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Conflicting claims problem associated with cost sharing of a network

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Abstract

A *minimum cost spanning tree* (*mcst*) problem analyzes the way to efficiently connect individuals to a source when they are located at different places. Once the efficient tree is obtained, the question on how allocating the total cost among the involved agents defines, in a natural way, a conflicting claims situation. For instance, we may consider the *endowment* as the *total cost of the network*, whereas for each individual her claim is the maximum amount she will be allocated, that is, her connection cost to the source. Obviously, we have a conflicting claims problem, so we can apply *claims rules* in order to obtain an allocation of the total cost. Nevertheless, the allocation obtained by using claims rules might not satisfy some appealing properties (in particular, it does not belong to the *core of the associated cooperative game*). We will define other natural claims problems that appear if we analyze the maximum and minimum amount that an individual should pay in order to support the minimum cost tree.

Keywords: Minimum cost spanning tree problem, Claims problem, Core
JEL classification: C71, D63, D71.

1. Introduction

We consider a situation in which some individuals located at different places want to be connected to a source in order to obtain a good or service.

There are some fixed costs of linking any two individuals, and of linking each individual to the source. Moreover, individuals do not mind being connected directly to the source, or indirectly through other individuals. There are several methods in order to obtain a way of connecting agents to the source so that *the total cost of the selected network is minimum* (we can use, for instance, Prim's algorithm). This situation is known as the *minimum cost spanning tree* problem and it is used to analyze different real-life issues, from telephone and cable TV to water supply networks.

The remaining important question is how this minimum cost should be allocated among the involved individuals. There is an extensive literature on this issue and several solutions have been proposed: *Bird (Bird, 1976), Kar (Kar, 2002), Folk (Bergantiños and Vidal-Puga, 2007), Cycle-complete (Trudeau, 2012), ...*

Our starting point to analyze the network cost sharing problem is by considering a claims problem associated to it:

> The *total minimal cost of the network*, C_m, has to be distributed among the n individuals involved in this network. The *cost of no cooperation* entails that each individual must pay the cost r_i of her connection to the source, so that the total amount to be paid in this case, $R = \sum r_i$, is greater or equal than C_m. Then, the pair (E, r):
>
> $$E = C_m, \quad r = (r_1, r_2, \dots, r_n),$$
>
> clearly defines a conflicting claims problem.[1] The total amount to be distributed, C_m, is the *endowment*, whereas the individual cost r_i is the claim.

By applying any of the well known claims rules[2] (*Proportional, (constrained) Egalitarian, Talmudian, Random Arrival, ...*) we obtain an allocation that efficiently allocates the cost C_m. Moreover, one of the conditions a claims rule must fulfill is *claim boundedness: no individual receives more than her claim*. This condition has an immediate and natural interpretation in our

[1] There is a difference with respect to the *classical* conflicting claims problem: in the usual setting agents *receive* a non-negative amount that does not exceed their claims, whereas in our approach agents *are charged* a non-negative amount lower than their individual cost. The formal problem has no difference.

[2] See, for instance, Thomson (2003).

minimal cost spanning tree context: *no individual should be allocated an amount greater that her individual cost to the source.*

Nevertheless, it is easy to obtain examples in which other *natural* condition in the network context fails: *the amount each individual is allocated should be greater than her minimum connection cost.*[3] To solve this drawback, we propose two different approaches:

1. Each individual pays initially the cost of no cooperation, that is her direct cost to the source. As cooperation entails a common profit, this profit is shared accordingly to a claims rule.
2. Each individual is allocated her minimum cost (the cost of her cheapest connection). The sum of these costs is lower than C_m and we propose the difference to be distributed accordingly to a claims rule.

In both cases, we need to define the appropriate conflicting claims problem. We will argue that these approaches are *dual* each other (in *claims' literature* terms) in the sense that the solution provided in one model by using a particular claims rule φ coincides with the solution provided in the other model by using the dual rule φ^d.

In this paper we set a bridge between the literature on conflicting claims problems and that of sharing the cost in network problems. Apart from providing new solutions to minimum cost spanning tree problems by using claims rules, we analyze the cooperative games associated to each problem.

2. Preliminaries

2.1. ***Minimum cost spanning tree problem***

A *minimum cost spanning tree* (hereafter *mcst*) problem involves a finite set of *agents*, $N = \{1, 2, \ldots, n\}$, who need to be connected to a *source* ω. We denote by $N_\omega = N \cup \{\omega\}$. The agents are connected by edges and for $i \neq j$, $c_{ij} \in \mathbb{R}_+$ represents the cost of the edge e_{ij} connecting agents $i, j \in N$. Following the notation in Kar (2002), c_{ii} represents the cost of connecting directly agent i to the source, for all $i \in N$. We denote by $\mathbf{C} = [c_{ij}]_{n\times n}$ the

[3] This condition is meaningful whenever the *non-property rights* approach is considered. In such a case, the proposed allocation does not belong to the core of the monotone irreducible cooperative game associated to the minimum cost spanning tree problem.

$n \times n$ symmetric cost matrix. The *mcst* problem is represented by the pair $(N_\omega, \mathbf{C})$. We denote by $\mathcal{N}$ the set of all *mcst* problems.

