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A note on discrete claims problems

José M. Giménez-Gómez · **Cori Vilella**

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Abstract In this note, we consider claims problems with indivisible goods. Specifically, by applying recursively the P-rights lower bound (Giménez-Gómez and Marco-Gil (2008)), we ensure the fulfillment of weak order preservation, considered by many authors as a minimal requirement of fairness. Moreover, we lead to the discrete constrained equal losses and the discrete constrained equal awards rules (Herrero and Martínez (2008a)). Finally, by the recursive double imposition of a lower and an upper bound, we obtain the average between them.

Keywords Claims problems · Indivisibilities · Order Preservation · Constrained equal awards rule · Constrained equal loses rule · Midpoint

1 Introduction.

When a firm goes bankrupt, and the remaining capital is not enough to satisfy its demands, how should the resources be divided among its creditors? This important issue of claims problems acquires an special interest in the actual global financial crisis. In this model (O'Neill (1982)) the amount to allocate and the demands of the creditors are perfectly divisible and homogeneous. However, there are many real-world situations where the amount to divide and the claims are indivisible and identical units. Thus, any solution assigns to each agent an integer number of units of this good. As an example consider a university that offers a certain number of research fellowships and each department, depending on its research level, claims a certain number of fellowships. Suppose that the aggregate demand is greater than the total amount of fellowships to divide among the departments. How many fellowships should be assigned to each department? Similarly, consider the case of waiting lists for surgery at hospitals, demand of organs to be transplanted, airport slots demanded

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by airlines, food or medical assistance in war or disaster situations, visas to potential immigrants. In all these situations the data of the problem, as well as the allocations, are all integer numbers.

Situations like these are studied under the framework of the so called claims problem (see Moulin (2002) and Thomson (2003, 2006) for surveys).

Usually, in the case of indivisible claims problems priority (rationing) methods are applied (Moulin (2000), Herrero and Martínez (2008a,b) and Chen (2012)). In contrast, we propose to implement the recursive method studied by Giménez-Gómez and Marco-Gil (2008). Specifically, we require that any allowed proposals of distribution must satisfy weak order preservation which is a discrete version of two basic axioms in the continuous case: order preservation and equal treatment of equals. In this context, we define lower and upper bounds on awards by ensuring the smallest and the highest quantities each agent can receive from the set of rules satisfying weak order preservation, respectively. Note that it is quite realistic to impose bounds on awards. In the formal definition of a claims problem solution there are already a lower and an upper bound since it is required that no agent receives more than his claim and less than zero. Moreover, many other bounds on awards have been proposed (O'Neill (1982), Herrero and Villar (2001), Moulin (2002), Moreno-Ternero and Villar (2004) and Dominguez (forthcoming)).

Since, in general the aggregate guaranteed amount by means of our *P-rights* will not exhaust the endowment, we propose and analyze its recursive application. Once each agent receives her *P-rights*, the problem is revised accordingly. Then, the so called *Recursive P-rights Process* proposes the recursive application of the *P-rights* in each recursive revised problem. As in Giménez-Gómez and Marco-Gil (2008) we lead to the discrete constrained equal loses rule and the discrete constrained equal awards rule. These results are similar to those obtained by Herrero and Martínez (2008a). The main difference is that our process recover the allocations that fulfill weak order preservation, a requirement that is not satisfied by all the possible divisions recommended by the mentioned rules.

Finally, by combining both the lower and the upper bounds recursively we obtain the average of the discrete constrained equal loses rule and the discrete constrained equal awards rule (as in Giménez-Gómez and Peris (2011) for the divisible case). Consequently, we have a rule for discrete claims problems that is invariant to awards and looses. The combination of this rules in the continuous case is also studied in Thomson (2007).

The paper is organized as follows: Section 2 presents the model. Section 3 and 4 provides our approaches and results. Section 5 contains final remarks. Appendices gather technical proofs.

2 The model.

A **discrete claims problem** is a vector $(E,c) \in \mathbb{Z}_{++} \times \mathbb{Z}^n_+$, where $\mathbb{Z}$ represents the set of integer numbers, E denotes the endowment and c is the vector of each agents' claim, c_i, for each $i \in N$, $N = \{1,...,i,...,n\}$, such that the aggregate demand is greater than the endowment, $C = \sum_{i \in N} c_i \geq E$. Without loss of generality we assume claims are increasingly ordered. Let $\mathscr{B}_{\mathscr{D}}$ represent the set of discrete claims problems.

A **rule** is a function, φ, which associates for each $(E,c) \in \mathscr{B}_{\mathscr{D}}$, a distribution of the endowment among the claimants, that satisfies the properties of $\varphi_i(E,c) \geq 0$ (non-negativity), $\varphi_i(E,c) \leq c_i$ (claim-boundedness) and $\sum_{i \in N} \varphi_i(E,c) = E$ (efficiency).

