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Decision Support

A proportional approach to claims problems with a guaranteed minimum

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ABSTRACT

In distribution problems, and specifically in bankruptcy issues, the *Proportional* (*P*) and the *Egalitarian* (*EA*) divisions are two of the most popular ways to resolve the conflict. Nonetheless, when using the egalitarian division, agents may receive more than her claim. We propose a compromise between the proportional and the egalitarian approaches by considering the restriction that no one receives more than her claim. We show that the *most egalitarian* compromise fulfilling this restriction ensures a minimum amount to each agent. We also show that this compromise can be interpreted as a process that works in two steps as follows: first, all agents receive an equal share up to the smallest claim if possible (egalitarian distribution), and then, the remaining estate (if any) is allocated proportionally to the remaining claims (proportional distribution). Finally, we obtain that the recursive application of this process finishes at the *Constrained Equal Awards* solution (*CEA*).

1. Introduction

A claims problem is a particular case of distribution problem, in which the amount to be distributed, the *estate E*, is not enough to cover the agents' claims on it. This model describes the situation faced by a court that has to distribute the net worth of a bankrupt firm among its creditors. But, it also corresponds with cost-sharing, taxation, or rationing problems. How should the scarce resources be allocated among its claimants? The formal analysis of situations like these, which originates in a seminal paper by O'Neill (1982), shows that a vast number of well-behaved solutions[1] have been defined for solving claims problems, being the *Proportional* and the *Equal Awards* (egalitarian) the two prominent concepts used in real world. The term well-behaved reflects the idea that the considered solutions might fulfill some principles of fairness, or appealing properties. A way of comparing solutions is given by the equity condition of Lorenz-dominance (see Dutta & Ray, 1989). A recent paper (Bosmans & Lauwers, 2011) compares the most usual bankruptcy rules in terms of Lorenz-dominance and analyzes those solutions that favor to smaller claimants relative to larger ones.

An illustrative example of claims problems is the fishing quotas reduction, in which the agent's claim can be understood as the previous captures, and the estate is the new (lower) level of joint captures. A similar example is given by milk quotas among the EU members.[2] In both examples, *proportionality* is the main principle used. Nevertheless, a minimal (*survival*) amount, guaranteed to each producer, should be fixed in order to ensure the profitability of fishing (milk) industries. That is, some part of the *estate* should be allocated in an *egalitarian way*. This idea is somewhat related to the axiom of *Sustainability* (see Herrero & Villar, 2002). As they mention,

> "Sustainability is a protective criterion for those agents with small claims. To illustrate this, consider the interpretation of a bankruptcy situation as a reduction in the fishing quotas. Here agent i's claim corresponds to her actual level of captures and the estate to be distributed to the new aggregate level of captures. Sustainable claims correspond to those levels of captures such that, if nobody else had a larger level, the aggregate new level of captures would not impose any rationing. Sustainability says that agents with sustainable claims should not be rationed after the change in the aggregate level of captures."

A similar situation can be found when a university distributes the budget to Departments. In this case, the resources are distributed proportionally to the number of Professors, students, subjects,

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[1] The reader is referred to Moulin (2002) and Thomson (2003) as surveys of this literature.

[2] Quotas were introduced in 1984. Each member state was given a reference quantity which was then allocated to individual producers. The initial quotas were not sufficiently restrictive as to remedy the surplus situation and so the quotas were cut in the late 1980s and early 1990s. Quotas will end on April 1, 2015.

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etc., but a minimal (fixed) amount is allocated to each regardless of size.

An alternative example of using the proportional approach is the way in which seats in the Spanish Parliament are allocated to each electoral district (province).[3] This is made proportionally to the population in each province, but a minimal number of seats (2) is guaranteed to each.[4] A similar situation is found in the US case: based on data from the decennial census, each state is allocated a proportion of the 435 seats in the United States House of Representatives, although each state is guaranteed a minimum of one seat, regardless of population.[5] The remaining seats are allocated one at a time, to the state with the highest priority number. This apportionment is based on the proportion of each state's population to that of the Fifty States together. We shall return to these examples later.

Although proportionality is the most used criterion,[6] whenever the smallest claim is very small compared with the largest one, a proportional division provides nearly nothing for this (these) small claimant(s). In this sense, the previous comments and examples show that real world, when applying proportional distributions, try to ensure an egalitarian (minimal) amount to each agent.

In this paper we will define a new solution concept that captures this behavior. This solution can be understood as a compromise between the proportional and the egalitarian distributions. In choosing this compromise, if we wanted to use the same weight on the proportional and the egalitarian distributions for each problem, the largest weight one could assign to the egalitarian distribution would be zero (otherwise for some problems an agent would receive an amount larger than her claim). So, we propose that the weight of each of the two distributions depends on the particular claims problem we are analyzing. In so doing, we define the weight used on the egalitarian distribution to be the highest weight such that the resulting vector satisfies the claims boundedness restriction.

Under an alternative view, we can differentiate between two different class of problems: the first class consists of problems where the per-capita estate is small relative to the smallest claim, $c_1 \geqslant E$ (a condition called in the literature as an *unsustainable* claim), whereas in the problems of the second class the smallest claim is *sustainable*. Then, if the claims problem is in the first category, the egalitarian distribution satisfies claims boundedness and all agents receive equal awards; if the claims problem falls in the second category, we first assign to each agent the smallest claim (egalitarian distribution), revise claims and estate accordingly, and then distribute the remaining estate proportionally to the revised claims (proportional solution). By this way, we define a new solution. Our main result, Proposition 3, shows that both approaches coincide in the same solution which we call α_{min} – *Egalitarian* solution.[7]

In short, our compromise solution:

- modifies the *Equal Awards* division, so that the proposal satisfies the claim-boundedness condition;
- modifies the *Proportional* division and considers a *minimal* amount that each agent should receive, which is endogenously determined in each particular problem (E, c);[8]
- provides a result that coincides with the one we would obtain if we assign to each agent this minimal amount, and distribute the remaining estate (if any) in a proportional way.

