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# CLOSED FORMULAE FOR THE STRONG METRIC DIMENSION OF LEXICOGRAPHIC PRODUCT GRAPHS

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**Abstract**

Given a connected graph $G$, a vertex $w \in V(G)$ strongly resolves two vertices $u, v \in V(G)$ if there exists some shortest $u - w$ path containing $v$ or some shortest $v - w$ path containing $u$. A set $S$ of vertices is a strong metric generator for $G$ if every pair of vertices of $G$ is strongly resolved by some vertex of $S$. The smallest cardinality of a strong metric generator for $G$ is called the strong metric dimension of $G$. In this paper we obtain several relationships between the strong metric dimension of the lexicographic product of graphs and the strong metric dimension of its factor graphs.

**Keywords:** strong metric dimension, strong metric basis, strong metric generator, lexicographic product graphs.

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## 1. Introduction

A vertex $v$ of a connected graph $G$ is said to *distinguish* two vertices $x$ and $y$ of $G$ if $d_G(v, x) \neq d_G(v, y)$, i.e., the distance between $v$ and $x$ is distinct from the distance between $v$ and $y$. A set $S \subset V(G)$ is said to be a *metric generator* for

$G$ if any pair of vertices of $G$ is distinguished by some element of $S$. A metric generator of the smallest cardinality is called a *metric basis*, and its cardinality the *metric dimension* of $G$. The problem of uniquely recognizing the position of an intruder in a network was the principal motivation of introducing the concept of metric generators in graphs by Slater in [16], where metric generators were called *locating sets*. The same concept was also introduced independently by Harary and Melter in [3], where the metric generators were called *resolving sets*. Several applications and theoretical studies about metric generators have been presented and published. In this sense, according to the amount of literature concerning this topic and all its close invariants, we restrict our references to those ones which are only citing papers that we really refer to in a non-superficial way.

Another invariant, more restricted than the metric dimension, was presented by Sebő and Tannier in [15], and studied further in several articles. That is, a vertex $w \in V(G)$ *strongly resolves* two vertices $u, v \in V(G)$ if $d_G(w,u) = d_G(w,v) + d_G(v,u)$ or $d_G(w,v) = d_G(w,u) + d_G(u,v)$, i.e., there exists some shortest $w-u$ path containing $v$ or some shortest $w-v$ path containing $u$. A set $S$ of vertices in a connected graph $G$ is a *strong metric generator* for $G$ if every two vertices of $G$ are strongly resolved by some vertex of $S$. The smallest cardinality of a strong metric generator for $G$ is called the *strong metric dimension* and is denoted by $\dim_s(G)$. A *strong metric basis* of $G$ is a strong metric generator for $G$ of cardinality $\dim_s(G)$.

The concept of strong metric generators and strong metric dimension was first presented in connection with the following. Given a (standard) metric generator $S$ of a graph $H$, the following question was asked in [15]: whenever $H$ is a subgraph of a graph $G$ and the vectors of distances of the vertices of $H$ relative to $S$ agree in both $H$ and $G$, is $H$ an isometric subgraph of $G$? Even though the vectors of distances relative to a metric generator for a graph distinguish all pairs of vertices in the graph, they do not uniquely determine all distances in a graph as was first shown in [15]. In order to deal with this question, it was observed in [15] that, if "metric generator" is replaced by a stronger notion, namely that of "strong metric generator", then the question above can be answered in the affirmative.

Further on, several works on strong metric dimension have been developed. For instance, some applications to combinatorial searching have been described in the article [15]. There were analyzed some problems concerning false coins arising from a connection between information theory and extremal combinatorics. In the same work, the authors have dealt with a combinatorial optimization problem related to finding "connected joins" in graphs. Moreover, several results about detection of false coins have been used to approximate the value of the strong metric dimension of some specific graphs, where we can recall the Hamming graphs. On the other hand, in [13] was proved that the problem of computing the strong metric dimension of graph is NP-hard. In concordance with this, several studies on a few interesting families of graphs have been presented. For instance, Cayley graphs were studied in [13], distance-hereditary graphs in [11], and convex polytopes in [4]. Also, some Nordhaus-Gaddum type results for the strong metric dimension of a graph and its complement are known from [17]. Besides the theoretical results related to the strong metric dimension, a mathematical programming model [4] and metaheuristic approaches [5, 12] for finding this parameter have been developed. The strong metric dimension of product graphs has previously been studied for the case of Cartesian product graphs and direct product graphs [14], strong product graphs [9, 10], corona product graphs and join graphs [7] and rooted product graphs [8]. For other more information we refer the reader to [6], as a short survey on the strong metric dimension. In this paper we study the strong metric dimension of lexicographic product graphs.

