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A New Approach for Bounding Awards in Bankruptcy Problems

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Abstract The solution for the 'Contested Garment Problem', proposed in the Babylonic Talmud, suggests that each agent should receive at least some part of the resources whenever the claim overcomes the available amount. In this context, we propose a new method to define lower bounds on awards, an idea that has underlied the theoretical analysis of claims problems from its beginning (O'Neill, 1982) to present day (Dominguez and Thomson, 2006). Specifically, starting from the fact that a society establishes its own set of commonly accepted principles, our proposal ensures to each agent the smallest amount she gets according to all the admissible rules. As in general this new bound will not exhaust the endowment, we analyze its recursive application for different sets of principles.

Keywords Bankruptcy problems · Bankruptcy rules · Lower bounds · Recursive process

1 Introduction

A claims problem is a situation where a group of agents claim more of a perfectly divisible resource (the endowment) than what is available. In this context, a rule prescribes how to share the endowment among the agents, according to the profile of claims. *How should the endowment be divided? Should each agent have a guaranteed level of awards?*

The main goal of the two approaches to study claims problems (the axiomatic and game theory methods) is to identify rules by means of appealing properties. Following this line, the establishment of lower bounds on awards has been found reasonable by many authors. In fact, the formal definition of a rule already includes both an upper and a lower bounds on awards by requiring that no agent receives more than her claim and less than zero. In 1982, O'Neill [16] provides another lower bound on awards called *Respect of Minimal Rights,*

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which requires that each agent receives at least what is left once the other agents have been fully compensated, or zero if this amount is negative. Herrero and Villar [10, 11] introduce the following two properties that bound awards. *Sustainability* says that if we truncate all claims at an agent i's claim and there is enough to honor all claims, then agent i's award should be equal to her claim. *Exemption* demands that agent i not be rationed when equal division provides her more than she claims. Moulin [14] defines a new restriction on awards, called *Lower Bound:* each agent receives at least the amount corresponding to the egalitarian division except those who demand less, in which case their claims are met in full. Moreno-Ternero and Villar [13] present a weaker notion of Moulin's *Lower Bound*, named *Securement*, which says that each agent should obtain at least the n-th part of her claim truncated at the endowment. Finally, Dominguez [7] proposes the *Min Lower Bound*, which modifies *Securement* by replacing each agent's claim by the smallest one.

Apart from *Respect of Minimal Rights*, a property that is implied by the definition of a rule, the other proposed lower bounds on awards have been justified by their own reasonability or appeal. In this paper, we propose a new definition along the line of O'Neill's proposal. Specifically, we define the agent's *P-rights* as the smallest amount recommended by all the rules satisfying a set of 'basic' requirements. This set of commonly accepted principles is formed by those properties that a specific society decides to apply in the resolution of claims problems. Then, we define the associated bound on awards, *Respect of P-rights*, by demanding that each agent should receive at least her *P-rights*.

In general the aggregate guaranteed amount by means of our *P-rights* will not exhaust the endowment, we propose and analyze its recursive application. Once each agent receives her *P-rights*, the problem is revised accordingly. Then, the so called *Recursive P-rights Process* proposes the recursive application of the *P-rights* in each recursive revised problem. The idea of recursion is not new, and indeed it has already been used in the context of claims problems by Alcalde et al. [1], order to generalize a proposal by Ibn Ezra, and by Dominguez and Thomson [8], whose starting point is Moreno-Ternero and Villar's concept of boundedness. Dominguez [7] also studies the behavior of the recursive application of a generic bound.

We apply our methodology to several sets of properties. We first propose the singleton P_1, whose only element is order preservation. We find that the *Recursive P-rights Process* leads to the *Constrained Equal Loss* rule. We then define the set P_2, by adding to order preservation the requirement of resource monotonicity and the midpoint property. We demonstrate that the *Recursive P-rights Process* leads to the *Constrained Egalitarian* rule, but only for two-agent problems.

Our previous results could be written as follows: 'For each two-agent problem, in the set of all admissible rules according to P_1 or P_2, the recursive application of the *P-rights* leads to the rule that provides greater awards to the agents with the largest claim'. Then, the generalization of this statement arises as a question naturally. We consider a new set of socially accepted requirements, P_3, consisting of super-modularity, resource monotonicity and the midpoint property. In this case, the rules that mark out the area of all admissible paths of awards for two-person problems are *Piniles'* rule and its dual. We show that the *Recursive P-rights Process* leads to a new admissible rule which is between *Piniles* and the *Dual of Piniles'* rules. We conclude that the generalization of the above statement is not possible. Therefore, we demonstrate that the plausibility of the recursive application of the *P-rights* for two-agent problems cannot in general be extended to the n-agent case. Specifically, we provide the result obtained by the *Recursive P-rights Process* for different three-agent problems for P_2 and P_3. The resulting rules do not satisfy the equity principles on which the recursive process is based.

The paper is organized as follows. Section 2 presents the model. Section 3 proposes our new approach for bounding awards and its recursive application. By using the previous ideas for P_1, Section 4 provides a new basis for the *Constrained Equal Losses* rule. Section 5 considers our process for two-agent problems when admissible rules are those satisfying properties in P_2 and P_3. Furthermore it shows the incompatibility of our process with some 'appealing' sets of equity principles. Section 6 presents our final remarks. Finally, the proofs are contained in the the appendices.

2 Preliminaries

We consider a group of agents having claims on a resource. A claims problem is a situation where the sum of the agents claims' are greater than the amount available. Each agent $i \in N$, $N = \{1, ..., i, ..., n\}$, has a claim c_i on the endowment, E, a perfectly divisible good. Formally,

Definition 1 A **claims problem** is a vector $(E,c) \in \mathbb{R}_{++} \times \mathbb{R}^n_+$ such that $E < \sum_{i \in N} c_i$.

Since the claims add up to more than the endowment, this should be rationed among agents.

Let $\mathscr{B}$ denote the set of all problems; given $(E,c) \in \mathscr{B}$, C denotes the sum of the agents' claims, $C = \sum_{i \in N} c_i$; L the total loss to distribute among the agents, $L = C - E$. Let $\mathscr{B}_0$ be the set of problems in which claims are increasingly ordered, that is problems with $c_i \leq c_j$ for $i < j$.

A rule associates within each problem a distribution of the endowment among the agents.

Definition 2 A **rule** is a function, $\varphi : \mathscr{B} \to \mathbb{R}^n_+$, such that for each $(E,c) \in \mathscr{B}$,

(a) $\sum_{i \in N} \varphi_i(E,c) = E$ (*Efficiency*) and
(b) $0 \leq \varphi_i(E,c) \leq c_i$ for each $i \in N$ (*Non-Negativity* and *Claim-Boundedness*).

Next are rules that will be used in the following sections, emphasizing their dual relations.

The *Constrained Equal Awards* rule (Maimonides, 12th century, among others) recommends equal awards to all agents subject to no-one receiving more than her claim.

Constrained Equal Awards rule, *CEA*: for each $(E,c) \in \mathscr{B}$ and each $i \in N$, $CEA_i(E,c) \equiv \min\{c_i, \mu\}$, where μ is chosen so that $\sum_{i \in N} \min\{c_i, \mu\} = E$.

Piniles' rule (Piniles [17]) provides, for each problem $(E,c) \in \mathscr{B}$, the awards that the *Constrained Equal Awards* rule recommends for $(E, c/2)$, when the endowment is less than the half-sum of the claims. Otherwise, firstly each agent receives her half-claim, then the *Constrained Equal Awards* rule is re-applied to the residual problem $(E - C/2, c/2)$.

Piniles' rule, *Pin*: for each $(E,c) \in \mathscr{B}$ and each $i \in N$,

$$Pin_i(E,c) \equiv \begin{cases} CEA_i(E, c/2) & \text{if } E \leq C/2 \\ \\ c_i/2 + CEA_i(E - C/2, c/2) & \text{if } E \geq C/2 \end{cases}$$

The next rule, introduced by Chun et al. [4], is inspired by the *Uniform* rule (Sprumont [19]), a solution to the problem of fair division when preferences are single-peaked. It makes the minimal adjustment in the formula for the *Uniform* rule, taking the half-claims as peaks and guaranteeing that awards are ordered in the same way as claims are.