The paper is organized as follows: Section 2 contains the preliminaries. Section 3 presents our solution concept. Sections 4 and 5 provide the axiomatic analysis, and in Section 6 we present some final comments. The appendix gathers the proofs.

2. Preliminaries: claims problems

Throughout the paper we will consider a set of agents $N = \{1, 2, \ldots, n\}$. Each agent is identified by her *claim*, c_i, $i \in N$, on the *estateE.* A **claims problem** appears whenever the estate is not enough to satisfy all the claims; that is, $\sum_{i=1}^{n} c_i > E$. Without loss of generality, we will order the agents according to their claims: $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_n$. The pair (E, c) represents the claims problem, and we will denote by $\mathcal{B}$ the set of all claims problems. A *claims rule* (**solution**) is a single valued function $\varphi : \mathcal{B}arrow\mathbb{R}^n_+$ such that, for each $i \in N$, $0 \leqslant \varphi_i(E, c) \leqslant c_i$, (**non-negativity** and **claim-boundedness**), and $\sum_{i=1}^{n} \varphi_i(E, c) = E$ (**efficiency**).

Many solution concepts have been defined in the literature aboutclaims problems (see for instance Thomson (2003) and Bosmans & Lauwers (2011)). The two most important criteria are the *Proportional* and the *Egalitarian* ones.

Definition 1. The *Proportional* solution, *P*. For each $(E, c) \in \mathcal{B}$ and each $i \in N$, $P_i(E, c) = \lambda c_i$, where $\lambda = \frac{E}{\sum_{i \in N} c_i}$.

Definition 2. The *Equal Awards* division, *EA*. For each $(E, c) \in \mathcal{B}$ and each $i \in N$, $EA_i(E, c) = \frac{E}{n}$.

It is easy to find examples in which the equal distribution of the *estate* exceeds some agent's claim.[9] In order to solve this situation the following modification of the *EA* division has been introduced.

Definition 3. The *Constrained Equal Awards* solution, *CEA*. For each $(E, c) \in \mathcal{B}$ and each $i \in N$, $CEA_i(E, c) \equiv \min\{c_i, \mu\}$, where μ is chosen so that $\sum_{i \in N} \min\{c_i, \mu\} = E$.

3. A proposal of solution: $\alpha_{\min}$ – Egalitarian

Given the *Proportional* and the *Egalitarian* divisions, we consider now the family of compromises:

$$\varphi_\alpha = \alpha P + (1 - \alpha) EA \quad \alpha \in [0, 1].$$

That is, given a claims problem (E, c) involving n agents,

$$(\varphi_\alpha)_i(E, c) = \alpha \frac{c_i E}{\sum_{i=1}^{n} c_i} + (1 - \alpha) \frac{E}{n} \quad \alpha \in [0, 1].$$

The following example computes this proposal for several values of α.

[3] This example involves *indivisibilities*, which is not a trivial issue (see, for instance, Moulin (2000)).

[4] In the case of Spanish Parliament, the allocation mechanism is as follows (Spanish LOREG, 2011, art. 162): (1) Congress is composed of three hundred and fifty Deputies. (2) Each province has a corresponding initial minimum of two deputies. (3) The remaining two hundred and forty-eight deputies are distributed among the provinces in proportion to its population, according to the following procedure: (a) Obtain a distribution fee obtained by dividing by two hundred forty-eight the total number of the legal population of peninsular and island provinces. (b) Allocate to each province as many deputies as resulting, in whole numbers, dividing the population of provincial law by the quota allocation. (c) The remaining deputies are distributed by assigning one to each of the provinces whose quotient obtained under paragraph before, have a higher decimal fraction.

[5] "Each State shall have at Least one Representative" (U.S. Const., art. I, 2, cl. 3.).

[6] "In western society, for example, the customary solution would be to split the asset in proportion to the claims", see Young (1994, p. 123).

[7] An interesting question that has been addressed to us is if we can do the same for any claims rule ψ instead of the *Proportional* one. We will see that it is not possible, in general, to extend our results.

[8] We will see that our proposal satisfies a lower bound on awards property.

[9] For instance, consider the claims vector $c = (20, 50, 60)$ and the estate $E = 100$.

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Example 1. Consider $(E, c) = (100, (40, 50, 70))$.

Claims	$\alpha = 0$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1$
40	100/3	31.25	29.17	27.08	25
50	100/3	32.81	32.29	31.77	31.25
70	100/3	35.94	38.54	41.15	43.75

As we have already mentioned, when $\alpha = 0$ the equal division may not satisfy the conditions of a solution (claim boundedness fails). In order to avoid this problem, we can obtain for each problem (E, c) the minimum value of $\alpha \in [0, 1]$ such that φ_α is a solution:

$$\alpha^*(E,c) = \min\{\alpha \in [0,1] \text{ such that for each } i \in N (\varphi_\alpha(E,c))_i \leqslant c_i\}.$$

Remark 1. Note that if we solve, for each agent $i \in N$, the equation

$$\alpha_i : (\varphi_\alpha(E,c))_i = c_i,$$

then

$$\alpha^*(E,c) = \max\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$$

Definition 4. The α_{min} – Egalitarian solution is defined for each claims problem (E, c) with $c_i > 0$ and for each $i \in N$, as:

$$\varphi_{\min}(E,c) = \varphi_{\alpha^*}(E,c)$$

where $\alpha^* = \alpha^*(E, c)$.

Note that α^* varies from a claims problem to another. However, by the way it is defined, the $\alpha_{\min}$– Egalitarian solution is continuous. Next, we consider a *consistent* extension of our solution in the presence of null claims, and we analyze the way of obtaining $\alpha^*(E, c)$.

