The equity core and the Lorenz-maximal allocations in the equal division core *

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Abstract

In this paper, we characterize the non-emptiness of the equity core (Selten, 1978) and provide a method, easy to implement, for computing the Lorenz-maximal allocations in the equal division core (Dutta-Ray, 1991). Both results are based on a geometrical decomposition of the equity core as a finite union of polyhedrons.

Keywords: Cooperative game, equity core, equal division core, Lorenz domination.
JEL classification: C71

1 Introduction

The notion of the equity core of a transferable utility coalitional game (a game, for short) was introduced by Selten (1978) as a weighted generalization of the equal division core (Selten, 1972). There are in the literature two main explanations for the equal division core. On one hand, Selten (1972) used this solution concept to explain outcomes of experimental cooperative games showing that the evidence suggests that equity considerations have a strong influence on observed payoff divisions. On the other hand, a much more theoretical approach is given by Dutta and Ray (1991) when they propose a solution which combines commitment for egalitarianism and selfish behavior. In that paper, the authors introduce the strong constrained egalitarian solution and show that this solution concept selects the Lorenz-maximal allocations in the equal division core. They also prove non-emptiness for N-superadditive games, a weaker condition than superadditivity. However, as far as we know, there is not a characterization for the existence of this solution. Moreover, in general,

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it is not immediate to find the Lorenz-maximal allocations in the equal division core of a game. Thus, the aim of the paper is twofold: characterize the existence of the equity core, which, in the particular case where all players have the same weight, gives a characterization of the existence of the equal division core, and provide a method, easy to implement, for computing the Lorenz-maximal allocations in the equal division core. Both results are based on a geometrical decomposition of the equity core as a finite collection of polyhedrons.

The paper is organized as follows. In Section 2, we introduce preliminaries and notation. Section 3 contains the decomposition theorem and the non-emptiness characterization result for the equity core. Section 4 provides a systematic method for computing the Lorenz-maximal allocations in the equal division core.

2 Preliminaries

The set of natural numbers $\mathbb{N}$ denotes the universe of potential players. By $N \subseteq \mathbb{N}$ we denote a finite set of players, in general $N = \{1, \ldots, n\}$. A *transferable utility coalitional game (a game)* is a pair (N, v) where $v : 2^N \longrightarrow \mathbb{R}$ is the characteristic function with $v(\emptyset) = 0$ and 2^N denotes the set of all subsets (coalitions) of N. We use $S \subset T$ to indicate strict inclusion, that is $S \subseteq T$ but $S \neq T$. By $|S|$ we denote the cardinality of the coalition $S \subseteq N$. The set of all games is denoted by Γ. Given a coalition $S \subset N, S \neq \emptyset$ and $(N, v) \in \Gamma$, we define the subgame (S, v_S) by $v_S(Q) := v(Q)$, for all $Q \subseteq S$.

Let $\mathbb{R}^N$ stand for the space of real-valued vectors indexed by N, $x = (x_i)_{i \in N}$, and for all $S \subseteq N$, $x(S) = \sum_{i \in S} x_i$, with the convention $x(\emptyset) = 0$. For each $x \in \mathbb{R}^N$ and $T \subseteq N$, x_T denotes the restriction of x to T: $x_T = (x_i)_{i \in T} \in \mathbb{R}^T$. Given two vectors $x, y \in \mathbb{R}^N$, $x \geq y$ denotes that $x_i \geq y_i$, for all $i \in N$, and $x > y$ denotes that $x_i > y_i$ for all $i \in N$. In addition, we define $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N \mid x \geq 0\}$ and $\mathbb{R}^N_{++} := \{x \in \mathbb{R}^N \mid x > 0\}$. By $z = \max\{x, y\}$, we denote the vector $z \in \mathbb{R}^N$ such that $z_i = \max\{x_i, y_i\}$, for all $i \in N$.

The *pre-imputation set* of a game (N, v) is defined by $X(N, v) := \{x \in \mathbb{R}^N \,|\, x(N) = v(N)\}$. A *solution* on a set Γ of games is a mapping σ which associates with any game (N, v) a subset $\sigma(N, v)$ of the set $X(N, v)$. Notice that the solution set $\sigma(N, v)$ is allowed to be empty. For a game (N, v), the set of *imputations* is given by $I(N, v) := \{x \in X(N, v) \,|\, x(i) \geq v(i),\text{ for all } i \in N\}$. The *core* of a game (N, v) is the set of those imputations where each coalition gets at least its worth, that is $C(N, v) := \{x \in X(N, v) \mid x(S) \geq v(S) \text{ for all } S \subseteq N\}$. The *equal division core* (Selten, 1972) is an extension of the core containing those imputations which can not be improved upon by the equal division allocation of any subcoalition. Formally, $EDC(N, v) := \left\{x \in I(N, v) \mid \text{for all } \emptyset \neq S \subseteq N, \text{ there is } i \in S \text{ with } x_i \geq \frac{v(S)}{|S|}\right\}$.

