on the network hold.

To summarize, in this work we have studied how noise in the weights of a complex network affects the critical threshold of a dynamical process. We have restricted our study to the wide family of processes where the threshold depends on the largest eigenvalue of the connectivity matrix. In this scenario, and using the well-known MFA, we have applied error propagation to derive analytical expressions for the mean and standard deviation of the threshold depending on the noise parameters and the moments of the degree distribution. We validated our results against numerical simulations, showing good agreement when the initial MFA holds. Moreover, the formalism allowed us to carefully examine the effect that the network structure plays in the amplification of the noise at the critical point. Surprisingly, we found a non-monotonous behavior of the critical uncertainty with respect to the heterogeneity of the underlying network. By considering the paradigmatic case of uncorrelated power-law networks, we found that networks with exponent $\gamma \approx 3$ ($\gamma \approx 2$) emerge as the structures that maximize (minimize) the uncertainty of the threshold, due to an interplay between the second and third moment of the degree distribution.

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Appendix A: Calculation of the mean

We can write the degrees and strengths in terms of the binary connections ($a_{ij} = 0$ or 1) and weights ($w_{ij} \in R$) of the connectivity matrix $\mathbf{C}$, i.e $k_i = \sum_{j=1}^N a_{ij}$ and $s_i = \sum_{j=1}^N a_{ij}w_{ij}$. For the average strength $\langle s \rangle$, we have

$$\langle s \rangle = \frac{1}{N}\sum_{i=1}^{N}(\sum_{j=1}^{N} a_{ij}w_{ij}). \tag{A1}$$

Note that we can write Eq.(A1) equivalently as $\langle s \rangle = (1/N)\sum_i \mu_i k_i$, where μ_i is the average weight of node i. For sufficiently large degree ($k_i \gg 1$), one can approximate $\mu_i = \mu$, and therefore $\langle s \rangle = \mu \langle k \rangle$. However, in general, it is important to keep the contribution of each node because each μ_i has a specific uncertainty depending on the degree of node i, and this affects the overall uncertainty on K_c. For the second moment $\langle s^2 \rangle$, we have

$$\begin{aligned}\langle s^2 \rangle &= \frac{1}{N}\sum_{i=1}^{N}(\sum_{j=1}^{N} a_{ij}w_{ij})^2 \\ &= \frac{1}{N}\sum_{i=1}^{N}(\sum_{j=1}^{N} a_{ij}w_{ij}^2 + \sum_{j\neq k}^{N} a_{ij}a_{ik}w_{ij}w_{ik}).\end{aligned} \tag{A2}$$

Noticing that $\sum_{j\neq k} a_{ij}a_{ik} = k_i^2 - k_i$, we obtain

$$\langle s^2 \rangle = \frac{1}{N}[\sum_{i=1}^{N} \mu_i^2 (k_i^2 - k_i) + \sum_{i=1}^{N} \langle w^2 \rangle_i k_i], \tag{A3}$$

where $\langle w^2 \rangle_i$ is the average second moment of the i-node. Plugging Eq.(A1) and Eq.(A3) into Eq.(2) in the main text, we obtain

$$K_c = \frac{\sum_{i=1}^{N} \mu_i k_i}{\sum_{i=1}^{N} \mu_i^2 (k_i^2 - k_i) + \sum_{i=1}^{N} \langle w^2 \rangle_i k_i}, \tag{A4}$$

which correspond to Eq.(3) in the main text.

