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An axiomatic justification of mediation in bankruptcy problems.

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Abstract

This paper provides a natural way of reaching an agreement between two prominent proposals in a bankruptcy problem. Particularly, using the fact that such problems can be faced from two different points of views, awards and losses, we justify the average of any pair of dual bankruptcy rules through the definition a *double recursive process*. Finally, by considering three posible sets of equity principles that a particular society may agree on, we retrieve the average of old and well known bankruptcy rules, the *Constrained Equal Awards* and the *Constrained Equal Losses* rules, *Piniles'* rule and its dual rule, and the *Constrained Egalitarian* rule and its dual rule.

Keywords: Bankruptcy problems, Midpoint, Bounds, Duality, Recursivity
JEL classification: C71, D63, D71.

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1. Introduction.

How should the scarce resources be allocated among its claimant? Such problems, where the available amount of a perfectly divisible good (or *estate*) is not enough to satisfy its aggregate claim, are known as bankruptcy problems. The formal analysis of situations like these, which originates in a seminal paper by O'Neill (1982), proposes a number of well-behaved rules. Each of these rules recommends a division of the resources among the agents, as a function of the estate and their claims. The term "well-behaved" reflects the idea that the considered rules might fulfil some principles of "fairness", or appealing properties. Moreover, these problems can be analyzed from two points of views: awards and losses. In the former case, claimants focus on the amount of money they get. In the latter case, they are worried about the quantity of incurred losses.

In this context, how can claimants reach an agreement on the way of distributing the resources? Traditionally, such conflicts, where two or more agents cannot decide the allocations, have been resolved trough the figure of an arbitrator. This arbitrator is a neutral third agent who, basing on the characteristics of each claimant and a set of reasonable properties, dictates the way of distributing the endowment. However, in such situations each agent knows the smallest amount of awards which she is willing to receive, and the necessity of conceding some part of the other agents' claims. This is the philosophy behind the mediation, which is an alternative dispute resolution, as used in law, applied in many and different issues (business and commercial, family and divorce, public policy, etc.) Concretely, as the "World Mediation Forum" defines it, mediation is a process by which a neutral third agent, called a mediator, helps agents in conflict negotiate a mutually acceptable agreement. Therefore, a bankruptcy problem involves a group of agents (represented by their claims), the estate, and an arbitrator or a mediator, who will be represented by the set of equity or fair principles, P, which will be applied in the resolution of the conflict.

On the other hand, the establishment of a lower bound on awards appears as a recent research line. Many authors consider reasonable that each agent has guaranteed a minimal level of awards (O'Neill (1982), Herrero and Villar (2001, 2002), Moulin (2002), Moreno-Ternero and Villar (2004), and Dominguez (2007)). To this respect, Jiménez-Gómez and Marco-Gil (2008) join all the previous ideas to analyze thoroughly the consequences of enriching the classical model of rationing with a third element, P. This element, called the *Legitimate Principles* set (which can be understood as the arbitrator or the mediator), is composed of basic fair principles commonly accepted by a group of agents to resolve a concrete family of such problems. Given this family, a new lower bound on awards appears by assigning to each agent the smallest awards she receives from those rules satisfying P. Yet, when combined its recursive application with a particular set of properties, Jiménez-Gómez and Marco-Gil (2008) retrieve well known bankruptcy rules: the *Constrained Equal Awards*, the *Constrained Equal Losses* or *Piniles'* rules.

In this paper, we analyze the figure of a mediator taking as our point of departure *Bankruptcy Problems with Legitimate Principles*. But, unlike what has been done, we study the consequences of establishing a warranty in awards (lower bound on awards), and the obligation of incurring minimum losses (lower bound on losses, or, equivalently, an upper bound on awards). The basic idea of this approach is motivated by the fact that *Fairness* hardly leads to a single point of view: the same distribution problem faced by two different agents, may, almost certainly, lead to the use of different distributional rules (Moulin (1988), Schokkaert and Overlaet (1989), Young (1987), among others). Particularly, as we have commented, two natural focus or different points of view appear in bankruptcy problems: awards and losses.

Regarding to all these ideas, firstly, we propose a new range in awards, where each agent:

a. should receive, at least, a minimum amount according to all the rules which satisfy some agreed fair properties;

b. should receive no more than the maximum amount provided by all these rules.

Secondly, we consider the recursive combination of both a lower bound and an upper bound on awards obtained by applying the above mentioned range, named *Double Boundedness Recursive Process*. Note that such processes have been used for introducing bankruptcy rules by Alcalde et al. (2005) and by Dominguez and Thomson (2006), or for studying the behavior of the recursive application of a generic bound (Dominguez (2007)).

Next, we establish some requirements on these problems:

a. for each problem we suppose that the set of rules satisfying P is defined by the set of distributions limited by two of such rules, called *Focal* rules;

b. each of these two *Focal* rules are dual to each other.