For any $x \in \mathbb{R}^N$, denote by $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ the vector obtained by rearranging from

x its coordinates in a non-decreasing order, that is, $\hat{x}_1 \leq \hat{x}_2 \leq \ldots \leq \hat{x}_n$. For any two vectors $y, x \in \mathbb{R}^N$, we say that y *Lorenz-dominates* x, $(y \succ_L x)$, if $\sum_{j=1}^{k} \hat{y}_j \geq \sum_{j=1}^{k} \hat{x}_j$, for every $k = 1, \ldots, n$, with at least one strict inequality. Given a coalition $\emptyset \neq S \subseteq N$ and a set $A \subseteq \mathbb{R}^S$, $E(A)$ denotes the set of allocations that are Lorenz undominated within A. Given a game (N, v), the *Lorenz-maximal allocations in the equal division core* is the set $E(EDC(N, v)) := \{x \in EDC(N, v) \,|\, \text{there is no}\, y \in EDC(N, v) \text{ such that } y \succ_L x\}$, which coincides with the set of *strong constrained egalitarian allocations* introduced by Dutta and Ray (1991).

A game with a non-empty core is called *balanced*. A game (N, v) is *convex* (Shapley, 1971) if, for every $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$. A game is *superadditive* if, for every $S, T \subseteq N$, $S \cap T = \emptyset$, $v(S) + v(T) \leq v(S \cup T)$. A family of nonempty coalitions $\{S_1, \ldots, S_m\}$, $S_i \subseteq N$ for all $i = 1, \ldots, m$, is a *partition* of N if (a) $S_i \cap S_j = \emptyset$ for all $i, j \in \{1, \ldots, m\}$, $i \neq j$, and (b) $\bigcup_{i=1}^{m} S_i = N$. A game is N-*superadditive* if for all partition $\{S_1, \ldots, S_m\}$ of N, it holds $v(S_1) + \ldots + v(S_m) \leq v(N)$.

Dutta and Ray (1989) define the *weak constrained egalitarian solution*, denoted by $DR(\cdot)$, and show that on the domain of convex games this solution picks the payoff vector that is obtained by the following algorithm: Let (N, v) be a convex game and $DR(N, v) = \{z\}$. **Step 1:** Define $v_1 = v$. Then find the unique coalition $S_1 \subseteq N$ such that for all $S \subseteq N$, (i) $\frac{v_1(S_1)}{|S_1|} \geq \frac{v_1(S)}{|S|}$, and (ii) if $\frac{v_1(S_1)}{|S_1|} = \frac{v_1(S)}{|S|}$ and $S \neq S_1$, then $|S_1| > |S|$. Uniqueness of such a coalition is guaranteed by convexity of (N, v). Then, for all $i \in S_1$, $z_i = \frac{v_1(S_1)}{|S_1|}$. **Step k:** Suppose that $S_1, \ldots, S_{k-1}$ have been defined. Let $N_k = N \setminus (S_1 \cup \ldots \cup S_{k-1})$ and (N_k, v_k) be the game defined as follows: $v_k(S) := v(S_1 \cup \ldots \cup S_{k-1} \cup S) - v(S_1 \cup \ldots \cup S_{k-1})$, for all $S \subseteq N_k$. It can be shown that (N_k, v_k) is convex. Then find the unique coalition $S_k \subseteq N_k$ such that for all $S \subseteq N_k$, (i) $\frac{v_k(S_k)}{|S_k|} \geq \frac{v_k(S)}{|S|}$, and (ii) if $\frac{v_k(S_k)}{|S_k|} = \frac{v_k(S)}{|S|}$ and $S \neq S_k$, then $|S_k| > |S|$. The uniqueness of such a coalition is guaranteed by the convexity of (N_k, v_k). Then, for all $i \in S_k$, $z_i = \frac{v_k(S_k)}{|S_k|} = \frac{v(S_1 \cup \ldots \cup S_k) - v(S_1 \cup \ldots \cup S_{k-1})}{|S_k|}$.

An ordering $\theta = (i_1, \ldots, i_n)$ of N, where $|N| = n$, is a bijection from $\{1, \ldots, n\}$ to N. We denote by $\mathcal{S}_N$ the set of all orderings of N. Given a game (N, v) and an ordering $\theta = (i_1, \ldots, i_n) \in \mathcal{S}_N$, we define the *marginal worth vector* associated to θ as the vector $m^\theta(v) \in \mathbb{R}^N$ which assigns to each player her marginal contribution in the order θ. Formally, $m^\theta_{i_1}(v) = v(\{i_1\})$ and $m^\theta_{i_k}(v) = v(\{i_1, \ldots, i_k\}) - v(\{i_1, \ldots, i_{k-1}\})$, for $k = 2, \ldots, n$.