Definition 5. If there are some zero claims, $c_1 = c_2 = \cdots = c_k = 0$, $c_j > 0$, for each $j > k$, we extend our solution in a *consistent* way:

$$\varphi_{\min}(E,c) = (\mathbf{0}, \varphi_{\min}(E,\bar{c})) \quad \mathbf{0} = (0,\ldots,0)_{1\times k} \quad \bar{c} = (c_{k+1},\ldots,c_n).$$

Proposition 1. *If the claim boundedness is fulfilled by the agent with lowest claim, it is fulfilled by each agent $i \in N$:*

$$(\varphi_\alpha(E,c))_1 \leqslant c_1 \Rightarrow (\varphi_\alpha(E,c))_i \leqslant c_i.$$

See the proof in the Appendix A1.

Remark 2. The result in the above proposition does not remain true if we use, in order to define φ_α, a solution ψ different from the *Proportional* one

$$\varphi_\alpha = \alpha\psi + (1-\alpha)EA \quad \alpha \in [0,1].$$

For instance, if we consider the problem $(E, c) = (90, (10, 12, 100))$ and a solution ψ such that $\psi(90, (10, 12, 100)) = (8, 11, 71)$, then the second agent is the one who defines $\alpha^*(E, c) = \frac{18}{19}$. See Section 6.

In the following result we obtain the exact expression of α^*.

Proposition 2. *Given a claims problem (E, c) the scalar α^* is:*

$$\alpha^*(E,c) = max\left\{0, \frac{C(E-nc_1)}{E(C-nc_1)}\right\} \quad C = \sum_{i=1}^{n} c_i$$

See the proof in the Appendix A2.

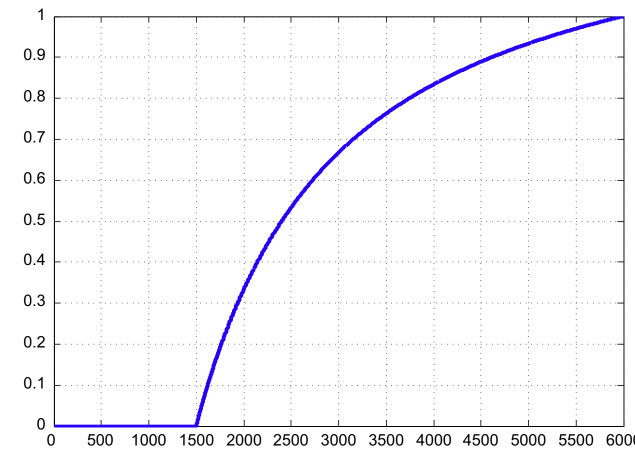

Fig. 1. $\alpha^*(E, c)$ as a function of E for fixed claims ($c = (500, 2000, 3500)$).

Remark 3. From the expression obtained in Proposition 2, it is immediate to see that, for $E \leqslant nc_1$ $\alpha^*(E, c) = 0$ and, for $E \geqslant nc_1$ $\alpha^*(E, c)$ is an strictly increasing and concave function of E for fixed claims vector c, as shown in Fig. 1.

Now, trying to facilitate the comparison with the main solutions in the literature, we compute our proposal for the next two examples taken from Bosmans and Lauwers (2011).[10]

Example 2. $(E, c) = (1500, (500, 2000, 3500))$.

c_i	*CEA*, $\varphi_{\min}$	*Pin*, *T*, *CE*	*AP*	*RA*, *MO*	*P*	*CEL*
500	500	250	214	166.7	125	0
2000	500	625	643	666.7	500	0
3500	500	625	643	666.7	875	1500

with $\alpha^*(E, c) = 0$.

Example 3. $(E, c) = (4500, (500, 2000, 3500))$.

c_i	*CEA*, *CE*	*Pin*	$\varphi_{\min}$	*P*	*RA*	*AP*	*T*	*MO*	*CEL*
500	500	500	500	375	333.3	285.72	250	166.7	0
2000	2000	1625	1500	1500	1333.3	1357.14	1375	1416.7	1500
3500	2000	2375	2500	2625	2333.3	2857.14	2875	2916.7	3000

with $\alpha^*(E, c) = \frac{8}{9}$.

Finally, in the following result, we find a precise expression of our solution which gives us an interesting interpretation: this solution assigns the minimal claim to any agent; thus it distributes the remaining estate $E' = E - nc_1$ in a proportional way among the agents with respect to the remaining claims $c'_i = c_i - c_1$. The proof is given in Appendix A3.

Proposition 3. *For each $(E, c) \in \mathcal{B}$, with $c_i > 0$ for each $i \in N$,*

$$\varphi_{min}(E,c) = \begin{cases} (E/n)\mathbf{1} & c_1 \geqslant E/n \\ \mathbf{c}^1 + P(E - nc_1, c - \mathbf{c}^1) & \text{otherwise} \end{cases}$$

[10] Hereinafter, *Pin*, *T*, *CE*, *AP*, *RA*, *MO*, and *CEL* will denote the *Piniles'*, *Talmud*, *Constrained Egalitarian*, *Adjusted Proportional*, *Random Arrival*, *Minimal Overlap and Constrained Equal Losses* solutions, respectively. See Thomson (2003) for their formal definitions.

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where $\mathbf{c}^1 = \begin{pmatrix} c_1 \\ \dots \\ c_1 \end{pmatrix}_{n\times 1}$ and $\mathbf{1} = \begin{pmatrix} 1 \\ \dots \\ 1 \end{pmatrix}_{n\times 1}$

The condition that splits both cases in Proposition 3 is known in the literature with the name of *sustainable claim*.[11] Note that if the smaller claim c_1 is not sustainable, $c_1 > E/n$, then no claim is sustainable. Therefore, the result in Proposition 3 can be stated as:

- If c_1 is *sustainable*, then $\varphi_{min}(E,c) = \mathbf{c}^1 + P(E - nc_1, c - \mathbf{c}^1)$.
- If c_1 is not *sustainable*, then $\varphi_{min}(E,c) = EA(E,c)$.

In Fig. 2 we represent the distribution of the *estate*, by depending on E, given by the $\alpha_{min}-$ Egalitarian solution.

Remark 4. It is important to mention that, as in Remark 2, the result in Proposition 3 is not true if we use a solution ψ different from the *Proportional* one. See Section 6.