We begin by giving some basic concepts and notations. Let $G = (V, E)$ be a simple graph. For two adjacent vertices $u$ and $v$ of $G$ we use the notation $u \sim v$ and, in this case, we say that $uv$ is an edge of $G$, i.e., $uv \in E$. The complement $G^c$ of $G$ has the same vertex set as $G$ and $uv \in E(G^c)$ if and only if $uv \notin E$. The *diameter* of $G$ is defined as

$$D(G) = \max_{u,v \in V}\{d_G(u, v)\}.$$

If $G$ is not connected, then we will assume that the distance between any two vertices belonging to distinct components of $G$ is infinity and, thus, its diameter is $D(G) = \infty$. For a vertex $v \in V$, the set $N_G(v) = \{u \in V :\ u \sim v\}$ is the open neighborhood of $v$ and the set $N_G[v] = N_G(v) \cup \{v\}$ is the closed neighborhood of $v$. Two vertices $x$, $y$ are called *true twins* if $N_G[x] = N_G[y]$. In this sense, a vertex $x$ is a *twin* if there exists $y \neq x$ such that they are true twins. We recall that a set $S$ is a *clique* in $G$, if the subgraph induced by $S$ is isomorphic to a complete graph. The *clique number* of a graph $G$, denoted by $\omega(G)$, is the number of vertices in a maximum clique in $G$. We refer to an $\omega(G)$-set in a graph $G$ as a clique of cardinality $\omega(G)$.

A set $S$ of vertices of $G$ is a *vertex cover* of $G$ if every edge of $G$ is incident with at least one vertex of $S$. The *vertex cover number* of $G$, denoted by $\alpha(G)$, is the smallest cardinality of a vertex cover of $G$. We refer to an $\alpha(G)$-set in a graph $G$ as a vertex cover set of cardinality $\alpha(G)$.

Recall that the largest cardinality of a set of vertices of $G$, no two of which are adjacent, is called the *independence number* of $G$ and is denoted by $\beta(G)$. We refer to a $\beta(G)$-set in a graph $G$ as an independent set of cardinality $\beta(G)$. The following well-known result, due to Gallai, states the relationship between the independence number and the vertex cover number of a graph.

**Theorem 1** (Gallai's Theorem). *For any graph $G$ of order $n$,*

$$\alpha(G) + \beta(G) = n.$$

A vertex $u$ of $G$ is *maximally distant* from $v$ if for every $w \in N_G(u)$, $d_G(v, w) \le d_G(u, v)$. If $u$ is maximally distant from $v$ and $v$ is maximally distant from $u$, then we say that $u$ and $v$ are *mutually maximally distant.* The *boundary* of $G = (V, E)$ is defined as

$$\partial(G) = \{u \in V : \text{ there exists } v \in V \text{ and } u, v \text{ are mutually maximally distant}\}.$$

We use the notion of strong resolving graph introduced by Oellermann and Peters-Fransen in [13]. The *strong resolving graph*[1] of $G$ is a graph $G_{SR}$ with vertex set $V(G_{SR}) = \partial(G)$ where two vertices $u, v$ are adjacent in $G_{SR}$ if and only if $u$ and $v$ are mutually maximally distant in $G$.

It was shown in [13] that the problem of finding the strong metric dimension of a graph $G$ can be transformed into the problem of computing the vertex cover number of $G_{SR}$.

**Theorem 2** [13]. *For any connected graph $G$,*

$$\dim_s(G) = \alpha(G_{SR}).$$

We will use the notation $K_n$, $C_n$, $N_n$ and $P_n$ for complete graphs, cycle graphs, empty graphs and path graphs on $n$ vertices, respectively. In this work, the remaining definitions will be given the first time that the concept appears in the text.

## 2. The Strong Metric Dimension of the Lexicographic Product of Graphs

The *lexicographic product* of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph $G \circ H$ with vertex set $V = V_1 \times V_2$ and two vertices $(a, b), (c, d) \in V$ are adjacent in $G \circ H$ if and only if either $ac \in E_1$ or ($a = c$ and $bd \in E_2$).

Note that the lexicographic product of two graphs is not a commutative operation. Moreover, $G \circ H$ is a connected graph if and only if $G$ is connected. For more information on structure and properties of the lexicographic product of graphs we suggest [2]. Nevertheless, we would point out the following known results.

[1]In fact, according to [13] the strong resolving graph $G'_{SR}$ of a graph $G$ has vertex set $V(G'_{SR}) = V(G)$ and two vertices $u, v$ are adjacent in $G'_{SR}$ if and only if $u$ and $v$ are mutually maximally distant in $G$. So, the strong resolving graph defined here is a subgraph of the strong resolving graph defined in [13] and can be obtained from the latter graph by deleting its isolated vertices.