Constrained Egalitarian rule, *CE*: for each $(E,c) \in \mathscr{B}$ and each $i \in N$,

$$CE_i(E,c) \equiv \begin{cases} CEA_i(E,c/2) & \text{if } E \leq C/2 \\ \max\{c_i/2, \min\{c_i, \delta\}\} & \text{if } E \geq C/2 \end{cases}$$

where δ is chosen so that $\sum_{i \in N} CE_i(E,c) = E$.

Given a rule φ, its dual distributes what is missing in the same way that φ divides what is available (Aumann and Maschler [2]).

The **dual** of φ, denoted by φ^d, is defined by setting for each $(E,c) \in \mathscr{B}$ and each $i \in N$, $\varphi_i^d(E,c) = c_i - \varphi_i(L,c)$.

It is straightforward to check that the duality operator is well defined, since for each $(E,c) \in \mathscr{B}$, $(L,c) \in \mathscr{B}$ and if φ satisfies *Efficiency*, *Non-Negativity* and *Claim-Boundedness*, so does φ^d.

The next rule, discussed by Maimonides (Aumann and Maschler [2]), is the dual of the *Constrained Equal Awards* rule (Herrero [9]). Specifically, it chooses the awards vector at which all agents incur equal losses, subject to no-one receiving a negative amount.

Constrained Equal Losses rule, *CEL*: for each $(E,c) \in \mathscr{B}$ and each $i \in N$, $CEL_i(E,c) \equiv \max\{0, c_i - \mu\}$, where μ is chosen so that $\sum_{i \in N} \max\{0, c_i - \mu\} = E$.

The *Dual of Piniles'* rule selects, for each problem $(E,c) \in \mathscr{B}$ when the endowment is less than the half-sum of the claims, the awards vector recommending by the *Constrained Equal Losses* rule for $(E,c/2)$,. Otherwise, each agent first receives her half-claim, then the *Constrained Equal Losses* rule is re-applied to the residual problem $(E - C/2, c/2)$.

Dual of Piniles' rule, *DPin*: for each $(E,c) \in \mathscr{B}$ and each $i \in N$,

$$DPin_i(E,c) \equiv \begin{cases} c_i/2 - \min\{c_i/2, \lambda\} & \text{if } E \leq C/2 \\ c_i/2 + (c_i/2 - \min\{c_i/2, \lambda\}) & \text{if } E \geq C/2 \end{cases}$$

where λ is such that $\sum_{i \in N} DPin_i(E,c) = E$.

The *Dual Constrained Egalitarian* rule, as the *CE* rule does for awards, gives the half-claims a central role and makes the minimal adjustment in the formula for the *Uniform* rule applied to the division of losses to guarantee that they are ordered in the same way as claims are.

Dual of Constrained Egalitarian rule, *DCE*: for each $(E,c) \in \mathscr{B}$ and each $i \in N$,

$$DCE_i(E,c) \equiv \begin{cases} c_i - \max\{c_i/2, \min\{c_i, \delta\}\} & \text{if } E \leq C/2 \\ c_i - \min\{c_i/2, \delta\} & \text{if } E \geq C/2 \end{cases}$$

where δ is chosen so that $\sum_{i \in N} DCE_i(E,c) = E$.

3 A new approach: bounding awards from equity principles

Respect of Minimal Rights is a consequence of *Efficiency*, *Non-Negativity* and *Claim Boundedness* together (Thomson [20]), the three conditions imposed by a rule (see Definition 2)[1]. Formally, it requires that each agent receives at least what is left of the endowment after the other agents have been fully compensated, or zero if this amount is negative.

Respect of Minimal Rights for each $(E,c) \in \mathscr{B}$ and each $i \in N$, $\varphi_i(E,c) \geq m_i(E,c) = \max\{E - \sum_{j \neq i} c_j, 0\}$.

In this section, following this O'Neill's proposal, we introduce a new method for bounding awards based on a set of principles that are commonly accepted by a society. We propose the following extension of a problem.

Definition 3 A **Claims Problem with Legitimate Principles** is a triplet (E,c,P_t), where $(E,c) \in \mathscr{B}$ and P_t is a set of properties on which a particular society has agreed.

Let P be the set of all subsets of properties of rules. Each $P_t \in P$ represents a specific society which will always apply such principles for solving its problems. Finally, let $\mathscr{B}_P$ be the set of all *Problems with Legitimate Principles*.

This modelling becomes really interesting if it is applied to some specific types of problems, since the more available information we have the easier it is to agree on these principles. For example, let $\mathscr{B}_P^T \subset \mathscr{B}_P$, the *Problems with Legitimate Principles* that represent the collection of a given amount of taxes in a community. In this case, *Progressively* (see Thomson [20]) may be commonly accepted. However, this property may not be reasonable in other situation.

For each *Problem with Legitimate Principles*, a society will consider as *socially admissible* any rule that satisfies the properties in P_t.

Definition 4 A socially admissible rule, or simply an **admissible rule**, is a function, $\varphi : \mathscr{B}_P \to \mathbb{R}_+^n$, such that its application in $\mathscr{B}$, $\varphi : \mathscr{B} \to \mathbb{R}_+^n$, is a rule satisfying all properties in P_t.

Let Φ denote the set of all rules and let $\Phi(P_t)$ be the subset of rules satisfying P_t.

Taking extended problems as a starting point, we propose a new lower bound on awards based on the application of the ordinary meaning of a guarantee. This bound, called *P-rights*, provides each agent the smallest amount recommended by all admissible rules. Formally,

Definition 5 Given (E,c,P_t) in $\mathscr{B}_P$, the **P-rights**, s, is for each $i \in N$,

$$s_i(E,c,P_t) = \inf_{\varphi \in \Phi(P_t)} \{\varphi_i(E,c)\}.$$

Now, we say that a rule *Respects P-rights* if, for each $P_t \in P$, each $(E,c) \in \mathscr{B}$ and each $i \in N$, $\varphi_i(E,c) \geq s_i(E,c,P_t)$.

Note that if P_t is the empty set, the *P-rights* corresponds with the concept of *Minimal Rights*.

[1] For each $i \in N$, if $m_i(E,c) > 0$ and $\varphi_i(E,c) < m_i(E,c)$ either $\sum_{i \in N} \varphi_i(E,c) < E$, contradicting *Efficiency*, or there is $j \neq i$ such that $\varphi_j(E,c) > c_j$, contradicting *Claim-Boundedness*. Otherwise, that is, if $m_i(E,c) = 0$, by *Non-Negativity* $\varphi_i(E,c) \geq m_i(E,c)$.

As in general, the sum of the agents' *P-rights* of a *Problem with Legitimate Principles* does not exhaust the endowment, a requirement of composition from the profile of these bounds arises in a natural way. It says that, the awards vector of each augmented problem should be equivalently obtainable either directly, or by first, assigning to each agent her lower bound on awards, second, adjusting claims down by these amounts, and third, applying the rule to divide the remainder. The following definition applies this idea to our bound on awards.

Definition 6 Given $P_t \in P$, a rule φ satisfies **P-rights First** if for each $(E,c) \in \mathscr{B}$ and each $i \in N$, $\varphi_i(E,c) = s_i(E,c,P_t) + \varphi_i(E - \sum_{i \in N} s_i(E,c,P_t), c - s(E,c,P_t))$.

Although many of the proposed lower bounds on awards are respected by most of the rules, composition from these lower bounds is quite demanding. For instance, *Respect of Minimal Rights* is satisfied by any rule, but none of the *Proportional, Constrained Equal Awards* or *Minimal Overlap* rules satisfy *Minimal Rights First* (Thomson [20]). Let us note that this kind of composition is equivalent to apply a recursive method from a lower bound on awards. In fact, this process has been used to generate new rules. The rule proposed by Dominguez and Thomson [8]. Their rule results from applying such a procedure to the *Securement* lower bound. Similarly, we define the recursive application of our *P-rights*, which we call the *Recursive P-rights Process*.

