THE VALUATION OF LIFE CONTINGENCIES: A SYMMETRICAL TRIANGULAR FUZZY APPROXIMATION[1]

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ABSTRACT

This paper extends the framework for the valuation of life insurance policies and annuities by Andrés-Sánchez and González-Vila (2012, 2014) in two ways. First, we allow various uncertain magnitudes to be estimated by means of fuzzy numbers. This applies not only to interest rates but also to the amounts to be paid out by the insurance company. Second, the use of symmetrical triangular fuzzy numbers allows us to obtain expressions for the pricing of life contingencies and their variability that are closely linked to standard financial and actuarial mathematics. Moreover, they are relatively straightforward to compute and understand from a standard actuarial point of view.

1. INTRODUCTION

Stochastic techniques are, beyond doubt, at the core of actuarial mathematics. However, in insurance decision-making problems, as well as in other areas related to economics and finance, much of the information is imprecise and vague, or relies heavily on subjective judgements and, so, it is not clearly measurable. For such information, the use of fuzzy set theory (FST) can represent a suitable alternative and/or a supplementary way to that of pure statistical methods as has been shown in De Witt (1982), Lemaire (1990), Ostaszewski (1993), Cummins and Derrig (1997), Andrés-Sánchez and Terceño (2003) and Shapiro (2004).

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[1] The authors acknowledge the useful comments of anonymous referees.

In the field of the financial pricing of insurance, FST has been used to model discount rates. Cummins and Derrig (1997), Derrig and Ostaszewski (1997) and Andrés-Sánchez (2014) do so in a non-life context, while Lemaire (1990), Ostaszewski (1993) and Betzuen *et al*. (1997) model discount rates for life insurance contingencies valuation. In these papers probabilities, however, are reduced to predefined frequencies and so, the financial pricing of insurance contracts is solved by applying the fuzzy financial mathematics developed by Buckley (1987). Anyway, when applying these methods, probabilistic information is lost because random magnitudes are reduced to their mathematical expectation.

Shapiro (2009) exposes the concept of fuzzy random variables (FRVs) with Actuarial Science in view. Similarly, Huang *et al.* (2009) develop an individual risk model in which the cost of accidents is estimated using fuzzy numbers (FNs), while the number of claims follows a Poisson process. In the field of life insurance, Andrés-Sánchez and González-Vila (2012, 2014) develop a methodology in which discount rates are fuzzy whereas the mortality is strictly random. In these papers, the stochastic modelling of life contingencies with deterministic discount rates and monetary amounts (see Wolthuis and Van Hoek (1986) for a complete description) is extended to cases in which the discount rates are fuzzy and, so, the outcomes (the present value of insured life contingencies) are fuzzy sets. All these developments also rely on the concept of FRVs. In their works, the authors do not assume any particular shape for fuzzy interest rates and consequently no closed expressions for those present values are developed. Likewise, only crisp unitary amounts are considered.

This paper extends the previous findings of Andrés-Sánchez and González-Vila (2012, 2014) in two ways. First, we also allow the insured amounts to be paid out by the insurance company to be uncertain and to be quantified with FNs, which is a more general framework. Note, that, in fact, these amounts may be linked to economic indexes, such as the consumer price index, a wage growth rate, etc. Likewise, if we are evaluating the underwriter's overall outcome of a policy, future maintenance costs, general and settlement expenses, etc. may not be known exactly *a priori*.

Second, we suppose that the amounts to be paid and the interest rates are fitted with symmetrical triangular fuzzy numbers (STFNs). Indeed, the use of these FNs is very common in the fuzzy literature. In a strictly actuarial context, we find Andrés-Sánchez and Terceño (2003), Shapiro (2013) and Heberle and Thomas (2014). Despite that under these hypothesis the present value of the analyzed life contingencies structures does not turn into a STFN, we will propose a STFN approximation that is relatively easy to implement and understand with standard actuarial skills since it relies on conventional statistical and financial concepts. In our opinion, the issue of maintaining the symmetrical triangular shape in the present value of insurance contract is relevant. Following Grzegorzewski and Pasternak-Winiarska (2014) complicated forms of FNs may cause unpleasant drawbacks in processing imprecise information modeled by these fuzzy structures including problems with calculations, computer implementation and in interpretation of the results. This is the reason why a suitable approximation of FNs is an interesting alternative to substitute the original "input" membership functions by another "outputs" which are simpler or more regular and hence more convenient for further tasks. In this sense Grzegorzewski and Mrówka (2005) indicate that triangular approximation can be considered a more complete kind of defuzzification than simply reducing a FN to a crisp

representative value given that performing defuzzification early may result in a loss of too much information and it is better to process fuzzy information as long as possible. Bearing in mind this idea, we are looking for simplification in the computational process and the interpretation of the results on the one hand but, on the other, we do not want to simplify too much the original information.