A *spanning tree* over N_ω is a non-oriented graph p with no cycles that connects all elements of N_ω. We can identify a spanning tree with a function $p : N \to N_\omega$ so that $p(i)$ is the agent (or the source) whom i connects, and defines the edges $e^p_{ij} = (i, p(i))$. In a spanning tree each agent is (directly or indirectly) connected to the source ω; that is, for all $i \in N$ there is some $t \in \mathbb{N}$ such that $(p \circ \ ..^{(t}. \circ p)(i) = p^t(i) = \omega$. Moreover, given the spanning tree p, there is a unique path from any i to the source for all $i \in N$, given by the edges $(i, p(i)), (p(i), p^2(i)), \ldots, (p^{t-1}(i), p^t(i) = \omega)$. The cost of building the spanning tree p is the total cost of the edges in this tree; that is,

$$C_p = \sum_{i=1}^{n} c_{ip(i)}$$

Prim (1957) provides an algorithm which solves the problem of connecting all agents to the source such that *the total cost of the network is minimum.*[4] The achieved solution, the *minimum cost spanning tree*, may not be unique. Denote by m a tree with minimum cost and by C_m its cost. That is,

$$C_m = \sum_{i=1}^{n} c_{im(i)} \leq C_p = \sum_{i=1}^{n} c_{ip(i)} \qquad \text{for all spanning tree } p$$

Once the *minimum cost spanning tree* is constructed, the important issue is how to allocate the associated cost C_m among the agents.[5]

A *sharing rule* is a function that proposes for any *mcst* problem $(N_\omega, \mathbf{C})$

[4] This algorithm has n steps. First, we select the agent with smallest cost to the source, i such that $c_{ii} \leq c_{jj}$, for all $j \in N$. In the second step, we select an agent in $N \setminus \{i\}$ with the smallest cost either directly to the source or to agent i, who is already connected. We continue until all agents are connected, at each step connecting an agent still not connected to a connected agent or to the source.

[5] Real word situations reveal that agents don't necessarily agree on how to distribute this cost and then the social optimum is not implemented, so a more expensive than necessary tree connects the agents to the source (see Bergantiños and Lorenzo (2004) for an example; see also Hernández et al. (2012) for a discussion about individual and social optimality).

an allocation[6] $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n_+$, such that

$$\sum_{i=1}^{n} \alpha_i = C_m.$$

Some sharing rules that can be applied in *mcst* problems are just adaptations of well known rules in general cost sharing problems. For instance the *proportional* solution, with respect to the stand alone cost c_{ii} (see, for instance, Kar (2002)) proposes the cost allocation

$$\alpha_i = \frac{c_{ii}}{\sum_{k=1}^{n} c_{kk}} C_m.$$

Remark 1. *It is important to note that this proportional solution is obtained with respect to the agents' costs of direct connection to the source, and the cost of the other edges are not taken into account.*

A different example of solution is the *egalitarian* one, in which the total cost C_m is equally divided among the agents,

$$\alpha_i = \frac{C_m}{n}.$$

This proposal may imply that some of the agents can be charged a cost greater than her direct cost to the source, c_{ii}, so this solution may fail to fulfill *individual rationality* (then, we should use a *constrained egalitarian* solution).

Many solutions have been defined in the *mcst* literature: for instance Bird (1976), Kar (2002), *Folk* (Bergantiños and Vidal-Puga, 2007), or *Cycle-complete* (Trudeau, 2012) solutions could be mentioned. Some of these solutions are defined as the Shapley value of a cooperative game obtained from the cost matrix in a *mcst* problem.[7]

In order to introduce a cooperative game associated to a *mcst* problem, we may find two different approaches, the dividing point being the existence or not of agents' *property rights* on their locations. The important question is:

[6] In some contexts the non-negativity condition is not required. We do not follow this approach. See the comments later about *property* and *non-property rights*.

[7] In order to facilitate the reading of the paper, definitions of some of these solutions are included in Appendix 1.

Given a coalition $S \subseteq N$, in order to obtain the minimal tree connecting the agents in S to the source ω, the agents outside S, can be used or not?

If the answer is not, then we are dealing with the *property rights approach* and, in the final cost sharing problem some allocations may be negative. This is the case of Kar solution (Kar, 2002).

We consider the second case, *the non-property rights approach*, so we allow agents to use other agent's locations. Then, the corresponding associated cooperative game is monotonic (it cannot be less expensive to add agents to a coalition) and therefore, *cost shares should be non-negative*. In this *non-property rights approach* the cooperative game is defined in the following way:

Given a coalition $S \subseteq N$,

$$v(S) = min\left\{C_m(T):\ S \subseteq T \subseteq N\right\}$$

where $C_m(T)$ is the minimal cost in the problem $(T_\omega, \mathbf{C}|_T)$.

The *core* associated to a *mcst* problem is then defined by:

$$co(N_\omega, \mathbf{C}) = \left\{\alpha \in \mathbb{R}^n : \sum_{i \in S} \alpha_i \leq v(S), \quad \forall S \subseteq N, \quad \sum_{i \in N} \alpha_i = v(N) = C_m\right\}.$$

It is clear that any allocation in the *core* satisfies *individual rationality*, a minimal requirement that any solution concept must fulfill:

Axiom 1. *Individual Rationality* ***(IR)****: Given a mcst* $(N_\omega, \mathbf{C})$*, an allocation* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ *is said to be individually rational if, for all* $i \in N$*,*

$$\alpha_i \leq c_{ii}.$$

Associated to a *mcst* problem $(N_\omega, \mathbf{C})$, the *irreducible cost matrix*[8] $\mathbf{C}^*$ is constructed as:

$$c^*_{ij} = max\left\{c_{i'j'} : (i', j') \in \tau^m_{ij}\right\}$$

[8] This matrix is used in order to define the *Folk* solution (Bergantiños and Vidal-Puga, 2007), and as an intermediate step to obtain the Cycle-complete solution (Trudeau, 2012).

where τ_{ij}^m is the network formed by the nodes in the unique path between i and j in the minimum cost spanning tree m. Now, by using matrix $\mathbf{C}^*$ instead of $\mathbf{C}$, a new cooperative game is defined in an analogous way, that we denote by v^*. It is immediate to note that

$$\emptyset \neq co(N_\omega, \mathbf{C}^*) \subseteq co(N_\omega, \mathbf{C}).$$

Some important properties related to the irreducible cost matrix are listed below.

Properties of the irreducible cost matrix (Bergantiños and Vidal-Puga, 2007).