Definition 7 For each $m \in \mathbb{N}$, the **Recursive P-rights Process** at the m-th step, RS^m, associates for each $(E,c,P_t) \in \mathscr{B}_P$ and each $i \in N$,

$$[RS^m(E,c,P_t)]_i = s_i(E^m,c^m,P_t),$$

$$\text{where } (E^1,c^1) \equiv (E,c) \text{ and for } m \geq 2,$$

$$(E^m,c^m) \equiv (E^{m-1} - \sum_{i \in N} s_i(E^{m-1},c^{m-1},P_t), c^{m-1} - s(E^{m-1},c^{m-1},P_t)).$$

According to this process, at the first step an agent receives her *P-rights* in the original problem. At the second step, we define a residual problem in which the endowment is what remains and the claims are adjusted down by the amounts just given. Then, each agent receives her *P-rights* in this residual problem, and so on. In general, it cannot be ensured that the sum of the amounts that agents receive in each step exhausts the endowment. If this occurs, we have defined the *Recursive P-rights* rule[2]. In this sense, the recursive application of the *Minimal Rights* fails this requirement, since from the second step on, each agents receives nothing.

Definition 8 The **Recursive P-rights** rule, φ^R, associates for each $(E,c,P_t) \in \mathscr{B}_P$ and each $i \in N$, $\varphi_i^R(E,c,P_t) = \sum_{m=1}^{\infty} [RS^m(E,c,P_t)]_i$, whenever

$$\sum_{i \in N} \left(\sum_{m=1}^{\infty} [RS^m(E,c,P_t)]_i \right) = E.$$

[2] Let us note that *Non-Negativity* and *Claim-Boundedness* are satisfied by construction. Moreover, as shown in Theorem 1, it is clear that whenever the *P-rights* provide a positive amount to some agent in each step, *Efficiency* is met.

At this point we should mention some contributions that have certain features in common with our approach. In the context of Nash's bargaining model, van Damme [22] uses the research on Nash equilibria of a non-cooperative game which is induced by a mechanism of successive concessions. Specifically, the agents' strategies are the choice of a rule among a set of reasonable ones. From van Damme's work, other mechanisms for bargaining and bankruptcy have been proposed. The *Unanimous Concessions* mechanism, provided by Marco et al. [12] and modified by Herrero [9] for its application to bankruptcy, is close to our *Recursive P-rights* process, but the starting point and analysis of the two are quite different (see also Chun [3] and Naeve-Steinweg [15]). Also for bargaining problems, Thomson [21] introduces and studies the concept of *Closedness Under Recursion* of a family of solutions, which means that the solution defined through the process is not only well-defined but also belongs to the family of solutions considered. This idea, although in a different framework, is close to our definition of admissible rule but the process he uses has no relation to ours.

Finally, Theorem 1 in Dominguez [7] shows the equivalence between the existence of a positive lower bound on awards and the *Efficiency* of the recursive process that such a bound defines. Therefore,

Theorem 1 *For each $m \in \mathbb{N}$ and each $(E,c,P_t) \in \mathscr{B}_P$, the Recursive P-rights rule exists whenever $s(E^m, c^m, P_t) > 0$.*

4 A minimal requirement of fairness

$$P_1 \equiv \{\text{order preservation}\}$$

Next, we apply our method to the singleton P_1 whose only element is order preservation. This property has been understood as a minimal requirement of fairness by many authors[3].

Order preservation (Aumann and Maschler [2]) requires that if agent i's claim is at least as large as agent j's claim, she should receive at least as much as agent j does; furthermore, agent i's loss should be at least as large as agent j's.

Order preservation: for each $(E,c) \in \mathscr{B}$ and each $i, j \in N$ such that $c_i \geq c_j$, $\varphi_i(E,c) \geq \varphi_j(E,c)$ and $c_i - \varphi_i(E,c) \geq c_j - \varphi_j(E,c)$.

Lemma 5 in Appendix 2 shows that the P_1*-rights* for agents 1 and n are given by the *Constrained Equal Losses* and the *Constrained Equal Awards* rules, respectively. As a direct consequence of this result, for two-agent problems, these two rules mark out the area of all the admissible rules in P_1. However, as it is shown in the next example, this fact cannot be generalized for problems with more than two agents.

Example 1 $N = \{1,2,3\}$ and (E,c,P_1), with $(E,c) = (49,(18,27,40)) \in \mathscr{B}$. Thus, we obtain that, $CEA(E,c) = \left(16\frac{1}{3}, 16\frac{1}{3}, 16\frac{1}{3}\right)$ and $CEL(E,c) = (6,15,28)$. In this case, by Lemma 5 in Appendix 2, $s_1(E,c,P_1) = 6$ and $s_3(E,c,P_1) = 16\frac{1}{3}$. However, for agent 2 neither of both awards is the smallest amount she can get according to P_1. For example, computing

[3] The requirement of order preservation could not be appropriate for other different contexts where the agents have absolute or relative priority. For instance, it can be easily found situations where some *secured* claims can have a higher priority or weight than *unsecured* ones.

the *Talmud* rule, T, $T(E,c) = \left(9, 13\frac{1}{2}, 26\frac{1}{2}\right)$.[4] Therefore, $T_2(E,c) = 13\frac{1}{2} < CEL_2(E,c) < CEA_2(E,c)$.

Next result shows the *Recursive P-rights* rule for P_1.

Theorem 2 *For each $(E,c,P_1) \in \mathscr{B}_P$, the Recursive P-rights rule is the Constrained Equal Losses rule, $\varphi^R(E,c,P_1)$ =CEL(E,c).*

Proof See Appendix 2.

To conclude this section, let us note that we have proved that the *Recursive P-rights* rule leads to the admissible rule which favours the largest claimant.

5 Other principles sets for two-agent problems

$$P_2 \equiv \{\text{order preservation, resource monotonicity, midpoint property}\}$$
$$P_3 \equiv \{\text{super-modularity, resource monotonicity, midpoint property}\}$$

In this section, we consider other possible sets of commonly accepted principles for two-agent problems. First, we propose the set P_2 obtained by adding to P_1 resource monotonicity and midpoint property. Next we deal with the meaning of the new properties.

Resource monotonicity (Curiel et al. [5], Young [23] and others) says that if the endowment increases, then all individuals should get at least what they received initially. In fact, no rule violating this property has been proposed. This property has been widely accepted.

Resource Monotonicity: for each $(E,c) \in \mathscr{B}$ and each $E' \in \mathbb{R}_+$ such that $C \geq E' > E$, $\varphi_i(E',c) \geq \varphi_i(E,c)$, for each $i \in N$.

Next, we require that if the endowment is equal to the sum of the half-claims, then all agents should receive their half-claim (Chun, Schummer and Thomson [4]). In this situation both gains and losses are equal. So this property treats the problem of dividing awards or losses equally, but only in a ver special case. In the words of Aumann and Maschler [2], ‘it is socially unjust for different creditors to be on opposite sides of the halfway point, $C/2$’.

Midpoint Property: for each $(E,c) \in \mathscr{B}$ and each $i \in N$, if $E = C/2$, then $\varphi_i(E,c) = c_i/2$.

Next theorem states that for two-agent problems the *Recursive P-rights* rule for P_2 leads to the *Dual of Constrained Egalitarian* rule. Note that in this case the *Constrained Egalitarian* and the *Dual of Constrained Egalitarian* rules mark out the area of all the admissible rules satisfying properties in P_2 for two-agent problems, result obtained from Lemma 6 in Appendix 3.

Theorem 3 *For each two-agent Problem with Legitimate Principles in $\mathscr{B}_P$ with $P = P_2$, the Recursive P-rights rule is the Dual of Constrained Egalitarian rule, $\varphi^R(E,c,P_2)$=DCE(E,c).*

Proof See Appendix 3.

[4] The *Talmud* rule (Aumann and Maschler [2]) assigns the awards that the *Constrained Equal Awards* rule recommends for (E, c/2), when the endowment is less than the half-sum of the claims. Otherwise, each agent receives her-half claim plus the amount provided by the *Constrained Equal Losses* rule when it is applied to the residual problem (E-C/2,c/2).

For the two sets of properties considered up to now, we have shown that the *Recursive P-rights* rule leads to the admissible rule which favours the largest claimants, for two-agent problems. Our results can therefore be interpreted as providing a new basis for old rules. This fact leads to a natural question, which is analysed next:

'For each two-agent problem and any appealing equity principle set, would its P-rights recursive application recover the admissible rule which favours the largest claimants?'