We structure the rest of our paper as follows. In section 2 we describe the concepts and instruments of FST used in our developments: FNs and FRVs. We then develop a STFN approximation for the present value derived from STFN cash-flows and discount rates with a straightforward financial interpretation. In sections 4 and 5 we introduce the use of FRV with STFN outcomes to price life contingencies. We conclude our paper with a summary of the main conclusions and possible extensions.

2. FUZZY NUMBERS AND FUZZY RANDOM VARIABLES

2.1. FUZZY NUMBERS AND FUZZY ARITHMETIC

In this section we describe the basic concepts of FST and FNs and so present the basic notation used throughout this paper. The basic concept on which FST is based is *fuzzy set*. A fuzzy set $\widetilde{A}$ can be defined as $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) | x \in X\}$, where $\mu_{\widetilde{A}}$ is known as the membership function and is a mapping from the referential set X to the interval [0,1], i.e. $\mu_{\widetilde{A}}: X \rightarrow [0,1]$. Therefore, 0 indicates non-membership in the fuzzy set $\widetilde{A}$ and 1 indicates absolute membership. Alternatively, a fuzzy set can be represented by its α-level sets or α-cuts. An α-cut is a crisp set A_α, where $A_\alpha = \{x \in X | \mu_{\widetilde{A}}(x) \geq \alpha\}, \forall \alpha \in (0,1]$, with the convention that $A_{\alpha=0}$ is the closure of the support of $\widetilde{A}$, i.e. all $x \in X$ that $\mu_{\widetilde{A}}(x) \geq 0$.

A fuzzy number is a fuzzy set $\widetilde{A}$ defined over the set of real numbers and it is a fundamental concept of FST for representing uncertain quantities. It is normal, i.e. $\max_{x \in X} \mu_{\widetilde{A}}(x) = 1$, and convex, that is, its α-cuts are closed and bounded intervals. So, they are $A_\alpha = [\underline{A}(\alpha), \overline{A}(\alpha)]$, where $\underline{A}(\alpha)$ ($\overline{A}(\alpha)$) are continuously increasing (decreasing) functions of the membership level $\alpha \in [0,1]$. A FN can be interpreted as a fuzzy quantity approximately equal to the real number for which the membership function takes the value 1. In this paper we use *symmetrical triangular fuzzy numbers*, which we denote as $\widetilde{A} = (A, r_A)$. The value A is the core (mode or center) and it can be understood as the most reliable value of the FN, i.e. $\mu_{\widetilde{A}}(A) = 1$. Likewise $r_A \geq 0$ is the spread or radius and indicates the variability of $\widetilde{A}$ respect its mode A. Thus, the membership function and its corresponding α-cuts are:

$$\mu_{\widetilde{A}}(x) = \max\left\{0, 1 - \frac{|x - A|}{r_A}\right\}$$

$$A_\alpha = [\underline{A}(\alpha), \overline{A}(\alpha)] = [A - r_A(1 - \alpha), A + r_A(1 - \alpha)] \qquad (1)$$

The *expected value* of the FN $\widetilde{A}$, $EV(\widetilde{A}; \lambda)$, is a representative real value of this FN that was developed in Campos and González (1989). This concept allows us to introduce the risk aversion of the decision maker with a coefficient $0 \leq \lambda \leq 1$ in such a way that:

$$EV(\tilde{A};\lambda) = (1-\lambda)\int_0^1 \underline{A}(\alpha)d\alpha + \lambda \int_0^1 \overline{A}(\alpha)d\alpha \qquad (2a)$$

In (2a), λ graduates the importance of the lower and upper extremes of A_α when defuzzifying $\tilde{A}$. So, the greater the risk aversion of the decision maker is, the greater λ is. For example, in a non-life claim reserving context Heberle and Thomas (2014) and Andrés-Sánchez (2014) use this criteria to defuzzify the value of reserves previously given by FNs in such a way that $\lambda > 0.5$ for a risk-averse criteria for reserving.