- $v^*(N) = v(N)$
- $c_{ij}^* \leq c_{ij}$, for all $i, j \in N$.
- v^* is concave, that is, for all $S, T \subset N$ and $i \in N$ such that $S \subset T$ and $i \notin T$,
$$v^*(S \cup \{i\}) - v^*(S) \geq v^*(T \cup \{i\}) - v^*(T).$$
- $v^*(S \cup \{i\}) - v^*(S) = \min_{j \in S \cup \{i\}}\{c_{ij}^*\}$, for all $S \subset N$ and $i \in N$, $i \notin S$.

From these properties we obtain the following result that establishes a *lower bound* in what each individual should be charged if the proposed solution is in the irreducible core.

Proposition 1. *Given a mcst problem* $(N_\omega, \mathbf{C})$

$$\alpha \in co(N_\omega, \mathbf{C}^*) \;\Rightarrow\; \alpha_i \geq \min\{c_{ik},\; k \in N\}$$

Proof. Let us consider $\alpha \in co(N_\omega, \mathbf{C}^*)$. Then,

$$v(N) = \sum_{k \in N} \alpha_k = v^*(N) = v^*(N \setminus \{i\}) + \min_{k \in N}\{c_{ik}^*\}$$

and core conditions imply

$$\sum_{k \in N,\, k \neq i} \alpha_k \leq v^*(N \setminus \{i\}) = \sum_{k \in N} \alpha_k - \min_{k \in N}\{c_{ik}^*\},$$

so $\min_{k\in N}\{c^*_{ik}\} \le \alpha_i$. Now, it is immediate to observe that the minimum cost of each individual coincides in the original and in the irreducible cost matrices. This fact completes the proof. ■

In what follows we will denote by c_{i_*} the minimum connection cost of individual i,

$$c_{i_*} = \min_{k\in N}\{c_{ik}\}$$

In order to illustrate our previous concepts, we will conclude this section with an example.

Example 1. *Let us consider the mcst problem defined by* $N = \{1, 2, 3\}$ *and the cost matrix*

$\mathbf{C} =$

1	4	2
4	10	5
2	5	20

where the main diagonal refers to the connection cost to the source for each agent. The minimum cost spanning tree is given by function m *defined as:*

$$m(1) = \omega \qquad m(2) = 1 \qquad m(3) = 1; \qquad C_m = c_{11} + c_{12} + c_{13} = 7,$$

$$\begin{array}{ccccc} 2 & & & & \\ & \searrow & & & \\ & & 1 & \longrightarrow & \omega \\ & \nearrow & & & \\ 3 & & & & \end{array}$$

The irreducible cost matrix is

$\mathbf{C}^* =$

1	4	2
4	4	4
2	4	2

The monotone cooperative game associated to the mcst problem is defined by the following characteristic function:

$v(\{1\}) = 1$, $v(\{2\}) = 5$, $v(\{3\}) = 3$, $v(\{1, 2\}) = 5$,
$v(\{1, 3\}) = 3$, $v(\{2, 3\}) = 7$, $v(\{1, 2, 3\}) = 7$.

The cooperative game associated to the irreducible matrix in the mcst problem is defined by the following characteristic function:

$v^*(\{1\}) = 1$, $v^*(\{2\}) = 4$, $v^*(\{3\}) = 2$, $v^*(\{1,2\}) = 5$,
$v^*(\{1,3\}) = 3$, $v^*(\{2,3\}) = 6$, $v^*(\{1,2,3\}) = 7$.

Table 1 presents the result of applying some usual sharing rules in this mcst problem.[9]

	α_1	α_2	α_3
Proportional	7/31	70/31	140/31
Egalitarian	7/3	7/3	7/3
Constrained Egalitarian	1	3	3
Bird	1	4	2
Folk	1	4	2

Table 1: Proposals given by rules with data in Example 1.

Remark 2. *Note that under the Proportional proposal, the first individual is allocated an amount below her minimum cost, $\alpha_1 < \min_{j \in N} c_{1j}$. As we have seen, this minimum cost should be a lower bound on what every individual must pay, and it will play an important role in our discussion.*

Also note that the Egalitarian, the Constrained Egalitarian and the Proportional proposals don't belong to the core $co(N_\omega, \mathbf{C})$.

2.2. ***Claims problems***

Given a finite set of agents $N = \{1, 2, \ldots, n\}$, a *conflicting claims problem* appears when some amount (a surplus, or a cost) should be distributed among a the agents, who claim more than the available endowment. Formally, a conflicting claims problem is defined as a vector $(E, r) \in \mathbb{R}_+ \times \mathbb{R}^n_+$, where

[9] Note that the *Egalitarian* rule allocates a payment to the first agent, α_1, greater than the cost of her direct connection to the source. This allocation is not individually rational. To avoid this problem, in some related situations, the notion of *Constrained Egalitarian* solution has been defined. The idea is to make the cost sharing as egalitarian as possible, restricted to no one pays more than what is "admissible" for her: the cost to be directly connected to the source.

E denotes the endowment and r is the vector of agents' claims, r_i, for each $i \in N$, such that the agents' aggregate demand is higher than the endowment, $\sum_{i \in N} r_i \geq E$. It is important to note that the claim r_i is the maximum amount individual i can be allocated.

A *claims rule* φ is a function that associates to each claims problem a distribution of the total endowment among the agents (*efficiency*), such that no-agent is allocated neither a negative amount (*non-negativity*) nor more than her claim (*claim-boundedness*):

$$0 \leq \varphi_i(E, r) \leq r_i \, , \qquad \sum_{i=1}^{n} \varphi_i(E, r) = E.$$

Many solution concepts have been proposed in the literature on claims problems. Apart from the *Proportional* rule (P), we must mention the *Constrained Equal Awards* (CEA), the *Constrained Equal Losses* (CEL), the *Talmudian* (T), or the *Random Arrival* (RA), rules.[10]

Given a claims problem (E, r), we can define its *dual* problem by considering the losses the agents have with respect to their claims. Let R denote the sum of the agents' claims, $R = \sum_{i \in N} r_i$, and L the total loss to be distributed among the agents with respect to the aggregate claims, $L = R - E$. Given a claims rule φ, its *dual rule* φ^d (Aumann and Maschler, 1985) shares losses in the same way that φ shares gains:

$$\varphi_i^d(E, r) = r_i - \varphi_i(L, r) \qquad i = 1, 2, \ldots, n.$$

The CEA and CEL rules are dual of each other, whereas the *Proportional* and *Talmudian* rules are *self-dual*, $\varphi^d = \varphi$.

