# Some Notes and Comments on the Efficient use of Information in Repeated Games with Poisson Signals.*

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**Abstract**

In the present paper we characterize the optimal use of Poisson signals to establish incentives in the "bad" and "good" news models of Abreu et al. (1991). In the former, for small time intervals the signals' quality is high and we observe a "selective" use of information; otherwise there is a "mass" use. In the latter, for small time intervals the signals' quality is low and we observe a "fine" use of information; otherwise there is a "non-selective" use.

JEL: C73, D82, D86.

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## 1 Introduction

Most economic situations of interest differ in the frequency of interaction between the involved parties. It tends to be the rule rather than the exception. This fact affects the provision of incentives and the value of these relations.

In the present paper we study the Abreu et al. (1991) repeated game information structure for varying frequencies of play other than the zero-limit case. It is not our goal to present a general theory, but rather we illuminate some conceptual points through a simple

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game. A small time interval favors coordination through discounting. However, the effect of time on the [Poisson] distribution of public signals is crucial and depends on how actions feedback into signals.[1]

The limit case often allows for a tractable analysis and provides meaningful insights. Yet, its limits have long been recognized. There is a clear need for a better understanding of what happens outside the limit. However, closed form payoff characterizations are difficult and conclusions must be based in numerical observations. This issue would not be relevant if we would not have found interesting and completely new trade-offs between information quantity and quality. Actually, this might be the most relevant issue in this literature.[2]

Our findings can be summarized as follows. We start characterizing the best symmetric equilibrium payoff in terms of the optimal use of Poisson signals.

In the **"bad" news model**[3] signals are more informative for small $\Delta$. In small but increasing $\Delta$, incentives are based on a large but decreasing set of signals choosen from the ones with higher quality. There is a **"selective"** use of information. The process continues until the signals' quality deteriorates and the deviation incentives through discounting become strong enough. The punishment decision becomes based on an increasing set of signals. There is a **"mass"** use of information.

In the **"good" news model**[4] the signals quality improves with $\Delta$. Away from the limit case, for sufficiently small but increasing $\Delta$ (in a small region), incentives are based on a small and decreasing set of signals of improved quality. In this scenario there is a **"fine"** use of information. For relatively larger time intervals, decisions are based on an increasing number of signals. In this region occurs a **"non-selective"** use of information.

A consequence of these trade-offs between discounting, information quality and quantity is that the payoffs are neither monotonic nor smooth in $\Delta$.

[1] Faingold and Sannikov (2011), Fudenberg and Levine (2007, 2009), Osório (2012, 2015) and Sannikov and Skrzypacz (2007, 2010) study the limit case with Brownian signals.

[2] The present paper also relates with the literature on information precision in repeated games, see Kandori (1992). Using a time varying information structure Kamada and Kominers (2010) show a informational paradox. Our argument is different but we also found that more information does not necessarily leads to better incentives and might be substituted by information quality, and vice versa.

[3] Extreme realizations are more likely to be interpreted as suggesting defection. Abreu et al. (1991) show that equilibrium payoffs above the static Nash can be sustained in the limit. Fudenberg and Levine (2007) present a Brownian "equivalent" model by assuming that a deviation increases the volatility of the process. They show that full efficiency is possible in the limit. We should note that the way Poisson and Brownian signals provide incentives is different (Fundenberg and Levine, 2007, 2009; Fudenberg and Olszewski, 2011; Osório, 2015). The Brownian "equivalent" full efficient result is possible because the diffusion is a frequent events process and reliable inference about extreme events is possible in very small time intervals.

[4] Extreme realizations are more likely to be interpreted as suggesting cooperation. Abreu et al. (1991) found that the equilibrium degenerates in the limit. Fudenberg and Levine (2007) suggest a Brownian "equivalent" formulation in which a deviation decreases the volatility. They show the existence of a non-trivial but not full efficient limit payoff.

| | $C$ | $D$ |
|---|---|---|
| $C$ | $\pi, \pi$ | $-(\pi' - \pi), \pi'$ |
| $D$ | $\pi', -(\pi' - \pi)$ | $0, 0$ |

Table 1: The Prisoners' Dilemma Stage Game Payoffs.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 characterizes the best symmetric payoff. Section 4 study the effect of varying $\Delta$. Section 5 discusses non-limit incentives and payoffs. All of the proofs are relegated to an appendix.