**Claim 3** [2]**.** *Let $G$ and $H$ be two non-trivial graphs such that $G$ is connected. Then the following assertions hold for any $a, c \in V(G)$ and $b, d \in V(H)$ such that $a \neq c$.*

(i) $N_{G \circ H}(a, b) = (\{a\} \times N_H(b)) \cup (N_G(a) \times V(H))$.

(ii) $d_{G \circ H}((a, b), (c, d)) = d_G(a, c)$.

(iii) $d_{G \circ H}((a, b), (a, d)) = \min\{d_H(b, d), 2\}$.

From the next lemmas we can describe the structure of the strong resolving graph of $G \circ H$.

**Lemma 4.** *Let $G$ be a connected non-trivial graph and let $H$ be a non-trivial graph. Let $a, b \in V(G)$ be such that they are not true twin vertices and let $x, y \in V(H)$. Then $(a, x)$ and $(b, y)$ are mutually maximally distant in $G \circ H$ if and only if $a$ and $b$ are mutually maximally distant in $G$.*

***Proof.*** Let $x, y \in V(H)$. We assume that $a, b \in V(G)$ are mutually maximally distant in $G$ and that they are not true twins. First of all, notice that $d_G(a, b) \geq 2$, (if $d_G(a, b) = 1$, then to be mutually maximally distant in $G$, they must be true twins). Hence, by Claim 3(i) we have that if $(c, d) \in N_{G \circ H}(b, y)$, then either $c = b$ or $c \in N_G(b)$. In both cases, by Claim 3(ii) we obtain $d_{G \circ H}((a, x), (c, d)) = d_G(a, c) \leq d_G(a, b) = d_{G \circ H}((a, x), (b, y))$. So, $(b, y)$ is maximally distant from $(a, x)$ and, by symmetry, we conclude that $(b, y)$ and $(a, x)$ are mutually maximally distant in $G \circ H$.

Conversely, assume that $(a, x)$ and $(b, y)$, $a \neq b$, are mutually maximally distant in $G \circ H$. If $c \in N_G(b)$, then for any $z \in V(H)$ we have $(c, z) \in N_{G \circ H}(b, y)$. Now, by Claim 3(ii) we obtain $d_G(a, c) = d_{G \circ H}((a, x), (c, z)) \leq d_{G \circ H}((a, x), (b, y)) = d_G(a, b)$. So, $b$ is maximally distant from $a$ and, by symmetry, we conclude that $b$ and $a$ are mutually maximally distant in $G$. ■

**Lemma 5.** *Let $G$ be a connected non-trivial graph, let $H$ be a graph of order $n \geq 2$, let $a, b \in V(G)$ be two distinct true twin vertices and let $x, y \in V(H)$. Then $(a, x)$ and $(b, y)$ are mutually maximally distant in $G \circ H$ if and only if both $x$ and $y$ have degree $n - 1$.*

***Proof.*** If $x \in V(H)$ has degree $n - 1$, then for any $y \in V(H)$ of degree $n - 1$ we have that $(a, x)$ and $(b, y)$ are true twins in $G \circ H$. Hence, $(a, x)$ and $(b, y)$ are mutually maximally distant in $G \circ H$.

Now, suppose that there exists $z \in V(H) \setminus N_H(x)$. By Claim 3(iii), it follows that $d_{G \circ H}((a, x), (a, z)) = 2$. Also, for every $y \in V(H)$, Claim 3(ii) gives $d_{G \circ H}((a, x), (b, y)) = 1$. Thus, we conclude that $(a, x)$ and $(b, y)$ are not mutually maximally distant in $G \circ H$. ■

In order to present our results we need to introduce some more terminology. Given a graph $G$, we define $G^*$ as the graph with vertex set $V(G^*) = V(G)$ such that two vertices $u, v$ are adjacent in $G^*$ if and only if either $d_G(u,v) \geq 2$ or $u, v$ are true twins. If a graph $G$ has at least one isolated vertex, then we denote by $G_-$ the graph obtained from $G$ by removing all its isolated vertices. In this sense, $G^*_-$ is obtained from $G^*$ by removing all its isolated vertices. Notice that $G^*$ satisfies the following straightforward properties.

**Remark 6.** Let $G$ be a connected graph of diameter $D(G)$, order $n$ and maximum degree $\Delta(G)$.

(i) If $\Delta(G) \leq n-2$, then $G^* \cong (K_1 + G)_{SR}$.
(ii) If $D(G) \leq 2$, then $G^*_- \cong G_{SR}$.
(iii) If $G$ has no true twins, then $G^* \cong G^c$.