4. Axiomatic analysis and comparison with other solutions

In this section we analyze our solution from an axiomatic point of view. First, next table summarizes the axiomatic comparative between the $\alpha_{min}-$ Egalitarian solution and the ones more directly related to it, *CEA* and *P*.

	φ_{min}	P	CEA
Order preservation	Yes	Yes	Yes
Resource monotonicity	Yes	Yes	Yes
Super-modularity	Yes	Yes	Yes
Order preservation under claims variations	Yes	Yes	Yes
Composition up	Yes	Yes	Yes
Composition down	Yes	Yes	Yes
Invariance under claims truncation	No	No	Yes
Self-duality	No	Yes	No
Midpoint property	No	Yes	No
Limited consistency	Yes	Yes	Yes
Reasonable lower bounds on awards	Yes	No	Yes

In order to check that the α_{min}-Egalitarian solution satisfies, or not, these properties, we formally give their definitions.

Order preservation (Aumann & Maschler, 1985) requires respecting the ordering of the claims: if agent *i's* claim is at least as large as agent *j's* claim, she should receive and lose at least as much as agent *j* does, respectively.

Order preservation: for each $(E,c) \in \mathcal{B}$, and each $i, j \in N$, such that $c_i \geqslant c_j$, then $\varphi_i(E,c) \geqslant \varphi_j(E,c)$, and $c_i - \varphi_i(E,c) \geqslant c_j - \varphi_j(E,c)$.

Resource monotonicity (Curiel, Maschler, & Tijs, 1987; Young, 1987) demands that if the endowment increases, then all individuals should get at least what they received initially.

Resource monotonicity: for each $(E,c) \in \mathcal{B}$ and each $E' \in \mathbb{R}_+$ such that $C > E' > E$, then $\varphi_i(E',c) \geqslant \varphi_i(E,c)$, for each $i \in N$.

Super-modularity (Dagan, Serrano, & Volij, 1997) requires that if the amount to divide increases, given two individuals, the one with the greater claim experiences a larger gain than the other.

Super-modularity: for each $(E,c) \in \mathcal{B}$, all $E' \in \mathbb{R}_+$ and each $i, j \in N$ such that $C > E' > E$ and $c_i \geqslant c_j$, then $\varphi_i(E',c) - \varphi_i(E,c) \geqslant \varphi_j(E',c) - \varphi_j(E,c)$.

[11] A claim c_i is said to be sustainable in (E,c) (see Herrero & Villar (2002)) if $\sum_{j=1}^{n} \min\{c_i, c_j\} \leqslant E$.

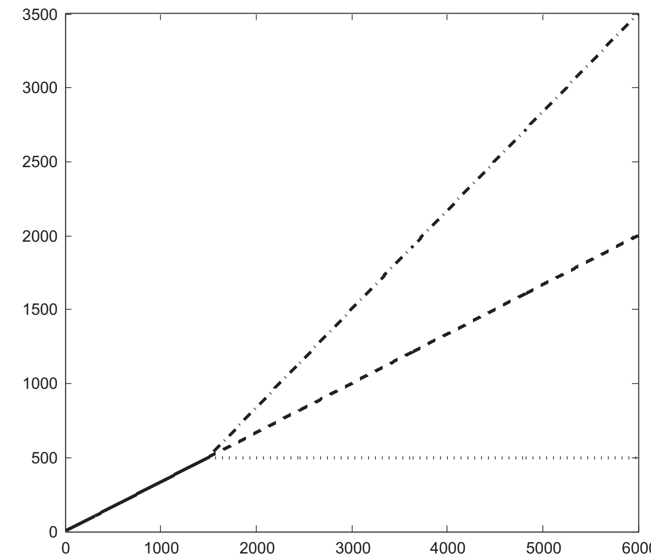

Fig. 2. The α_{min}-Egalitarian solution. The horizontal axis represents different levels of the *estate* E, and vertical axis denotes the amount each agent receives according their claims, $c = (500, 2000, 3500)$. The solid black line represents the egalitarian distribution of the estate our proposal obtains when $E \leqslant 1500$. From this point on, our proposal recommends the pointed-dashed lines for agents 1, 2, 3, from bottom to top, respectively.

Reasonable lower bounds on awards (Moreno-Ternero & Villar, 2004; Dominguez & Thomson, 2006) ensures that each individual receives at least the minimum of (i) her claim divided by the number of individuals and (ii) the amount available divided by the number of individuals.

Reasonable lower bounds on awards: for each $(E,c) \in \mathcal{B}$ and each $i \in N$, $\varphi_i(E,c) \geqslant \frac{\min\{c_i, E\}}{n}$.

Order preservation under claims variations (Thomson, 2006) requires that if the claim of some individual decreases, given two other individuals, the one with the greater claim experiences a larger gain than the other.

Order preservation under claims variations: for each $k \in N$, each pair (E,c) and $(E,c') \in \mathcal{B}$, with $c' = (c'_k, c_{-k})$ and $c'_k < c_k$ and each pair i and $j \in N\backslash k$ with $c_i \leqslant c_j$, $\varphi_i(E,c') - \varphi_i(E,c) \leqslant \varphi_j(E,c') - \varphi_j(E,c)$.[12]

Composition down requires that if, after the resources are distributed, they are reduced, a solution recommends the same allocation if we (i) cancel the initial distribution and apply the solution in the new situation or (ii) consider the initial awards as agents' claims on the revised problem and apply the solution to this new problem.

Composition down: for each $(E,c) \in B$, each $i \in N$, and each $0 \leqslant E' \leqslant E$, $\varphi_i(E',c) = \varphi_i(E', \varphi(E,c))$.

Composition up shows the opposite situation to composition down. If, after the resources are distributed, they are increased, a solution recommends the same allocation if we (i) cancel the initial distribution and apply the solution in the new situation or (ii) let agents keep their initial awards, adjust claims down by these amounts, and reapply the solution to divide only the increment of the estate with these adjusted claims.

Composition up: for each $(E',c) \in B$, each $i \in N$, and each $0 \leqslant E \leqslant E'$, $\varphi_i(E',c) = \varphi_i(E,c) + \varphi_i(E' - E, c - \varphi(E,c))$.