# 2 The Model

We consider the prisoners' dilemma stage game payoffs in Table 1, with $\pi' > \pi > 0$. The profile $(C, C)$ returns the best symmetric payoff.

At moments in time $0, \Delta, 2\Delta, ...$, players simultaneously take their actions. In the subsequent period, an imperfect signal about these actions is commonly observed.

The common discount factor is $\delta \equiv e^{-r\Delta}$, where $r \in (0, \infty)$ denotes the discount rate.[5]

The public signals follow an homogeneous Poisson process with rate parameter $\beta$ in case of mutual cooperation $(C, C)$, and $\mu$ in case of defection $(C, D)$ or $(D, C)$. The probability of occurrence of a particular number of news or events $k = 0, 1, ...$, are, respectively,

$$p_k \equiv (\beta\Delta)^k e^{-\beta\Delta}/k! \text{ and } q_k \equiv (\mu\Delta)^k e^{-\mu\Delta}/k!. \tag{1}$$

Let $\Pi$ denotes the set of events that suggest cooperation, see the example 2. The probability of observing any number $k \in \Pi$ is given by $\sum_{k\in\Pi} p_k$ in case of cooperation, and by $\sum_{k\in\Pi} q_k$ in case of defection.

As in Abreu et al. (1991), we consider the **"bad" news model** $(\mu > \beta)$ : large values of $k$ (many bad news) are more likely to be interpreted as signaling defection. The **"good" news model** $(\beta > \mu)$ : low values of $k$ (few good news) are more likely to be interpreted as signaling defection. Cooperation has the opposite interpretation.

We look at profiles of strategies that form a *perfect public equilibrium (PPE)*.[6]

[5] We restrict to the simplest setting. The results generalize straightforwardly for more general discount factors, payoffs and games.

[6] A strategy is public if it depends only on the public history [of signals] and not on the private history [of signals and individual actions]. Given a public history, a profile of public strategies that induces Nash equilibrium on the continuation game from that time on is called a PPE.

# 3 The Best Symmetric Equilibrium

The Poisson is a discrete probability distribution, therefore, $\alpha - grim$ strategies are required to make the enforceability constraint (3) to bind. After the occurrence of a given event $k = 0, 1, ...,$ players coordinate the punishment decision on a public random device, which effectively punishes with probability $\alpha_k \in [0, 1]$, and forgives otherwise. Therefore, the set of effective punishment probabilities is defined as $\alpha \equiv \{\alpha_0, \alpha_1, ..., \alpha_\infty\}$.

The normalized value of the infinitely repeated game is given by

$$v = (1 - \delta)\,\pi + \delta v \textstyle\sum_{k \in \Pi} p_k\,(1 - \alpha_k)\,. \tag{2}$$

It is a convex combination between the instantaneous cooperative payoff and the continuation value. This structure is enforceable if

$$v \geq (1 - \delta)\,\pi' + \delta v \textstyle\sum_{k \in \Pi} q_k\,(1 - \alpha_k)\,. \tag{3}$$

To study the non-limit case we need to complete Propositions 1 and 2 in Abreu et al. (1991) by specifying the optimal choice of the signals in $\Pi$ and effective punishment probabilities $\alpha$ for any frequency of play.

**Proposition 1** *The best symmetric payoff is given by*

$$v = \pi - (\pi' - \pi)/(l_K - 1), \tag{4}$$

*where*

$$l_K = \frac{1 - \left(\sum_{k \in \Pi} q_k + (1 - \alpha_K) q_K\right)}{1 - \left(\sum_{k \in \Pi} p_k + (1 - \alpha_K) p_K\right)}, \tag{5}$$

*and*

$$\alpha_K = 1 - \frac{\pi\left(1 - \delta \sum_{k \in \Pi} q_k\right) - \pi'\left(1 - \delta \sum_{k \in \Pi} p_k\right)}{\delta\left(\pi q_K - \pi' p_K\right)} \in (0, 1]\,, \tag{6}$$

*where* $\Pi = \{0, 1, ..., K - 1\}$ *and* $\alpha = \{0, ..., 0, \alpha_K, 1, ..., 1\}$ *for the "bad" news model, and* $\Pi = \{K + 1, ..., \infty\}$ *and* $\alpha = \{1, ..., 1, \alpha_K, 0, ..., 0\}$ *for the "good" news model.*