3 Decomposition theorem and existence

In this section, we show that the equity core can be decomposed as the union of a finite collection of polyhedrons. Making use of this decomposition result, we characterize the non-emptiness of the equity core. The equity core is an asymmetric extension of the equal division core in which players may have different weights. Formally, given a vector of weights $w \in \mathbb{R}^N_{++}$, the *equity core* w.r.t. w is defined as follows:

$$EC^w(N,v) := \left\{ x \in I(N,v) \,|\, \text{for all } \; \emptyset \neq S \subseteq N \, \text{there is } \; i \in S \text{ with } \; x_i \geq \frac{v(S)}{w(S)} w_i \right\}.$$

Since the equity core is a compact extension of the core, balancedness gives a first condition to guarantee non-emptiness (Bondareva, 1963 and Shapley, 1967). However, the equity core can be non-empty even if the core is empty. Indeed, consider the following three-player game: $v(\{i\}) = 0$, for all $i \in \{1,2,3\}$, and $v(S) = 1$, otherwise. It is not difficult to see that for $w = (1,1,1)$, $EC^w(N,v) = EDC(N,v) = \{(0.5, 0.5, 0), (0.5, 0, 0.5), (0, 0.5, 0.5)\}$ and the core is empty.

As we have commented before, for N-superadditive games both the strong constrained egalitarian solution and the equal division core are non-empty. Next we show that N-superadditivity is sufficient to guarantee non-emptiness of the equity core.

Theorem 3.1. *Let (N,v) be a N-superadditive game. Then, $EC^w(N,v) \neq \emptyset$ for any vector of weights $w \in \mathbb{R}^N_{++}$.*

Proof: Let (N,v) be a N-superadditive game, $w \in \mathbb{R}^N_{++}$ and $\{B_1, \ldots, B_m\}$ a partition of N such that

$$\frac{v(B_k)}{w(B_k)} = \max_{\emptyset \neq C \in Q_k} \left\{ \frac{v(C)}{w(C)} \right\},$$

where $Q_1 = 2^N$ and $Q_k = 2^{N \setminus \{B_1 \cup \ldots \cup B_{k-1}\}}$, for $k = 2, \ldots, m$. Define the vector $x \in \mathbb{R}^N$ as follows:

$$x_i := \frac{v(B_j)}{w(B_j)} \, w_i, \text{ for all } i \in B_j \; \text{ and all } j = 1, \ldots, m.$$

Take $\emptyset \neq S \subseteq N$ and $k = \min\{j \in \{1, \ldots, m\} \mid S \cap B_j \neq \emptyset\}$. Notice that $S \in Q_k$. Take $i \in S \cap B_k$, then

$$x_i = \frac{v(B_k)}{w(B_k)} \, w_i = \max_{\emptyset \neq C \in Q_k} \left\{ \frac{v(C)}{w(C)} \right\} w_i \geq \frac{v(S)}{w(S)} \, w_i. \tag{1}$$

Next we prove that the vector $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \mathbb{R}^N$, defined by $\tilde{x}_i := x_i + \left(\frac{v(N) - (v(B_1) + \ldots + v(B_m))}{w(N)} \right)$ w_i for all $i \in N$, belongs to $EC^w(N,v)$. Efficiency follows from the definition of the vector x taking into account that $x(B_j) = v(B_j)$, for all $j = 1, \ldots, m$. By N-superadditivity, $v(B_1) + \ldots + v(B_m) \leq v(N)$, therefore $\tilde{x}_i \geq x_i$, for any player $i \in N$. Then, from expression (1), for all $\emptyset \neq S \subseteq N$, there is $i \in S$ such that $\tilde{x}_i \geq x_i \geq \frac{v(S)}{w(S)} \, w_i$. Thus, we conclude that $\tilde{x} \in EC^w(N,v)$. □

The next example shows that the N-superadditivity is not necessary to guarantee non-emptiness of the equity core.

Example 1. *Let* (N, v) *be a three-player game, where* $N = \{1, 2, 3\}$*,* $v(\{i\}) = 0$*, for all* $i = 1, 2, 3$*,* $v(\{1, 2\}) = 2$*,* $v(\{1, 3\}) = v(\{2, 3\}) = 0$ *and* $v(N) = 1$*.*
This game is not N*-superadditive since* $v(\{1, 2\}) + v(\{3\}) > v(\{1, 2, 3\})$*. The equity core w.r.t any* $w = (w_1, w_1, w_3) \in \mathbb{R}^N_{++}$ *is*

$$EC^w(N, v) = \left\{ x \in I(N, v) \mid x_1 \geq \frac{2}{w_1 + w_2}\, w_1 \text{ or } x_2 \geq \frac{2}{w_1 + w_2}\, w_2 \right\}.$$

Notice that at least one of the values must be lower or equal than 1*, since otherwise we get a contradiction. Therefore, we have two possibilities: (a)* $\frac{2}{w_1+w_2}\, w_1 \leq 1$ *and (b)* $\frac{2}{w_1+w_2}\, w_2 \leq 1$*. If case (a) holds, then* $\left(\frac{2}{w_1+w_2}\, w_1, 1 - \frac{2}{w_1+w_2}\, w_1, 0\right) \in EC^w(N, v)$*. In case (b) we have* $\left(1 - \frac{2}{w_1+w_2}\, w_2, \frac{2}{w_1+w_2}\, w_2, 0\right) \in EC^w(N, v)$*. Thus, for any vector of weights* $w \in \mathbb{R}^N_{++}$*, the equity core is non-empty.*

Our objective is now to characterize non-emptiness. To this end, first we show that the equity core can be decomposed as the union of simple polyhedrons. In order to find these polyhedrons we define the *proportional share worth vectors*.