Let us consider the set of commonly accepted principles, P_3, consisting of super-modularity, resource monotonicity and midpoint property. This set is more restrictive than P_2 since P_3 is obtained from P_2 by substituting order preservation for a strengthener version, super-modularity, i.e., $P_3 \supset P_2$.

Super-modularity (Dagan et al. [6]) demands that, when the endowment increases, if agent i's claim is at least as large as agent j's claim, share of the increment should be at least as large as agent j's.

Super-Modularity: for each $(E,c) \in \mathscr{B}$, each $E' \in \mathbb{R}_+$ and each $i, j \in N$ such that $C \geq E' > E$ and $c_i \geq c_j$, $\varphi_i(E',c) - \varphi_i(E,c) \geq \varphi_j(E',c) - \varphi_j(E,c)$.

Apart from the *Constrained Egalitarian* rule and its dual, all of the rules that have been introduced in the literature satisfy super-modularity.

Now, note that in P_3 the rules that mark out the area of all the admissible rules for two-agent problems are *Pin* and *DPin*. Moreover, we can easily show that the *Recursive* P_3*-rights* rule leads to a new rule that lies between *Piniles'* and the *Dual of Piniles'* rules. As a consequence, we conclude that although the *Recursive* P_3*-rights* rule is well defined, the answer to the above question is negative.

Remark 1 For each two-agent *Problem with Legitimate Principles* in $\mathscr{B}_P$ with $P = P_3$, the *Recursive P-rights* rule does not lead to neither *Piniles'* nor *Dual of Piniles'* rules.

Therefore, the natural following step consists of analysing the sets of commonly accepted principles for n-agents problems. It would probably not be difficult to find a society that willingly accepts either P_2 or P_3, and which considers our *Recursive P-rights Process* to be relatively 'natural'. However, we are sure that the result of this combination would not be accepted by any member of such a society. Specifically, as next results show, our process provides a rule that does not satisfy one of the equity principles upon which the society initially agreed to found its decisions.

Proposition 1 *For* P_2 *with* $n > 2$*, the Recursive P-rights rule does not satisfy resource monotonicity.*

Proof See Appendix 4.

Remark 2 For P_3, the *Recursive P-rights* rule does not satisfy super-modularity.

Obviously, the first proposition implies that our rule does not recover the *Dual Constrained Egalitarian* rule. Moreover, although, by Remark 1, we know that the *Dual of Piniles'* rule is not what comes out of our process, the second remark says that the result may not be admissible in general[5].

[5] The proof of Remark 2 can be easily obtained in the line of the proof of Proposition 1.

6 Final remarks

Finally, we remark that our approach can be rewritten for losses by using the idea of duality. Because all the considered properties are *Self-Dual*, P_1, P_2 and P_3 will be the same when focusing on losses[6]. Moreover, let us note that a rule, φ, is admissible if and only if its dual, φ^d, is also admissible. Specifically, by considering (L,c,P_t) for each (E,c,P_t) with $P_t \in \{P_1\}$ we have that

$$s_i(L,c,P_t) = \inf_{\varphi\in\Phi(P_t)}\{\varphi_i(L,c)\} = \inf_{\varphi\in\Phi(P_t)}\{c_i - \varphi_i^d(E,c)\} = c_i - \sup_{\varphi\in\Phi(P_t)}\{\varphi_i(E,c)\}.$$

Thus, our process applied to losses is equivalent to the following. Firstly determine the agents' upper bound on awards by searching for the supremum of what they are answered among all the admissible rules in P_t. Now, revise each agent's claim by this agent's upper bound and if the sum of the revised claims is greater than the endowment, follow the recursive process until the sum of the revised claims is equal to the endowment.

Therefore, if for each $(E,c) \in \mathscr{B}$ we consider its associated distribution of losses, that is the problem (L,c), and the *P-rights* is applied recursively, it can be easily shown that (i) the *Constrained Equal Awards* rule is obtained for P_1; (ii) the *Constrained Egalitarian* rule is obtained for two-agent problems for P_2; (iii) a new rule for the two-agent problems for P_3 is defined; and, (iv) equivalent results to Propositions 1 and 2 are met.

The following issues remain open: the study of the *Dual of Piniles'* and the *Dual of Constrained Egalitarian* rules from a strategic point of view; a complete analysis of the *Recursive P_3-rights* rule; the search for new procedures that ensure compatibility with socially accepted equity principles; and the analysis of conditions on the legitimate principle sets that guarantee such principles are upheld when applying our recursive process.

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[6] Self-Duality requires invariance regarding the perspective from which the problem is thought, that is, dividing 'what is available' or 'what is missing'. Formally, two properties, $\mathscr{P}$ and $\mathscr{P}'$, are **dual** if whenever a rule, φ, satisfies $\mathscr{P}$, its dual, φ^d, satisfies $\mathscr{P}'$. A property, $\mathscr{P}$, is **Self-Dual** when it coincides with its dual.

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APPENDIX 1 *General Remarks*

We present three remarks which are used in the proofs of the Appendices 2 and 3. Henceforth, $m \in \mathbb{N}$ denotes the m-*th* step of the *Recursive P-rights Process* (see Definition 7).

First for any *Problem with Legitimate Principles*, the total loss to distribute is the same at every step of the *Recursive P-rights Process.*

Remark 3 For each $(E,c,P_t) \in \mathscr{B}_P$ and each $m \in \mathbb{N}$, $L^m = L$.

Proof Let $(E,c,P_t) \in \mathscr{B}_P$ and $m \in \mathbb{N}$. Then,

$$L^m = C^m - E^m = \sum_{i \in N} \left(c_i - \sum_{k=1}^{m} s_i(E^k, c^k, P_t) \right) - \left(E - \sum_{i \in N} \sum_{k=1}^{m} s_i(E^k, c^k, P_t) \right) = C - E = L.$$

■

Second for each $P \in \{P_1, P_2, P_3\}$, the order of the agents' claims remains the same along the *Recursive P-rights Process*.

Remark 4 For each $(E,c,P_t) \in \mathscr{B}_P$ with $P_t \in \{P_1, P_2, P_3\}$, and each $i, j \in N$ if $c_i^m \le c_j^m$, then $c_i^{m+1} \le c_j^{m+1}$.

Proof Let $(E,c,P_t) \in \mathscr{B}_P$ with $P_t \in \{P_1, P_2, P_3\}$, $c_i^m \le c_j^m$ and φ^*, φ' belonging to $\Phi(P_t)$.

Since, for each $P_t \in \{P_1, P_2, P_3\}$, all the admissible rules satisfy order preservation, for each $\varphi \in \Phi(P_t), c_i^m - \varphi_i(E^m, c^m) \le c_j^m - \varphi_j(E^m, c^m)$ so that,

(a) If $s_i^m(E,c,P_t) = \varphi_i^*(E^m, c^m)$ and $s_j^m(E,c,P_t) = \varphi_j^*(E^m, c^m)$, by *Order Preservation,* $c_i^m - s_i^m(E^m, c^m, P_t) \le c_j^m - s_j^m(E^m, c^m, P_t)$. Therefore, $c_i^{m+1} \le c_j^{m+1}$.

(b) If $s_i^m(E,c,P_t) = \varphi_i^*(E^m, c^m)$ and $s_j^m(E,c,P_t) = \varphi_j'(E^m, c^m)$, by Definition 5, $\varphi_j'(E^m, c^m) \le \varphi_j^*(E^m, c^m)$, so that, $c_i^m - \varphi_i^*(E^m, c^m) \le c_j^m - \varphi_j^*(E^m, c^m) \le c_j^m - \varphi_j'(E^m, c^m)$. Therefore, $c_i^{m+1} \le c_j^{m+1}$.

■

Third for each $P_t \in \{P_1, P_2, P_3\}$, the sum of the amounts that agents are assigned by the *Recursive P-rights Process* is the entire endowment.

Remark 5 For each $(E, c, P_t) \in \mathscr{B}_P$ with $P_t \in \{P_1, P_2, P_3\}$, $\sum_{i \in N} \left(\sum_{m=1}^{\infty} [RS^m(E, c, P_t)]_i \right) = E$.