Appendix B: Calculation of the variance

The propagation of uncertainty of a non-linear function of the random variables as Eq.(3) requires to use a truncated Taylor expansion[23]. Up to second-order, and in the notation used in the main text, the approximate variance of the function is given by

$$(\delta K_c)^2 \approx J_0^T \mathbf{V} J_0 + \frac{1}{2}\,\mathrm{Tr}[(\mathbf{H_0}\mathbf{V})^2] \tag{B1}$$

where the Jacobian vector and the Hessian matrix are evaluated at the mean values of the random variables $\vec{\mu}$ and $\langle \vec{w^2} \rangle$. The Jacobian of the system in Eq.(3) is

$$J = (\frac{\partial K_c}{\partial \mu_1}, ..., \frac{\partial K_c}{\partial \mu_N}, \frac{\partial K_c}{\partial \langle w^2 \rangle_1}, ..., \frac{\partial K_c}{\partial \langle w^2 \rangle_N}). \quad \text{(B2)}$$

First, we compute the partial derivatives in Eq.(B2) explicitly from Eq.(3), obtaining

$$\frac{\partial K_c}{\partial \mu_i} \approx \frac{1}{N} \frac{k_i(\mu^2 \langle k^2 \rangle + \sigma^2 \langle k \rangle) - 2\mu^2 (k_i^2 - k_i)\langle k \rangle}{(\mu^2 \langle k^2 \rangle + \sigma^2 \langle k \rangle)^2},$$
$$\frac{\partial K_c}{\partial \langle w^2 \rangle_i} \approx -\frac{1}{N} \frac{k_i \mu \langle k \rangle}{(\mu^2 \langle k^2 \rangle + \sigma^2 \langle k \rangle)^2}, \quad \text{(B3)}$$

where the sign $\approx$ stands for assuming, in good approximation, that the input parameters μ and σ^2 are the actual mean values of the random variables $\vec{\mu}$ and $\vec{\sigma}^2 = \langle \vec{w^2} \rangle - \vec{\mu}^2$.

The Hessian matrix, the square matrix of the second-order partial derivatives of the function in Eq.(3) can be directly obtained by taking derivates from Eq.(B3). After some algebra, and defining $Q = \mu^2 \langle k^2 \rangle + \sigma^2 \langle k \rangle$, we obtain

$$\frac{\partial^2 K_c}{\partial \mu_i \partial \mu_j} \approx \frac{1}{N^2 Q^3} [Q(2\mu(k_j^2 - k_j)k_i - (2 + 2\delta_{ij}\mu(k_i^2 - k_i)k_j)) - (k_i - 8\mu^3 \langle k \rangle (k_i^2 - k_i)(k_j^2 - k_j)). \quad \text{(B4)}$$

The Hessian matrix of our system is symmetric, such that $\partial^2 K_c / \partial \mu_i \partial \langle w^2 \rangle_j = \partial^2 K_c / \partial \langle w^2 \rangle_i \partial \mu_j$. We obtain

$$\frac{\partial^2 K_c}{\partial \mu_i \partial \langle w^2 \rangle_j} \approx \frac{1}{N^2 Q^3} [-Q k_i k_j + 4\mu^2 \langle k \rangle k_j (k_i^2 - k_i)], \quad \text{(B5)}$$

and for the last term we have

$$\frac{\partial K_c}{\partial \langle w^2 \rangle_i \partial \langle w^2 \rangle_j} \approx \frac{2\mu k_i k_j \langle k \rangle}{N^2 Q^3}. \quad \text{(B6)}$$

For the covariance matrix, we can obtain explicit expression for the entries $(\mathbf{V})_{ij}$ when the noise in the weights is assumed gaussian and uncorrelated. By assumption, the network is symmetric and so it will be the covariance matrix, which can be written in block form as

$$\mathbf{V} = \left(\begin{array}{c|c} \mathbf{v}_{\mu}{}^2 & \mathbf{v}_{\mu,\langle \mathbf{w^2} \rangle} \\ \hline \mathbf{v}_{\mu,\langle \mathbf{w^2} \rangle} & \mathbf{v}_{\langle \mathbf{w^2} \rangle}{}^2 \end{array} \right),$$

where $\mathbf{v}_{\mu}{}^2$, $\mathbf{v}_{\mu,\langle \mathbf{w^2} \rangle}$ and $\mathbf{v}_{\langle \mathbf{w^2} \rangle}{}^2$ are symmetric matrices in $R^{N \times N}$ that capture each covariance term between the two random variables $(\mu_i, \langle w^2 \rangle_i)$ of all nodes. Explicitly