Definition. Let (N, v) be a game, $w \in \mathbb{R}^N_{++}$ a vector of weights and $\theta = (i_1, \ldots, i_n) \in \mathcal{S}_N$. We define the proportional share worth vector w.r.t. w and θ, denoted by $\bar{x}^\theta_w(v) \in \mathbb{R}^N$, as follows:

$$\bar{x}^\theta_{w,i_k}(v) := \max_{S \in P_{i_k}} \left\{ \frac{v(S)}{w(S)} \right\} w_{i_k}, \text{ for } k = 1, \ldots, n,$$

where $P_{i_1} := \{S \subseteq N \mid i_1 \in S\}$ and $P_{i_k} := \{S \subseteq N \mid i_1, \ldots, i_{k-1} \notin S,\ i_k \in S\}$, for $k = 2, \ldots, n$.

Remark 3.2. Notice that for all $\theta = (i_1, \ldots, i_n) \in \mathcal{S}_N$, the set $\{P_{i_1}, \ldots, P_{i_n}\}$ forms a partition of the set $2^N \setminus \emptyset$. In addition, for all $i \in N$, $\bar{x}^\theta_{w,i}(v) \geq v(\{i\})$, and for any non-empty coalition $S \subseteq N$, there is a player $i \in S$ such that $\bar{x}^\theta_{w,i}(v) \geq \frac{v(S)}{w(S)}\, w_i$. However, in general $\bar{x}^\theta_w(v)$ is not an efficient vector and hence it does not belong to the equity core. Let us denote by $\delta_{\bar{x}^\theta_w} = v(N) - \bar{x}^\theta_w(v)(N)$ the increment or decrement for the vector $\bar{x}^\theta_w(v)$ to reach efficiency.

Definition. Let (N, v) be a game, $w \in \mathbb{R}^N_{++}$ a vector of weights and $\theta = (i_1, \ldots, i_n) \in \mathcal{S}_N$. We define the polyhedron generated by the proportional share worth vector $\bar{x}^\theta_w(v)$, denoted by $\Delta^{\bar{x}^\theta_w}(v)$, as the convex hull of all $\bar{x}^\theta_w(v) + \delta_{\bar{x}^\theta_w}\, e_i$, where e_i is the i-th canonical vector of $\mathbb{R}^N$, for any $i \in N$. That is, $\Delta^{\bar{x}^\theta_w}(v) := \text{convex}\, \{\bar{x}^\theta_w(v) + \delta_{\bar{x}^\theta_w}\, e_i, \text{ for all } i \in N\}$.

To characterize non-emptiness we only need to work with an special kind of polyhedrons, those generated by the proportional share worth vectors associated to $\theta \in \mathcal{S}_N$ with $\delta_{\bar{x}^\theta_w} \geq 0$, and minimal with respect to the usual order in $\mathbb{R}^N$. Given a game (N, v), $w \in \mathbb{R}^N_{++}$ and $\theta \in \mathcal{S}_N$ such that $\delta_{\bar{x}^\theta_w} \geq 0$, it is easy to see that

$$\Delta^{\bar{x}^\theta_w}(v) = \{x \in X(N, v) \mid x \geq \bar{x}^\theta_w(v)\}. \tag{2}$$

Lemma 3.3. *Let (N,v) be a game, $w \in \mathbb{R}^N_{++}$ and $\bar{x}^{\theta}_w(v)$, $\bar{x}^{\theta'}_w(v)$ the proportional share worth vectors w.r.t. $\theta, \theta' \in \mathcal{S}_N$ respectively, such that $\delta_{\bar{x}^{\theta}_w} \geq 0$, $\delta_{\bar{x}^{\theta'}_w} \geq 0$. Then, the following statements are equivalent:*

1. $\bar{x}^{\theta}_w(v) \leq \bar{x}^{\theta'}_w(v)$.
2. $\Delta^{\bar{x}^{\theta'}_w}(v) \subseteq \Delta^{\bar{x}^{\theta}_w}(v)$.