Then, the *Double Boundedness Recursive Process* will correspond with the average of them. This result has two consequences. On the one hand, we provide a new justification of a convex combination (the middle point) of two extreme and opposite ways of distributing the endowment. On the other hand, we obtain a new method for rationing the resources which is invariant to the point of view used (awards and losses).

Finally, we apply these results to three different *Legitimate Principles* sets, providing new basis for the average of old bankruptcy rules.

The paper is organized as follows: Section 2 presents the preliminaries. Section 3 provides our new approach. Sections 4 and 5 contain our main results and apply them on different *Legitimate Principles* sets, respectively. Section 6 summarizes our conclusions. Finally, the Appendix gathers technical proofs.

2. Preliminaries.

A **bankruptcy problem** is a pair $(E, c) \in \mathbb{R}_+ \times \mathbb{R}^n_+$, such that $\sum_{i \in N} c_i \geq E$, where each agent $i \in N$, $N = \{1, ..., i, ..., n\}$, has a claim c_i on the estate or endowment, E, which represents the quantity of a perfectly divisible good that should be rationed among the agents.

For notational convenience, $\mathcal{B}$ will denote the set of all bankruptcy problems; and C the sum of the agents' claims, $C = \sum_{i \in N} c_i$.

Each bankruptcy problem can be faced from two points of views: awards and losses. Thus, we have two focal positions, depending on whether we worry about the awards we receive or the amount of our demand that is not satisfied. In this latter case, we consider the **dual bankruptcy problem**, which is the pair $(L, c) \in \mathbb{R}_+ \times \mathbb{R}^n_+$, such that L will denote the total amount of losses to distribute among the agents, $L = C - E$, and $\sum_{i \in N} c_i > L$.

In this context, a rule is a function, $\varphi : \mathcal{B} \rightarrow \mathbb{R}^n_+$, such that for each $(E, c) \in \mathcal{B}$, (a) $\sum_{i \in N} \varphi_i(E, c) = E$ (*efficiency*) and (b) $0 \leq \varphi_i(E, c) \leq c_i$ for each $i \in N$ (*non-negativity* and *claim-boundedness*).

Given a rule φ, its dual rule shares out losses in the same way that φ divides the endowment (Aumann and Maschler (1985)).

The **dual** of φ, denoted by φ^d, assigns for each $(E, c) \in \mathcal{B}$ and each $i \in N$, $\varphi_i^d(E, c) = c_i - \varphi_i(L, c)$.

It is straightforward to check that for each rule, φ, its dual rule is well defined, since given that $(E, c) \in \mathcal{B}$, $(L, c) \in \mathcal{B}$ and given that φ satisfies *efficiency*, *non-negativity* and *claim-boundedness*, the same will apply for φ^d. It is also clear that $(\varphi^d(E, c))^d = \varphi(E, c)$.

Additionally, if a rule recommends the same allocation when dividing awards and losses, it is called *Self-Dual*.

A rule φ is **Self-Dual**, if for each $(E, c) \in \mathcal{B}$ and each $i \in N$, $\varphi_i(E, c) = c_i - \varphi_i(L, c)$. [2]

In this paper we will focus on three particular rules: the *Constrained Equal Awards*, *Piniles'* and the *Constrained Egalitarian* rules, and their dual rules.

The **Constrained Equal Awards** rule, *CEA*, (Maimonides 12th Century, among others) recommends, for each $(E, c) \in \mathcal{B}$, the vector $(\min\{c_i, \mu\})_{i \in N}$, where μ is chosen so that $\sum\limits_{i \in N} \min\{c_i, \mu\} = E$.

Piniles' rule, *Pin*, (Piniles (1861)) provides, for each $(E, c) \in \mathcal{B}$, the vector $(CEA_i(E, c/2))_{i \in N}$, if $E \leq C/2$; and $(c_i/2 + CEA_i(E - C/2, c/2))_{i \in N}$, if $E \geq C/2$.

The **Constrained Egalitarian** rule, *CE*, (Chun et al. (2001)) chooses, for each $(E, c) \in \mathcal{B}$, the vector $(CEA_i(E, c/2))_{i \in N}$, if $E \leq C/2$; and $(\max\{c_i/2, \min\{c_i, \delta\}\})_{i \in N}$, if $E \geq C/2$, where δ is chosen so that $\sum\limits_{i \in N} CE_i(E, c) = E$.

Note that the *Constrained Equal Losses* rule, *CEL,* (Aumann and Maschler (1985)) is the dual of the *Constrained Equal Awards* rule (Herrero and Villar (2001)). Moreover, *DPin* and *DCE* will denote the *Dual of Piniles'* and the *Dual of Constrained Egalitarian* rules, respectively.

[2] That is, $\varphi^d(E, c) = \varphi(E, c)$.