**Lemma 7.** *Let $G$ be a connected non-trivial graph. Let $x, y \in V(H)$ be two distinct vertices of a graph $H$ and let $a \in V(G)$. Then $(a,x)$ and $(a,y)$ are mutually maximally distant vertices in $G \circ H$ if and only if $x$ and $y$ are adjacent in $H^*$.*

***Proof.*** By Claim 3(iii), $d_{G\circ H}((a,x),(a,y)) \leq 2$ and, by Claim 3(i), if $c \neq a$, then $(c,w) \in N_{G\circ H}(a,x)$ if and only if $c \in N_G(a)$. Hence, $(a,x)$ and $(a,y)$ are mutually maximally distant if and only if either $(a,x)$ and $(a,y)$ are true twins in $G \circ H$ or $(a,x)$ and $(a,y)$ are not adjacent in $G \circ H$.

On one hand, by the definition of the lexicographic product, $(a,x)$ and $(a,y)$ are not adjacent in $G \circ H$ if and only if $x$ and $y$ are not adjacent in $H$. Moreover, by Claim 3(i), $(a,x)$ and $(a,y)$ are true twins in $G \circ H$ if and only if $x$ and $y$ are true twins in $H$. Therefore, the result follows. ■

**Proposition 8.** *Let $G$ be a connected graph of order $n \geq 2$ and let $H$ be a non-complete graph of order $n' \geq 2$. If $G$ has no true twin vertices, then*

$$(G \circ H)_{SR} \cong (G_{SR} \circ H^*) \cup \bigcup_{i=1}^{n-|\partial(G)|} H^*_-.$$

***Proof.*** We assume that $G$ has no true twin vertices. By Lemmas 4 and 7, we have the following facts.

- For any $a \notin \partial(G)$ it follows that $(G \circ H)_{SR}$ has a subgraph, say $H_a$, induced by $(\{a\} \times V(H)) \cap \partial(G \circ H)$ which is isomorphic to $H^*_-$.
- For any $b \in \partial(G)$, we have that $(G \circ H)_{SR}$ has a subgraph, say $H_b$, induced by $(\{b\} \times V(H)) \cap \partial(G \circ H)$ which is isomorphic to $H^*$.
- The set $(\partial(G) \times V(H)) \cap \partial(G \circ H)$ induces a subgraph in $(G \circ H)_{SR}$ which is isomorphic to $G_{SR} \circ H^*$.
- For any $a \notin \partial(G)$ and any $b \in \partial(G)$ there are no edges of $(G \circ H)_{SR}$ joining vertices belonging to $H_a$ with vertices belonging to $H_b$.

- For any distinct vertices $a_1, a_2 \notin \partial(G)$ there are no edges of $(G \circ H)_{SR}$ joining vertices belonging to $H_{a_1}$ with vertices belonging to $H_{a_2}$.

Therefore, the result follows. ■

Figure 1 shows the graph $P_4 \circ P_3$ and its strong resolving graph. Notice that $(P_3)^*_- \cong K_2$, $(P_3)^* \cong K_2 \cup K_1$ and $(P_4)_{SR} \cong K_2$. So, $(P_4 \circ P_3)_{SR} \cong K_2 \circ (K_2 \cup K_1) \cup K_2 \cup K_2$.

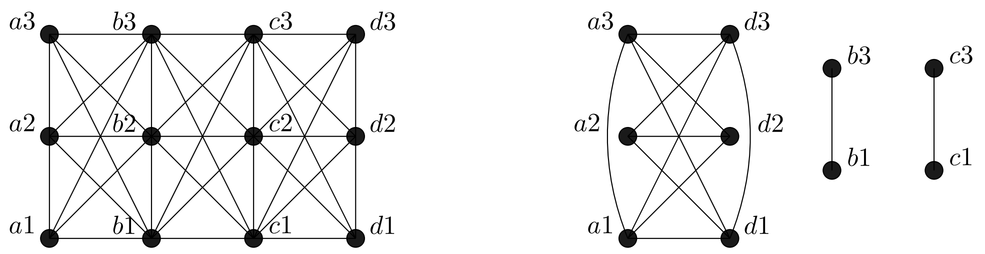


Figure 1. The graph $P_4 \circ P_3$ and its strong resolving graph.

The following well-known result will be a useful tool in determining the strong metric dimension of lexicographic product graphs.