So, for a STFN $\tilde{A} = (A, r_A)$ it is straightforward to check that:

$$EV(\tilde{A};\lambda) = A + r_A\left(\lambda - \frac{1}{2}\right) \qquad (2b)$$

Let $f(\cdot)$ be a continuous real valued function of n-real variables $x_j, j = 1,2,\dots,n$, and let $\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n$ n FNs. Then Zadeh's *extension principle* in Zadeh (1965) allows us to define a FN $\tilde{B}$ induced by the FNs $\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n$ through $f(\cdot)$ as $\tilde{B} = f(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n)$.

Although it is usually impossible to obtain the membership function of $\tilde{B}$, it is often possible to obtain a closed expression for its α-cuts, B_α. If $f(\cdot)$ is increasing with respect to the first m variables, $m \le n$, and decreases in the last $n - m$ variables, Buckley and Qu (1990) demonstrate that:

$$B_\alpha = [\underline{B}(\alpha), \overline{B}(\alpha)] = \Big[f\left(\underline{A}_1(\alpha), \underline{A}_2(\alpha), \dots, \underline{A}_m(\alpha), \overline{A}_{m+1}(\alpha), \overline{A}_{m+2}(\alpha), \dots \overline{A}_n(\alpha)\right),$$

$$f\left(\overline{A}_1(\alpha), \overline{A}_2(\alpha), \dots \overline{A}_m(\alpha), \underline{A}_{m+1}(\alpha), \underline{A}_{m+2}(\alpha), \dots, \underline{A}_n(\alpha)\right)\Big] \qquad (3)$$

When $f(\cdot)$ is simply a linear combination of its variables $\sum_{j=1}^n k_j x_j, k_j \in \Re, j = 1,2,\dots,n$, the result of evaluating this function with $\tilde{A}_j = \left(A_j, r_{A_j}\right), j = 1,2,\dots,n$, is a STFN $\tilde{B} = (B, r_B)$, where:

$$B = \sum_{j=1}^n k_j A_j, r_B = \sum_{j=1}^n |k_j| r_{A_j}$$

Notice that whereas the center of $\tilde{B}$ is simply the linear combination of $A_j, j = 1,2,\dots,n$, the spread of $\tilde{B}$ is the sum of the spreads of $k_j \tilde{A}_j, j = 1,2,\dots,n$. Since this spread is positive, the absolute value of $k_j, \forall j$, has to be used. For example, if we consider $f(x_1, x_2) = 2x_1 - 4x_2$, $\tilde{A}_1 = (3,2)$ and $\tilde{A}_2 = (6,3)$, the FN $\tilde{B}$ induced by the FNs $\tilde{A}_1$ and $\tilde{A}_2$ through $f(\cdot)$ is the STFN $(B, r_B) = 2 \cdot (3,2) - 4 \cdot (6,3)$ with $B = 2 \cdot 3 - 4 \cdot 6 = -18$ and $r_B = 2 \cdot 2 + |-4| \cdot 3 = 16$.

However, the result of evaluating non-linear functions with STFNs is not a STFN. In any case, if $f(\cdot)$ is a real-valued function derivable and continuously increasing (decreasing) with respect to the first (last) m $(n - m)$ variables, it admits a STFN approximation that is based on the approximation to non-linear operations with L-R FNs developed in Dubois and Prade (1993, p. 131). It is built up from the first-order Taylor polynomial expansion from the 1-cut to any α-cut. So, let us approximate $\underline{B}(\alpha)$ in (3) from $B(1)$ using the Taylor expansion to the first degree with $\underline{B}(\alpha) \approx \underline{B}(1) + \frac{d\underline{B}(1)}{d\alpha}(\alpha - 1)$. If we name the vector

comprising the centers of $\tilde{A}_j, j = 1,2, \ldots, n, A_C = (A_1, A_2, \ldots, A_n)$, this Taylor expansion is equivalent to:

$$\underline{B}(\alpha) = f(\underline{A}_1(\alpha), \underline{A}_2(\alpha), \ldots, \underline{A}_m(\alpha), \overline{A}_{m+1}(\alpha), \overline{A}_{m+2}(\alpha), \ldots \overline{A}_n(\alpha)) \approx$$

$$\approx f(A_C) - \sum_{j=1}^{m} \frac{\partial f(A_C)}{\partial x_j} r_{A_j}(1 - \alpha) + \sum_{j=m+1}^{n} \frac{\partial f(A_C)}{\partial x_j} r_{A_j}(1 - \alpha)$$

$$= f(A_C) - \sum_{j=1}^{n} \left| \frac{\partial f(A_C)}{\partial x_j} \right| r_{A_j}(1 - \alpha)$$

Analogously, for $\overline{B}(\alpha)$ we find:

$$\overline{B}(\alpha) = f\left(\overline{A}_{m+1}(\alpha), \overline{A}_{m+2}(\alpha), \ldots \overline{A}_n(\alpha), \underline{A}_1(\alpha), \underline{A}_2(\alpha), \ldots, \underline{A}_m(\alpha) \right) \approx$$

$$\approx f(A_C) + \sum_{j=1}^{n} \left| \frac{\partial f(A_C)}{\partial x_j} \right| r_{A_j}(1 - \alpha)$$

So, $\tilde{B} \approx (B, r_B)$ where:

$$B = f(A_C) \tag{4a}$$

$$r_B = \sum_{j=1}^{n} \left| \frac{\partial f(A_C)}{\partial x_j} \right| r_{A_j} \tag{4b}$$

In an actuarial context, STFNs and a STFN approximation to any arithmetic operation with STFNs are used in Andrés-Sánchez and Terceño (2003) to price life and non-life insurance, whereas Heberle and Thomas (2014) and Andrés-Sánchez and Terceño (2003) apply it in non-life reserving problems.

2.2. FUZZY RANDOM VARIABLES

In many real situations, uncertainty is caused by a variety of factors: randomness, hazard, vagueness, inaccuracy, imprecision, etc. Stochastic variability can be described by the use of probability theory, while other types of uncertainty, such as vagueness and imprecision, can be captured with the use of fuzzy sets (Viertl and Hareter (2004)). The concept of FRVs combines both random and fuzzy uncertainty.

Roughly speaking, a FRV can be defined as a random variable (RV) whose outcomes are not real numbers (or vectors) but FNs (or fuzzy sets defined on $\Re^n$). It was initially proposed by Kwakernaak (1978, 1979), who extended the concept of RVs on $\Re$ to the case where the realizations are FNs. Kruse and Meyer (1987) subsequently enhanced Kwakernaak's concepts. Likewise, Puri and Ralescu (1986) conceptualized FRVs as a fuzzification of a random set. However, when outcomes are mapped to the real line both definitions coincide, as demonstrated in Krätchmer (2001). In an insurance context, Shapiro (2009) presented a general reflection on the potential uses of FRVs in Actuarial Science. In more specific problems, Huang *et al.* (2009) developed a non-life risk model in which the number of claims follows a Poisson distribution, while their monetary values

are triangular FNs. Later, Shapiro (2013) models future lifetime as a FRV. Finally, Andrés-Sánchez and González-Vila (2012, 2014) apply FRVs to price life contingencies under the hypothesis of fuzzy interest rate and random behavior of mortality.

To define a discrete FRV, we consider a probability space defined by $(\Omega, \mathcal{A}, P)$ where Ω is a discrete space of elementary events $\{\omega_j\}$, $j = 1, 2, \ldots, n$, $\mathcal{A}$ is the σ-algebra of subsets of Ω and P is a probability measure on Ω. Additionally, we consider the Borel measurable space $(\mathfrak{R}, \mathcal{B})$ and we name the set of all FNs defined on $\mathfrak{R}$ as $F(\mathfrak{R})$. The mapping $\tilde{X} : \Omega \to F(\mathfrak{R})$, where $\forall \omega_j \in \Omega$ the fuzzy outcome is $\tilde{X}_j$ with α-cuts $X_{j_\alpha} = \left[\underline{X_j}(\alpha), \overline{X_j}(\alpha) \right]$, is called a *fuzzy random variable $\tilde{X}$* if:

$$\forall B \in \mathcal{B}, \forall \alpha \in [0,1], \left\{ \omega_j \in \Omega \mid X_{j_\alpha} \cap B \neq \emptyset \right\} \in \mathcal{A}$$