For instance, in the "bad" news model the most informative events about cooperation are the lower magnitude ones because $p_k/q_k > p_{k+1}/q_{k+1}$ for all $k$. Moreover, if the observation $k = K - 1$ is considered as signaling cooperation, then all the other numbers of smaller magnitude must signal cooperation as well (i.e., $\alpha_k = 0$ for all $k = 0, 1, ..., K - 1$). Similarly, if the observation $k = K + 1$ suggests defection, so does all the other numbers of higher magnitude (i.e., $\alpha_k = 1$ for all $k = K + 1, K + 2, ...$). Finally, the observation $k = K$ implies that punishment occurs with probability $\alpha_K$. This decision is taken by a public randomization device that depends on the players incentives (enforceability must bind).

A similar reasoning applies to the "good" news model, in which case $p_{k+1}/q_{k+1} > p_k/q_k$ for all $k$, i.e., the larger magnitude events are the most informative about cooperation.[7]

In order to get a better intuition consider the following example.

**Example 2** *Let $\pi = 2$, $\pi' = 3$, $r = 0.1$ and $\Delta = 5$.*

***"Bad" news model*** *with $\mu = 1.7 > \beta = 1$ : in equilibrium we have the cutoff signal $K = 9$, $\Pi = \{0, 1, ..., 8\}$ and $\alpha = \{0, ..., 0, 0.33, 1, ..., 1\}$ (for completeness $v = 1.87$). In other words, the events $k = 10, 11, ...,$ suggest defection and are punished with probability one, while the events $k = 0, 1, ..., 8,$ suggest cooperation and are punished with zero probability. The event $k = 9$ is punished with probability $\alpha_9 = 0.33$.*

***"Good" news model*** *with $\beta = 3 > \mu = 1$ : in equilibrium we have the cutoff signal $K = 4$, $\Pi = \{5, 6, ..., \infty\}$ and $\alpha = \{1, 1, 1, 1, 0.34, 0, ..., 0\}$ (for completeness $v = 1.99$). In other words, the events $k = 0, 1, 2, 3,$ suggest defection and are punished with probability one, while the events $k = 5, 6, ...,$ suggest cooperation and are punished with zero probability. The event $k = 4$ is punished with probability $\alpha_4 = 0.34$.*

Following the discussion, the cutoff event $k = K$ draws the frontier between signals suggesting cooperation and punishment.[8]

**Definition 3** *We say that a non-trivial equilibrium exists if $\alpha_K \in (0, 1]$ enforces cooperation.*

Note that for a given $K$, as we vary $r$ or $\Delta$, so thus the punishment probability $\alpha_K$ in the interval $[0, 1]$. We have the following continuity property.

**Corollary 4** *In the "bad" news model, $\alpha_K = 0 \Leftrightarrow \alpha_{K+1} = 1$ and $\alpha_K = 1 \Leftrightarrow \alpha_{K-1} = 0$.*

*In the "good" news model, $\alpha_K = 1 \Leftrightarrow \alpha_{K+1} = 0$ and $\alpha_K = 0 \Leftrightarrow \alpha_{K-1} = 1$.*

For instance, in the "bad" news model, monitoring with the set $\Pi = \{0, 1, ..., K - 1\}$ and punish the event $k = K$ with probability zero, is the same as monitoring with the set $\Pi = \{0, 1, ..., K\}$ and punish the event $k = K + 1$ with probability one.

[7] The expressions $1 - \left(\sum_{k\in\Pi} q_k + q_K (1 - \alpha_K)\right)$ and $1 - \left(\sum_{k\in\Pi} p_k + p_K (1 - \alpha_K)\right)$ are the equivalent to the Abreu et al. (1991) probabilities of correct and mistaken punishment, respectively. Therefore, $l_K$ is the likelihood of correct detection of defective behavior.

[8] Note that $K$ depends on $r$ and $\Delta$ (and the other parameters of the model). To shorten on notation, we denote it without this explicit dependence.

# 4 Varying the Time Interval $\Delta$

In order to understand how information is used to provide incentives we study the behavior of the function $\alpha_K$ for varying $\Delta$.