[12] We write (c'_k, c_{-k}) for the claims vector obtained from c by replacing c_k by c'_k.

Limited consistency states that adding an agent with a zero claim does not change the awards of the individuals already present. Obviously, if $(E,(c_1,c_2,\ldots,c_n))$ is a claims problem involving n individuals, then $(E,(0,c_1,c_2,\ldots,c_n))$ is a problem with $n+1$ individuals.

Limited consistency: for each $(E,c)\in\mathcal{B}$ and each $i\in N$, $\varphi_i(E,c)=\varphi_i(E,(0,c_1,\ldots,c_n))$.

Next Proposition, whose proof is given in Appendix A4, shows that the α_{min}-Egalitarian solution fulfills the above mentioned properties.

Proposition 4. *The α_{min}-Egalitarian solution fulfills Order preservation, Resource monotonicity, Super-modularity, Reasonable lower bounds on awards, Order preservation under claims variations, Composition up, Composition down and Limited consistency.*

Remark 5. Note that there is a property our solution fulfills that is not satisfied by the *Proportional* solution: *Reasonable lower bounds on awards*. This is the part that the *EA* division brings to our solution. The drawback is that some properties *P* fulfills are lost. Next we show some of them.[13]

Self-Duality implies that a solution recommends the same allocation when dividing awards and losses. Given a claims problem (E,c), losses are defined by the difference among the estate and the claims, $L=\sum_{i\in N}c_i-E$.

Self-duality: for each $(E,c)\in\mathcal{B}$ and each $i\in N$, $\varphi_i(E,c)=c_i-\varphi_i(L,c)$.

Midpoint Property ensures to each agent half of her claim when the estate equals half of the aggregate claim.

Midpoint Property: for each $(E,c)\in\mathcal{B}$ and each $i\in N$, if $E=C/2$, then $\varphi_i(E,c)=c_i/2$.

Invariance under claims truncation tells us that the part of a claim that is above the resources should not be taken into account.

Invariance under claims truncation: for each $(E,c)\in\mathcal{B}$ and each $i\in N$, $\varphi_i(E,c)=\varphi_i(E,\min\{c_i,E\}_{i\in N})$.

The following example shows that the α_{min}-Egalitarian solution does not satisfy these properties.

Example 4. Consider $(E,c)=(2000,(500,2000,3500))$. Then

$$\varphi_{min}(E,c)=(500,666.66,833.33).$$

$(L,c)=(4000,(500,2000,3500))$, and $\varphi_{min}(L,c)=(500,1333.33,2166.66)$. So, $c-\varphi_{min}(L,c)=(0,727.28,1272.73)\neq\varphi_{min}(E,c)$, not satisfying *Self-duality*.

Midpoint property implies $\varphi(E,c)=(250,1000,1750)\neq\varphi_{min}(E,c)$.

For $(E,c')=(2000,(500,2000,2000))$, $\alpha_{min}(E,c')=(500,750,750)\neq\varphi_{min}(E,c)$, not satisfying *Invariance under claims truncation*.

Finally, we introduce an operation for solutions that will help us to analyze the iterative application of the α_{min}-Egalitarian solution. We name this operation *Self-composition*, since it is related to the *Self-consistency* property (see for instance Grahn & Voorneveld, 2002).[14] In particular, *Self-composition* proposes a "recursive" distribution of the resources starting from agent 1. Formally,

Definition 6. Self-composition: for each $(E,c)\in\mathcal{B}$, and each m, $1\leqslant m\leqslant n$, the Self-composition of degree m is defined by:

$$\varphi^m(E,c)=(\varphi_1(E^1,c^1),\ldots,\varphi_{m-1}(E^{m-1},c^{m-1}),\varphi_m(E^m,c^m),\ldots,\varphi_n(E^m,c^m)),$$

where $(E^1,c^1)=(E,c)$ and for each $k>1$,

$$E^k=E^{k-1}-\varphi_{k-1}(E^{k-1},c^{k-1});\quad c^k=(0,\ldots,0,c_k,\ldots,c_n).$$

For instance, the *Self-composition* of degree 2 for some solution, φ^2, is obtained in the following way: first, agent 1 receives the amount recommended for her by $\varphi(E,c)$; then we solve the new problem in which the *estate* is reduced in the amount given to agent 1, and this agent has no claim anymore. That is, $\varphi^2(E,c)=(\varphi_1(E,c),\varphi_2(E^2,c^2),\varphi_3(E^2,c^2),\ldots,\varphi_n(E^2,c^2))$ where $E^2=E-\varphi_1(E,c)$; $c^2=(0,c_2,\ldots,c_n)$.

It is immediate to observe that if a solution is *Self-consistent*, then the *Self-composition* of any degree coincides with the own function (in some sense, it is *idempotent*); i.e., if φ satisfies *Self-consistency*, then for each $(E,c)\in\mathcal{B}$ and each m,

$$\varphi^m(E,c)=\varphi(E,c).$$

Next result, which can be straightforwardly obtained from Proposition 3, shows that if we compute the *Self-composition* of degree $n-1$ of the α_{min}-Egalitarian solution, we obtain the *CEA* solution.

Proposition 5. *The Self-composition of degree $n-1$ of the α_{min}-Egalitarian solution retrieves the CEA solution, where n is the number of agents.*

The result in the above Proposition may be understood as a recursive process (for a solution φ) which can be described as follows. Assume the agents are ordered so that $c_1\leqslant c_2\leqslant\ldots\leqslant c_n$. The solution φ applied to the original problem (E,c) only determines the share of agent 1, who in turn leaves with this share. The estate is reduced accordingly and the updated problem, say (E^2,c^2) for agents $2,3,\ldots,n$ is now used only to determine the share of agent 2. Agent 2 then leaves with this share and the estate is again reduced to construct (E^3,c^3) for agents $3,4,\ldots,n$. This recursive process is used to determine the share of every agent. The result shows that this recursive process, when applied to φ_{min}, produces the *CEA* allocation.