Proof: The implication $1 \to 2$ follows straightforward from expression (2). Next we prove $2 \to 1$. Assuming $\Delta^{\bar{x}^{\theta'}_w}(v) \subseteq \Delta^{\bar{x}^{\theta}_w}(v)$, we deduce that $\bar{x}^{\theta'}_w(v) + \delta_{\bar{x}^{\theta'}_w}\, e_k \in \Delta^{\bar{x}^{\theta}_w}(v)$, for all $k \in N$. Hence, again from expression (2), $\bar{x}^{\theta'}_w(v) + \delta_{\bar{x}^{\theta'}_w}\, e_k \geq \bar{x}^{\theta}_w(v)$. Finally, take $j \in N$ and $k \neq j$, then $\bar{x}^{\theta'}_{w,j}(v) \geq \bar{x}^{\theta}_{w,j}(v)$, getting the result. □

Definition. Let (N,v) be a game and $w \in \mathbb{R}^N_{++}$. We define the set of minimal proportional share worth vectors as follows:

$$\mathcal{M}^w(v) := \{\bar{x}^{\theta}_w(v) \,|\, \theta \in \mathcal{S}_N,\, \delta_{\bar{x}^{\theta}_w} \geq 0 \,\text{and there is no }\; \theta' \text{ such that } \bar{x}^{\theta'}_w(v) \leq \bar{x}^{\theta}_w(v)\}.$$

Now we have all the tools to state a decomposition theorem for the equity core in terms of the above polyhedrons.

Theorem 3.4. *Let (N,v) be a game and $w \in \mathbb{R}^N_{++}$ a vector of weights. Then,*

$$EC^w(N,v) = \bigcup_{\bar{x}^{\theta}_w(v) \in \mathcal{M}^w(v)} \Delta^{\bar{x}^{\theta}_w}(v).$$

Proof: We first prove that $EC^w(N,v) \subseteq \bigcup_{\bar{x}^{\theta}_w(v) \in \mathcal{M}^w(v)} \Delta^{\bar{x}^{\theta}_w}(v)$. Take $x \in EC^w(N,v)$. We construct a specific order $\theta \in \mathcal{S}_N$ such that $x \geq \bar{x}^{\theta}_w(v)$. This order θ is generated by the following algorithm. We choose a coalition $S_1 \in 2^N$, $S_1 \neq \emptyset$, such that $\frac{v(S_1)}{w(S_1)} = \max_{\emptyset \neq C \in 2^N} \left\{\frac{v(C)}{w(C)}\right\}$. Having chosen S_1, since $x \in EC^w(N,v)$, there exists a player i_1 such that $x_{i_1} \geq \frac{v(S_1)}{w(S_1)}\, w_{i_1} = \max_{\emptyset \neq C \in 2^N} \left\{\frac{v(C)}{w(C)}\right\} w_{i_1}$. Second, choose $S_2 \in 2^{N \setminus \{i_1\}}$, $S_2 \neq \emptyset$, such that $\frac{v(S_2)}{w(S_2)} = \max_{\emptyset \neq C \in 2^{N \setminus \{i_1\}}} \left\{\frac{v(C)}{w(C)}\right\}$. As before, since $x \in EC^w(N,v)$, there exists a player $i_2 \in S_2$ such that $x_{i_2} \geq \frac{v(S_2)}{w(S_2)}\, w_{i_2} = \max_{\emptyset \neq C \in 2^{N \setminus \{i_1\}}} \left\{\frac{v(C)}{w(C)}\right\} w_{i_2}$. Following this process we obtain an ordering $\theta = (i_1, i_2, \ldots, i_n) \in \mathcal{S}_N$ such that

$$x \geq \bar{x}^{\theta}_w(v). \tag{3}$$

Since $x \in X(N,v)$, from (3) it follows that $\delta_{\bar{x}^{\theta}_w} \geq 0$. Hence, from expression (2) we have $x \in \Delta^{\bar{x}^{\theta}_w}(v)$. If $\bar{x}^{\theta}_w(v) \in \mathcal{M}^w(v)$, we are finished. If not, we can find an order θ' such that $\bar{x}^{\theta'}_w(v) < \bar{x}^{\theta}_w(v)$ with $\bar{x}^{\theta'}_w(v) \in \mathcal{M}^w(v)$. But from Lemma 3.3, $\Delta^{\bar{x}^{\theta}_w}(v) \subseteq \Delta^{\bar{x}^{\theta'}_w}(v)$, thus $x \in \Delta^{\bar{x}^{\theta'}_w}(v)$.

To show the reverse inclusion, take $x \in \Delta^{\bar{x}^\theta_w}(v)$, where $\Delta^{\bar{x}^\theta_w}(v)$ is generated by $\bar{x}^\theta_w(v) \in \mathcal{M}^w(v)$. Then, from expression (2), $x \in X(N,v)$ and $x \geq \bar{x}^\theta_w(v)$. Recall that for all $\theta = (i_1,\ldots,i_n) \in \mathcal{S}_N$, the set $\{P_{i_1},\ldots,P_{i_n}\}$ as described in Definition 2 forms a partition of the set $2^N \setminus \{\emptyset\}$ (see Remark 3.2). Now take $S \in 2^N \setminus \{\emptyset\}$ and $i_r \in S$ be the first player in S w.r.t. the ordering θ. Then, $S \in P_{i_r}$ and so $x_{i_r} \geq \bar{x}^\theta_{w,i_r}(v) = \max_{C \in P_{i_r}} \left\{ \frac{v(C)}{w(C)} \right\} w_{i_r} \geq \frac{v(S)}{w(S)} w_{i_r}$. Hence, we conclude that $x \in EC^w(N,v)$. $\square$

A direct consequence of Theorem 3.4 is a characterization of the non-emptiness of the equity core.