3. A claims problem associated to a *mcst* problem.

To each *mcst* problem $(N_\omega, \mathbf{C})$, $N_\omega = \{1, 2, \ldots, n\}$, $\mathbf{C} = (c_{ij})$ symmetric matrix, a claims problem may be associated in a natural way by considering

[10] For formal definitions, properties and references see, for instance, Moulin (2002) and Thomson (2003). In order to facilitate the reading of the paper, definitions of these solutions are included in Appendix 1.

the pair (E, r) defined by:

$$r \equiv (c_{11}, c_{22}, \ldots, c_{nn}), \quad E \equiv C_m$$

Under this construction, claims are the individuals' costs to get the source directly, that is the maximum amount they can be allocated in the sharing of the common cost, and E reflects the total cost of the *mcst*. Note that by construction of m, it is satisfied that $\sum_{i=1}^{n} r_i \geq E$, and the claims problem is well defined.

Remark 3. *In this context, the claim-boundedness property coincides with individual rationality (Axiom 1) of the provided allocation, so claims rules always define an individually rational cost sharing of C_m.*

Furthermore, it must be noticed that in this model, as with the Proportional solution for mcst problems, we only take into account the individuals' costs of connecting directly to the source ω, and other cost connections are not considered. The same thing happens when the proportional solution for mcst problems is defined.

We can now apply claims rules to the induced conflicting claims problem (E, r) and obtain allocations of the minimum cost C_m. To illustrate this fact, in Example 1 we obtain $(E, r) = (7, (1, 10, 20))$. Table 2 summarizes the results of applying the main claims rules.

	α_1	α_2	α_3
P	$7/31$	$70/31$	$140/31$
CEA	1	3	3
CEL	0	0	7
T	$2/4$	$13/4$	$13/4$
RA	$1/3$	$10/3$	$10/3$

Table 2: Proposals given by *claims rules* with data in Example 1.

It must be noticed that none of these solutions belongs to the core of the cooperative game associated to the *mcst* problem. This drawback leads us to define a new claims problem associated to a *mcst*.

A question arises at this point: is the above mentioned drawback caused by the choice of the claims vector, $r_i = c_{ii}$? It could be checked that other natural reference points neither provide allocations in the core. For instance, if we choose $r_i = v(i)$, or $r_i = v^*(i)$, the obtained allocations do not belong to the core. In view of this fact, we consider two alternative approaches to distribute the total minimum cost C_m :

- First we consider that each agent is initially allocated her maximum possible cost c_{ii}. That's why we call this model the *pessimistic* approach. Then, the savings obtained throughout cooperation are distributed.

- The second model proposes that each agent pays initially her minimum cost (we call this situation the *optimistic* approach). The remaining cost is then distributed among the agents.

Next subsections develop these approaches.

3.1. ***The pessimistic claims problem: sharing the benefits of cooperation.***

We consider that, as in the previous section, each individual takes into account her direct cost to the source, c_{ii}. Then, under no cooperation, the total cost of connecting individuals to the source is $C_\omega = \sum_{i \in N} c_{ii}$. If individuals cooperate, they can connect at the minimal cost C_m. Then, the *mcst* problem can be seen as the way of sharing the benefits from cooperation $E_p \equiv C_\omega - C_m$.

Definition 1. *Given a claims vector r, $r_i \geq 0$, $\sum_{i \in N} r_i \geq E_p$, and a claims rule $\varphi(\cdot, \cdot)$, we associate the following allocation to the mcst problem:*

$$(\alpha_p^\varphi)_i = c_{ii} - \varphi_i(E_p, r) \qquad i = 1, 2, \ldots, n.$$

We denote by α_p^φ the sharing rule so defined.

Now we need to choose each individual claim r_i in order to apply a claims rule. Note that the proposal given by such a claims rule is the benefit from the *status quo* point, and the maximum benefit of an individual is just her claim, that is, for all i the maximum value for α_i is $c_{ii} - r_i$. So, if we want the provided allocation to be in the irreducible core then, after Proposition 1, we know that

$$\alpha_i = c_{ii} - \varphi_i(E_p, r) \geq c_{i_*} \Rightarrow r_i \leq c_{ii} - c_{i_*} \qquad \text{for all } i \in N.$$

Moreover, as (E_p, r) must be a claims problem, $\sum_{i \in N} r_i \geq E_p = C_\omega - C_m$. A sufficient condition to obtain this inequality is by considering $c_{ii} - c_{im(i)} \leq r_i$ and then

$$c_{ii} - c_{im(i)} \leq r_i \leq c_{ii} - c_{i_*} \qquad \text{for all } i \in N.$$

If the lower bound is taken as the claims vector, $r_i = c_{ii} - c_{im(i)}$, then the claims problem (E_p, r) is *degenerate*, $E_p = \sum_{i \in N} r_i$, and any claims rule proposes the solution $\varphi_i(E_p, r) = r_i$, that is $\alpha_i = c_{im(i)}$ which coincides with the Bird solution[11] of the *mcst* problem.

In the remaining of the section we will use the upper bound as the claims vector. We will name the *residual claim* of individual i to the maximum amount s_i she can gain from cooperation,

$$s_i \equiv c_{ii} - c_{i_*} \qquad \text{for all } i \in N.$$

and we call (E_p, s) the *pessimistic claims problem* associated to the *mcst* problem. In order to observe the allocations provided by different claims rules, let us observe the following example.