**Proposition 5** *Suppose that* $\pi(\mu + r) > \pi'(\beta + r)$. *In the non-trivial "bad" news equilibrium,* $\partial\alpha_K/\partial\Delta < 0$ *for small* $\Delta$, *and* $\partial\alpha_K/\partial\Delta > 0$ *for large* $\Delta$, *with* $K > 0$.

*In the non-trivial "good" news equilibrium,* $\partial\alpha_0/\partial\Delta < 0$ *for small* $\Delta > 0$, *and* $\partial\alpha_K/\partial\Delta > 0$ *for large* $\Delta$, *with* $K < \infty$.

**Corollary 6** *In a non-trivial equilibrium, while* $\partial\alpha_K/\partial\Delta < 0$ *the cardinality of the set* $\Pi$ *weakly increases, while if* $\partial\alpha_K/\partial\Delta > 0$ *the cardinality of the set* $\Pi$ *weakly decreases when* $\Delta$ *increases.*

These results show the effect of time on the use of information quantity for the provision of incentives. The full picture requires the consideration of information quality effects.

Note a trade-off between payoffs and incentives. All the rest constant, in the ideal scenario the larger the cardinality of the set $\Pi$, less mistaken punishment, and therefore, higher payoffs.

# 5 Non-limit Incentives and Payoffs

When $r$ decreases players become more patient, we can add more events to the set $\Pi$, and payoffs improve monotonically.[9] For varying $\Delta$, the result is less clear cut. We may observe in simultaneous an increases in the punishment probability and payoffs, and vice versa. Such it is the result of a mixture of trade-offs between discounting, information quality and quantity. These aspects are tractable in the limit case in Abreu et al. (1991), but are complex for other values of $\Delta$.

[insert figure about here]

**"Bad" news model** - Signals are more informative for small $\Delta$. When $\Delta$ is small the deviation incentives are low and the most likely signals are the low magnitude ones, for that reason $K$ is small (e.g., for $\Delta = 0.05$ we have $K = 1$ and $\Pi = \{0\}$). The punishment decision is based on a large number of events, i.e., the most unlikely and more informative [high magnitude] events.

[9] Proposition 4 in Abreu et al. (1991), shows a full efficient result for the "bad" news model. Note that the limit $r \downarrow 0$, implies that $K \uparrow \infty$ and $K = 0$, in the "bad" and "good" news models, respectively.

Note that, as $\Delta$ increases, incentives are based on a decreasing set of signals choosen from the higher quality ones. Information quantity (number of signals) is exchanged for information precision. We observe a **"selective"** use of information. In this region the less informative but more likely signals (the lower magnitude ones) are used to create value (e.g., for $\Delta = 2$ we have $K = 5$ and $\Pi = \{0, ..., 4\}$). For that reason the payoffs tend to increase.

The process continues until the signals' quality deteriorates and the deviation incentives through discounting are strong enough. In this case, incentives are maintained through an increasing use of information quantity (i.e., the punishment decision is based on a large set of signals). Incentives are provided through a **"mass"** use of information. The set $\Pi$ decreases and payoffs fall.

For sufficiently large $\Delta$ the incentives collapse and the equilibrium degenerates. Figure 3 shows the punishment probability and the signals used to create value for varying $\Delta$.[10]

**"Good" news model** - The low magnitude events are the most informative and signals quality improves with $\Delta$. In spite of the low deviation incentives, for small time intervals information quality is so low that the equilibrium degenerates. The difficulty is the lack of observed signals. Moreover, the most likely event is $k = 0$ and defection makes it even more likely to occur. For that reason monitoring collapses.

For sufficiently small $\Delta$ but away from the limit, it might be possible to sustain non-trivial equilibria. The punishment probability decreases together with an improvement in quality. This scenario occurs in a small region, e.g., the point $\Delta = 0.25$ in Figure 4. The low and specific number of employed signals corresponds to a situation of **"fine"** [very specialized] information usage.

For relatively larger time intervals, the punishment decision is based on a small but increasing number of low magnitude signals. For instance, from $\Delta = 4$ to $\Delta = 5$, we move from $K = 3$ to $K = 4$. Note that in spite of $\Pi$ decreases from $\{4, ..., \infty\}$ to $\{5, ..., \infty\}$, payoffs may increase because of the signals improved quality. In this region we observe a **"non-selective"** information usage.