Next, we introduce some properties of rules which, subsequently, will be interpreted as *Legitimate Principles*. Moreover, we present the notion of *duality* and *Self-Duality* between properties.

Resource Monotonicity (Curiel et al. (1987), Young (1987), among others) demands that if the endowment increases, then all individuals should get at least what they received initially. No rule violating this property has been proposed, otherwise we could have situations where increased endowment would cause disadvantages to certain agents.

Resource Monotonicity: for each $(E, c) \in \mathcal{B}$ and each $E' \in \mathbb{R}_+$ such that $C > E' > E$, then $\varphi_i(E', c) \geq \varphi_i(E, c)$, for each $i \in N$.

Order Preservation (Aumann and Maschler (1985)) requires respecting the ordering of the claims: if agent $i's$ claim is at least as large as agent $j's$ claim, she should receive and loss at least as much as agent j, does respectively. This property is satisfied by all proposed rules, and it has been understood by many authors as a minimal requirement of fairness (see for instance Thomson (2003)).

Order Preservation: for each $(E, c) \in \mathcal{B}$, and each $i, j \in N$, such that $c_i \geq c_j$, then $\varphi_i(E, c) \geq \varphi_j(E, c)$, and $c_i - \varphi_i(E, c) \geq c_j - \varphi_j(E, c)$.

A *Super-Modular* rule (Dagan et al. (1997)) allocates each additional dollar in an 'order preserving' manner. In other words, when the endowment increases, agents with higher claims receive a greater part of the increment than those with lower claims. Apart from the *Constrained Egalitarian rule*, all of the main rules proposed in the literature satisfy this property.

Super-Modularity: for each $(E, c) \in \mathcal{B}$, all $E' \in \mathbb{R}_+$ and each $i, j \in N$ such that $C > E' > E$ and $c_i \geq c_j$, then $\varphi_i(E', c) - \varphi_i(E, c) \geq \varphi_j(E', c) - \varphi_j(E, c)$.

Midpoint Property (Chun et al. (2001)) requires that if the estate is equal to the sum of the half-claims, then all agents should receive their half-claim. Robust arguments have been established for this property. Aumann and Maschler (1985) argue that 'it is socially unjust for different creditors to be on opposite sides of the halfway point, $C/2$'. Note that this property treats the problem of dividing awards or losses equally.

Midpoint Property: for each $(E, c) \in \mathcal{B}$ and each $i \in N$, if $E = C/2$, then $\varphi_i(E, c) = c_i/2$.

The dual relation defined between rules has been carried to the concept of property. In this sense, given two properties, we say that they are dual of each other if whenever a rule satisfies one of them, its dual satisfies the other.

Two properties, $\mathcal{P}$ and $\mathcal{P}^d$, are **dual** if whenever a rule, φ, satisfies $\mathcal{P}$, its dual, φ^d, satisfies $\mathcal{P}^d$.

It is worth noting that all the principles we have introduced are invariant to the perspective from which the problem is thought, that is, they do not change when dividing "what is available" or "what is missing", so , they are *Self-Dual*. Formally:

A property, $\mathcal{P}$, is **Self-Dual** when it coincides with its dual.

In the following table we can observe the properties which are fulfilled by the rules considered in this paper.

	CEA	*CEL*	*Pin*	*DPin*	*CE*	*DCE*
Resource Monotonicity	Yes	Yes	Yes	Yes	Yes	Yes
Order Preservation	Yes	Yes	Yes	Yes	Yes	Yes
Super-Modularity	Yes	Yes	Yes	Yes	No	No
Resource Monotonicity	Yes	Yes	Yes	Yes	Yes	Yes
Midpoint	No	No	Yes	Yes	Yes	Yes

Table 1: Proposed properties and the considered rules.

To close this section, and in order to compare the proposals given by the different rules, we will use the well-known *Lorenz (equity) criterion* (Lorenz (1905)), which will be considered as the general equity principle. Specifically, Arin (2007) and Dutta and Ray (1989), among others, consider that this criterion captures the idea that the desirable social goal is to treat everybody as evenly as possible.

Let $A = \{x \in \mathbb{R}^N : x \geq 0\}$, and for each vector $x \in A$, we denote by $\Pi(x)$ the vector that results from x by permuting the coordinates in such a way that $\Pi_1(x) \leq \Pi_2(x) \leq ... \leq \Pi_n(x)$. Let $x, y \in \mathbb{R}^N$, we say that x **Lorenz dominates** y, denoted by $x \succ_L y$, if $\Pi_1(x) \geq \Pi_1(y)$, $\Pi_1(x) + \Pi_2(x) \geq \Pi_1(y) + \Pi_2(y)$, and so on, with at least one strict inequality. Note that, given

$x, y \in \mathbb{R}^N$, we do not impose on these vectors the condition $\sum_{i \in N} x_i = \sum_{i \in N} y_i$ in order to apply the *Lorenz* domination criterion (see Arin (2007)).