Proof Given that for each $P_t \in \{P_1, P_2, P_3\}$ the *P-rights* always provide a positive amount to certain agents in each step, *Efficiency* of the *Recursive P-rights Process* straightforwardly comes from Theorem 1.

■

APPENDIX 2 ***Proof of Theorem 2.***

The proof is based on five lemmas, but before presenting them, we note the following two facts. We assume throughout this Appendix, without loss of generality, that $(E, c) \in \mathscr{B}_0$.

Fact 1 *For each $(E, c) \in \mathscr{B}_0$ and each $i \in N$, $CEL_i(E, c) = \max\{0, c_i - \mu\}$, where μ is such that $\sum_{i \in N} \max\{0, c_i - \mu\} = E$.*

Therefore, μ can be understood as the losses incurred by the agents who receive positive amounts by applying the CEL rule. A straightforward way to compute this rule, which will be useful later on, is as follows.

For each $(E, c) \in \mathscr{B}_0$ and each $i \in N$, the loss imposed on agent i by CEL is

$$\gamma_i = \min\{c_i, \alpha_i\},$$

where

$$\alpha_i = \left(L - \sum_{j<i} \gamma_j \right) / (n - i + 1).$$

Therefore, for each $i \in N$,

$$CEL_i(E, c) = c_i - \gamma_i.$$

Fact 2 *By Fact 1 and Remark 3 we have:*

(a) For each $(E, c) \in \mathscr{B}_0$, and each $i \in N$, if $\gamma_i = c_i$, then for each $j < i$, $\gamma_j = c_j$.

(b) For each $(E, c) \in \mathscr{B}_0$, and each $i \in N$, if $\gamma_i = \alpha_i$, then $\alpha_i = \mu$ and for each $j > i$, $\alpha_j = \alpha_i$. Therefore $\gamma_i = \mu$.

(c) For each $m \in \mathbb{N}$ and each $i, j \in N$ such that $j < i$, α_i^m only depends on the initial problem, (E, c), and on agent j's claim at step m.

Next, we provide the five lemmas on which Theorem 2 is based.

The first lemma says that the losses incurred by the agents who receive positive amounts by applying the *CEL* rule is fixed in any step of the *Recursive P-rights Process* for P_1.

Lemma 1 *For each (E, c, P_1) with $(E, c) \in \mathscr{B}_0$ and each $m \in \mathbb{N}$, $\mu^{m+1} = \mu^m$ where μ^m, μ^{m+1} solve $\sum_{i \in N} CEL_i(E^m, c^m) = E^m$ and $\sum_{i \in N} CEL_i(E^{m+1}, c^{m+1}) = E^{m+1}$, respectively (see definition of the CEL rule in Fact 1).*

Proof Let agent i be the first agent who receives a positive amount at step $m \in \mathbb{N}$ according to the *CEL* rule, i.e., (i) $CEL_i(E^m, c^m) > 0$ and (ii) for each $j < i$, $CEL_j(E^m, c^m) = 0$. By (i) and Fact 2, $c_i^m > \mu^m = \alpha_i^m$. Given (ii) and Definition 7 of the *Recursive P-rights Process* for P_1 at the m-th step, $c_j^{m+1} = c_j^m$. By Fact 2-(c), $\alpha_i^{m+1} = \alpha_i^m = \mu^m < c_i^m$. Furthermore,

$$\begin{aligned} c_i^{m+1} &= c_i^m - \min_{\varphi \in \Phi(P_1)} \{\varphi_i(E^m, c^m)\} \\ &\geq c_i^m - CEL_i(E^m, c^m) = c_i^m - (c_i^m - \mu^m) \\ &= \mu^m = \alpha_i^{m+1}. \end{aligned}$$

Therefore, by Remark 4 and Fact 2-(b), $\gamma_i^{m+1} = \alpha_i^{m+1} = \mu^{m+1}$.

■

From now on, μ will denote μ^m, for each $m \in \mathbb{N}$.

The second lemma states that if at some step $m \in \mathbb{N}$ the agent i's *P-rights* for P_1 is $CEL_i(E^m, c^m)$, then at each step after step m, her *P-rights* for P_1 is zero.

Lemma 2 *For each $(E, c) \in \mathscr{B}_0$, and each $i \in N$ if there is $m \in \mathbb{N}$ such that*

$$s_i(E^m, c^m, P_1) = CEL_i(E^m, c^m)$$

then, for each $h \in \mathbb{N}$

$$s_i(E^{m+h}, c^{m+h}, P_1) = 0.$$

Proof We show that if $s_i(E^m, c^m, P_1) = CEL_i(E^m, c^m)$ then,

$$s_i(E^{m+1}, c^{m+1}, P_1) = CEL_i(E^{m+1}, c^{m+1}) = 0.$$

Let $(E, c) \in \mathscr{B}_0$ and $m \in \mathbb{N}$, be such that

$$s_i(E^m, c^m, P_1) = CEL_i(E^m, c^m) = c_i^m - \min\{c_i^m, \mu\}.$$

Then,

$$c_i^{m+1} = c_i^m - CEL_i(E^m, c^m) = c_i^m - (c_i^m - \min\{c_i^m, \mu\}) = \min\{c_i^m, \mu\}.$$

Therefore,

$$\begin{aligned} CEL_i(E^{m+1}, c^{m+1}) &= c_i^{m+1} - \min\left\{c_i^{m+1}, \mu\right\} \\ &= \min\{c_i^m, \mu\} - \min\{\min\{c_i^m, \mu\}, \mu\} \\ &= \min\{c_i^m, \mu\} - \min\{c_i^m, \mu\} = 0. \end{aligned}$$

By Fact 1, if $CEL_i(E^m, c^m) = 0$, then, for each $h \in \mathbb{N}$, $CEL_i(E^{m+h}, c^{m+h}) = 0$, so the agent i's *P-rights* for P_1 is, from this step on, zero.

■

The next lemma establishes that, if agent i's *P-rights* for P_1 is, at each step, a different amount from that provided by the *CEL* rule, then the total amount received by this agent is at most her award as calculated by the *CEL* rule applied to the initial problem.

Lemma 3 *For each $(E, c) \in \mathscr{B}_0$, and each $i \in N$, if, for each $m \in \mathbb{N}$,*

$$s_i(E^m, c^m, P_1) = \varphi_i(E^m, c^m) \neq CEL_i(E, c), \text{ then}$$

$$\varphi_i^R(E, c, P_1) = \sum_{k=1}^{\infty} s_i(E^k, c^k, P_1) \leq CEL_i(E, c).$$

Proof. Let $(E, c) \in \mathscr{B}_0$ and $i \in N$. If for each $m \in \mathbb{N}$, $s_i(E^m, c^m, P_1) = \varphi_i(E^m, c^m) \neq CEL_i(E, c)$, then by Definition 5,

$$s_i(E^m, c^m, P_1) < CEL_i(E^m, c^m) = c_i^m - \mu = c_i - \sum_{k=1}^{m-1} s_i(E^k, c^k, P_1) - \mu,$$

so that

$$s_i(E^m, c^m, P_1) + \sum_{k=1}^{m-1} s_i(E^k, c^k, P_1) < c_i - \mu,$$

that is,

$$\sum_{k=1}^{m} s_i(E^k, c^k, P_1) < CEL_i(E, c).$$

Therefore

$$\lim_{n\to\infty} \sum_{k=1}^{m} s_i(E^k, c^k, P_1) \leq CEL_i(E, c)$$

■

The fourth lemma says that if at some step $m \in \mathbb{N}$, an agent's *P-rights* for P_1 is the amount provided by the *CEL* rule for the problem (E^m, c^m), then the total amount received by this agent up to that step is given by the *CEL* rule applied to the initial problem.