The α_{min}-Egalitarian solution does not satisfy self-consistency (otherwise, self-composition could not retrieve the *CEA* solution). But it satisfies a weaker version that we call *backwards consistency*. This condition requires that if the agent with largest claim leaves with his part, none of the other agents takes advantage.

Definition 7. Backwards Consistency: for each $(E,c)\in\mathcal{B}$,

$$\varphi(E,c)=(\varphi(E-\varphi_n(E,c),(c_1,c_2,\ldots,c_{n-1},0)),\varphi_n(E,c))$$

It is obvious that Self-consistency implies Backwards-consistency, but the converse is not true as shows the following result in which we prove that the α_{min}-Egalitarian solution satisfies this property. The proof is given in Appendix A5.

Proposition 6. *The α_{min}-Egalitarian solution satisfies Backwards-consis- tency.*

5. Lorenz dominance

An interesting tool to compare the behavior of solution concepts is that of Lorenz dominance. Let $\mathbb{R}^n_+$ be the set of positive n-dimensional vectors $x=(x_1,x_2,\ldots,x_n)$ ordered from small to large, i.e., $0<x_1\leqslant x_2\leqslant\ldots\leqslant x_n$. Let x and y be in $\mathbb{R}^n_+$. We say that x Lorenz dominates y, $x\succ_L y$, if for each $k=1,2,\ldots,n-1$: $x_1+x_2+\cdots+x_k\geqslant y_1+y_2+\cdots+y_k$ and $x_1+x_2+\cdots+x_n=y_1+y_2+\cdots+y_n$. If x Lorenz dominates y and $x\neq y$, then at least one of these $n-1$ inequalities

[13] It must be noticed that the main reason for not satisfying these properties is that *EA*, taken as a function, does not satisfy them.

[14] **Self-consistency:** for each $(E,c)\in\mathcal{B}$, each $S\subseteq N$ and each $i\in S$, then $\varphi_i(E,c)=\varphi_i(\sum_{k\in S}\varphi_k(E,c),c_{|S})$.

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is a strict inequality. The following definition extends the notion of Lorenz dominance to bankruptcy solutions.

Definition 8. Given two solutions φ and ψ it is said that φ Lorenz dominates ψ, $\varphi \succ_L \psi$, if for any claims problem (E, c) the vector $\varphi(E,c)$ Lorenz dominates $\psi(E,c)$.

Lorenz domination is a criterion used to check whether a solution is more favorable to smaller claimants relative to larger claimants. So, in some sense, a Lorenz dominant solution can be understood as more equitable. In a recent paper, Bosmans and Lauwers (2011) obtain a Lorenz dominance comparison among several solutions and they obtain that *CEA* is the more equitable solution, in the sense that it Lorenz dominates any other solution. More precisely, the dominance relation they obtain is as follows:

$$CEA \succ_L CE \succ_L Pin \succ_L P \succ_L CEL$$

Then, the *Proportional* solution only dominates *CEL*, which is the most favorable solution for larger claimants relative to smaller ones (so, the less equitable one).[15]

Among the solutions analyzed in Bosmans and Lauwers (2011), only *CEA* dominates the α_{min}-Egalitarian solution. Next result shows the Lorenz relationships between our solution and the ones on that paper.

Proposition 7

(a) *The α_{min}-Egalitarian solution Lorenz dominates P and CEL.*
(b) *There is no Lorenz domination between the α_{min}-Egalitarian solution and CE, Pin, RA, MO, T, and AP solutions.*

Part (b), with respect to *CE* and *Pin* is directly obtained from Examples 2 and 3. Moreover, Example 3 shows a claims problem in which the α_{min}-Egalitarian solution Lorenz dominates *RA*, *MO*, *T* and *AP*. Next example shows a case in which these solutions are not Lorenz dominated by the α_{min}-Egalitarian solution.

Example 5. Let $(E,c) = (20,(2,20,40))$. Then,

c_i	$\varphi_{\min}$	$RA = MO$	AP	T
2	2	0.66	0.96	1.9
20	6.5	9.66	9.52	9.5
40	11.5	9.66	9.52	9.5

Proof of part (a) is given in Appendix A6.

6. Final comments

In this paper we have proposed a compromise between the two most important and well-known ways of solving distribution problems: the *Proportional* and the *Egalitarian*. Moreover, we have analyzed the properties of this new solution and defined a recursive process, *Self-composition*, which allows us to recover the *Constrained Equal Awards* solution, by using our solution.

A natural question arises at this point: if we consider an alternative solution concept (e.g. Talmud solution, *T*) and we define in an analogous way

$$\varphi_\alpha^T = \alpha T + (1-\alpha) EA \qquad \alpha \in [0,1],$$

can we obtain with φ_{min}^T all the results we have obtained with φ_{min}? The answer is negative, as we have yet mentioned. The main result, that shows the equivalence between finding the α^* and applying φ_{min}, or assigning to each agent the smallest claim and distribute the remaining estate by using the Talmud solution, is no longer true, as the following example shows:

Example 6. Consider $(E,c) = (4500,(500,2000,3500))$. Then, $T(E,c) = (250,1375,2875)$, $EA(E,c) = (1500,1500,1500)$, $\alpha^* = 0.8$ and we obtain: $\varphi_{min}^T = (500, 1400, 1600)$ But, if we compute $(500,500,500) + T(3000,(0,1500,3000)) = (500,500,500) + (0,750,-2250) = (500,1250,2750)$, which is a different result.

Note that the α_{min}-Egalitarian solution can be also understood as a kind of *"Constrained Proportional"* solution in the sense that it can be used to ensure a minimum amount to any agent. Suppose that a small amount $\tilde{c} < c_1$ must be received by each agent.[16] What remains of the *estate*, if any, is shared proportionally among all agents. Then, given a claims problem (E,c) this distribution can be obtained by using the α_{min}- Egalitarian solution in the following way:

$$\varphi(E,c) := \varphi_{min}(E + \tilde{c}, c^*) \qquad c^* = (c_0 = \tilde{c}, c_1, \ldots, c_n)$$

where only the last $n-$ components of the α_{min}-Egalitarian solution are considered. This interpretation can be used, as we have mentioned in the Introduction, to obtain the distribution of seats in Spanish Parliament among districts. The Spanish system guarantees two seats to any district. The other seats are distributed to districts proportional to the population. Then, by applying the α_{min}-Egalitarian solution with $\tilde{c} = 2$ we obtain the actual distribution of seats.