Example 2. *Let us consider the mcst problem defined by* $N = \{1, 2, 3\}$ *and the cost matrix*

$$\mathbf{C} = \begin{array}{|c|c|c|} \hline 4 & 1 & 2 \\ \hline 1 & 10 & 3 \\ \hline 2 & 3 & 20 \\ \hline \end{array}$$

The minimum cost spanning tree is given by function m *defined as:*

$$m(1) = \omega \qquad m(2) = 1 \qquad m(3) = 1; \qquad C_m = c_{11} + c_{12} + c_{13} = 7,$$

$$\begin{array}{cccc} 2 & & & \\ & \searrow & & \\ & & 1 & \longrightarrow \quad \omega \\ & \nearrow & & \\ 3 & & & \end{array}$$

[11] It is possible that there exist several minimum cost trees. In this case, we consider the Bird solution associated to one of them, say m. Formally, the Bird solution is defined as an average of the trees associated with Prim's algorithm.

The monotone cooperative game associated to the mcst problem is defined by the following characteristic function:

$v(\{1\}) = 4,\ v(\{2\}) = 5,\ v(\{3\}) = 6,\ v(\{1,2\}) = 5,$
$v(\{1,3\}) = 6,\ v(\{2,3\}) = 7,\ v(\{1,2,3\}) = 7.$

And, finally, the cooperative game associated to the irreducible matrix in the mcst problem is defined by the following characteristic function:

$v^*(\{1\}) = 4,\ v^*(\{2\}) = 4,\ v^*(\{3\}) = 4,\ v^*(\{1,2\}) = 5,$
$v^*(\{1,3\}) = 6,\ v^*(\{2,3\}) = 6,\ v^*(\{1,2,3\}) = 7.$

Table 3 presents the result of applying some sharing rules in mcst problems.

	α_1	α_2	α_3
Proportional	28/34	70/34	140/34
Egalitarian	7/3	7/3	7/3
Bird	4	1	2
Folk	13/6	13/6	16/6

Table 3: Proposals given by rules with data in Example 2.

In order to apply our pessimistic model, we first compute the benefits of cooperation $E_p = C_\omega - C_m = 27$*. On the other hand,* $c_* = (1,1,2)$*, so* $s = (3,9,18)$*. Table 4 shows the results obtained by applying different claims rules,* $(\alpha_p^\varphi)_i = c_{ii} - \varphi_i(27,(3,9,18))$*.*

	α_1	α_2	α_3
P	13/10	19/10	38/10
CEA	1	1	5
CEL	2	2	3
T	2	2	3
RA	2	2	3

Table 4: Proposals given by *claims rules* with data in Example 2.

Remark 4. *Note that, in this example, the proposals given by* CEA*,* CEL*, Talmudian, or Random Arrival rules agree with the Folk solution in that* *individuals* 1 *and* 2 *should pay the same sharing of the total cost. However, in these cases, the agents with lower* s_i *take more advantage from cooperation, especially in the* CEA *proposal.*

3.2. ***The optimistic claims problem: distributing the remaining cost.***

Now, instead of initially considering the cost c_{ii} of connecting each agent directly to the source, we use c_{i_*}, the minimal connection cost; that is, the agent would choose her cheapest link. Then we assume that each individual pays her corresponding minimum amount c_{i_*}, so the total amount paid is $C^{\min} = \sum_{i \in N} c_{i_*}$. Then, the remaining cost, $E_o = C_m - C^{\min}$ is distributed accordingly to some claims rule.

Definition 2. *Given a claims vector* r, $r_i \geq 0$, $\sum_{i \in N} r_i \geq E_o$, *and a claims rule* $\varphi(\cdot, \cdot)$, *we associate the following allocation to the mcst problem:*

$$(\alpha_o^{\varphi})_i = c_{i*} + \varphi_i(E_o, r) \qquad i = 1, 2, \ldots, n.$$

We denote by α_o^{φ} *the sharing rule so defined.*

We can argue as in the pessimistic model in order to choose the particular conflicting claims problem (E_o, s) defined by:

$$s_i \equiv c_{ii} - c_{i_*}, \quad s \equiv (s_1, s_2, \ldots, s_n), \quad E_o \equiv C_m - C^{\min}.$$

Obviously, this conflicting claims problem is well defined, since $\sum_{i=1}^{n} s_i \geq E_o$. That is, we associate to the *mcst* problem $(N_\omega, \mathbf{C})$, the allocation:

$$(\alpha_o^{\varphi})_i = c_{i_*} + \varphi_i(E_o, s) \qquad i = 1, 2, \ldots, n.$$

Example 3. *With the data in Example 2,* $c_* = (1, 1, 2)$, *so* $E_o = 3$, *and* $s = (3, 9, 18)$. *Table 5 shows the results obtained by applying this model to the problem in Example 2.*

3.3. ***Duality***

If we observe the solutions in Tables 4 and 5, for the *mcst* problem in Example 2, and denote by α_p^{φ} and α_o^{φ} the allocations provided, respectively, by the pessimistic and optimistic models (both with claims $s_i = c_{ii} - c_{i_*}$), then the pessimistic allocation associated to φ coincides with the optimistic one associated to the dual claims rule φ^d. So, with both proposals we obtain the same family of allocations. Next Proposition proves this fact in the general case.

	α_1	α_2	α_3
P	13/10	19/10	38/10
CEA	2	2	3
CEL	1	1	5
T	2	2	3
RA	2	2	3

Table 5: Proposals given by *claims rules* with data in Example 2.