Each claims problem can be faced from two points of view: those of awards and those of losses. Thus, we have two focal positions depending on whether we are focused on, awards or the unsatisfied demand. In the latter case, given a claims problem (E,c) we consider the **dual claims problem**, which is defined by the pair $(\sum_{i\in N} c_i - E, c) \in \mathbb{Z}_+ \times Z^n_+$.

Given a rule, φ, its dual Aumann and Maschler (1985)) distributes losses in the same way as φ divides the endowment. Formally, the **dual** rule of φ, denoted by φ^d, assigns the following distribution for each $(E,c) \in \mathscr{B}_{\mathscr{D}}$ and each $i \in N$, $\varphi^d_i(E,c) = c_i - \varphi_i(\sum_{i\in N} c_i - E, c)$.

The dual of a rule is a well-defined rule, if whenever $(E,c) \in \mathscr{B}_{\mathscr{D}}$, then $(\sum_{i\in N} c_i - E, c) \in \mathscr{B}_{\mathscr{D}}$. Because φ satisfies efficiency, non-negativity and claim-boundedness, the same conditions apply to φ^d. It is clear that $(\varphi^d)^d = \varphi$.

Next, we introduce the rules on which we will focus in this paper. First, we define the constrained equal awards and the constrained equal loses rules in the continuous case. Let $\mathscr{B}$ be the set of claims problems in which the resource is perfectly divisible.

The **constrained equal awards** rule, CEA, (Maimonides 12th Century, among others) recommends, for each $(E,c) \in \mathscr{B}$, the vector $(\min\{c_i,\mu\})_{i\in N}$, where μ is chosen so that $\sum_{i\in N} \min\{c_i,\mu\} = E$.

The **constrained equal losses** rule, CEL: for each $(E,c) \in \mathscr{B}$ and each $i \in N$, $CEL_i(E,c) \equiv \max\{0, c_i - \mu\}$, where μ is chosen so that $\sum_{i\in N} \max\{0, c_i - \mu\} = E$.

In order to introduce the CEA and the CEL in the discrete case (indivisible goods) hereinafter, we denote by CEA^z and CEL^z the integer part of the allocation proposed by CEA and CEL rules in the continuous case. Here is an illustration.

Example 1 Let $(E,c) = (9,(2,6,8))$. In this case, $CEA(E,c) = (2,3.5,3.5)$, and $CEL(E,c) = (0,3.5,5.5)$. Then, $CEA^z(E,c) = (2,3,3)$, and $CEL^z(E,c) = (0,3,5)$.

Herrero and Martínez (2004) propose that the discrete constrained equal awards rule, $DCEA$, recommends to each agent the integer part of the CEA, CEA^z, and the remained estate, $E' = E - \sum_{i\in N} CEA^z(E,c) > 0$, is distributed following a priority order σ among the agents whose assignment is not an integer. Let us denote this set of agents by $Q(CEA;E,c)$. According to the priority order we give one unit to each of the claimants in $Q(CEA;E,c)$ with the highest priority until what remains of the endowment is distributed. Formally,

Definition 1 The **discrete constrained equal awards** rule, $DCEA$: for each $(E,c) \in \mathscr{B}_{\mathscr{D}}$ and each $i \in N$, $DCEA_i(E,c) = CEA^z_i(E,c) + 1$ if i is in the list of the E' elements with highest priority order in $Q(CEA;E,c)$; or $CEA^z_i(E,c)$, otherwise.

Similarly for the discrete constrained equal loses rule, $DCEL$, let $E'' = E - \sum_{i\in N} CEL^z(E,c) > 0$ and $Q(CEL;E,c)$ be the set of agents whose assignment in CEL is not an integer. Again according to the priority order, we give one unit to each of claimants in $Q(CEL;E,c)$ with the highest priority until E'' is distributed.

Definition 2 The **discrete constrained equal losses** rule, $DCEL$: for each $(E,c) \in \mathscr{B}_{\mathscr{D}}$ and each $i \in N$, $DCEL_i(E,c) = CEL^z_i(E,c) + 1$ if i is in the list of E'' elements with highest priority order in $Q(CEL;E,c)$; or $CEL^z_i(E,c)$, otherwise.

Example 2 Following the above example, with priority order $\sigma_1 \geq \sigma_2 \geq \sigma_3$, $DCEA(E,c) = (2,4,3)$, and $DCEL(E,c) = (0,4,5)$.

Finally, we consider a society in which the distribution of the endowment is based on a set of basic properties or fair principles.[1] Note that the more properties are required by a society, the fewer the number of admissible rules. Next, we formally present such problems and the definitions of their associated rules, as introduced by Giménez-Gómez and Marco-Gil (2008).