Given a set $S \subseteq A$, a vector $x \in S$ is **Lorenz Maximal in** S if there is no other vector $y \in S$ such that, $y \succ_L x$.

Given two rules f and g, we say that f Lorenz-Gains dominates g if for each $(E, c) \in \mathcal{B}$, $f(E, c) \succ_L g(E, c)$. And a rule f is **Lorenz-Gains Maximal, LGM,** if there is no other rule g such that, for each $(E, c) \in \mathcal{B}$, $g(E, c) \succ_L f(E, c)$. Analogously, f Lorenz-Losses dominates g if for each $(E, c) \in \mathcal{B}$, $f(L, c) \succ_L g(L, c)$. And a rule f is **Lorenz-Losses Maximal, LLM**, if there is no other rule g such that, for each $(E, c) \in \mathcal{B}$, $g(L, c) \succ_L f(L, c)$.

3. The model.

As we have mentioned, the main goal of the Axiomatic approach, a formal method to study bankruptcy problems, is to identify rules by means of appealing properties. Regarding to this, we are interested in analyzing bankruptcy problems where all the allowed rules satisfy a '*Legitimate Principles'* set, denoted by P. That is, we consider a society[3] which agrees on that the distribution of the endowment must be based on a set of basic properties or fair principles. Note that, the more properties required by a society, the less number of allowed rules. For example, suppose that a society demands efficiency, claim-boundedness and non-negativity. Then, it will permit any rule. Now, suppose that some society agrees on demanding *Self-Duality* and *Composition up*. Then, the *Proportional* rule[4] will be the only admissible rule.

Next, we present such problems formally, and the definition of their associated rules, introduced by Jiménez-Gómez and Marco-Gil (2008).

Definition 1. *A* ***Bankruptcy Problem with Legitimate Principles*** *is a triplet* (E, c, P_t)*, where* $(E, c) \in \mathcal{B}$ *and* P_t *is a fixed set of principles on which a particular society has agreed.*

[3]With society, we mean the group of agents involved in each problem.

[4]See Thomson (2003) for formal definitions of *Composition up* and the *Proportional* rule.

Henceforth, let P be the set of all subsets of properties of rules. Each $P_t \in P$ represents a specific society which will always apply such principles for solving its problems. Finally, let $\mathcal{B}_P$ be the set of all *Problems with Legitimate Principles*.

In this context, an *Admissible* rule for a society that has agreed on P_t is a rule satisfying all these properties.

Definition 2. *An **Admissible rule**, is a function, $\varphi : \mathcal{B}_P \to \mathbb{R}^n_+$, such that its application in $\mathcal{B}$, $\varphi : \mathcal{B} \to \mathbb{R}^n_+$, is a rule satisfying all properties for P_t.*

Let Φ denote the set of all rules and let $\Phi(P_t)$ be the subset of rules satisfying P_t.

As we have mentioned, we introduce two bounds:

- the *P-Safety*, which is a lower bound on awards defined by Jiménez-Gómez and Marco-Gil (2008); and
- the *P-Ceiling*, which is an upper bound on awards.

Particularly, the former corresponds with the smallest amount that each agent could get according to the application of the *Admissible* rules; whereas the latter ensures that each agent's awards are confined to the maximum amount among them. Formally,

Definition 3. *Given (E, c, P_t) in $\mathcal{B}_P$, the **P-Safety,** s, is for each $i \in N$,*

$$s_i(E, c, P_t) = \min_{\varphi \in \Phi(P_t)} \{\varphi_i(E, c)\}.$$

Definition 4. *Given $(E, c, P_t) \in \mathcal{B}_P$, the **P-Ceiling,** ce, is for each $i \in N$,*

$$ce_i(E, c, P_t) = \max_{\varphi \in \Phi(P_t)} \{\varphi_i(E, c)\}.$$

Following the Axiomatic approach, the way of rationing the endowment should fulfil some fair principles [5]. Moreover, many authors require that a rule should guarantee a level of awards to each agent. By duality, the same idea can be applied on losses. That is, each agent should have a limit on the awards she can get. Then, it appears in a "natural way" a process where each

[5] These principles define the set of *Admissible* rules for a given society

agent should receive, at least, a minimum amount, and, at most, a maximum amount according to all the *Admissible* rules. To this respect, we define the *Double Boundedness Recursive Process* as the procedure in which at each step, every agent's claim is truncated by her *P-Ceiling*, and each of them receives her *P-Safety*. This idea, although with differences in the procedure, has been introduced in bargaining problems by Marco-Gil et al. (1995) with the definition of the *Unanimous-Concession* mechanism. At a first step, their process guarantees to each agent the minimum amount according to a set of agreed solutions, which would be the disagreement point at the following step, and straightforwardly determine the solution of the (meta) bargaining process.