Proposition 2. *Given a mcst problem* $(N_\omega, \mathbf{C})$, *for any claims rule* φ, *if we consider the claims vector* s, $s_i = c_{ii} - c_{i_*}$, *for all* $i \in N$, *then*

$$\alpha_p^{\varphi}(N_\omega, \mathbf{C}) = \alpha_o^{\varphi^d}(N_\omega, \mathbf{C}).$$

Proof. We know that

$$(\alpha_p^{\varphi})_i = c_{ii} - \varphi_i(E_p, s), \qquad E_p = \sum_{i \in N} c_{ii} - C_m, \quad s_i = c_{ii} - c_{i_*}.$$

On the other hand,

$$\varphi_i(E_p, s) = s_i - \varphi_i^d\left(\sum_{i \in N} s_i - E_p, s\right) = s_i - \varphi_i^d(E_o, s).$$

Then,

$$(\alpha_p^{\varphi})_i = c_{ii} - \varphi_i(E_p, s) = c_{ii} - \big(s_i - \varphi_i^d(E_o, s)\big) =$$
$$= c_{i_*} + \varphi_i^d(E_o, s) = (\alpha_o^{\varphi^d})_i.$$

■

4. Core selection

Although in Examples 1 and 2 the obtained allocations using claims rules belong to the core of the monotone cooperative game (N, v) associated to the *mcst* problem, this fact is not always true.[12]

[12] An example with $n = 7$ individuals is provided in Appendix 2.

As shown in the next result (the obvious proof is omitted) these solutions satisfy the stand alone condition (*Axiom 1*) and the lower bound that determines the minimum each individual must pay in order to ensure that the cost allocation belongs to the irreducible core.

Proposition 3. *For any mcst problem* $(N_\omega, \mathbf{C})$, *any claims rule* φ, *and any claims vector* r,

$$c_{i_*} \leq (\alpha_p^\varphi(N_\omega, \mathbf{C}))_i \leq c_{ii},$$
$$c_{i_*} \leq (\alpha_o^\varphi(N_\omega, \mathbf{C}))_i \leq c_{ii}.$$

A way to obtain allocations in the core of the monotone cooperative game (N, v) associated to the *mcst* problem consists of analyzing the relationship between this cooperative game and the one defined by O'Neill in claims problems.

4.1. ***Relationship between the cooperative games***

Associated to a conflicting claims problem (E, r) the *O'Neill*'s cooperative game is defined by:

$$u(S) = \max\left\{E - \sum_{i \notin S} r_i, 0\right\}$$

Is it possible to find a claims vector r such that the monotone cooperative game associated to the network $(N_\omega, \mathbf{C})$ coincides with the *O'Neill*'s cooperative game of the claims problem (C_m, r)? The next example shows that in general the answer is negative.

Example 4. *Let us consider the mcst problem defined by* $N = \{1, 2, 3\}$ *and the cost matrix*

$\mathbf{C} =$

1	0	2
0	1	2
2	2	2

There exist several possibilities for choosing the mcst, all of them with the cost $C_m = 3$. *We may choose, for instance,* $m(1) = \omega$, $m(2) = 1$, $m(3) = 1$. *The minimal cost connection for each agent is* $c_{1*} = 0$, $c_{2*} = 0$, $c_{3*} = 2$. *The monotone cooperative game is given by the following characteristic function:*

$$v(\{1\}) = 1,\ v(\{2\}) = 1,\ v(\{3\}) = 2,\ v(\{1, 2\}) = 1,$$

$$v(\{1,3\}) = 3,\ v(\{2,3\}) = 3,\ v(\{1,2,3\}) = 3.$$

It is easy to observe that there is not $r = (r_1, r_2, r_3)$ such that $u(S) = v(S)$ for all $S \subseteq N$, where $u(S)$ is the characteristic function associated to the claims problem (E_p, r). On the one hand, the following relation must be fulfilled: $r_1 + r_2 = 1$. But we also obtain $r_1 = 0$ and $r_2 = 0$, which is not possible.

In this mcst problem, Bird's and Folk solutions coincide, $B(N_\omega, \mathbf{C}) = F(N_\omega, \mathbf{C}) = (0.5, 0.5, 2)$. It must be noticed that $E_p = E_o = 1$ and $s = (1, 1, 0)$, so for any claims rule φ both the pessimistic and optimistic allocations also coincide with this solution, $\alpha_p^\varphi(N_\omega, \mathbf{C}) = \alpha_o^\varphi(N_\omega, \mathbf{C}) = (0.5, 0.5, 2)$, that belongs to the core of the monotone cooperative game.

Then, instead of looking for a claims vector such that both cooperative games coincide, we explore conditions that relate their characteristic functions with the costs of directly connecting each individual to the source. In so doing, the following result provides a sufficient condition for the allocation defined by the pessimistic model to be in the core.

Proposition 4. *If there is a claims vector r, with $\sum_{i \in N} r_i \geq E_p$, such that for all $S \subseteq N$, $\sum_{i \in S} c_{ii} \leq u(S) + v(S)$, then for any claims rule φ, $\alpha_p^\varphi(N_\omega, \mathbf{C}) = c_\omega - \varphi(E_p, r)$ belongs to $co(N_\omega, \mathbf{C})$, where $c_\omega = (c_{11}, c_{22}, \ldots, c_{nn})$.*

Proof. For any claims rule φ the next condition is satisfied

$$\sum_{i \in S} \varphi_i(E_p, r) \geq u(S) \qquad \text{for all } \ S \subseteq N.$$

Then, if we consider $\alpha_i = c_{ii} - \varphi_i(E_p, r)$ for all $i \in N$,

$$\sum_{i \in S} \alpha_i = \sum_{i \in S} c_{ii} - \sum_{i \in S} \varphi_i(E_p, r) \leq \sum_{i \in S} c_{ii} - u(S) \leq v(S)$$

and $\alpha_p^\varphi(N_\omega, \mathbf{C}) \in co(N_\omega, \mathbf{C})$. ■

Next result shows that a claims vector r fulfilling the conditions in Proposition 4 always exist.