Finally, for sufficiently high $\Delta$ the equilibrium degenerates. The informational gains do not compensate the fall in discounting. Figure 4 shows the punishment probability and the signals used to provide incentives for varying $\Delta$.[11]

[10] Figure 1 illustrates the "bad" news model payoffs for varying $\Delta$. They increase for $K = 1$. They have a u-shape for $K = 2, ..., 9$. They are concave decreasing for $K = 10$. The largest payoff is achieved with $\alpha_7 = 0$ or $\alpha_8 = 1$, i.e., at the coordinate $(\Delta, v) = (4.35, 1.88)$.

[11] Figure 2 illustrates the respective "good" news model payoffs for varying $\Delta$. The payoffs increase for $K = 0$, They presents a u-shape for $K = 1, ..., 13$, and are convex decreasing for $K = 14, ..., 19$. The largest payoff is achieved at $\alpha_9 = 1$ or $\alpha_{10} = 0$, i.e., the coordinate $(\Delta, v) = (8.43, 1.99)$. This observation is consistent with Abreu et al. (1991), i.e., time lags favor the provision of incentives and payoffs. Fudenberg et al. (2014) exploit this argument to show a non-limit folk theorem.

**Final comments**

1) In both models, for small (respectively, large) $\Delta$ the punishment probability is negatively (respectively, positively) sloped. Therefore, a shift in the slope must occur at some point as stated in Proposition 5.

2) In a given time interval, an event $k$ may be unlikely for one of two reasons. Either, time has passed and other larger magnitude events have become more likely, or the other way around, i.e., it has not passed enough time to make that signal likely. Therefore, each event has an interval where it is more likely to occur. This aspect is a particularity of the Poisson process.

3) Surprisingly, payoffs are neither monotonic nor smooth in $\Delta$. For a given $K$, as $\Delta$ increases, we may observe an early decrease and a later increase in payoffs, creating a u-shape. It is a consequence of the fact that $K$ is discrete, and the complex trade-offs between discounting, information quality and quantity, and the associated effects on the punishment probability. Nonetheless, there is an overall trend for a niche (massive) use of information to improve (decrease) payoffs. The selective and non-selective uses of information may reproduce both trends.

# Appendix: Proofs of Lemmas and Propositions

**Proof of Propositions 1.** Replace the LHS of (3) by (2) to obtain,

$$v \geq (1-\delta)\left(\pi' - \pi\right)/\delta \textstyle\sum_{k\in\Pi}\left(p_k - q_k\right)\left(1-\alpha_k\right). \tag{7}$$

Our objective is to $\max_{\alpha=\{\alpha_0,\alpha_1,\ldots\}}$ (2) subject to (7) binding and $\alpha_k \in (0,1]$. In the "*bad" news model* we have $p_k > q_k$ for small $k$ and $p_k < q_k$ for large $k$. In the former (latter, respectively), the observation of a small (big, respectively) event $k$ is more likely if there is cooperation (defection, respectively). Our objective is the largest no punishment probability without breaking incentives. Therefore, starting with $k = 0$ and while $p_k > q_k$, we add numbers to $\Pi$. Consequently, the denominator on the RHS of (7) increases and the associated ratio decreases. When $p_k < q_k$, the addition of every new signal decreases the denominator on the RHS and increases the ratio in (7). At a certain point inequality (7) fails for $\Pi = \{0,1,...,K\}$, but it is satisfied for $\Pi = \{0,1,...,K-1\}$. Therefore, we must find some $\alpha_K \in (0,1]$ such that (7) binds. Moreover, $\alpha_k = 1$ for $k > K$ and $\alpha_k = 0$ for $k < K$. In the "good" news model the argument is similar. In this case, $p_k < q_k$ for small $k$ and $p_k > q_k$ for large $k$. Start from the largest magnitude $k$ (infinite) adding events to $\Pi$. At a certain point the inequality (7) holds for $\Pi = \{K+1,...,\infty\}$, but fails for $\Pi = \{K,...,\infty\}$ with $0 \leq K$.

Therefore, in a binding equilibrium we must have $\alpha_K \in (0,1]$, $\alpha_k = 0$ for $k > K$, and $\alpha_k = 1$ for $k < K$.