Wang and Zhang (1992) demonstrate that any FRV $\tilde{X}$ defines, $\forall \alpha \in [0,1]$, an infima RV $\underline{X}(\alpha)$ and a suprema RV $\overline{X}(\alpha)$, whose realizations are, respectively, $\left\{ \underline{X_j}(\alpha) \right\}_{j=1,2,\ldots,n}$ and $\left\{ \overline{X_j}(\alpha) \right\}_{j=1,2,\ldots,n}$. These RVs allow us to bound the distribution function of the FRV $F_{\tilde{X}}(x) = P(\tilde{X} \leq x)$.

Concretely, if we symbolize as $F_{\underline{X}(\alpha)}(x)$ and $F_{\overline{X}(\alpha)}(x)$, $\forall \alpha \in [0,1]$, the distribution functions of the RVs $\underline{X}(\alpha)$ and $\overline{X}(\alpha)$ obtained from $\tilde{X}$, we define *the couple of the distribution functions* of the RVs infima and suprema for that membership level, $F_{\tilde{X}}(x)_\alpha = \left\{ \underline{F_{\tilde{X}}(x)}_\alpha, \overline{F_{\tilde{X}}(x)}_\alpha \right\}$, as:

$$\underline{F_{\tilde{X}}(x)}_\alpha = P(\overline{X}(\alpha) \leq x) = F_{\overline{X}(\alpha)}(x) \tag{5a}$$

$$\overline{F_{\tilde{X}}(x)}_\alpha = P(\underline{X}(\alpha) \leq x) = F_{\underline{X}(\alpha)}(x) \tag{5b}$$

Likewise, we can define the *ε-quantile* of $\tilde{X}$, $Q^\varepsilon(\tilde{X})$, as the minimum quantity that allows $F_{\tilde{X}}\left(Q^\varepsilon(\tilde{X}) \right) \geq \varepsilon$, i.e. $Q^\varepsilon(\tilde{X}) = F_{\tilde{X}}^{-1}(\varepsilon)$. Again, with the RVs $\underline{X}(\alpha)$ and $\overline{X}(\alpha)$ obtained from $\tilde{X}$, $\forall \alpha \in [0,1]$, we bound the ε-quantile with the couple $Q^\varepsilon(\tilde{X})_\alpha = \left\{ \underline{Q^\varepsilon(\tilde{X})}_\alpha, \overline{Q^\varepsilon(\tilde{X})}_\alpha \right\}$, where:

$$\underline{Q^\varepsilon(\tilde{X})}_\alpha = \left\{ \min x \mid F_{\underline{X}(\alpha)}(x) \geq \varepsilon \right\} \tag{6a}$$

$$\overline{Q^\varepsilon(\tilde{X})}_\alpha = \left\{ \min x \mid F_{\overline{X}(\alpha)}(x) \geq \varepsilon \right\} \tag{6b}$$

The *mathematical expectation* of a FRV is a FN $E(\tilde{X})$ whose α-cuts are $E(\tilde{X})_\alpha = \left[E\left(\underline{X}(\alpha) \right), E\left(\overline{X}(\alpha) \right) \right]$. To obtain its defuzzified value (e.g. to rank FRVs from their mathematical expectations), López-Díaz and Gil (1998) propose using the concept of expected value of a FN, developed by Campos and González (1989), given its desirable properties for fuzzy decision problems. López-Díaz and Gil (1998) show that the fundamentals of the fuzzy utility function can be established by means of an axiomatic development of the fuzzy expected utility.