Definition 5. *For each $m \in \mathbb{N}$, the* ***Double Boundedness Recursive Process****, DBR^m, at the m-th step associates for each $(E, c, P_t) \in \mathcal{B}_P$ and each $i \in N$,*

$$[DBR(E^m, c^m, P_t)]_i = s_i(E^m, c^m, P_t),$$
$$\text{where } (E^1, c^1) \equiv (E, c) \text{ and for } m \geq 2,$$
$$E^m \equiv (E^{m-1} - \sum_{i \in N} s_i(E^{m-1}, c^{m-1}, P_t)$$
$$c_i^m = ce_i(E^{m-1}, c^{m-1}, P_t) - s_i(E^{m-1}, c^{m-1}, P_t).$$

Returning to the example introduced in Section 1, this mechanism naturally occurs in a mediation process. In this sense, if agents consider a set of fair properties, they will coincide that each agent should have guaranteed the minimum amount of the resources according to the accepted principles, and should not receive more than the maximum amount recommended by the *Admissible* rules. So, the recursive application of these two bounds may be applied.

Note that this process is not always efficient, but we can easily see that it does whenever the *P-Safety* always provides a positive amount to certain agents in each step[6]. In such cases we call it the *Double Recursive* rule. Formally,

Definition 6. *The* ***Double Recursive*** *rule, φ^{DR}, associates for each $(E, c, P_t) \in \mathcal{B}_P$ and each $i \in N$, $\varphi_i^{DR}(E, c, P_t) = \sum_{m=1}^{\infty} [DBR(E^m, c^m, P_t)]_i$.*

[6]See Dominguez (2007).

4. Main results.

In this section, we take as starting point situations where discrepancy for sharing the estate is considered by means of the existence of two fixed focal rules representing two prominent proposals. Such approach was introduced by Gadea-Blanco et al. (2010) in a more general framework from a cooperative point of view, under the name of *Bifocal distribution problems.* Particularly, we consider two rules, f and g, called **Focal** rules, which mark out the area of all the admissible paths of awards, φ_i, satisfying properties for P_t, that is, for each $(E, c, P_t) \in \mathcal{B}_{\mathcal{P}}$, and each $i \in N$ $\min\{f_i(E, c, P_t), g_i(E, c, P_t)\} \leq \varphi_i(E, c, P_t) \leq \max\{f_i(E, c, P_t), g_i(E, c, P_t)\}$.

Then, the next result shows that the *Double Recursive* rule for P_t can be defined as the average of the two *Focal* rules if they are *dual* to each other.

Theorem 7. *For each* $(E, c, P_t) \in \mathcal{B}_P$*, such that for each* $i \in N$*, if any Admissible rule,* φ*, fulfils that:* $\min\{f_i(E, c, P_t), g_i(E, c, P_t)\} \leq \varphi_i(E, c, P_t) \leq \max\{f_i(E, c, P_t), g_i(E, c, P_t)\}$*, and* $f(E, c, P_t) = c - g(L, c, P_t)$*, then,*

$$\varphi^{DR}(E, c, P_t) = \frac{f(E, c, P_t) + g(E, c, P_t)}{2}.$$

Proof. See Appendix 1.

In our framework, the two *Focal* rules arise in a natural way, as any distribution in bankruptcy problems can be observed by focusing either on awards or on losses. Other contexts where discrepancy over the proposed allocation results in two focal points are the surplus sharing problem (see Moulin (1988) who argues that the equal and proportional sharing rules are the two focal solutions of this problem and, indeed, the only ones satisfying a reasonable set of properties), or problems of fairness and envy-free allocations (see Varian (1974) who proposes two focal distributions: an income-fair and a wealth-fair allocations basing on abilities).

The next theorem, which follows straightforwardly from the proof of Theorem 7, establishes that, whenever there are two *Focal* rules which are dual to each other, the final allocation provided by the *Double Recursive* rule will correspond with the average of the *P-Safety* and the *P-Ceiling* are *dual.*

Theorem 8. *For each* $(E, c, P_t) \in \mathcal{B}_P$*, such that for each* $i \in N$*, if any Admissible rule,* φ*, fulfils that:* $\min\{f_i(E, c, P_t), g_i(E, c, P_t)\} \leq \varphi_i(E, c, P_t) \leq \max\{f_i(E, c, P_t), g_i(E, c, P_t)\}$*, and* $f(E, c, P_t) = c - g(L, c, P_t)$*, then,*

$$\varphi^{DR}(E, c, P_t) = \frac{s(E, c, P_t) + ce(E, c, P_t)}{2}.$$

A direct consequence of the above results is that, in our context, the *Double Recursive* rule satisfies efficiency, non-negativity and claim-boundedness, and also satisfies *Self-Duality* (Thomson and Yeh (2008)), which means that it provides the same allocation of the endowment when distributing awards or losses.