**Theorem 9** [1]**.** *For any graphs $G$ and $H$ of order $n$ and $n'$, respectively,*

$$\alpha(G \circ H) = n\alpha(H) + n'\alpha(G) - \alpha(G)\alpha(H).$$

**Theorem 10.** *Let $G$ be a connected graph of order $n \geq 2$ and let $H$ be a graph of order $n' \geq 2$. If $G$ has no true twin vertices, then the following assertions hold.*

(i) *If $D(H) \leq 2$, then* $\dim_s(G \circ H) = n \cdot \dim_s(H) + n' \cdot \dim_s(G) - \dim_s(G)\dim_s(H)$.

(ii) *If $D(H) > 2$, then* $\dim_s(G \circ H) = n \cdot \dim_s(K_1 + H) + n' \cdot \dim_s(G) - \dim_s(G)\dim_s(K_1 + H)$.

***Proof.*** By Theorem 2 and Proposition 8 we have,

$$\dim_s(G \circ H) = \alpha(G_{SR} \circ H^*) + (n - |\partial(G)|)\alpha(H^*_-)$$

and, by Theorem 9 we have

$$\dim_s(G \circ H) = |\partial(G)|\alpha(H^*) + n'\alpha(G_{SR}) - \alpha(G_{SR})\alpha(H^*) + (n - |\partial(G)|)\alpha(H^*_-). \tag{1}$$

Now, if $D(H) \leq 2$, then $\alpha(H^*) = \alpha(H^*_-) = \alpha(H_{SR})$ and, as a result,

$$\dim_s(G \circ H) = n\alpha(H_{SR}) + n'\alpha(G_{SR}) - \alpha(G_{SR})\alpha(H_{SR}).$$

Also, and if $D(H) > 2$, then $\alpha(H^*) = \alpha(H^*_-) = \alpha((K_1 + H)_{SR})$, so

$$\dim_s(G \circ H) = n\alpha((K_1 + H)_{SR}) + n'\alpha(G_{SR}) - \alpha(G_{SR})\alpha((K_1 + H)_{SR}).$$

Therefore, by Theorem 2 we conclude the proof. ■

Note that the case where $H$ is not connected is also considered in Theorem 10, because we are assuming that if $H$ is not connected, then $D(H) = \infty > 2$.

Now we show some particular examples of graphs $G$ without true twin vertices where $\dim_s(G)$ is easy to compute or known.

(1) For any complete $k$-partite graph $G = K_{p_1,p_2,\dots,p_k}$ such that $p_i \geq 2$, $i \in \{1,2,\dots,k\}$, we have $G_{SR} \cong \bigcup_{i=1}^{k} K_{p_i}$. Hence, $\dim_s(G) = \sum_{i=1}^{k}(p_i - 1)$.

(2) For any tree $T$ with $l(T)$ leaves, $T_{SR} \cong K_{l(T)}$, so $\dim_s(T) = l(T) - 1$.

(3) The strong resolving graph of any cycle graph is $(C_{2k})_{SR} \cong \bigcup_{i=1}^{k} K_2$ or $(C_{2k+1})_{SR} \cong C_{2k+1}$. So, $\dim_s(C_{2k}) = k$ and $\dim_s(C_{2k+1}) = k + 1$.

(4) The strong resolving graph of any grid graph $P_r \Box P_t$ is $(P_r \Box P_t)_{SR} = K_2 \cup K_2$. Thus, $\dim_s(P_r \Box P_t) = 2$.

(5) For any connected graph $G_1$ of order $n_1$ and any graph $G_2$, the *corona graph* $G_1 \odot G_2$ is obtained by taking one copy of $G_1$ and $n_1$ copies of $G_2$ and then adding all edges between the $i$-th vertex of $G_1$ and every vertex of the $i$-th copy of $G_2$. It was shown in [7] that if $n_1 \geq 2$ and $G_2$ is a triangle free graph of order $n_2 \geq 2$ and maximum degree $\Delta(H) \leq n_2 - 2$, then $\dim_s(G_1 \odot G_2) = n_1 n_2 - 2$.

Using the preceding results and the above known values for several families of graphs, we can obtain the strong metric dimension of several combinations of lexicographic product of two graphs. We leave the computations to the reader.

According to Theorem 10(i), for any connected graph $G$ without true twin vertices, $\dim_s(G \circ K_{n'}) = n(n'-1) + \dim_s(G)$. Now we will show that this formula holds for any connected graph $G$.