Definition 3 A **discrete claims problem with legitimate principles** is a triplet (E,c,P), where $(E,c) \in \mathscr{B}_D$, and P is a set of properties upon which a particular society has agreed.

Henceforth, let $\mathscr{B}_{DP}$ be the set of all discrete claims problems with legitimate principles.

An admissible rule for a society that has agreed on P is a rule satisfying all these properties.

Definition 4 An **admissible** rule is a function, $\varphi : \mathscr{B}_D \rightarrow \mathbb{R}^n_+$ satisfying all properties in P. Let $\Phi(P)$ denote the set of admissible rules for P.

Given the set of admissible rules, we obtain the P-Rights (Giménez-Gómez and Marco-Gil (2008)) and the P-Utopia (Giménez-Gómez and Peris (2011)) as the minimal and the maximal awards for each agent given the legitimate principles (properties) upon which the society has agreed, respectively. Formally, this is determined as follows:

Definition 5 Given $(E,c,P) \in \mathscr{B}_{DP}$, the **discrete P-rights,** dpr, is for each $i \in N$, $dpr_i(E,c,P) = \min_{\varphi \in \Phi(P)} \{\varphi_i(E,c)\}$.

Definition 6 Given $(E,c,P) \in \mathscr{B}_{DP}$, the **discrete P-utopia,** dpu, is for each $i \in N$, $dpu_i(E,c,P) = \max_{\varphi \in \Phi(P)} \{\varphi_i(E,c)\}$.

Many authors agree that a minimal requirement of fairness is captured by the property of Order Preservation in the perfect divisible claims problems.In our context, let $P = \{WOP\}$ be the set whose only element is weak order preservation. This property requires that if agent i's claim is larger than agent j's claim, he should receive al least as much as agent j and agent i should also lose al least as much as agent j. Furthermore, if two agents have equal claims, then they should receive amounts that differ, at most, by one unit.

Weak order preservation, WOP**:** For each $(E,c) \in \mathscr{B}_{\mathscr{D}}$ and each pair $i, j \in N$:

(i) If $c_i > c_j$, then $\varphi_i(E,c) \geq \varphi_j(E,c)$ and $c_i - \varphi_i(E,c) \geq c_j - \varphi_j(E,c)$;
(ii) If $c_i = c_j$, then $\mid \varphi_i(E,c) - \varphi_j(E,c) \mid \leq 1$.

Note that in this extended framework, we do not allow those allocations that fail the agreed upon legitimate principles. For instance, in Example 2 depending on the priority order, we can obtain an allocation that gives more to lower claimants, so in that case, WOP is violated.

[1] With society, we mean the group of agents involved in each problem.

3 The recursive discrete P-rights rule.

In this section, we apply the approach introduced by Giménez-Gómez and Marco-Gil (2008) but in the context of discrete claims problems: the recursive discrete P-rights process. Specifically, in this process, at the first step each agent will receive her P- rights of the original problem. At the second step, the estate is what remains and the claims are adjusted down by the amounts just assigned. Then, each agent receives her P-rights in this residual problem, and so on.

Definition 7 For each $m \in \mathbb{N}$, at the m-th step, the **recursive discrete P-rights process**, RD^m, associates with each $(E,c,P) \in \mathscr{B}_{DP}$ and each $i \in N$, the amount

$$[RD^m(E,c,P)]_i = dpr_i(E^m,c^m,P),$$

$$\text{where } (E^1,c^1) \equiv (E,c) \text{ and for } m \geq 2,$$

$$(E^m,c^m) \equiv (E^{m-1} - \sum_{i \in N} dpr_i(E^{m-1},c^{m-1},P), c^{m-1} - dpr(E^{m-1},c^{m-1},P)).$$

This process is efficient whenever the discrete P-rights provides a positive value to some agent at each step[2]. Based on it, we define the next rule:

Definition 8 The **recursive discrete P-rights** rule, φ^{RD}, associates with each $(E,c,P) \in \mathscr{B}_{DP}$ and each $i \in N$, $\varphi_i^{RD}(E,c,P) = \sum_{m=1}^{\infty} [RD^m(E,c,P)]_i$.

Note that, since $P = \{WOP\}$, at each step among all the possible allocations proposed by any rule, we only consider the ones satisfying WOP. If that gives more than one possibility, there is more than one admissible allocation, then we select the one that respects the priority order.

Theorem 1 *For each $(E,c,P) \in \mathscr{B}_{DP}$, the recursive discrete P-rights rule is the discrete constrained equal losses rule, $\varphi^{RD}(E,c,P) = DCEL(E,c)$, satisfying WOP.*

Proof. See Appendix 1.