Proposition 5. *Given a mcst problem $(N_\omega, \mathbf{C})$ there exists a claims vector r, with $\sum_{i \in N} r_i \geq E_p$, such that $\sum_{i \in S} c_{ii} \leq u(S) + v(S)$, where $u(S)$ is the characteristic function associated to the claims problem (E_p, r).*

Proof. For each $i \in N$, consider $r_i = c_{ii} - c_{im(i)}$. Then, $\sum_{i\in N} r_i = E_p$, and for all $S \subseteq N$,

$$u(S) = \sum_{i\in S} r_i = \sum_{i\in S} c_{ii} - \sum_{i\in S} c_{im(i)}.$$

As the solution, $\alpha_i = c_{im(i)}$, belongs to the core, $\sum_{i\in S} c_{im(i)} \leq v(S)$ and then

$$u(S) + v(S) \geq \sum_{i\in S} c_{ii},$$

the required inequality. ■

Previous propositions lead to the following conclusion.

Theorem 1. *Given a mcst problem $(N_\omega, \mathbf{C})$ there exists a claims vector r, with $\sum_{i\in N} r_i \geq E_p$, such that the allocations provided by the pessimistic or the optimistic model, by using the claims vector r, belong to the core of the monotone cooperative game.*

Finally, in the next result we show sufficient conditions defined in terms of the *mcst* cost matrix, ensuring that the allocation provided by α_p^φ belong to the core when the claims vector is s, with $s_i = c_{ii} - c_{i*}$, for all $i \in N$.

Theorem 2. *Let $(N_\omega, \mathbf{C})$ a mcst problem such that for all $S \subseteq N$:*

$$v(S) \geq \sum_{i\in S} c_{ii} + C_m - \sum_{i\in S} c_{i*} - \sum_{i\notin S} c_{ii}. \quad (1)$$

Then, for any claims rule φ, the allocations provided by the pessimistic model, $\alpha_p^\varphi(N_\omega, \mathbf{C}) = c_\omega - \varphi(E_p, s)$, belong to the core of the monotone cooperative game.

Proof. It is sufficient to prove that condition in equation (1) implies the condition in Proposition 4. On the one hand, for all $S \subseteq N$,

$$u(S) = \max\left\{E_p - \sum_{i\notin S} s_i, 0\right\} = \max\left\{C_\omega - C_m - \sum_{i\notin S}(c_{ii} - c_{i*}), 0\right\} =$$

$$= \max\left\{\sum_{i\notin S} c_{ii} + \sum_{i\in S} c_{i*} - C_m, 0\right\}.$$

Then,

$$u(S) + v(S) \geq \sum_{i \notin S} c_{ii} + \sum_{i \in S} c_{i*} - C_m + v(S) \geq$$

and by equation (1)

$$\geq \sum_{i \in S} c_{ii} - v(S) + v(S) = \sum_{i \in S} c_{ii}.$$

■

By applying duality, we obtain the corresponding result for the optimistic model. We denote by $c_* = (c_{1*}, c_{2*}, \ldots, c_{n*})$.

Theorem 3. *Let $(N_\omega, \mathbf{C})$ a mcst problem such that for all $S \subseteq N$:*

$$v(S) \geq \sum_{i \in S} c_{ii} + C_m - \sum_{i \in S} c_{i*} - \sum_{i \notin S} c_{ii}.$$

Then, for any claims rule φ, the allocations provided by the optimistic model, $\alpha_o^\varphi(N_\omega, \mathbf{C}) = c_* + \varphi(E_o, s)$, belong to the core of the monotone cooperative game.

4.2. ***Other properties***

Apart from the important property of *core selection* (discussed in the previous section) other properties have been analyzed in the literature about *mcst* problems.

The properties of *Symmetry* and *Positivity* are always satisfied by our pessimistic and optimistic allocations, independently[13] of the claims rule being considered.

SYMMETRY: A solution α for *mcst* problems is said to satisfy *Symmetry* if, for each problem $(N_\omega, \mathbf{C})$, if individuals $i, j \in N$ are such that $c_{ik} = c_{jk}$, for all $k \in N$, then

$$\alpha_i(N_\omega, \mathbf{C}) = \alpha_j(N_\omega, \mathbf{C}).$$

POSITIVITY: A solution α for *mcst* problems is said to satisfy *Positivity* if, for each problem $(N_\omega, \mathbf{C})$, and all $i \in N$, then

$$\alpha_i(N_\omega, \mathbf{C}) \geq 0.$$

[13] In order to obtain *symmetry*, this condition must be required to the claims rule. Most of claims rules satisfy symmetry.

It is immediate to observe that the pessimistic allocation fulfills both properties.

Proposition 6. *The pessimistic cost allocation* $\alpha_p^{\varphi}(N_\omega, \mathbf{C}) = c_\omega - \varphi(E_p, s)$, *satisfies Symmetry and Positivity for every claims rule* φ *satisfying symmetry.*

The *Cost Monotonicity* property (considered a compelling requirement) is satisfied for some claims vector r, but it can not be ensured for $r = s$. This property is stated as follows:

COST MONOTONICITY: A solution α for *mcst* problems is said to satisfy *Cost Monotonicity* if, for any pair of problems $(N_\omega, \mathbf{C})$, $(N_\omega, \mathbf{C}')$, and $i, j \in N$:

$$c_{ij} < c'_{ij} \text{ and } c_{lk} = c'_{lk} \quad \forall\, (l,k) \neq (i,j) \Rightarrow \alpha_i(N_\omega, \mathbf{C}) \leq \alpha_i(N_\omega, \mathbf{C}').$$

Proposition 7. *A claims vector* r *exists, with* $\sum_{i \in N} r_i \geq E_p$, *such that the pessimistic cost allocation* $\alpha_p^{\varphi}(N_\omega, \mathbf{C}) = c_\omega - \varphi(E_p, r)$, *satisfies Cost Monotonicity for every claims rule* φ.