Lemma 4 *For each* $(E, c) \in \mathscr{B}_0$ *and each* $i \in N$, *if there is* $m^* \in \mathbb{N}$, $m^* > 1$, *such that* $s_i(E^{m^*}, c^{m^*}, P_1) = CEL_i(E^{m^*}, c^{m^*})$ *and* $s_i(E^{m^*-1}, c^{m^*-1}, P_1) = \varphi_i(E^{m^*-1}, c^{m^*-1}) \neq CEL_i(E^{m^*-1}, c^{m^*-1})$, *then*

$$\sum_{k=1}^{m^*} s_i(E^k, c^k, P_1) = CEL_i(E, c).$$

Proof Let $(E, c) \in \mathscr{B}_0$. We have

$$s_i(E^{m^*}, c^{m^*}, P_1) = CEL_i(E^{m^*}, c^{m^*}) \text{ and}$$
$$s_i(E^{m^*-1}, c^{m^*-1}, P_1) = \varphi_i(E^{m^*-1}, c^{m^*-1}) \neq CEL_i(E^{m^*-1}, c^{m^*-1}).$$

Since $\varphi_i(E^{m^*-1}, c^{m^*-1}) < CEL_i(E^{m^*-1}, c^{m^*-1})$, $CEL_i(E^{m^*-1}, c^{m^*-1}) > 0$. Therefore $c_i^{m^*-1} > \mu$ and by Lemma 1, $c_i^{m^*} \geq \mu$. Then, at step m^*, agent i has received

$$\begin{aligned}
\sum_{k=1}^{m^*} s_i(E^k, c^k, P_1) &= \sum_{k=1}^{m^*-1} s_i(E^k, c^k, P_1) + CEL_i(E^{m^*}, c^{m^*}) \\
&= \sum_{k=1}^{m^*-1} s_i(E^k, c^k, P_1) + \left[c_i^{m^*} - \min\left\{c_i^{m^*}, \mu\right\}\right] \\
&= \sum_{k=1}^{m^*-1} s_i(E^k, c^k, P_1) + \left[\left(c_i - \sum_{k=1}^{m^*-1} s_i(E^k, c^k, P_1)\right) - \min\left\{c_i^{m^*}, \mu\right\}\right] \\
&= c_i - \min\left\{c_i^{m^*}, \mu\right\} = c_i - \mu.
\end{aligned}$$

Therefore,

$$\sum_{k=1}^{m^*} s_i(E^k, c^k, P_1) = CEL_i(E,c).$$

■

The last lemma shows that the *P-rights* for agents 1 and n, when considering P_1, correspond to the *CEL* and *CEA* rules, respectively.

Lemma 5 *For each* $(E,c,P_1) \in \mathscr{B}_{\mathscr{P}}$, $s_1(E,c,P_1) = CEL_1(E,c)$ *and* $s_n(E,c,P_1) = CEA_n(E,c)$.

Proof First we show that $s_1(E,c,P_1) = CEL_1(E,c)$. Let (E,c,P_1) with $(E,c) \in \mathscr{B}_0$. We consider the two following cases:
- $CEL_1(E,c) = 0$. By *Non-Negativity*, $s_1(E,c,P_1) = CEL_1(E,c)$.
- $CEL_1(E,c) > 0$. By the definition of the *CEL* rule, $c_1 - CEL_1(E,c) = c_j - CEL_j(E,c)$ for each $j \neq 1$. Let us suppose that there is $\varphi \in \Phi(P_1)$ such that $\varphi_1(E,c) < CEL_1(E,c)$. By *Efficiency* for some $j \neq 1$ $\varphi_j(E,c) > CEL_j(E,c)$. Then $c_1 - \varphi_1(E,c) > c_j - \varphi_j(E,c)$, contradicting order preservation. Therefore, $s_1(E,c,P_1) = CEL_1(E,c)$.

Second, using a similar reasoning to the previous one it can be straightforwardly obtained that $s_n(E,c,P_1) = CEA_n(E,c)$.

■

Proof of Theorem 2.
Let $(E,c) \in \mathscr{B}_0$. There are two cases.

Case a: All claims are equal. Then, by definition of *P-rights* for P_1, each agent receives the same amount and the entire endowment is distributed at the first step. Therefore, $\varphi^R(E,c,P_1) = CEL(E,c)$.

Case b: There are at least two agents whose claims differ. By Lemma 5, $s_1(E,c,P_1) = CEL_1(E,c)$. Furthermore, by Lemmas 2 and 4, for each agent $r \in N$ who at some step $m \in \mathbb{N}$, receives $CEL_r(E^m, c^m)$ as *P-rights* for P_1, we have $\varphi_r^R(E,c,P_1) = CEL_r(E,c)$. Moreover, for each agent $l \neq r$, by Lemma 3, $\varphi_l^R(E,c,P_1) \leq CEL_l(E,c)$. Then, since $\varphi^R(E,c,P_1)$ exhausts the endowment, by Remark 5, $\varphi^R(E,c,P_1) = CEL(E,c)$.

■

APPENDIX 3 ***Proof of Theorem 3.***

Next we present a lemma and a fact, which the proof of Theorem 3 is based on. We assume throughout this Appendix, without loss of generality, that $(E,c) \in \mathscr{B}_0$.

The lemma shows that the *P-rights* for agents 1 and n, when considering P_2, correspond to the *DCE* and *CE* rules, respectively.

Lemma 6 *For each* $(E,c,P_2) \in \mathscr{B}_{\mathscr{P}}$, $s_1(E,c,P_2) = DCE_1(E,c)$ *and* $s_n(E,c,P_2) = CE_n(E,c)$.

Proof First we show that $s_1(E,c,P_2) = DCE_1(E,c)$. Given (E,c,P_2) with $(E,c) \in \mathscr{B}_0$, if $E = C/2$ by the midpoint property $s_1(E,c,P_2) = DCE_1(E,c)$. Next we consider the rest of the possibilities.
Case a: $E < C/2$. Let us consider the following subcases:
- $DCE_1(E,c) = 0$. By *Non-Negativity*, $s_1(E,c,P_2) = DCE_1(E,c)$.

• $DCE_1(E,c) > 0$ and $DCE_j(E,c) = c_j/2$ for each $j \neq 1$. Let us suppose that there is $\varphi \in \Phi(P_2)$ such that $\varphi_1(E,c) < DCE_1(E,c)$. By *Efficiency* for some $j \neq 1$, $\varphi_j(E,c) > c_j/2$. By the midpoint property, $\varphi(C/2,c) = c/2$, then $\varphi_j(E,c) > \varphi_j(C/2,c)$, contradicting resource monotonicity. Therefore $s_1(E,c,P_2) = DCE_1(E,c)$.
• $DCE_1(E,c) > 0$ and $DCE_j(E,c) \neq c_j/2$ for each $j \neq 1$. By the *DCE* rule definition, $c_1 - DCE_1(E,c) = c_j - DCE_j(E,c)$, for each $j \neq 1$. Let us suppose that there is $\varphi \in \Phi(P_2)$ such that $\varphi_1(E,c) < DCE_1(E,c)$. By *Efficiency* for some $j \neq 1$, $\varphi_j(E,c) > DCE_j(E,c)$. Then $c_1 - \varphi_1(E,c) > c_j - \varphi_j(E,c)$, contradicting order preservation. Therefore $s_1(E,c,P_2) = DCE_1(E,c)$.
• $DCE_1(E,c) > 0$ and there are S,T, $\varnothing \neq S \subset N\backslash\{1\}$ and $\varnothing \neq T \subset N\backslash\{1\}$ such that for each $l \in S$, $DCE_l(E,c) \neq c_l/2$, and for each $k \in T$, $DCE_k(E,c) = c_k/2$. By the *DCE* rule definition, $c_1 - DCE_1(E,c) = c_j - DCE_j(E,c)$, for each $j \neq 1$. Let us suppose that there is $\varphi \in \Phi(P_2)$ such that $\varphi_1(E,c) < DCE_1(E,c)$. By *Efficiency* for some $j \neq 1$, $\varphi_j(E,c) > DCE_j(E,c)$. Then, if $j \in S$, $c_1 - \varphi_1(E,c) > c_j - \varphi_j(E,c)$, contradicting order preservation. If $j \in T$, by the midpoint property, $\varphi(C/2,c) = c/2$, then $\varphi_j(E,c) > \varphi_j(C/2,c)$, contradicting resource monotonicity. Therefore $s_1(E,c,P_2) = DCE_1(E,c)$.
Case b: $E > C/2$. Let us consider the following subcases:
• $DCE_1(E,c) = c_1/2$. Let us suppose that there is $\varphi \in \Phi(P_2)$ such that $\varphi_1(E,c) < c_1/2$. By the midpoint property, $\varphi(C/2,c) = c/2$, then $\varphi_1(E,c) < \varphi_1(C/2,c)$, contradicting resource monotonicity. Therefore $s_1(E,c,P_2) = DCE_1(E,c)$.
• $DCE_1(E,c) > c_1/2$. By the *DCE* rule definition, $c_1 - DCE_1(E,c) = c_j - DCE_j(E,c)$, for each $j \in N\backslash\{1\}$. Let us suppose that there is $\varphi \in \Phi(P_2)$ such that $\varphi_1(E,c) < DCE_1(E,c)$. By *Efficiency* for some $j \neq 1$, $\varphi_j(E,c) > DCE_j(E,c)$, then $c_1 - \varphi_1(E,c) > c_j - \varphi_j(E,c)$, contradicting order preservation. Therefore $s_1(E,c,P_2) = DCE_1(E,c)$.