Since the CEL rule (Aumann and Maschler (1985)) is the dual of the CEA rule (Herrero and Villar (2001)), the next result is obtained in a straightforward way.

Corollary 1 *For each $(E,c,P) \in \mathscr{B}_{DP}$, $c - \varphi^{RD}(L,c,P) = DCEA(E,c)$, satisfying WOP.*

4 The double recursive discrete rule.

In this section, given the set of admissible rules, we apply the double imposition of the minimal and the maximal awards that each agent must receive according to P (Giménez-Gómez and Peris (2011)).

Definition 9 For each $m \in \mathbb{N}$ at the m-th step, the **double boundedness recursive discrete process**, $DBRD^m$, associates with each $(E,c,P) \in \mathscr{B}_{DP}$ and each $i \in N$,

[2] See Dominguez (forthcoming).

$$[DBRD(E^m,c^m,P)]_i = dpr_i(E^m,c^m,P),$$
$$\text{where } (E^1,c^1) \equiv (E,c) \text{ and for } m \geq 2,$$
$$E^m \equiv E^{m-1} - \sum_{i\in N} dpr_i(E^{m-1},c^{m-1},P)$$
$$c_i^m = dpu_i(E^{m-1},c^{m-1},P) - dpr_i(E^{m-1},c^{m-1},P).$$

Definition 10 The **double recursive discrete** rule, φ^{DRD}, associates with each $(E,c,P) \in \mathscr{B}_{DP}$ and each $i \in N$, $\varphi_i^{DRD}(E,c,P) = \sum_{m=1}^{\infty} [DBRD^m(E,c,P)]_i$.

Now, we assume that the society is concerned about equity. Then we use the well-known Lorenz (equity) criterion, which is considered a general equity principle (Dutta and Ray (1989), Arin (2007)), to delimit the set of admissible rules to those that are between the most and the least egalitarian ways of distributing the resources. Note that under $P = \{WOP\}$, these extreme rules are the CEA and CEL rules, respectively (Thomson (2010)). In this context, by focusing on awards and losses respectively, an admissible rule raises in a natural way as those ones included within the area between the CEA and CEL satisfying WOP.

Definition 11 An **admissible rule** is a function, $\varphi : \mathscr{B}_D \to \mathbb{R}^n_+$, satisfying all properties in P such that $\min\{DCEA_i(E,c), DCEL_i(E,c)\} \leq \varphi_i(E,c) \leq \max\{DCEA_i(E,c), DCEL_i(E,c)\}$.

Next result states that the recursive double imposition of the dpr and the dpu that satisfy WOP recommends the midpoint allocation between $DCEA$ and $DCEL$. The average of this rules in the continuous case is also studied in Thomson (2007).

Theorem 2 *For each $(E,c,P) \in \mathscr{B}_{DP}$, with $P = \{WOP\}$,*

$$\varphi^{DRD}(E,c,P) = \frac{DCEA(E,c) + DCEL(E,c)}{2}.$$

Proof. See Appendix 2.

5 Final remarks.

On one hand we have lead to the discrete constrained equal losses rule, when focusing on awards, and the discrete constrained equal awards rule, when focusing on losses. To do that we have applied in a recursive way the lower bound dpr obtained by the requirement of WOP.

On the other hand, by the double impositions of dpr and dpu, we have lead to the midpoint between $DCEA$ and $DCEL$ rules. This result, can be extended to any problem where two dual rules delimit the set of admissible allocations, as shown by Giménez-Gómez and Peris (2011) for perfectly divisible claims problems.

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Appendix 1: Proof of Theorem 1.

Remark 1 For each $(E,c,P) \in \mathscr{B}_{DP}$ and each $m \in \mathbb{N}$, $L^m = L$.

Proof. Let $(E,c,P) \in \mathscr{B}_{DP}$. Then,

$$L^m = C^m - E^m = \sum_{i\in N}\left(c_i - \sum_{k=1}^{m} dpr_i(E^k,c^k,P)\right) - \left(E - \sum_{i\in N}\sum_{k=1}^{m} dpr_i(E^k,c^k,P)\right) = C - E = L.$$

■

First fact establishes that for P, the sum of the awards given by the recursive discrete P-rights rule is the entire estate.

Fact 1 *For each* $(E,c,P) \in \mathscr{B}_{DP}$, $\sum_{i\in N}\left(\sum_{m=1}^{\infty}[RD^m(E,c,P)]_i\right) = E$.

The second fact gives a useful way to compute the CEL rule.