**Proposition 11.** *For any connected non-trivial graph $G$ of order $n \geq 2$ and any integer $n' \geq 2$,*

$$(G \circ K_{n'})_{SR} \cong (G_{SR} \circ K_{n'}) \cup \bigcup_{i=1}^{n-|\partial(G)|} K_{n'}.$$

***Proof.*** Notice that $(K_{n'})^* \cong K_{n'}$ and, by Lemma 7, for any $a \in V(G)$, the subgraph of $(G \circ K_{n'})_{SR}$ induced by $(\{a\} \times V(K_{n'})) \cap \partial(G \circ K_{n'})$ is isomorphic to $K_{n'}$. Also, from Lemmas 4 and 5, the subgraph of $(G \circ K_{n'})_{SR}$ induced by $(\partial(G) \times V(K_{n'})) \cap \partial(G \circ K_{n'})$ is isomorphic to $G_{SR} \circ K_{n'}$. Moreover, for $a \notin \partial(G)$ and $b \in \partial(G)$ there are not edges of $(G \circ K_{n'})_{SR}$ joining vertices belonging to $\{a\} \times V(K_{n'})$ with vertices belonging to $\{b\} \times V(K_{n'})$. Therefore, the result follows. ■

**Theorem 12.** *For any connected non-trivial graph $G$ of order $n \geq 2$ and any integer $n' \geq 2$,*

$$\dim_s(G \circ K_{n'}) = n(n' - 1) + \dim_s(G).$$

***Proof.*** From Theorem 2 and Proposition 11 we have,

$$\dim_s(G \circ K_{n'}) = \alpha(G_{SR} \circ K_{n'}) + (n - |\partial(G)|)(n' - 1).$$

By using Theorem 9 and applying Theorem 2 again,

$$\begin{aligned}\dim_s(G \circ K_{n'}) &= |\partial(G)|(n' - 1) + n'\alpha(G_{SR}) - \alpha(G_{SR})(n' - 1)\\ &\quad + (n - |\partial(G)|)(n' - 1) = n(n' - 1) + \dim_s(G).\end{aligned}$$ ■

We have studied the case in which the second factor in the lexicographic product is a complete graph. Since this product is not commutative, we now consider the case in which the first factor is a complete graph.

**Proposition 13.** *Let $n \geq 2$ be an integer and let $H$ be a graph of order $n' \geq 2$. If $H$ has maximum degree $\Delta(H) \leq n' - 2$, then*

$$(K_n \circ H)_{SR} \cong \bigcup_{i=1}^{n} H^*.$$

***Proof.*** We assume that $H$ has maximum degree $\Delta(H) \leq n' - 2$. Notice that $H^*$ has no isolated vertices and, by Lemma 7, for any $a \in V(K_n)$, the subgraph $(K_n \circ H)_{SR}$ induced by $(\{a\} \times V(H)) \cap \partial(K_n \circ H)$ is isomorphic to $H^*$.

Also, by Lemma 5, for any distinct $a, b \in V(K_n)$ and any $x, y \in V(H)$, the vertices $(a, x)$ and $(b, y)$ are not mutually maximally distant in $K_n \circ H$. Therefore, the result follows. ■

**Theorem 14.** *Let $n \geq 2$ be an integer and let $H$ be a graph of order $n' \geq 2$ and maximum degree $\Delta(H) \leq n' - 2$.*

(i) *If $D(H) = 2$, then* $\dim_s(K_n \circ H) = n \cdot \dim_s(H)$.

(ii) *If $D(H) > 2$, then* $\dim_s(K_n \circ H) = n \cdot \dim_s(K_1 + H)$.

***Proof.*** By Theorems 2 and 13 we have, $\dim_s(K_n \circ H) = n \cdot \alpha(H^*)$. Hence, if $D(H) = 2$, then $\dim_s(K_n \circ H) = n \cdot \alpha(H_{SR})$ and if $D(H) > 2$, then $\dim_s(K_n \circ H) = n \cdot \alpha((K_1 + H)_{SR})$. Therefore, by Theorem 2 we conclude the proof. ■

For the particular case of empty graphs $H = N_{n'} = (K_{n'})^c$, Theorem 14 leads to the next corollary, which is straightforward because $K_n \circ N_{n'} \cong K_{n',n',\ldots,n'}$, is a complete $n$-partite graph, and so $(K_n \circ N_{n'})_{SR} \cong \bigcup_{i=1}^{n} K_{n'}$.

**Corollary 15.** *For any integers $n, n' \geq 2$,* $\dim_s(K_n \circ N_{n'}) = n(n' - 1)$.

We define the *TF-boundary* of a non-complete graph $G = (V, E)$ as a set $\partial_{TF}(G) \subseteq \partial(G)$, where $x \in \partial_{TF}(G)$ whenever there exists $y \in \partial(G)$, such that $x$ and $y$ are mutually maximally distant in $G$ and $N_G[x] \neq N_G[y]$ (which means that $x, y$ are not true twins). The *strong resolving TF-graph* of $G$ is a graph $G_{SRS}$ with vertex set $V(G_{SRS}) = \partial_{TF}(G)$, where two vertices $u, v$ are adjacent in $G_{SRS}$ if and only if $u$ and $v$ are mutually maximally distant in $G$ and $N_G[x] \neq N_G[y]$. Since the strong resolving TF-graph is a subgraph of the strong resolving graph, an instance of the problem of transforming a graph into its strong resolving TF-graph forms part of the general problem of transforming a graph into its strong resolving graph. From [13], it is known that this general transformation is polynomial. Thus, the problem of transforming a graph into its strong resolving TF-graph is also polynomial.