The *Double Recursive* rule proposes the midpoint between the two rules which represent extreme and opposite ways of sharing awards among claimants according to the imposed requirements. So, in other words, it could be said that the rationing of the endowment obtained by the recursive double imposition of the *P-Safety* and the *P-Ceiling* neither favor nor hurts to any agent in particular. Following Thomson and Yeh (2008),

> *'When two rules express opposite points of views on how to solve a bankruptcy problem, it is natural to compromise between them by averaging.*

5. Applications.

In this section we consider three possible choices of *Legitimate Principles* to apply the approach introduced previously. These properties have been selected since they have been understood by many authors as minimal requirements of fairness, and, moreover, they are satisfied by most of the rules proposed in the literature Thomson (2003)). Furthermore, all the used properties are *Self-Dual*.

As we have mentioned in Section 2, following Arin (2007), Dutta and Ray (1989), Cowell (2000) and Lambert (2001), and with the aim of determining the two *Focal* rules, the discrepancy for sharing the resources is considered by means of the existence of two fixed rules based on the *Lorenz criterion*. That is, we combine the two focus, awards and losses, that 'naturally' arises in bankruptcy problems, with the two most egalitarian rules according to these points of views, i.e., the *Lorenz-Gains Maximal* and the *Lorenz-Losses Maximal* for each (E, c, P_t).

Specifically, we consider the three following *Legitimate Principles* sets,

$$P_1 = \{\textit{Efficiency, Claim-boundedness, Non-Negativity, Lorenz criterion}\}$$
$$P_2 = P_1 \cup \{\textit{Resource Monotonicity, Super-Modularity, Midpoint Property}\}$$
$$P_3 = P_1 \cup \{\textit{Resource Monotonicity, Midpoint Property}\}$$

At this point, Lorenz comparisons of bankruptcy rules from the awards point of view can be found in Bosmans and Lauwers (2011) and Thomson (2007). With their results together with duality, we obtain that the *Focal* rules that mark out the region of the *Admissible* rules for P_1, P_2, and P_3, are the pairs (CEA, CEL), $(Pin, Dpin)$, and (CE, DCE), respectively. So, *Bankruptcy Problems with Legitimate Principles* for each of these principles sets are well-defined, being their elements triplets, such that, for each $(E, c) \in \mathcal{B}$,

$$(E, c, P_1) \text{ with } \textit{Focal} \text{ rules } CEA \text{ and } CEL,$$
$$(E, c, P_2) \text{ with } \textit{Focal} \text{ rules } Pin \text{ and } DPin,$$
$$(E, c, P_3) \text{ with } \textit{Focal} \text{ rules } CE \text{ and } DCE.$$

If we apply the result in Theorem 7 to the three different *Legitimate Principles* sets mentioned previously, we obtain:

Corollary 9. *For each $(E, c, P_1) \in \mathcal{B}_P$, the Double Recursive rule is the average of the Constrained Equal Awards and the Constrained Equal Losses rules.*

Corollary 10. *For each $(E, c, P_2) \in \mathcal{B}_P$, the Double Recursive rule is the average of Piniles' and the Dual of Piniles' rules.*

Corollary 11. *For each $(E, c, P_3) \in \mathcal{B}_P$, the Double Recursive rule is the average of the Constrained Egalitarian and the Dual Constrained Egalitarian rules.*

Note that these corollaries imply that the allocation proposed by our new procedure is *Admissible* with $P_t \in \{P_1, P_2, P_3\}$, since, by Thomson and Yeh (2008), the *Double Recursive* rule preserves *Resource Monotonicity, Super-Modularity* and the *Midpoint* property.

However, the *Double Recursive* rule fails some properties, even when the two *Focal* rules fulfil them (Thomson and Yeh (2008)), such that *Composition Down* (Moulin (2000)), *Composition Up* (Young (1987)), and *Consistency* (Young (1987)). For example, we can see that if we add to the set P_1 the property of *Consistency,* the two *Focal* rules remain to be CEA and CEL, but the average of these rules does not satisfy consistency. Then, the natural question is '*Would a society have any argument to apply this new rule?*' The answer could be affirmative since, although a society could have reasons to not agree on applying the result of the *Double Recursive* rule, since it may fail some properties, this way of distributing the endowment has been defended by many authors as a natural way to agree on a 'middle' allocation between two extreme ways of rationing (see, for instance, Thomson and Yeh (2008)). Moreover, another reason to use this method is that we know exactly the result of the procedure and the properties which are satisfied by the final allocation. Finally, as we have commented, this is a procedure which is naturally applied in mediation.

6. Conclusions.

In this paper we observe that in contexts where two *Focal* positions appear, the application of a recursive method (which can be understood as the proposal of an arbitrator or a mediator) retrieves the midpoint between these two focus. This fact, apart from its own logic, allows to anticipate the result. Moreover, whenever the average of these *Focal* rules fulfils the properties on which the context is based, then the *Double Recursive* rule leads to an *Admissible* allocation. In this context, the following issues thus remain open: the search for a characterization of the convex combination of rules, and in particular, for the average; and the analysis of conditions on the legitimate principle sets that guarantee such principles are upheld when applying our recursive process with more than two *Focal* rules.