Fact 2 *For each* $(E,c) \in \mathscr{B}$ *and each* $i \in N$, *the loss imposed on agent* i *by CEL is* $CEL_i(E,c) = c_i - \gamma_i$, *where* $\gamma_i = \min\{c_i, \alpha_i\}$ *and* $\alpha_i = (L - \sum_{j<i}\gamma_j)/(n-i+1)$.

Fact 3 *By Fact 2 and Remark 1 we get:*

(a) For each $(E,c) \in \mathscr{B}$ *and each* $i \in N$, *if* $\gamma_i = c_i$, *then for each* $j < i$, $\gamma_j = c_j$.

(b) For each $(E,c) \in \mathscr{B}$ *and each* $i \in N$, *if* $\gamma_i = \alpha_i$, *then* $\alpha_i = \mu$, *and for each* $j > i$, $\alpha_j = \alpha_i$. *Therefore* $\gamma_i = \mu$.

(c) At each $m \in \mathbb{N}$ *and for each* $i \in N$, α_i^m *only depends on the initial problem,* (E,c), *and on agent* j'*s claim, for each* $j < i$

Here, we denote by $(c_i - \min\{c_i,\mu\})^z$ the integer part of $(c_i - \min\{c_i,\mu\})$. Then the integer part of the award is the following:

Lemma 1 *For each* $(E,c) \in \mathscr{B}_D$ *and each* $i \in N$,

$$CEL_i^z(E,c) = c_i - \min\{c_i,\tilde{\mu}\} \quad \text{where} \quad \tilde{\mu} = \begin{cases} \mu, & \text{if } \mu \in \mathbb{Z}; \\ \mu^z + 1, & \text{if } \mu \notin \mathbb{Z}. \end{cases}$$

Proof. Let $(E,c) \in \mathscr{B}_D$ and $i \in N$. By definition $CEL_i^z(E,c) = (c_i - \min\{c_i,\mu\})^z$. We distinguish two cases:

Case 1: $\min\{c_i,\mu\} = c_i$. Since $\tilde{\mu} \geq \mu$ then, $CEL_i^z(E,c) = (c_i - \min\{c_i,\mu\})^z = c_i - \min\{c_i,\tilde{\mu}\} = 0$.

Case 2: $\min\{c_i,\mu\} = \mu$. We have two possibilities:
2.1 If $\mu \in \mathbb{Z}$, then $\tilde{\mu} = \mu \in \mathbb{Z}$ and $CEL_i^z(E,c) = (c_i - \min\{c_i,\mu\})^z = (c_i-\mu)^z = c_i - \mu = c_i - \tilde{\mu} = c_i - \min\{c_i,\tilde{\mu}\}$.
2.2 If $\mu \notin \mathbb{Z}$, then $\tilde{\mu} = \mu^z + 1 > \mu$ and $CEL_i^z(E,c) = (c_i - \min\{c_i,\mu\})^z = (c_i-\mu)^z$. Since $\mu \notin \mathbb{Z}$ we have $\mu^z < \mu < \mu^z + 1$ and $\mu = \mu^z + \xi$ where $\xi \in (0,1)$. Thus $c_i - (\mu^z+1) < c_i - \mu < c_i - \mu^z$ and $(c_i-\mu)^z = (c_i - (\mu^z+\xi))^z = c_i - (\mu^z+1) = c_i - \tilde{\mu}$. Moreover, $c_i > \mu$ and $\mu \notin \mathbb{Z}$ so, we have $c_i \geq \mu^z + 1 = \tilde{\mu}$. Thus, $c_i - (\mu^z+1) = c_i - \tilde{\mu} = c_i - \min\{c_i,\tilde{\mu}\}$. Therefore, $CEL_i^z(E,c) = c_i - \min\{c_i,\tilde{\mu}\}$.

■

As a consequence of the above lemma we have the discrete constrained equal losses rule.

Remark 2 For each $(E,c) \in \mathscr{B}_D$ and each $i \in N$,

$$DCEL_i(E,c) = c_i - \min\{c_i, \bar{\mu}\}$$

where $\bar{\mu} = \begin{cases} \tilde{\mu} - 1, & \text{if } i \text{ is one of the } E'' \text{ elements with highest priority order in } Q(CEL;E,c); \\ \tilde{\mu}, & \text{otherwise.} \end{cases}$

Lemma 2 *For each (E,c,P) with $(E,c) \in \mathscr{B}_D$ and each $m \in \mathbb{N}$, $\bar{\mu}^{m+1} = \bar{\mu}^m$ where $\bar{\mu}^m$ solve $\sum_{i\in N} CEL_i^z(E^m,c^m) = E^m$ and $\bar{\mu}^{m+1}$ solves $\sum_{i\in N} CEL_i^z(E^{m+1},c^{m+1}) = E^{m+1}$.*

Proof. By Lemma 1, Giménez-Gómez and Marco-Gil (2008) and Remark 2. ∎

From now on, μ and $\bar{\mu}$ denote μ^m and $\bar{\mu}^m$ respectively, for each $m \in \mathbb{N}$.