Theorem 3.5. *Let (N,v) be a game and $w \in \mathbb{R}^N_{++}$ a vector of weights. Then, the following statements are equivalent:*

1. $EC^w(N,v) \neq \emptyset$.
2. *There exists $\theta \in \mathcal{S}_N$ such that $\bar{x}^\theta_w(v)(N) \leq v(N)$.*

As we have already mentioned, the equity core coincides with the equal division core when all the players have the same weight. Moreover, since the equal division core is a compact set, the non-emptiness is equivalent to the non-emptiness of the set $E(EDC(N,v))$. Hence, as a consequence of the above theorem a characterization of the non-emptiness of both the equal division core and the $E(EDC(N,v))$ is getting.

Corollary 3.6. *Let (N,v) be a game and $w = (1,\ldots,1) \in \mathbb{R}^N$ the vector of weights. Then, the following statements are equivalent:*

1. *$EDC(N,v) \neq \emptyset$ and $E(EDC(N,v)) \neq \emptyset$.*
2. *There exists $\theta \in \mathcal{S}_N$ such that $\bar{x}^\theta_w(v)(N) \leq v(N)$.*

Next, we give a four-player glove market game to illustrate the above decomposition result and to check the non-emptiness of the equal division core.

Example 2. *Let (N,v) be a game with $N = \{1,2,3,4\}$ and $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{4\}) = 0$, $v(\{1,2\}) = v(\{3,4\}) = 0$, $v(\{1,3\}) = v(\{1,4\}) = v(\{2,3\}) = v(\{2,4\}) = 1$, $v(\{1,2,3\}) = v(\{1,2,4\}) = v(\{1,3,4\}) = v(\{2,3,4\}) = 1$, and $v(\{1,2,3,4\}) = 2$.*

As the reader may check, the set of minimal share worth vectors is $\mathcal{M}(v) = \{x = (0.5,0.5,0,0),\ y = (0,0,0.5,0.5)\}$, and the equal division core is the union of the corresponding two polyhedrons, $EDC(N,v) = \Delta^x(v) \cup \Delta^y(v)$.

In Figure 1 we represent the core and the equal division core of this game in the efficiency hyperplane (of dimension 3). The equal division core corresponds to the two shadowed pyramides and the core is the discontinuous black segment. The set of Lorenz-maximal allocations in the equal division core is the intersection point between the two pyramides, $E(EDC(N,v)) = \{(0.5, 0.5, 0.5, 0.5)\}$.

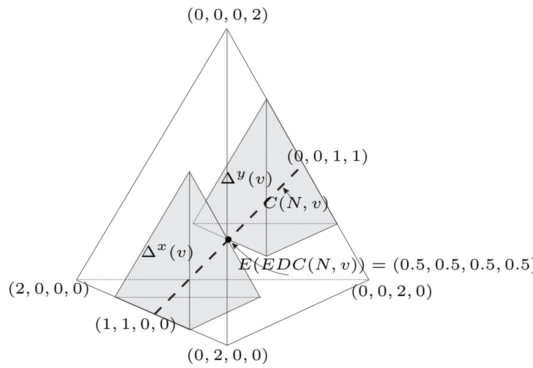

Figure 1: Decomposition of the equal division core corresponding to Example 2.

4 Finding the Lorenz-maximal allocations in the equal division core

In this section, we provide a systematic procedure for computing the Lorenz-maximal allocations in the equal division core based on its geometrical decomposition as a finite union of polyhedrons. For this purpose we use the definitions and results stated in Section 3 when all players have the same weight.

To locate the Lorenz-maximal allocations in the equal division core it is enough to see that in each polyhedron $\Delta^x(v)$ there is a unique Lorenz-maximal element. As we prove in Lemma 4.1, this element is quite similar to the one reported by the *constrained equal awards rule* for bankruptcy problems (see Moulin, 2002 or Thomson, 2003).

Definition. Let (N, v) be a game and $x \in X^*(N, v)$. The vector $y_x \in \mathbb{R}^N$ is defined as $y_{x,i} := \max\{x_i, \lambda\}$, for all $i \in N$, where λ is chosen so as to satisfy $\sum_{j \in N} \max\{x_j, \lambda\} = v(N)$.

The next result states that y_x Lorenz dominates every other element in the polyhedron $\Delta^x(v)$.