The *variance* measure of a FRV admits a fuzzy definition, in the same way as the expectation (Kruse and Meyer (1987)) or a real value (Körner (1997), Näther (2000) and Feng *et al.* (2001)). Following Couso and Dubois (2009) considerations, Andrés-Sánchez and González-Vila (2012) propose using the definition in Feng *et al.* (2001) for the pricing of life contingencies. In this paper the variance of $\widetilde{X}$, $V(\widetilde{X})$, is defined as:

$$V(\widetilde{X}) = \frac{1}{2}\int_0^1 \left[V\left(\underline{X}(\alpha)\right) + V\left(\overline{X}(\alpha)\right) \right] d\alpha \qquad (7)$$

where $V(\cdot)$ of RVs stands for the usual variance operator. Of course, the standard deviation is $SD(\widetilde{X}) = \sqrt{V(\widetilde{X})}$.

Let us examine some results when the outcomes of the FRVs are STFNs, i.e., the jth outcome is $\tilde{X}_j = \left(X_j, r_{X_j} \right)$. If we symbolize as X the RV whose realizations are the centers of $\tilde{X}_j$, i.e. X_j, and as r_X the RV that can be built up from the spreads, r_{X_j}, $\forall j = 1,2,\dots,n$, the FRV $\widetilde{X}$ can be represented as $\widetilde{X} = (X, r_X)$.

Example 1: Let us consider a probability space defined by $(\Omega, \mathcal{A}, P)$ where Ω is a discrete space of elementary events $\{\omega_1, \omega_2\}$. We define the FRV $\widetilde{X}$ as follows:

$$\begin{aligned}
\widetilde{X}: \Omega &\rightarrow & F(\mathfrak{R}) \\
\omega_1 &\rightarrow & \widetilde{X}(\omega_1) = (2, 0.01) = \tilde{X}_1 \\
\omega_2 &\rightarrow & \widetilde{X}(\omega_2) = (3, 0.005) = \tilde{X}_2
\end{aligned}$$

being the α-cuts of these FNs, considering (1):

$$X_{1_\alpha} = [2 - 0.01(1 - \alpha), 2 + 0.01(1 - \alpha)]$$

$$X_{2_\alpha} = [3 - 0.005(1 - \alpha), 3 + 0.005(1 - \alpha)]$$

As explained above, $\widetilde{X}$ defines, $\forall \alpha \in [0,1]$, an infima RV $\underline{X}(\alpha)$ and a suprema RV $\overline{X}(\alpha)$, whose realizations are, respectively, $\{2 - 0.01(1 - \alpha),\ 3 - 0.005(1 - \alpha)\}$ and $\{2 + 0.01(1 - \alpha),\ 3 + 0.005(1 - \alpha)\}$.

Furthermore, it is possible to define the RVs X and r_X, whose realizations are, respectively, $\{2, 3\}$ and $\{0.01, 0.05\}$ and, in this way, the FRV can be represented by $\widetilde{X} = (X, r_X)$.

From this representation of this FRV, whose outcomes are a STFN, it follows that the lower and upper RVs for a given α-level are:

$$\underline{X}(\alpha) = X - r_X(1 - \alpha)$$

$$\overline{X}(\alpha) = X + r_X(1 - \alpha)$$

As discussed above, the α-cuts of the mathematical expectation of the FRV $\widetilde{X}$ can be obtained from $E\left(\underline{X}(\alpha)\right)$ and $E\left(\overline{X}(\alpha)\right)$. Since $E\left(\underline{X}(\alpha)\right) = E(X) - E(r_X)(1-\alpha)$, with similar expression for $E\left(\overline{X}(\alpha)\right)$, it turns out that the mathematical expectation of $\widetilde{X}$ is also a STFN with:

$$E(\widetilde{X}) = \left(E(X), E(r_X)\right)$$

And consequently, considering, (2), the expected value of the mathematical expectation is:

$$EV(E(\widetilde{X}); \lambda) = E(X) + E(r_X)\left(\lambda - \tfrac{1}{2}\right) \tag{8}$$

For the variance, from (7) we can write:

$$V(\widetilde{X}) = \frac{1}{2}\int_0^1 [V(X - r_X(1-\alpha)) + V(X + r_X(1-\alpha))]d\alpha$$

$$= \frac{1}{2}\int_0^1 \left\{ E\left[(X - r_X(1-\alpha))^2 - \left(E(X) - E(r_X)(1-\alpha)\right)^2\right] \right.$$

$$\left. + E\left[(X + r_X(1-\alpha))^2 - \left(E(X) + E(r_X)(1-\alpha)\right)^2\right]\right\} d\alpha = V(X) + \frac{1}{3}V(r_X) \tag{9}$$

Example 2: Consider that in example 1 the probabilities are $p(\omega_1) = 0.6$ and $p(\omega_2) = 0.4$. The mathematical expectation of $\widetilde{X}$ is $E(\widetilde{X}) = (2.4, 0.008)$ and its variance $V(\widetilde{X}) = 0.240002$.