Lemma 3 *For each $(E,c,P) \in \mathscr{B}_{DP}$, if there is $m \in \mathbb{N}$ such that $dpr_i(E^m,c^m,P) = DCEL_i(E^m,c^m)$. Then, for each $h \in \mathbb{N}$, $dpr_i(E^{m+h},c^{m+h},P) = 0$.*

Proof. We show that if $dpr_i(E^m,c^m,P) = DCEL_i(E^m,c^m)$ then $dpr_i(E^{m+1},c^{m+1},P) = DCEL_i(E^{m+1},c^{m+1}) = 0$.
Let $(E,c) \in \mathscr{B}_D$ and $m \in \mathbb{N}$, be such that $dpr_i(E^m,c^m,P) = DCEL_i(E^m,c^m) = c_i^m - \min\{c_i^m, \bar{\mu}\}$. Then, ${c_i}^{m+1} = {c_i}^m - DCEL_i(E^m,c^m) = {c_i}^m - (c_i^m - \min\{c_i^m,\bar{\mu}\}) = \min\{c_i^m,\bar{\mu}\}$. Thus
$DCEL_i(E^{m+1},c^{m+1}) = c_i^{m+1} - \min\{c_i^{m+1},\bar{\mu}\} = 0$. By Fact 2, if $DCEL_i(E^{m+1},c^{m+1}) = 0$ then, for each $h \in \mathbb{N}$, $dpr_i(E^{m+1},c^{m+1},P) = DCEL_i(E^{m+h},c^{m+h}) = 0$.

Lemma 4 *For each $(E,c,P) \in \mathscr{B}_{DP}$, and each $i \in N$. If for all $m \in \mathbb{N}$, $dpr_i(E^m,c^m,P) = \varphi_i(E^m,c^m) \neq DCEL_i(E,c)$, then*

$$\varphi_i^{RD}(E,c,P) = \sum_{k=1}^{\infty} dpr_i(E^k,c^k,P) \leq DCEL_i(E,c).$$

Proof. Let $(E,c,P) \in \mathscr{B}_{DP}$ and $i \in N$. Suppose that for each $m \in \mathbb{N}$, $dpr_i(E^m,c^m,P) = \varphi_i(E^m,c^m) \neq DCEL_i(E,c)$. By Lemma 1, for each $m \in \mathbb{N}$, $DCEL_i(E^m,c^m) = c_i^m - \min\{c_i^m,\bar{\mu}\}$ and by Definition 5, $dpr_i(E^m,c^m,P) \leq DCEL_i(E^m,c^m) = (c_i^m - \mu)^z = c_i^m - \bar{\mu} = c_i - \sum_{k=1}^{m-1} dpr_i(E^k,c^k,P) - \bar{\mu}$. Thus, $dpr_i(E^m,c^m,P) + \sum_{k=1}^{m-1} dpr_i(E^k,c^k,P) \leq c_i - \bar{\mu} = DCEL_i(E,c)$, that is $\sum_{k=1}^{m} dpr_i(E^k,c^k,P) \leq DCEL_i(E,c)$. Therefore, $\lim_{m\to\infty} \sum_{k=1}^{m} dpr_i(E^k,c^k,P) \leq DCEL_i(E,c)$. ∎

Lemma 5 *For each $(E,c) \in \mathscr{B}_D$ and each $i \in N$, if there is $m^* \in \mathbb{N}$, $m^* > 1$, such that $dpr_i(E^{m^*},c^{m^*},P) = DCEL_i(E^{m^*},c^{m^*})$ and $dpr_i(E^{m^*-1},c^{m^*-1},P) = \varphi_i(E^{m^*-1},c^{m^*-1}) \neq DCEL_i(E^{m^*-1},c^{m^*-1})$, then*

$$\sum_{k=1}^{m^*} dpr_i(E^k,c^k,P) = DCEL_i(E,c).$$

Proof. Let $(E,c) \in \mathscr{B}_D$ and $m^* \in \mathbb{N}$, $m^* > 1$ be such that $dpr_i(E^{m^*}, c^{m^*}, P) = DCEL_i(E^{m^*}, c^{m^*})$ and $dpr_i(E^{m^*-1}, c^{m^*-1}, P) = \varphi_i(E^{m^*-1}, c^{m^*-1}) \neq DCEL_i(E^{m^*-1}, c^{m^*-1})$. Since $\varphi_i(E^{m^*-1}, c^{m^*-1}) < DCEL_i(E^{m^*-1}, c^{m^*-1})$, $DCEL_i(E^{m^*-1}, c^{m^*-1}) > 0$.
By Lemma 1 $DCEL_i(E^{m^*}, c^{m^*}) = c_i^{m^*} - \min\{c_i^{m^*}, \bar{\mu}\}$. Since $DCEL_i(E^{m^*-1}, c^{m^*-1}) > 0$, then $c_i^{m^*-1} > \bar{\mu}$. By Lemma 2, $c_i^{m^*} \geq \bar{\mu}$. Then, at step m^*, agent i has received