Second, using a similar reasoning to the previous one it can be straightforwardly obtain that $s_n(E,c,P_2) = CE_n(E,c)$.

■

The following fact provides two conditions that will be used in the proof of Theorem 3.

Fact 3 *Let $(E,c) \in \mathscr{B}_0$ a two-agent problem. By Lemma 6, at each step $m \in \mathbb{N}$, $s_1(E^m,c^m,P_2) = DCE_1(E^m,c^m)$. Therefore, next inequality characterizes the fact that agent 1 is guaranteed nothing at each step $m \in \mathbb{N}$*

$$s_1(E^m,c^m,P_2) = 0 \Leftrightarrow E^m \leq \min\{c_2^m - c_1^m, c_2^m/2\}. \tag{1}$$

Now, applying (1) for $m = 2$ and substituting, in terms of the problem at step $m-1$, the expressions of E^m and c_i^m for each $i \in N$, that is

$$E^m = E^{m-1} - s_1(E^{m-1},c^{m-1},P_2) - s_2(E^{m-1},c^{m-1},P_2)$$

and

$$c_i^m = c_i^{m-1} - s_i(E^{m-1},c^{m-1},P_2),$$

we have next inequality, that we call Condition 2

$$s_1(E^2,c^2,P_2) = 0 \Leftrightarrow E \leq \min\begin{cases} c_2 - c_1 + 2s_1(E,c,P_2), \\ c_2/2 + s_2(E,c,P_2)/2 + s_1(E,c,P_2) \end{cases} \tag{2}$$

Proof of Theorem 3
For each two-agent problem $(E,c) \in \mathscr{B}_0$, by Lemma 6 and *Efficiency,* at each step $m \in \mathbb{N}$, $s(E^m, c^m, P_2) = DCE(E^m, c^m)$. Given this, we show that agent 1's *P-rights* for P_2 at each step $m \geq 2$, is zero, so agent 1's *Recursive P-rights* rule for P_2 is the *Dual of Constrained Egalitarian* rule. Then, since $\varphi^R(E,c,P_2)$ exhausts the endowment, given Remark 5, $\varphi^R(E,c,P_2) = DCE(E,c)$.

If $c_1 = c_2$, by the Definition of the *Recursive P-rights* rule for P_2, each agent i receives the same amount at the initial step, and if $c_1 \neq c_2$, with $E = (c_1+c_2)/2$ by the midpoint property, each agent i receives her half-claim, $c_i/2$. Therefore, in both cases, at the initial step the endowment is exhausted, and $\varphi^R(E,c,P_2) = DCE(E,c)$.

When $c_1 \neq c_2$ there are three cases.

Case 1: $s_1(E,c,P_2) = 0$.
Then, by Condition 1 for $m = 1$, $E \leq \min\{c_2 - c_1, c_2/2\}$. Now, in the following step Condition 2 states that

$$s_1(E^2,c^2,P_2) = 0 \Leftrightarrow E \leq \min\{c_2 - c_1, c_2/2 + s_2(E,c,P_2)/2\},$$

which follows from

$$E \leq \min\{c_2 - c_1, c_2/2\}.$$

Thus, $s_1(E^2,c^2,P_2) = 0$. Applying these conditions for each step $m > 2$, we obtain that $s_1(E^m,c^m,P_2) = 0$. So, $\varphi_1^R(E,c,P_2) = 0$. Therefore, by Remark 3, $\varphi^R(E,c,P_2) = (0,E) = DCE(E,c)$.

In Cases 2 and 3, we will show that at $m = 2$ agent 1's *P-rights* for P_2 is zero. Case 1 can then be applied to the residual *Problem with Legitimate Principles*, so from $m = 2$ on, $s_1(E^{m+h}, c^{m+h}, P_2) = 0$, for each $h \in \mathbb{N}$, and $\varphi_1^R(E,c,P_2) = s_1(E,c,P_2)$.

Case 2: $s_1(E,c,P_2) > 0$, and $c_2/2 \geq c_2 - c_1$.

Case 2.1: $c_2 - c_1 \leq E \leq c_1$. Then, $s_1(E,c,P_2) = (E + c_1 - c_2)/2$ and $s_2(E,c,P_2) = E/2$. Now, substituting these expressions in Condition 2,

$$s_1(E^2,c^2,P_2) = 0 \Leftrightarrow E \leq 2c_1,$$

which is true, as in this region, $E \leq c_1$. Therefore,

$$\varphi^R(E,c,P_2) = ((E + c_1 - c_2)/2, (E - c_1 + c_2)/2) = DCE(E,c).$$

Case 2.2: $c_1 \leq E \leq (c_1+c_2)/2$. Then, $s_1(E,c,P_2) = E - c_2/2$ and $s_2(E,c,P_2) = E - c_1/2$. Now, substituting these expressions in Condition 2,

$$s_1(E^2,c^2,P_2) = 0 \Leftrightarrow E \geq c_1,$$

which is obviously fulfilled in this region. Therefore,

$$\varphi^R(E,c,P_2) = (E - c_2/2, c_2/2) = DCE(E,c).$$

Case 2.3: $(c_1+c_2)/2 \leq E \leq [(c_1+c_2)/2] + [(c_2-c_1)/2] = c_2$. Then, $s_1(E,c,P_2) = c_1/2$ and $s_2(E,c,P_2) = c_2/2$. Again by substituting these expressions in Condition 2,

$$s_1(E^2,c^2,P_2) = 0 \Leftrightarrow E \leq \min\{c_2, (3c_2/4) + c_1/2\}.$$

On the one hand $E \leq c_2$ is fulfilled since c_2 is the Estate-upper bound of this region. On the other hand, in Case 2 $c_2/2 \geq c_2 - c_1$ which implies $c_1/2 \geq c_2/4$ and $(3c_2/4)+(c_1/2) \geq c_2$ then, again by the Estate-upper bound of this region, $E \leq (3c_2/4)+(c_1/2)$ is true. Therefore,

$$\varphi^R(E,c,P_2) = (c_1/2, E - c_1/2) = DCE(E,c).$$

Case 2.4: $c_2 \leq E \leq 2c_1$. Then, $s_1(E,c,P_2) = (E+c_1-c_2)/2$ and $s_2(E,c,P_2) = E/2$. Now, substituting these expressions in Condition 2,

$$s_1(E^2,c^2,P_2) = 0 \Leftrightarrow E \leq 2c_1,$$

which is obviously fulfilled in this region. Therefore,

$$\varphi^R(E,c,P_2) = ((E+c_1-c_2)/2, (E-c_1+c_2)/2) = DCE(E,c).$$

Case 2.5: $2c_1 \leq E$. Then, $s_1(E,c,P_2) = (E+c_1-c_2)/2$ and $s_2(E,c,P_2) = E - c_1$. Here, the substitution of these expressions in Condition 2 does not imply any restriction, so that,

$$\varphi^R(E,c,P_2) = ((E+c_1-c_2)/2, (E-c_1+c_2)/2) = DCE(E,c).$$

Case 3: $s_1(E,c,P_2) > 0$, and $c_2/2 \leq c_2 - c_1$.
Case 3.1: $c_2/2 \leq E \leq (c_1+c_2)/2$. Then, $s_1(E,c,P_2) = E - c_2/2$ and $s_2(E,c,P_2) = E - c_1/2$. Now, substituting these expressions in Condition 2,