Now, optimal behavior implies that (2) and (3) become

$$v = (1-\delta)\,\pi + \delta v \left(\sum_{k\in\Pi} p_k + p_K (1-\alpha_K)\right),$$

and

$$v \geq (1-\delta)\,\pi' + \delta v \left(\sum_{k\in\Pi} q_k + q_K (1-\alpha_K)\right),$$

respectively. The solution to this system of two equations (with the latter holding with equality) and two unknowns ($v$ and $\alpha_K$) returns (4) (with (5)), and (6). ■

**Lemma 7** *In the "bad" news model, if*

$$1 < (\pi/\pi')e^{-(\mu-\beta)\Delta}\,(\mu/\beta)^K < \mu/\beta, \tag{8}$$

*then* $\alpha_K = 1$ *enforces cooperation but not* $\alpha_{K-1} = 0$.

*In the "good" news model, if*

$$1 < (\pi/\pi')e^{(\beta-\mu)\Delta}\,(\mu/\beta)^K < \beta/\mu, \tag{9}$$

*then* $\alpha_K = 1$ *enforces cooperation but not* $\alpha_{K+1} = 0$.

**Proof of Lemma 7.** A large discount rate implies a low discount factor. To keep the players' incentives the monitoring becomes more strict and the expected payoff decreases monotonically, Proposition 3 in Abreu et al. (1991). It implies that

$$\partial\alpha_K/\partial r = (\pi' - \pi)\,\Delta/\delta\,(\pi q_K - \pi' p_K) > 0,$$

for some $\alpha_K \in (0,1]$ and $K > 0$. In the former model, when $r$ increases the value of $K$ moves in the $K-1$ direction. When $\partial\alpha_K/\partial r > 0$ is no longer true the non-trivial equilibrium fails. At that point, we observe a shift from $\partial\alpha_K/\partial r > 0$ to $\partial\alpha_{K-1}/\partial r < 0$, or from $\partial\alpha_K/\partial r > 0$ to $\partial\alpha_{K+1}/\partial r < 0$, in the "bad" and"good" news models, respectively. When the equilibrium fails to exist the sign of the above derivative reverses, i.e., $\pi q_{K-1} < \pi' p_{K-1}$. Otherwise, by Corollary 4 and monotonicity of Proposition 3 in Abreu et al. (1991), we would have a contradiction. After some algebra we obtain (8). In the "good" news model the argument is similar. There exist a $K+1$ such that $\alpha_{K+1}$ is decreases with $r$, i.e., we have $\pi q_K > \pi' p_K$ and $\pi q_{K+1} < \pi' p_{K+1}$. After some algebra we obtain (9). ■

**Proof of Proposition 5.** Condition (8) and (9) establish the $K$ value associated with the transition between the non-trivial and trivial equilibrium for increasing $r$. In spite that variations in $r$ and $\Delta$ have different implications, for sufficiently large $\Delta$ in the infinitesimal neighborhood between a non-trivial and a trivial equilibrium, the monitoring technology must be tightening. Consequently, the same conditions of Lemma 7 obtained for large $r$ must be true for large $\Delta$, i.e., $\partial\alpha_K/\partial\Delta > 0$.

The next step is to consider the $\alpha_K$ behavior for small $\Delta$ and link it with large $\Delta$ via Lemma 7. Start by notice that we cannot have a $K = 0$ in a "bad" news equilibrium, since $\alpha_0 = 1 + (\pi' - \pi)/\delta(\pi q_0 - \pi' p_0) > 1$, because $\pi q_0 > \pi' p_0$. Otherwise, we have an impossibility, see the proof of Lemma 7. In the case $K = 1$, we have,

$$\alpha_1 = 1 - \left(\pi\left(1 - \delta e^{-\mu\Delta}\right) - \pi'\left(1 - \delta e^{-\beta\Delta}\right)\right)/\delta\left(\pi q_1 - \pi' p_1\right),$$