$$\begin{aligned}
\sum_{k=1}^{m^*} dpr_i(E^k, c^k, P) &= \sum_{k=1}^{m^*-1} dpr_i(E^k, c^k, P) + DCEL_i(E^{m^*}, c^{m^*}) \\
&= \sum_{k=1}^{m^*-1} dpr_i(E^k, c^k, P) + (c_i^{m^*} - \min\left\{c_i^{m^*}, \bar{\mu}\right\}) \\
&= \sum_{k=1}^{m^*-1} dpr_i(E^k, c^k, P) + c_i - \sum_{k=1}^{m^*-1} dpr_i(E^k, c^k, P) - \min\left\{c_i^{m^*}, \bar{\mu}\right\} \\
&= c_i - \min\left\{c_i^{m^*}, \bar{\mu}\right\} = c_i - \bar{\mu} = DCEL_i(E,c).
\end{aligned}$$

■

Lemma 6 *For each* $(E,c,P) \in B_{DP}$, $dpr_1(E,c,P) = DCEL_1(E,c)$ *and* $dpr_n(E,c,P) = DCEA_n(E,c)$.

Proof. First we show that $dpr_1(E,c,P) = DCEL_1(E,c)$. For each problem (E,c,P) with $(E,c) \in \mathscr{B}_D$, consider the two following cases:

- $DCEL_1(E,c) = 0$. By non-negativity, $dpr_1(E,c,P) = DCEL_1(E,c)$.
- $DCEL_1(E,c) > 0$. By the $DCEL$ rule definition and WOP, $c_1 - DCEL_1(E,c) \leq c_j - DCEL_j(E,c)$ for each $j \neq 1$. Let us suppose that there is $\varphi \in \Phi(P)$ such that $\varphi_1(E,c) < DCEL_1(E,c)$. By efficiency for some $j \neq 1$ $\varphi_j(E,c) > DCEL_j(E,c)$. Then $c_1 - \varphi_1(E,c) > c_j - \varphi_j(E,c)$, contradicting WOP. Therefore, $dpr_1(E,c,P) = DCEL_1(E,c)$.

Second, by using a similar reasoning to the previous one it is straightforward that $dpr_n(E,c,P) = DCEA_n(E,c)$.

■

Proof of Theorem 1.
Let $(E,c) \in \mathscr{B}_D$. There are two cases.

Case a: All agents claim the same amount. Then, by definition of discrete P-rights for P, $\varphi_i(E,c) - \varphi_j(E,c) \leq 1$ and the entire estate is distributed at the first step. Therefore, $\varphi^{RD}(E,c,P) = DCEL(E,c)$.

Case b: There are at least two agents whose claims differ. By Lemma 6, $dpr_1(E,c,P) = DCEL_1(E,c)$. Furthermore, by Lemmas 3 and 5, for each agent $r \in N$ who at some step $m \in \mathbb{N}$, receives $DCEL_r(E^m, c^m)$ as dpr for P, we have $\varphi_r^{RD}(E,c,P) = DCEL_r(E,c)$. Moreover, for each agent $l \neq r$, by Lemma 4, $\varphi_l^{RD}(E,c,P) \leq DCEL_l(E,c)$. By Remark 1 $\varphi^{RD}(E,c,P)$ exhausts the estate, then $\varphi^{RD}(E,c,P) = DCEL(E,c)$.

■

Appendix 2: Proof of Theorem 2.

This proof is based on a fact, two lemmas, and a remark. Notice that it is also based on that any admissible rule under WOP is between the $DCEA$ and the $DCEL$ rules.

The next fact says that the dpr and the dpu are dual to each other.

Fact 4 *For each* $(E,c,P) \in \mathscr{B}_{DP}$*,* $dpr(E,c,P) = c - dpu(L,c,P)$.

The following lemma shows that at any step $m \in \mathbb{N}$, $m > 1$, the sum of the dpr and the dpu coincides with the sum of the claims.