$$s_1(E^2,c^2,P_2) = 0 \Leftrightarrow E \geq c_1,$$

inequality fulfilled as in this region $c_2/2 \leq c_2 - c_1$, implying $c_1 \leq c_2/2$. Therefore,

$$\varphi^R(E,c,P_2) = (E - c_2/2, c_2/2) = DCE(E,c).$$

Case 3.2: $(c_1+c_2)/2 \leq E \leq c_1 + c_2/2$. Then $s_1(E,c,P_2) = c_1/2$ and $s_2(E,c,P_2) = c_2/2$. Now, substituting these expressions in Condition 2,

$$s_1(E^2,c^2,P_2) = 0 \Leftrightarrow E \leq \min\{c_2, (3c_2/4)+(c_1/2)\}.$$

Both inequalities $E \leq c_2$ and $E \leq 3c_2/4 + c_1/2$ are satisfied as in this region $c_2/2 \leq c_2 - c_1$, which implies $c_1 \leq c_2/2$. Therefore,

$$\varphi^R(E,c,P_2) = (c_1/2, E - c_1/2) = DCE(E,c).$$

Case 3.3: $c_1 + c_2/2 \leq E \leq c_2$. $s_1(E,c,P_2) = c_1/2$ and $s_2(E,c,P_2) = E - c_1$. Now, substituting these expressions in Condition 2

$$s_1(E^2,c^2,P_2) = 0 \Leftrightarrow E \leq c_2,$$

which is the Estate-upper bound in this region. Therefore,

$$\varphi^R(E,c,P_2) = (c_1/2, E - c_1/2) = DCE(E,c).$$

Case 3.4: $c_2 \leq E$. Then, $s_1(E,c,P_2) = (E+c_1-c_2)/2$ and $s_2(E,c,P_2) = E - c_1$. Here, the substitution of these expressions in Condition 2 does not imply any restriction, so that,

$$\varphi^R(E,c,P_2) = ((E+c_1-c_2)/2, (E-c_1+c_2)/2) = DCE(E,c).$$

■

APPENDIX 4 ***Proof of Proposition 1.***

Let us consider the problem $(E,c) = (21,(5,19\frac{1}{2},20)) \in \mathscr{B}$, and the following rule, φ^*:

Case a) For each (E,c) such that $c_3 - c_2 \leq \frac{3}{16}c_1$ and $c_3 - c_2 \leq c_2 - c_1$,

$$\varphi^*(E,c) = \begin{cases} (0,0,E) & \text{if } 0 \leq E \leq c_3 - c_2 \\ \left(\frac{E-(c_3-c_2)}{3}, \frac{E-(c_3-c_2)}{3}, \frac{E+2(c_3-c_2)}{3}\right) & \text{if } c_3 - c_2 \leq E \leq 6(c_3-c_2) \\ \left(\frac{E}{2} - \frac{4}{3}(c_3-c_2), \frac{E}{2} - \frac{4}{3}(c_3-c_2), \frac{8}{3}(c_3-c_2)\right) & \text{if } 6(c_3-c_2) \leq E \leq 8(c_3-c_2) \\ \left(\frac{E}{3}, \frac{E}{3}, \frac{E}{3}\right) & \text{if } 8(c_3-c_2) \leq E \leq \frac{3}{2}c_1 \\ \left(\frac{c_1}{2}, \frac{c_1}{2}, E - c_1\right) & \text{if } \frac{3}{2}c_1 \leq E \leq \frac{3}{2}c_1 + c_3 - c_2 \\ \left(\frac{c_1}{2}, \frac{E-(c_3-c_2)}{2} - \frac{c_1}{4}, \frac{E+(c_3-c_2)}{2} - \frac{c_1}{4}\right) & \text{if } \frac{3}{2}c_1 + c_3 - c_2 \leq E \leq \frac{c_1}{2} + c_2 \\ \left(\frac{c_1}{2}, E - \frac{c_1+c_3}{2}, \frac{c_3}{2}\right) & \text{if } \frac{c_1}{2} + c_2 \leq E \leq \frac{C}{2} \\ CE(E,c) & \text{if } E \geq \frac{C}{2} \end{cases}$$

Case b) Otherwise, $\varphi^*(E,c) \equiv CE(E,c)$

Note that, it is easy to check that, not only φ^* is an admissible rule for P_2, but also, for each of the following claims problems in which we apply it, φ^* recommends the smallest amount for agent 2 among all the admissible rules for P_2. By Lemma 6, we know that for each $(E,c,P_2) \in B_P$, $s_1(E,c,P_2) = DCE_1(E,c)$ and $s_3(E,c,P_2) = CE_3(E,c)$. Taking into account these facts, next we compute some steps of the *Recursive P_2-rights Process* for the previously defined claims problem.

Step $\mathbf{m=1}$: $(E^1,c^1) = (21,(5,19\frac{1}{2},20)$, $CE(E^1,c^1) = \left(\frac{10}{4}, \frac{37}{4}, \frac{37}{4}\right)$, $DCE(E^1,c^1) = \left(\frac{5}{4}, \frac{39}{4}, 10\right)$, and $\varphi^*(E^1,c^1) = \left(\frac{5}{2}, 9, \frac{19}{2}\right)$. Then,

$$s(E^1,c^1,P_2) = \left(\frac{5}{4}, 9, 9\frac{1}{4}\right).$$

Step $\mathbf{m=2}$: $(E^2,c^2) = \left(\frac{3}{2}, \left(\frac{15}{4}, 10\frac{1}{2}, 10\frac{3}{4}\right)\right)$, $CE(E^2,c^2) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $DCE(E^2,c^2) = \left(0, \frac{1}{8}, \frac{7}{8}\right)$, and $\varphi^*(E^2,c^2) = \left(\frac{5}{12}, \frac{5}{12}, \frac{2}{3}\right)$. Then,

$$s(E^2,c^2,P_2) = (0, \frac{5}{12}, \frac{1}{2}).$$

Step $\mathbf{m=3}$: $(E^3,c^3) = \left(\frac{7}{12}, \left(\frac{15}{4}, 10\frac{1}{12}, 10\frac{1}{4}\right)\right)$, $CE(E^3,c^3) = \left(\frac{7}{36}, \frac{7}{36}, \frac{7}{36}\right)$, $DCE(E^3,c^3) = \left(0, \frac{5}{24}, \frac{3}{8}\right)$, and, $\varphi^*(E^3,c^3) = \left(\frac{5}{36}, \frac{5}{36}, \frac{11}{36}\right)$. Then,

$$s(E^3,c^3,P_2) = \left(0, \frac{5}{36}, \frac{7}{36}\right).$$

Step m = 4 : $\left(E^4,c^4\right)=\left(\frac{1}{4},\left(\frac{15}{4},9\frac{17}{18},10\frac{1}{18}\right)\right)$, $CE(E^4,c^4)=(\frac{1}{12},\frac{1}{12},\frac{1}{12})$, $DCE(E^4,c^4)=(0,\frac{5}{72},\frac{13}{72})$, and $\varphi^*(E^4,c^4)=\left(\frac{5}{108},\frac{5}{108},\frac{17}{108}\right)$. Then,

$$s(E^4,c^4,P_2)=\left(0,\frac{5}{108},\frac{1}{12}\right).$$

Therefore,

$$\begin{aligned}\varphi^R(21,(5,19\frac{1}{2},20),P_2) &= \sum_{k=1}^{4} s(E^k,c^k,P_2)+\sum_{k=5}^{\infty} s(E^k,c^k,P_2) = \\ &= (\frac{5}{4},9\frac{65}{108},10\frac{1}{36})+\sum_{k=5}^{\infty} s(E^k,c^k,P_2).\end{aligned}$$

Now, let us consider the problem $(E',c)=(22\frac{1}{2},(5,19\frac{1}{2},20))$. By the midpoint property,

$$\varphi^R(22\frac{1}{2},(5,19\frac{1}{2},20),P_2)=(2\frac{1}{2},9\frac{3}{4},10).$$

Since by Definition 5 for each $m\in\mathbb{N}$ and each $i\in N$ $s_i(E^m,c^m,P_2)\geq 0$, the two previous distributions contradict resource monotonicity as the highest agent receives less when the endowment increases.

■