which has an asymptote at $\Delta_a = \ln(\pi\mu/\pi'\beta)/(\mu - \beta) > 0$ for $\pi\mu > \pi'\beta$. Therefore, if $\Delta \uparrow \Delta_a$ we have $\alpha_1 \to \pm\infty$, and if $\Delta \downarrow 0$ we have $\alpha_1 \to r(\pi' - \pi)/(\pi\mu - \pi'\beta)$, which is smaller than the unit for $\pi(\mu + r) > \pi'(\beta + r)$ implying $\pi\mu > \pi'\beta$. To know if it converges from above or below, differentiate $\alpha_1$ with respect to $\Delta$ and let $\Delta \downarrow 0$ to obtain that $\alpha_1$ is increasing at that point if inequality (10) in Footnote 12 is reversed, and decreasing otherwise.[12] In the latter case, $\alpha_1$ is monotonically increasing in $\Delta \in (0, \Delta_1) \subseteq (0, \Delta_a)$, with the value of $\Delta_1$ such that $\alpha_1(\Delta_1) = 1$, and a trivial equilibrium in the interval $(\Delta_1, \Delta_2)$ with the value of $\Delta_2$ such that $\alpha_2(\Delta_2) = 1$. In the former case, $\alpha_1$ decreases monotonically in $\Delta \in (0, \Delta_a)$, and hits zero at some point in this interval, i.e., $\partial\alpha_1/\partial\Delta < 0$. In both cases, for any $K \geq 2$ while $\Delta$ is sufficiently small we have $\partial\alpha_K/\partial\Delta < 0$, and $\alpha_K$ has an asymptote at $\Delta \downarrow 0$ with $\alpha_K \uparrow \infty$ and at $\Delta \uparrow \Delta_K$ with $\alpha_K \downarrow -\infty$. The result holds assuming monotonicity. Following Corollary 4 and Lemma 7, for $\Delta$ sufficiently large there must be a shift to $\partial\alpha_K/\partial\Delta > 0$. In this case $\alpha_K > 0$ has an u-shape with $\alpha_K = 1$ at the extreme points. By Lemma 7, at a certain point enforceability fails. Finally, note that when $\pi(\mu + r) < \pi'(\beta + r)$ the equilibrium degenerates for small $\Delta \downarrow 0$, but it might exist for large $\Delta$.

In the "good" news model, $\alpha_0$ is given by,

$$\alpha_0 = \left(1 - e^{-r\Delta}\right)(\pi' - \pi)/e^{-r\Delta}\left(\pi e^{-\mu\Delta} - \pi' e^{-\beta\Delta}\right),$$

[12] The statement of Proposition 5 is complete if

$$\frac{\pi\pi'\left(\left(r^2 + \beta\mu\right)(\beta + \mu) + 2r\left(\beta^2 + \mu^2\right)\right)}{\beta\left(\pi'(\beta + r)\right)^2 + \mu\left(\pi(\mu + r)\right)^2} < 1. \tag{10}$$

Otherwise, $\alpha_1 \downarrow r(\pi' - \pi)/(\pi\mu - \pi'\beta)$ as $\Delta \downarrow 0$, and $\alpha_1$ increases. This case is special, the equilibrium alternates between time intervals where enforceability fails.

and has a zero root for $\Delta \downarrow 0$, with $\partial\alpha_0/\partial\Delta = -r$ at the limit. Therefore, $\alpha_0 \notin (0, 1]$ for small $\Delta$. However, for $\Delta$ larger than the asymptote point $\Delta_a = \ln(\pi'/\pi)/(\beta - \mu) > 0$, we have $\alpha_0 > 0$. To show that $\partial\alpha_0/\partial\Delta < 0$ at $\alpha_0 = 1$ for small $\Delta > \Delta_a$ and that $\partial\alpha_0/\partial\Delta > 0$ at $\alpha_0 = 1$ for large $\Delta > \Delta_a$. Note that $\alpha_0 \uparrow \infty$ if $\Delta \downarrow \Delta_a$. Now, take $\Delta \uparrow \infty$ and observe that $\alpha_0 \uparrow \infty$. Since $\alpha_0$ is continuous and differentiable in $\Delta \in (\Delta_a, \infty)$, then $\alpha_0$ must have a u-shape and a minimum in this interval. If at the minimum $\alpha_0 < 1$, then we must have two values of $\Delta > \Delta_a$ such that $\alpha_0 = 1$. Following Corollary 4 and while enforceability holds (Lemma 7) $\partial\alpha_K/\partial\Delta > 0$ for $K > 0$. Note that if the minimum value of $\alpha_0$ is larger than one, then enforceability fails for all $\Delta$. ■

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