Lemma 7 *For each* $(E,c,P) \in \mathscr{B}_{DP}$*, and* $m \in \mathbb{N}, m > 1$*,*

$$\sum_{i \in N} [dpr_i(E^m,c^m,P) + dpu_i(E^m,c^m,P)] = C^m.$$

Proof. Let $(E,c,P) \in \mathscr{B}_{DP}$, and $m \in \mathbb{N}, m > 1$. Then, $dpr_i(E,c,P) = \min\{DCEA_i(E,c,), DCEL_i(E,c)\}$, and $dpu_i(E,c,P) = \max\{DCEA_i(E,c), DCEL_i(E,c)\}$. By Fact 4, the next expression comes straightforwardly.

$$\sum_{i \in N} \left[\frac{dpu_i(E^m,c^m,P) + dpr_i(E^m,c^m,P)}{2}\right] = E^m.$$

Finally,

$$\begin{aligned}
E^m &= E^{m-1} - \sum_{i \in N} dpr_i(E^{m-1},c^{m-1},P) = \\
&= \sum_{i \in N} \left[\frac{dpu_i(E^{m-1},c^{m-1},P) + dpr_i(E^{m-1},c^{m-1},P)}{2}\right] - \\
&- \sum_{i \in N} dpr_i(E^{m-1},c^{m-1},P) = \\
&= \sum_{i \in N} \left[\frac{dpu_i(E^{m-1},c^{m-1},P) - dpr_i(E^{m-1},c^{m-1},P)}{2}\right] = C^m/2,
\end{aligned}$$

by the definition of the double boundedness recursive discrete process.

■

The following remark is a direct consequence of Lemma 7.

Remark 3 For each $(E,c,P) \in \mathscr{B}_{DP}$, and $m \in \mathbb{N}, m > 1$, $E^m = L^m = C^m/2$.

Proof. Let each $(E,c,P) \in \mathscr{B}_{DP}$, and $m \in \mathbb{N}$, $m > 1$. We know that, $L^m = C^m - E^m$. By Lemma 7, $E^m = C^m/2$. Therefore, $L^m = C^m - C^m/2 = C^m/2$.

■

Finally, the next lemma says that each agent's claim at each step different for step 1 coincides with the sum of dpr and dpu.

Lemma 8 *For each* $(E,c,P) \in \mathscr{B}_{DP}$*, such that* $m \in \mathbb{N}, m > 1$*,*

$$c_i^m = dpu_i(E^m,c^m,P) + dpr_i(E^m,c^m,P).$$

Proof. Let $(E,c,P) \in \mathscr{B}_{DP}$, and $m \in \mathbb{N}$, $m > 1$. By Remark 3 $L^m = E^m$, so $dpr_i(E^m,c^m,P) = dpr_i(L^m,c^m,P)$. By duality, $dpu_i(E^m,c^m,P) = c_i^m - dpr_i(L^m,c^m,P) = c_i^m - dpr_i(E^m,c^m,P)$, then, $c_i^m = dpu_i(E^m,c^m,P) + dpr_i(E^m,c^m,P)$.

■

Proof of Theorem 2.

Let $(E,c,P) \in \mathscr{B}_{DP}$,. For each $m \in \mathbb{N}$,

$$\varphi_i^{DRD}(E,c,P) = dpr_i(E,c,P) + \sum_{m=2}^{\infty} dpr_i(E^m,c^m,P).$$

By the definition of the double boundedness recursive discrete process,

$$\sum_{m=2}^{\infty} c_i^m = \sum_{m=2}^{\infty} \left[dpu_i(E^{m-1},c^{m-1},P) - dpr_i(E^{m-1},c^{m-1},P)\right] =$$

$$= dpu_i(E^m,c^m,P) + \sum_{m=2}^{\infty} dpu_i(E^m,c^m,P) - dpr_i(E^m,c^m,P) -$$

$$\sum_{m=2}^{\infty} dpr_i(E^m,c^m,P).$$

By Lemma 8, $\sum_{m=2}^{\infty} c_i^m = \sum_{m=2}^{\infty} [dpu_i(E^m,c^m,P) + dpr_i(E^m,c^m,P)]$. So,
$dpu_i(E,c,P) + \sum_{m=2}^{\infty} dpu_i(E^m,c^m,P) - dpr_i(E,c,P) - \sum_{m=2}^{\infty} dpr_i(E^m,c^m,P) = \sum_{m=2}^{\infty} [dpu_i(E^m,c^m,P) + dpr_i(E^m,c^m,P)]$. Thus,
$\sum_{m=2}^{\infty} dpr_i(E^m,c^m,P) = (dpu_i(E,c,P) - dpr_i(E,c,P))/2$. Therefore,

$$\varphi_i^{DRD}(E,c,P) = dpr_i(E,c,P) + \frac{dpu_i(E,c,P) - dpr_i(E,c,P)}{2} =$$

$$\frac{DCEA_i(E,c) + DCEL_i(E,c)}{2}.$$

■

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