Finally, we want to point out a possible way of extending our results by considering a more general class of problems, with claims and constraints simultaneously (see Bergantiños & Lorenzo (2008)).

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Appendix A

A.1. Proof of Proposition 1

For each $(E,c) \in \mathcal{B}$ and given an agent $i \neq 1 \in N$, $(\varphi_{min}(E,c))_i = (1-\alpha^*)\frac{E}{n} + \alpha^* \frac{c_i E}{C} = c_1 - \alpha^* \; \frac{c_1 E}{C} + \alpha^* \frac{c_i E}{C} = c_i + \; \left(\frac{\alpha^* E}{C} - 1\right)(c_i - c_1) \leqslant c_i$ $\square$

A.2. Proof of Proposition 2

From Proposition 1, α^* is the solution of the equation:

$$\alpha P_1 + (1-\alpha)\frac{E}{n} = c_1.$$

That implies

$$\alpha = \frac{\frac{E}{n} - c_1}{\left(\frac{1}{n} - \frac{c_1}{C}\right)E} = \frac{C(E - nc_1)}{E(C - nc_1)} \qquad C = \sum_{i=1}^{n} c_i.$$

[15] See Bosmans and Lauwers (2011) for additional relationships.

[16] Such situations can be found, for instance, in the distribution of a heritage; or the State's guarantee of a minimum retirement pension; fixing a minimal fishing quota, or milk quota; …

By observing that denominator is always positive, it is immediate to obtain that this fraction is less or equal than 1. On the other hand, it is negative whenever $\frac{E}{n} \leqslant c_1$ and, in this case, $\alpha^* = 0$. □

A.3. Proof of Proposition 3

Given a claims problem $(E, c) \in \mathcal{B}$, it is clear that whenever $c_1 \geqslant E/n$ then $\alpha^*(E, c) = 0$ and $\varphi_{min}(E, c) = CEA(E, c) = E/n$.

Suppose now that $c_1 < E/n$. Then, for each $i \in N$, see Proposition 2,

$$\begin{aligned}(\varphi_{min}(E,c))_i &= \alpha^* P_i(E,c) + (1-\alpha^*)EA_i(E,c)\\ &= \frac{C(E-nc_1)}{E(C-nc_1)}\frac{Ec_i}{\sum_{j=1}^n c_j} + \left(1 - \frac{C(E-nc_1)}{E(C-nc_1)}\right)\frac{E}{n}\\ &= \frac{E-nc_1}{C-nc_1}c_i + \frac{c_1(C-E)}{C-nc_1} = c_1 + (E-nc_1)\frac{c_i-c_1}{C-nc_1}\\ &= c_1 + P_i(E-nc_1, c-\mathbf{c}^1). \qquad \square\end{aligned}$$

A.4. Proof of Proposition 4

In order to check this result, note that for each $(E, c) \in \mathcal{B}$, if $c_1 \geqslant \frac{E}{n}$, then the φ_{min} distributes the estate as the *EA* solution, which satisfies all properties. Otherwise,

$$\varphi_{min}(E,c) = \mathbf{c}^1 + P(E-nc_1, c-\mathbf{c}^1).$$

That is, each agent receives the smallest claim c_1 and the remaining estate $E_1 = E - nc_1$ is distributed in a proportional way among the other agents. Then, *Order Preservation* is obvious. With respect to *Resource monotonicity* the only unclear case is whenever

$$c_1 < \frac{E'}{n} \quad \text{and} \quad c_1 \geqslant \frac{E}{n}.$$

Then,

$$\varphi_{min}(E,c) = \frac{E}{n}, \quad \varphi_{min}(E',c) = \mathbf{c}^1 + P(E'_1, c-\mathbf{c}^1).$$

and the property is fulfilled. A similar reasoning can be made with *Super-modularity* and *Composition down*. Regarding to *Composition up* the only unclear case is whenever $c_1 > \frac{E'}{n}$ and $c_1 \leqslant \frac{E}{n}$. But, in this case, $\varphi_{min}(E', c) = \varphi_{min}(E', \varphi_{min}(E, c)) = \frac{E'}{n}$, and the property is fulfilled.

Reasonable lower bounds on awards is satisfied, since

$$(\varphi_{min}(E,c))_i \geqslant min\left\{\frac{E}{n}, c_1 + P_i(E_1, c-c^1)\right\} \geqslant \frac{min\{c_i, E\}}{n}.$$

Finally, in order to prove that our solution fulfills *Order preservation under claims variations* consider two claims problems (E, c), $(E, c') \in \mathcal{B}$, such that $c' = (c'_k, c_{-k})$, $c'_k < c_k$, and consider $i, j \in N \backslash k$ with $c_i \leqslant c_j$. We have the following possibilities:

(1) If $c_1 \geqslant c'_1 \geqslant \frac{E}{n}$, then the α_{min} distributes the estate as the *CEA* solution, which satisfies *Order preservation under claims truncation*.