An interesting example of a strong resolving TF-graph is obtained from the corona graph $G \odot K_{n'}$, $n' \geq 2$, where $G$ has order $n \geq 2$. Notice that any two distinct vertices belonging to any two copies of the complete graph $K_{n'}$ are mutually maximally distant, but if they are in the same copy, then they are also true twins. Thus, in this case $\partial_{TF}(G \odot K_{n'}) = \partial(G \odot K_{n'})$, while we have that $(G \odot K_{n'})_{SR} \cong K_{nn'}$ and $(G \odot K_{n'})_{SRS}$ is isomorphic to a complete $n$-partite graph $K_{n',n',\ldots,n'}$.

**Proposition 16.** *Let $G$ be a connected non-complete graph of order $n \geq 2$ and let $H$ be a graph of order $n' \geq 2$. If $H$ has maximum degree $\Delta(H) \leq n' - 2$, then*

$$(G \circ H)_{SR} \cong (G_{SRS} \circ H^*) \cup \bigcup_{i=1}^{n-|\partial_{TF}(G)|} H^*.$$

***Proof.*** We assume that $H$ has maximum degree $\Delta(H) \leq n' - 2$. Notice that $H^*$ has no isolated vertices and, by Lemma 7, for any $a \in V(G)$, the subgraph $(G \circ H)_{SR}$ induced by $(\{a\} \times V(H)) \cap \partial(G \circ H)$ is isomorphic to $H^*$.

Also, by Lemma 5, if two distinct vertices $a, b$ are true twins in $G$ and $x, y \in V(H)$, then $(a, x)$ and $(b, y)$ are not mutually maximally distant in $G \circ H$. So, from Lemmas 4 and 7 we deduce that the subgraph of $(G \circ H)_{SR}$ induced by $(\partial_{TF}(G) \times V(H)) \cap \partial(G \circ H)$ is isomorphic to $G_{SRS} \circ H^*$. Moreover, for $a \notin \partial_{TF}(G)$ and $b \in \partial_{TF}(G)$ there are no edges of $(G \circ H)_{SR}$ joining vertices belonging to $\{a\} \times V(H)$ with vertices belonging to $\{b\} \times V(H)$. Therefore, the result follows. ■

Figure 2 shows the graph $(K_1 + (K_1 \cup K_2)) \circ P_4$ and its strong resolving graph. Notice that $(P_4)^* \cong P_4$ and $(K_1 + (K_1 \cup K_2))_{SRS} \cong P_3$. So, $((K_1 + (K_1 \cup K_2)) \circ P_4)_{SR} \cong (P_3 \circ P_4) \cup P_4$.

**Theorem 17.** *Let $G$ be a connected non-complete graph of order $n \geq 2$ and let $H$ be a graph of order $n' \geq 2$ and maximum degree $\Delta(H) \leq n' - 2$.*

(i) *If $D(H) = 2$, then* $\dim_s(G \circ H) = n \cdot \dim_s(H) + n' \cdot \alpha(G_{SRS}) - \alpha(G_{SRS}) \dim_s(H)$.

(ii) *If $D(H) > 2$, then* $\dim_s(G \circ H) = n \cdot \dim_s(K_1 + H) + n' \cdot \alpha(G_{SRS}) - \alpha(G_{SRS}) \dim_s(K_1 + H)$.

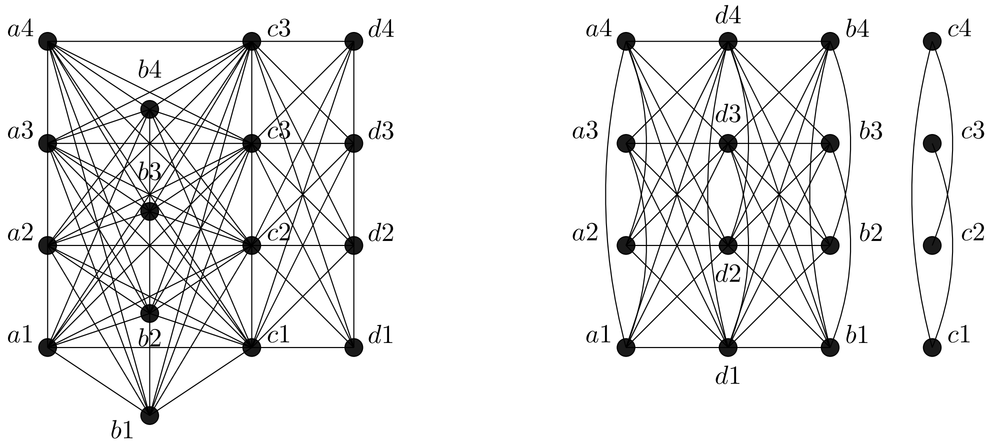


Figure 2. The graph $(K_1 + (K_1 \cup K_2)) \circ P_4$ and its strong resolving graph.