$$(\mathbf{v}_{\mu}{}^2)_{ij} = \frac{\sigma^2}{k_i}(\delta_{ij} + \frac{a_{ij}}{k_j}), \quad \text{(B7)}$$

$$(\mathbf{v}_{\mu,\langle \mathbf{w^2} \rangle})_{ij} = \frac{2\mu\sigma^2}{k_i}(\delta_{ij} + \frac{a_{ij}}{k_j}), \quad \text{(B8)}$$

$$(\mathbf{v}_{\langle \mathbf{w^2} \rangle}{}^2)_{ij} = \frac{2\sigma^2(2\mu^2 + \sigma^2)}{k_i}(\delta_{ij} + \frac{a_{ij}}{k_j}). \quad \text{(B9)}$$

The first term in the sums is the contribution of the diagonal entries. The gaussian variances (σ^2 and $2\sigma^2(2\mu^2 + \sigma^2)$) and covariance ($2\mu\sigma^2$) of a single weight w_{ij} drawn from (μ, σ^2) are divided by the number of elements (the degree k_i) involved in computing the averages μ_i and $\langle w^2 \rangle_i$. The second term accounts for the non-diagonal entries. If two nodes (i, j) are neighbours, i.e. $a_{ij} = 1$, then we have to add an additional correlation due to the presence of the shared weight, which is divided by the product of their degrees (k_i and k_j).

For the first order expansion, we can compute explicitly $(\delta K_c)^2$ in terms of the noise parameters (μ, σ) and the moments of the degree distribution. We can write Eq.(5) as

$$(\delta K_c)^2 \approx \sum_{i=1}^{N} \sum_{j=1}^{N} [(\frac{\partial K_c}{\partial \mu_i})(\frac{\partial K_c}{\partial \mu_j})(\sigma_{\mu}{}^2)_{ij}, \quad \text{(B10)}$$
$$+ (\frac{\partial K_c}{\partial \langle w^2 \rangle_i})(\frac{\partial K_c}{\partial \langle w^2 \rangle_j})(\sigma_{\langle \mathbf{w^2} \rangle}{}^2)_{ij}, \quad \text{(B11)}$$
$$+ 2(\frac{\partial K_c}{\partial \mu_i})(\frac{\partial K_c}{\partial \langle w^2 \rangle_j})(\sigma_{\mu,\langle \mathbf{w^2} \rangle})_{ij}], \quad \text{(B12)}$$

and after some algebra, we obtain

$$(\delta K_c)^2 \approx \frac{2\sigma^2 \langle k \rangle}{N Q^4} [Q^2 - 4\mu^2 \langle k^2 \rangle Q + 2\mu^2(2\mu^2 + \sigma^2)\langle k \rangle^2 + 2\mu^4(\langle k \rangle \langle k^3 \rangle + \langle k^2 \rangle (\langle k^2 \rangle - 4\langle k \rangle) + 2\langle k \rangle^2) + 8\mu^4 \langle k \rangle (\langle k^2 \rangle - \langle k \rangle)], \quad \text{(B13)}$$

where we have used that $\sum_i \sum_j a_{ij} k_i k_j = N \langle k^2 \rangle^2 / \langle k \rangle$. Simplifying further, we get the resulting Eq.(6) in the main text. Explicitly,

$$(\delta K_c)^2 \approx a[\mu^4(2\langle k \rangle \langle k^3 \rangle - \langle k^2 \rangle^2) - 2\mu^2\sigma^2(\langle k \rangle \langle k^2 \rangle - \langle k \rangle^2) + \sigma^4 \langle k \rangle^2] \quad \text{(B14)}$$

with $a = 2\sigma^2 \langle k \rangle / [N(\mu^2 \langle k^2 \rangle + \sigma^2 \langle k \rangle)^4]$.