(2) If $c_1 \geqslant \frac{E}{n} > c'_1$, then $k = 1$ and

$$(\varphi_{min})_i(E,c) = \frac{E}{n} \quad (\varphi_{min})_i(E,c') = c'_1 + \frac{E-nc'_1}{\sum_{i\in N\backslash 1}(c_i - c'_1)}(c_i - c'_1).$$

So, for each pair $i, j \in N\backslash 1$ with $c_i \leqslant c_j$,

$$\begin{aligned}&[(\varphi_{min})_i(E,c') - (\varphi_{min})_i(E,c) \leqslant (\varphi_{min})_j(E,c') - (\varphi_{min})_j(E,c)]\\ &\iff \left[c'_1 + \frac{E-nc'_1}{\sum_{i\in N\backslash 1}(c_i-c'_1)}(c_i-c'_1) - \frac{E}{n}\right.\\ &\qquad \left.\leqslant c'_1 + \frac{E-nc'_1}{\sum_{i\in N\backslash 1}(c_j-c'_1)}(c_j-c'_1) - \frac{E}{n}\right]\\ &\iff [c_i - c'_1 \leqslant c_j - c'_1] \iff c_i \leqslant c_j.\end{aligned}$$

(3) If $c_1 \leqslant \frac{E}{n}$, then

$$(\varphi_{min})_i(E,c) = c_1 + \frac{E-nc_1}{\sum_{i\in N\backslash 1}(c_i-c_1)}(c_i-c_1)$$

(3.1) If $k = 1$, for each pair $i, j \in N\backslash 1$ with $c_i \leqslant c_j$,

$$\begin{aligned}&\left[(\varphi_{min})_i(E,c') - (\varphi_{min})_i(E,c) \leqslant (\varphi_{min})_j(E,c') - (\varphi_{min})_j(E,c)\right]\\ &\iff \left[c'_1 + \frac{E-nc'_1}{\sum_{i\in N\backslash 1}(c_i-c'_1)}(c_i-c'_1) - c_1\right.\\ &\qquad - \frac{E-nc_1}{\sum_{i\in N\backslash 1}(c_i-c_1)}(c_i-c_1)\\ &\qquad \leqslant c'_1 + \frac{E-nc'_1}{\sum_{i\in N\backslash 1}(c_j-c'_1)}(c_j-c'_1) - c_1\\ &\qquad \left.- \frac{E-nc_1}{\sum_{i\in N\backslash 1}(c_j-c_1)}(c_j-c_1)\right]\\ &\iff \left[\frac{E-nc'_1}{\sum_{i\in N\backslash 1}(c_i-c'_1)}(c_i-c'_1) - \frac{E-nc_1}{\sum_{i\in N\backslash 1}(c_i-c_1)}(c_i-c_1)\right.\\ &\qquad \left.\leqslant \frac{E-nc'_1}{\sum_{i\in N\backslash 1}(c_j-c'_1)}(c_j-c'_1) - \frac{E-nc_1}{\sum_{i\in N\backslash 1}(c_j-c_1)}(c_j-c_1)\right]\\ &\iff \left[\frac{E-nc_1}{\sum_{i\in N\backslash 1}(c_j-c_1)}(c_j-c_i) \leqslant \frac{E-nc'_1}{\sum_{i\in N\backslash 1}(c_j-c'_1)}(c_j-c_i)\right]\\ &\iff c'_1 \leqslant c_1.\end{aligned}$$

(3.2) If $k \neq 1$, then

$$(\varphi_{min})_i(E,c) = c_1 + \frac{E-nc_1}{\sum_{i\in N\backslash 1}(c_i-c_1)}(c_i-c_1)$$

$$(\varphi_{min})_j(E,c) = c_1 + \frac{E-nc_1}{\sum_{i\in N\backslash 1}(c_i-c_1)}(c_j-c_1),$$

and the property is fulfilled.

Clearly, by the way we have defined our consistent extension (see Remark 4), the α_{min}-Egalitarian solution fulfills *Limited consistency*. □

A.5. Proof of Proposition 5

Consider a claims problem $(E, c) \in \mathcal{B}$.

(1) If $c_1 \leqslant \frac{E}{n}$, and we name $(x_1, x_2, \ldots, x_n) = \varphi_{min}(E, c)$

$$x_i = c_1 + \frac{c_i - c_1}{C - c_1}(E - nc_1); \quad C = \sum_{i=1}^n c_i;$$

$$E' = E - x_n = (n-1)c_1 + (E - nc_1) - \frac{c_n - c_1}{C - nc_1}(E - nc_1);$$

$$c' = (c_1, c_2, \ldots, c_n - 1); \quad C' = C - c_n; \quad c_1 \leqslant \frac{E'}{n-1}.$$

Then,

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$$(\varphi_{min})_i(E', c') = c_1 + \frac{c_i - c_1}{C' - c_1}(E' - (n-1)c_1), \quad i = 1, 2, \ldots, n-1,$$

which coincides with x_i.

(**2**) If $c_1 > \frac{E}{n}$, then $\varphi_{min}(E, c) = EA(E, c) = \frac{E}{n}$ and the property is fulfilled. □

A.6. Proof of Proposition 6

(a) For each $(E, c) \in \mathcal{B}$ and each $i \in N$, it follows from Bosmans and Lauwers (2011) that φ_{min} Lorenz dominates *CEL*. In order to prove that it also dominates the proportional solution *P*, some notation will help. Given a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ we define the partial sums vector:

$$\mathbf{z}_x = (x_1, x_1 + x_2, \ldots, x_1 + x_2 + \ldots + x_n)$$

Then, $\mathbf{x} \succ_L \mathbf{y} \quad \Leftrightarrow \quad \mathbf{x} \neq \mathbf{y}$ and $(\mathbf{z}_x)_i \geqslant (\mathbf{z}_y)_i$. Now denote:

$$\mathbf{x} = EA(E, c) \quad \mathbf{y} = P(E, c)$$

We know that $\mathbf{x} \succ_L \mathbf{y}$, so $(\mathbf{z}_x)_i \geqslant (\mathbf{z}_y)_i$. For each $\alpha \in [0, 1]$,

$$\alpha(\mathbf{z}_y)_i + (1-\alpha)(\mathbf{z}_x)_i \geqslant \alpha(\mathbf{z}_y)_i + (1-\alpha)(\mathbf{z}_y)_i = (\mathbf{z}_y)_i.$$

We conclude that $(\mathbf{z}_{\varphi min}(E, c))_i \geqslant (\mathbf{z}_y)_i$ and then $\varphi_{min}(E, c) \succ_L P(E, c)$. □

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