***Proof.*** By Theorem 2 and Proposition 16 we have,

$$\dim_s(G \circ H) = \alpha(G_{SRS} \circ H^*) + (n - |\partial_{SR}(G)|)\alpha(H^*)$$

and, by Theorem 9, we have

$$\begin{aligned} \dim_s(G \circ H) &= |\partial(G)|\alpha(H^*) + n'\alpha(G_{SRS}) - \alpha(G_{SRS})\alpha(H^*) \\ &\quad + (n - |\partial_{SR}(G)|)\alpha(H^*). \end{aligned} \tag{2}$$

Now, if $D(H) = 2$, then $\alpha(H^*) = \alpha(H_{SR})$ and, if $D(H) > 2$, then $\alpha(H^*) = \alpha((K_1 + H)_{SR})$. Hence, if $D(H) = 2$, then

$$\dim_s(G \circ H) = n\alpha(H_{SR}) + n'\alpha(G_{SRS}) - \alpha(G_{SRS})\alpha(H_{SR}),$$

and if $D(H) > 2$, then

$$\dim_s(G \circ H) = n\alpha((K_1 + H)_{SR}) + n'\alpha(G_{SRS}) - \alpha(G_{SRS})\alpha((K_1 + H)_{SR}).$$

Therefore, by Theorem 2 we conclude the proof. ■

We consider now the case of empty graphs $N_{n'} = (K_{n'})^c$.

**Corollary 18.** *Let $G$ be a connected non-complete graph of order $n \geq 2$ and let $n' \geq 2$ be an integer. Then*

$$\dim_s(G \circ N_{n'}) = n(n' - 1) + \alpha(G_{SRS}).$$

*In particular, if $G$ has no true twin vertices, then*

$$\dim_s(G \circ N_{n'}) = n(n'-1) + \dim_s(G).$$

As one might expect, if $G$ has no true twin vertices and $H$ has maximum degree $\Delta(H) \le n'-2$, then both Theorem 10 and Theorem 17 lead to the same result.

**Theorem 19.** *Let $G$ be a connected graph of order $n \ge 2$ and let $H$ be a graph of order $n' \ge 2$ and maximum degree $\Delta(H) \le n'-2$. Then the following assertions hold.*

(i) *If $H$ has no true twin vertices, then*

$$\dim_s(G \circ H) = (n - \alpha(G_{SRS}))(n' - \omega(H)) + n'\alpha(G_{SRS}).$$

(ii) *If neither $G$ nor $H$ have true twin vertices, then*

$$\dim_s(G \circ H) = (n - \dim_s(G))(n' - \omega(H)) + n' \dim_s(G).$$

***Proof.*** First of all, notice that by Theorem 1, $\alpha(H^c) = n' - \beta(H^c) = n' - \omega(H)$. Also, since $\Delta(H) \le n'-2$, we have $H^* = H^*_-$ and, if $H$ has no true twin vertices, then $H^* = H^c$. Hence, equality (2) leads to (i). Moreover, if $G$ has no true twin vertices, then equality (1) leads to (ii). ∎

## Conclusion and Open Problems

We have studied the strong metric dimension of the lexicographic product of graphs $G \circ H$ such that

- $H$ is any non-trivial graph and $G$ has no true twins.
- $G$ is any connected graph and $H$ is a non-trivial graph having maximum degree at most its order minus two.

On the other hand, we notice that the strong resolving graph of a graph plays a very important role for computing the strong metric dimension of graphs (this fact can be also noted in the articles [8, 10, 14]). According to this it would be desirable to describe the strong resolving graph of other families of graphs. Such problem was already mentioned (but not remarked) in the article [13]. For instance, there was opened a question concerning characterizing all the graphs such that its strong resolving graphs are isomorphic to a bipartite graph. The main motivation for this question arises from the fact that the vertex cover number can be computed in polynomial time for bipartite graphs. Thus, according to Theorem 2, we can also compute the strong metric dimension. On the other hand, it is perhaps possible to find another interesting application of the strong resolving graph.

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