Proof. It is sufficient to consider the claims vector in Proposition 5. Then, the pessimistic cost allocation always coincide with the allocation $\alpha_i = c_{im(i)}$, that fulfills Cost Monotonicity. ■

5. Final comments

The current paper has explored a bridge between two independent problems that have been extensively analyzed in the literature: *minimum cost spanning tree* and *conflicting claims* problems. Specifically, it presents two new ways of distributing the cost of a network among the agents. The first one (that we call the pessimistic model) takes the cost to the source as the departure point. Then, it distributes the gains of cooperation. The second one (named optimistic model) departs from each agent's minimum connection cost, and it distributes the additional cost. We prove that these approaches are dual of each other.

The benefit, or the additional cost, are distributed among the individuals by using claims rules. Hereby, once the endowment and the claims vector are determined, a particular claims rule is applied. In this context, we have shown that it is possible to find a claims vector such that, for any claims rule, the provided allocation fulfills *core selection, symmetry, positivity, and claims monotonicity.*

Acknowledgments:

To be added.

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APPENDIX 1. SOLUTIONS

Networks solutions

- BIRD SOLUTION *(Bird, 1976) If there is only one mcst m, each agent pays the link she uses to be connected to the source:*

$$\alpha_i = c_{im(i)}$$

It is possible the existence of several spanning trees $m_1, m_2, \ldots, m_k$ with the same minimum cost C_m. In this case, the Bird solution is defined as the average of the Bird costs associated to any minimum cost spanning tree:

$$\alpha_i = \frac{\sum_{r=1}^{k} c_{im_r(i)}}{k}.$$

- FOLK SOLUTION *(Bergantiños and Vidal-Puga, 2007) This solution is defined as:*

$$F_i = Sh_i(N, v^*).$$

where $v*$ is the cooperative game associated to the irreducible cost matrix $\mathbf{C}^*$.

Claims solutions

- PROPORTIONAL SOLUTION, *P. For each $(E, r) \in \mathcal{B}$ and each $i \in N$,*

$$P_i(E, r) = \lambda r_i,$$

where $\lambda = \dfrac{E}{\sum\limits_{i \in N} r_i}$.

- CONSTRAINED EQUAL AWARDS SOLUTION, *CEA. For each $(E, r) \in \mathcal{B}$ and each $i \in N$,*

$$CEA_i(E, r) \equiv \min\{r_i, \mu\},$$

where μ is chosen so that $\sum\limits_{i \in N} \min\{r_i, \mu\} = E$.

- CONSTRAINED EQUAL LOSSES SOLUTION, *CEL. For each $(E, r) \in \mathcal{B}$ and each $i \in N$,*

$$CEL_i(E, r) \equiv \max\{0, r_i - \mu\},$$

where μ is chosen so that $\sum\limits_{i \in N} \max\{0, r_i - \mu\} = E$.

- Talmudian solution, T. *For each* $(E,r)\in\mathcal{B}$ *and each* $i\in N$,

$$\textit{if } E \leq \frac{\left(\sum_{i\in N} r_i\right)}{2}, \quad T_i(E,r) \equiv CEA_i\left(E, \frac{r}{2}\right); \ \textit{otherwise},$$

$$T_i(E,r) = \frac{r_i}{2} + CEL\left(E - \frac{\left(\sum_{i\in N} r_i\right)}{2}, \frac{r}{2}\right).$$

- Random arrival solution, RA. *For each,* $(E,r)\in\mathcal{B}$ *and each* $i\in N$,

$$RA_i(E,r) \equiv Sh_i(u) \qquad u(S) = max\left\{E - \sum_{k\notin S} r_k, 0\right\}$$

APPENDIX 2. Example whit proposals not in the core

Example 5. *Let us consider the mcst with* $n=7$ *individuals defined by the cost matrix*

$\mathbf{C} =$

1	0	1	2	3	3	3
0	1	1	2	3	3	3
1	1	1	2	3	3	3
2	2	2	2	3	3	3
3	3	3	3	3	2	2
3	3	3	3	2	3	1
3	3	3	3	2	1	3

There exist several trees with minimum cost $C_m = 10$. *For instance,*

$$m(1)=\omega;\ m(2)=1;\ m(3)=2;\ m(4)=3;\ m(5)=4;\ m(6)=7;\ m(7)=6.$$

The vector of direct costs to the source is

$$c_\omega = (c_{11}, c_{22}, \ldots, c_{77}) = (1,1,1,2,3,3,3).$$

The vector of minimum connection costs is

$$c_* = (c_{1_*}, c_{2_*}, \ldots, c_{7_*}) = (0,0,1,2,2,1,1)$$

and then

$$s = (1, 1, 0, 0, 1, 2, 2), \text{ where } s_i = c_{ii} - c_{i_*}.$$

Moreover, $C_\omega = 14$ *and* $E_p = C_\omega - C_m = 4$. *So, the claims problem to be solved is* $(E_p, s) = (4, (1, 1, 0, 0, 1, 2, 2))$, *and the corresponding solution to the mcst problem is*

$$\alpha_i = c_{ii} - \varphi_i(E_p, s).$$

Then, if we consider $\varphi = CEA$, *the proposed sharing of* C_m *is*

$$\alpha^{CEA} = (0.2, 0.2, 1, 2, 2.2, 2.2, 2.2).$$

This allocation does not belong to the core of the monotone cooperative game associated to the mcst problem since

$$\alpha_6^{CEA} + \alpha_7^{CEA} = 4.4 > v(6, 7) = 4.$$

A similar situation appears if we set $\varphi = CEL$, *or* $\varphi = P$, *the proportional claims rule, in which cases we obtain, respectively*

$$\alpha^{CEL} = (0.6, 0.6, 1, 2, 2.6, 1.6, 1.6).$$

$$\alpha^P = (0.43, 0.43, 1, 2, 2.43, 1.855, 1.855).$$

None of them belong to the core.[14]

[14] Although none of these solutions belong to the core, they provide *reasonable* sharings of the minimum cost C_m.