Lemma 4.1. *Let (N, v) be a game and $x \in X^*(N, v)$. Then, y_x Lorenz dominates every other element in $\Delta^x(v)$.*

Proof: From definition $y_x \in \Delta^x(v)$. To prove $y_x \succ_L t$, for all $t \in \Delta^x(v)$, $t \neq y_x$, we define the game (N, v_x) as $v_x(S) := x(S)$, if $S \subset N$, and $v_x(N) := v(N)$. Notice that (N, v_x)

is convex and $C(N, v_x) = \Delta^x(v)$. Thus, since for convex games the weak constrained egalitarian allocation Lorenz dominates every other point in the core (Dutta and Ray, 1989), we must see that $DR(N, v_x) = \{y_x\}$. Assume, without loss of generality, $x_1 \geq x_2 \geq \ldots \geq x_n$. If $x_1 \leq \frac{v_x(N)}{|N|}$, then $DR(N, v_x) = \left\{y_x = \left(\frac{v_x(N)}{|N|}, \ldots, \frac{v_x(N)}{|N|}\right)\right\}$. Otherwise, take $k \in \{1, \ldots, n-1\}$, $n \geq 2$, and define the vector $y^k := \left(x_1, \ldots, x_k, \frac{v(N)-(x_1+\ldots+x_k)}{n-k}, \ldots, \frac{v(N)-(x_1+\ldots+x_k)}{n-k}\right)$. Observe that $y_x = y^{k^*}$, where $k^* = \min\{k \in \{1, \ldots, n-1\} \mid y^k \geq x\}$. Let $\mathcal{P} = \{S_1, S_2, \ldots, S_m\}$ be the partition of N obtained by means of the Dutta and Ray(1989) algorithm to compute the weak constrained egalitarian allocation of (N, v_x) and take $DR(N, v_x) = \{z\}$. Since $x_1 > \frac{v_x(N)}{|N|}$, $m \geq 2$. In this case, $S_1 = \{i \in N \mid x_i \geq x_k \text{ for all } k \in N\}$ and $S_h = \{i \in N \setminus S_1 \cup \ldots \cup S_{h-1} \mid x_i \geq x_k \text{ for all } k \in N \setminus S_1 \cup \ldots \cup S_{h-1}\}$ for $h = 2, \ldots, m-1$. That is, S_1 is formed by those players with the maximum payoff at x. Then, removing players of S_1, coalition S_2 is formed in a similar way, and so on until the last but one element of the partition, S_{m-1}. Moreover, $z_i = x_i$, for all $i \in S_h$ and all $h = 1, \ldots, m-1$, and $z_i = \frac{v(N)-\sum_{i \in N \setminus S_m} x_i}{|S_m|}$, for all $i \in S_m$. Hence, $z = y^k$, where $k = |S_1 \cup \ldots \cup S_{m-1}|$. Now suppose that k is not minimal and denote by $k^* = \min\{r \in \{1, \ldots, n-1\} \mid y^r \geq x\}$. Then, $y^{k^*} = \left(x_1, \ldots, x_{k^*}, \frac{v(N)-(x_1+\cdots+x_{k^*})}{n-k^*}, \ldots, \frac{v(N)-(x_1+\cdots+x_{k^*})}{n-k^*}\right)$. By the minimality of k^*, we have $z_i \leq y_i^{k^*}$ for all $i \in \{1, \ldots, k^*, \ldots, k\}$. Moreover, for all $i > k$, since $i \in S_m$ and $k \in S_{m-1}$, we have $z_i < z_k = x_k \leq y_k^{k^*} = y_i^{k^*}$. Then, $z(N) < y^{k^*}(N) = v(N)$, a contradiction. Hence, $z = y_x$. $\square$

Combining the above two lemmas, and taking into account the transitivity of the Lorenz relation, the next result follows straightforward. Notice that the set of Lorenz-maximal elements in a compact set is not generally finite (see, for instance, Example 4 in Dutta and Ray, 1989).

Theorem 4.2. *Let (N, v) be a game. The set of Lorenz-maximal allocations in the equal division core is finite. Moreover, $E(EDC(N, v)) = E\{y_x \mid x \in \mathcal{M}(v)\}$.*

Now from Theorem 4.2 one can compute the Lorenz-maximal allocations in the equal division core as follows:
step 1. Find all elements of $\mathcal{M}(v)$;
step 2. Compute y_x for each $x \in \mathcal{M}(v)$;
step 3. Find the Lorenz-maximal elements in $\{y_x \mid x \in \mathcal{M}(v)\}$.

Let us show two examples to illustrate how this procedure works.

Example 3. *Let (N, v) be a game with $N = \{1, 2, 3\}$ and $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1, 2\}) = v(\{1, 3\}) = 100$, $v(\{2, 3\}) = 0$, and $v(\{1, 2, 3\}) = 125$.*

Since $\arg\max_{S \neq \emptyset} \frac{v(S)}{|S|} = \{\{1, 2\}, \{1, 3\}\}$, we must consider all orderings of N. Let denote by θ_{ijk} the ordering (i, j, k). Then, $x^{\theta_{123}} = x^{\theta_{132}} = (50, 0, 0)$, $x^{\theta_{213}} = (50, 50, 0)$, $x^{\theta_{231}} =$

$x^{\theta_{321}} = (0, 50, 50)$, and $x^{\theta_{312}} = (50, 0, 50)$. Thus, $\mathcal{M}(v) = \{x^{\theta_{123}}, x^{\theta_{231}}\}$, and the candidates to be Lorenz-maximal allocations in the equal division core are $y_{x^{\theta_{123}}} = (50, 37.5, 37.5)$ and $y_{x^{\theta_{231}}} = (25, 50, 50)$. Since $y_{x^{\theta_{123}}} \succ_L y_{x^{\theta_{231}}}$, we have $E(EDC(N, v)) = \{y_{x^{\theta_{123}}}\}$.