APPENDIX 1. ***Proof of Theorem 8.***

The proof of this result is based on a fact, two lemmas and a remark. The fact tells us that whenever there are two *Focal* rules, which are dual to each other, then the *P-Safety* and the *P-Ceiling* are dual to each other, too.

Fact 1. *For each* $(E, c, P_t) \in \mathcal{B}_P$*, such that for each* $i \in N$*, if any Admissible rule,* φ*, fulfils that:* $\min\{f_i(E, c, P_t), g_i(E, c, P_t)\} \leq \varphi_i(E, c, P_t) \leq \max\{f_i(E, c, P_t), g_i(E, c, P_t)\}$*, and* $f(E, c, P_t) = c - g(L, c, P_t)$*, then, the P-Safety and the P-Ceiling are dual, i.e.,*

$$s(E, c, P_t) = c - ce(L, c, P_t).$$

The following lemma shows that, whenever there are two *Focal* rules, which are dual to each other, in any step $m \in \mathbb{N}$, $m > 1$, the sum of the *P-Safety* and the *P-Ceiling* coincides with the sum of the claims.

Lemma 12. *For each* $(E, c, P_t) \in \mathcal{B}_P$*, such that for each* $i \in N$*, if any Admissible rule,* φ*, fulfils that:* $\min\{f_i(E, c, P_t), g_i(E, c, P_t)\} \leq \varphi_i(E, c, P_t) \leq \max\{f_i(E, c, P_t), g_i(E, c, P_t)\}$*, and* $f(E, c, P_t) = c - g(L, c, P_t)$*, and* $m \in \mathbb{N}, m > 1$*,*

$$\sum_{i \in N} [ce_i(E^m, c^m, P_t) + s_i(E^m, c^m, P_t)] = C^m.$$

Proof. Let each $(E, c, P_t) \in \mathcal{B}_P$, such that for each $i \in N$, any *Admissible* rule, φ, fulfils that: $\min\{f_i(E, c, P_t), g_i(E, c, P_t)\} \leq \varphi_i(E, c, P_t) \leq \max\{f_i(E, c, P_t), g_i(E, c, P_t)\}$, $f(E, c, P_t) = c - g(L, c, P_t)$, and $m \in \mathbb{N}, m > 1$. Then, by the Definitions 3 and 4, these two *Focal* rules define the *P-Safety* and the *P-Ceiling* of each agent for a set of properties P_t, i.e.,

$s_i(E, c, P_t) = \min\{f_i(E, c, P_t), g_i(E, c, P_t)\}$, and

$ce_i(E, c, P_t) = \max\{f_i(E, c, P_t), g_i(E, c, P_t)\}$.

By Fact 1, for each agent we are adding the two *Focal* rules. So next expression comes straightforwardly.

$$\sum_{i \in N} \left[\frac{ce_i(E^m, c^m, P_t) + s_i(E^m, c^m, P_t)}{2} \right] = E^m.$$

Finally, we know that

$$\begin{aligned}
E^m &= E^{m-1} - \sum_{i\in N} s_i(E^{m-1}, c^{m-1}, P_t) = \\
&= \sum_{i\in N} \left[\frac{ce_i(E^{m-1}, c^{m-1}, P_t) + s_i(E^{m-1}, c^{m-1}, P_t)}{2}\right] - \\
&- \sum_{i\in N} s_i(E^{m-1}, c^{m-1}, P_t) = \\
&= \sum_{i\in N} \left[\frac{ce_i(E^{m-1}, c^{m-1}, P_t) - s_i(E^{m-1}, c^{m-1}, P_t)}{2}\right] = C^m/2,
\end{aligned}$$

by the definition of the *Double Boundedness Recursive Process*. **q.e.d.**

The following remark is a direct consequence of Lemma 12 and it says that for each *Bankruptcy Problem with Legitimate Principles*, whenever there are two *Focal* rules, which are dual to each other, and at any step $m \in \mathbb{N}, m > 1$, the half of the claims sum at every step of the *Double Boundedness Recursive Process* coincides with both the endowment and the total loss at every step of the process.

Remark 1. *For each* $(E, c, P_t) \in \mathcal{B}_P$, *such that for each* $i \in N$, *if any Admissible rule,* φ, *fulfils that:* $\min\{f_i(E, c, P_t), g_i(E, c, P_t)\} \leq \varphi_i(E, c, P_t) \leq \max\{f_i(E, c, P_t), g_i(E, c, P_t)\}$, *and* $f(E, c, P_t) = c - g(L, c, P_t)$, *and* $m \in \mathbb{N}, m > 1$, $E^m = L^m = C^m/2$.