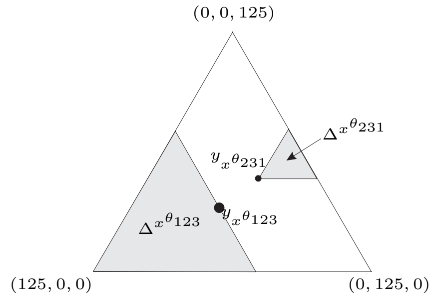

Figure 2: The union of the two shadowed triangles corresponds to the equal division core and $E(EDC(N, v)) = \{y_{x^{\theta_{123}}}\}$ in Example 3.

Example 4. *(Dutta and Ray, 1991) Let (N, v) be a game with $N = \{1, 2, 3, 4, 5\}$ and $v(\{1\}) = 0.96$, $v(\{2\}) = 0.70$, $v(\{3\}) = 0.70$, $v(\{4\}) = 0.00$, $v(\{1, 2\}) = 1.66$, $v(\{1, 3\}) = 1.66$, $v(\{1, 4\}) = 2.00$, $v(\{2, 3\}) = 1.40$, $v(\{2, 4\}) = 1.81$, $v(\{3, 4\}) = 1.80$, $v(\{1, 2, 3\}) = 2.36$, $v(\{1, 2, 4\}) = 2.70$, $v(\{1, 3, 4\}) = 2.70$, $v(\{2, 3, 4\}) = 2.85$, and $v(\{1, 2, 3, 4\}) = 3.81$. Player 5 is a dummy player, thus $v(S \cup \{5\}) = v(S)$, for all $S \subseteq \{1, 2, 3, 4\}$.*

Since $\arg\max_{S \neq \emptyset} \left\{ \frac{v(S)}{|S|} \right\} = \{1, 4\}$, we must consider only those orderings in which either player 1 or 4 comes first. Let θ_{ijklm} be the ordering (i, j, k, l, m). A routine calculus shows that $\mathcal{M}(v) = \left\{ x^{\theta_{12345}}, x^{\theta_{12435}}, x^{\theta_{13245}}, x^{\theta_{13425}}, x^{\theta_{14235}}, x^{\theta_{41235}} \right\}$, where

$$\begin{aligned}
x^{\theta_{12345}} &= (1.000, 0.950, 0.900, 0.000, 0.000), \\
x^{\theta_{12435}} &= (1.000, 0.950, 0.700, 0.900, 0.000), \\
x^{\theta_{13245}} &= (1.000, 0.905, 0.950, 0.000, 0.000), \\
x^{\theta_{13425}} &= (1.000, 0.700, 0.950, 0.905, 0.000), \\
x^{\theta_{14235}} &= (1.000, 0.700, 0.700, 0.950, 0.000), \\
x^{\theta_{41235}} &= (0.960, 0.700, 0.700, 1.000, 0.000),
\end{aligned}$$

and

$$\begin{aligned}
y_{x^{\theta_{12345}}} &= (1.0000, 0.9500, 0.9000, 0.4800, 0.4800), \\
y_{x^{\theta_{12435}}} &= (1.0000, 0.9500, 0.7000, 0.9000, 0.2600), \\
y_{x^{\theta_{13245}}} &= (1.0000, 0.9050, 0.9500, 0.4775, 0.4775), \\
y_{x^{\theta_{13425}}} &= (1.0000, 0.7000, 0.9500, 0.9050, 0.2550), \\
y_{x^{\theta_{14235}}} &= (1.0000, 0.7000, 0.7000, 0.9500, 0.4600), \\
y_{x^{\theta_{41235}}} &= (0.9600, 0.7000, 0.7000, 1.0000, 0.4500).
\end{aligned}$$

Thus, $E(EDC(N, v)) = \left\{ y_{x^{\theta_{12345}}},\ y_{x^{\theta_{14235}}} \right\}$. Notice that, the solution reported by Dutta and Ray (1991) is $E(EDC(N, v)) = \{(1, 0.95, 0.9, 0.48, 0.48), (1, 0.9, 0.95, 0.48, 0.48), (1, 0.7, 0.7, 0.95, 0.46)\}$.

However, it turns out that $(1, 0.9, 0.95, 0.48, 0.48)$ is not an element of the equal division core since $\frac{v(\{2,4\})}{2} = 0.905$ and both players in coalition $\{2, 4\}$ receive less than 0.905. Hence, $(1, 0.9, 0.95, 0.48, 0.48) \notin E(EDC(N, v))$.

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