Proof. Let each $(E, c, P_t) \in \mathcal{B}_P$, such that for each $i \in N$, any *Admissible* rule, φ, fulfils that: $\min\{f_i(E, c, P_t), g_i(E, c, P_t)\} \leq \varphi_i(E, c, P_t) \leq \max\{f_i(E, c, P_t), g_i(E, c, P_t)\}$, $f(E, c, P_t) = c - g(L, c, P_t)$, and $m > 1 \in \mathbb{N}$. We know that, $L^m = C^m - E^m$. By Lemma 12, $E^m = C^m/2$. Therefore, $L^m = C^m - C^m/2 = C^m/2$. **q.e.d.**

Finally, next lemma says that, whenever there are two *Focal* rules, which are dual to each other, each agent's claim at each step different of the initial one coincides with sum of both the lower and upper bound on awards.

Lemma 13. *For each* $(E, c, P_t) \in \mathcal{B}_P$*, such that for each* $i \in N$*, if any Admissible rule,* φ*, fulfils that:* $\min\{f_i(E,c,P_t), g_i(E,c,P_t)\} \leq \varphi_i(E,c,P_t) \leq \max\{f_i(E,c,P_t), g_i(E,c,P_t)\}$*, and* $f(E,c,P_t) = c - g(L,c,P_t)$*, and* $m > 1 \in \mathbb{N}$*,*

$$c_i^m = ce_i(E^m, c^m, P_t) + s_i(E^m, c^m, P_t).$$

Proof. Let each $(E, c, P_t) \in \mathcal{B}_P$, such that for each $i \in N$, any *Admissible* rule, φ, fulfils that: $\min\{f_i(E,c,P_t), g_i(E,c,P_t)\} \leq \varphi_i(E,c,P_t) \leq \max\{f_i(E,c,P_t), g_i(E,c,P_t)\}$, $f(E,c,P_t) = c - g(L,c,P_t)$, and $m > 1 \in \mathbb{N}$, by Remark 1 we know that for $m > 1 \in \mathbb{N}$, $L^m = E^m$, so, $s_i(E^m, c^m, P_t) = s_i((L^m, c^m, P_t)^d)$. By duality $ce_i(E^m, c^m, P_t) = c_i^m - s_i((L^m, c^m, P_t)) = c_i^m - s_i(E^m, c^m, P_t)$, then, $c_i^m = ce_i(E^m, c^m, P_t) + s_i(E^m, c^m, P_t)$. **q.e.d.**

Proof of Theorem 8.

Let $(E, c, P_t) \in \mathcal{B}_P$, such that for each $i \in N$, $\min\{f_i(E,c,P_t), g_i(E,c,P_t)\} \leq \varphi_i(E,c,P_t) \leq \max\{f_i(E,c,P_t), g_i(E,c,P_t)\}$, $f(E,c,P_t) = c - g(L,c,P_t)$, for each $i \in N$, and each $m \in \mathbb{N}$,

$$\varphi_i^{DR}(E,c,P_t) = s_i(E,c,P_t) + \sum_{m=2}^{\infty} s_i(E^m, c^m, P_t).$$

By the definition of the *Double Boundedness Recursive Process*,

$$\begin{aligned}
\sum_{m=2}^{\infty} c_i^m &= \sum_{m=2}^{\infty} \left[ce_i(E^{m-1}, c^{m-1}, P_t) - s_i(E^{m-1}, c^{m-1}, P_t) \right] = \\
&= ce_i(E^m, c^m, P_t) + \sum_{m=2}^{\infty} ce_i(E^m, c^m, P_t) - s_i(E^m, c^m, P_t) - \\
&- \sum_{m=2}^{\infty} s_i(E^m, c^m, P_t).
\end{aligned}$$

By Lemma 13,

$$\sum_{m=2}^{\infty} c_i^m = \sum_{m=2}^{\infty} \left[ce_i(E^m, c^m, P_t) + s_i(E^m, c^m, P_t) \right].$$

So,

$$ce_i(E,c,P_t) + \sum_{m=2}^{\infty} ce_i(E^m,c^m,P_t) - s_i(E,c,P_t) - \sum_{m=2}^{\infty} s_i(E^m,c^m,P_t) =$$

$$= \sum_{m=2}^{\infty} \left[ce_i(E^m,c^m,P_t) + s_i(E^m,c^m,P_t)\right].$$

Thus,

$$\sum_{m=2}^{\infty} s_i(E,c,P_t) = \left(ce_i(E^m,c^m,P_t) - s_i(E,c,P_t)\right)/2.$$

Therefore,

$$\varphi_i^{DR}((E,c),P_t) = s_i(E,c,P_t) + \frac{ce_i(E,c,P_t) - s_i(E,c,P_t)}{2}$$

$$= \frac{s_i(E,c,P_t) + ce_i(E,c,P_t)}{2}.$$ **q.e.d.**

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