Note that in this section, as well as throughout the paper, an RV is denoted with a bold letter and a FN with a non-bold letter with the superscript "~". Thus, a FRV is denoted with a bold and superscripted letter.

3. PRESENT VALUE WITH SYMMETRICAL TRIANGULAR FUZZY NUMBERS

In this section we describe how to compute the present value of a stream of cash flows when they and the interest rate are given by a STFN. Let us begin with the crisp set of cash flows $F_1, F_2, \ldots, F_n \geq 0$ whose maturities are $t_1, t_2, \ldots, t_n$ years, respectively. To value them we use the interest rate $i \geq 0$. If we denote by $F = (F_1, F_2, \ldots, F_n)$, the present value of F is the function:

$$PV(F, i) = \sum_{j=1}^{n} F_j(1 + i)^{-t_j} \tag{10}$$

with:

$$\frac{\partial PV(F, i)}{\partial F_j} = (1 + i)^{-t_j} > 0 \tag{11a}$$

$$\frac{\partial PV(F,i)}{\partial i} = -\frac{\sum_{j=1}^{n} F_j t_j (1+i)^{-t_j}}{1+i} < 0 \qquad (11b)$$

Actuaries are familiar with asset-liability management (ALM) (see for example Luckner *et al.* (2003)) as a bunch of techniques to protect surplus against changes in interest rates. The essential idea is to measure the price sensitivity to interest rate changes of both liabilities and the assets supporting them. Macauley duration is one of the measures of this sensitivity. It is defined as the weighted average term to maturity of the cash flows from an asset or liability portfolio. It can also be interpreted as the elasticity of the present value (10) to changes in current interest rates. By measuring and equating asset and liability durations (with any of the possible tools of immunization), the price sensitivity of surplus to interest rates becomes small and thereby the risk interest is reduced.

As far as we are concerned, due to the importance of considering the duration in the ALM, the introduction of this measure in our approach may be very useful.

Of course, the Macaulay duration can also be understood as a function of F and i as:

$$D(F,i) = \frac{\sum_{j=1}^{n} F_j t_j (1+i)^{-t_j}}{\sum_{j=1}^{n} F_j (1+i)^{-t_j}} = -\frac{\frac{\partial PV(F,i)}{\partial i}(1+i)}{PV(F,i)}$$

$$\frac{\partial PV(F,i)}{\partial i} = -PV(F,i)\frac{D(F,i)}{1+i} \qquad (11c)$$

Financial mathematics with fuzzy parameters was developed in the late 1980s and early '90s by such authors as Kaufmann (1986), Buckley (1987) and Li Calzi (1990). The problem consists, primarily, in evaluating the basic expression of the present value (10) (and also the accumulated value function) when interest rate(s) and cash flows are FNs.

In our case, we evaluate the net present value of fuzzy cash flows $\tilde{F}_j, j = 1,2,\dots,n$, where $\tilde{F}_j = \left(F_j, r_{F_j}\right)$ and the fuzzy interest rate $\tilde{\imath} = (i, r_i)$. Likewise, we denote $\tilde{F} = \left(\tilde{F}_1, \tilde{F}_2, \dots, \tilde{F}_n\right)$ and name by $F_C = (F_1, F_2, \dots, F_n)$ and $r_F = \left(r_{F_1}, r_{F_2}, \dots, r_{F_n}\right)$ the vectors comprising the centers and radios of $\tilde{F}_j, j = 1,2,\dots,n$, respectively. So, the fuzzy present value is the FN $\widetilde{PV}$ induced by $\tilde{F}$ and $\tilde{\imath}$ through the function $PV(\cdot)$, i.e. $\widetilde{PV} = PV(\tilde{F}, \tilde{\imath})$. Its α-cuts representation is, from (3) and (10):

$$PV_\alpha = \left[\underline{PV}(\alpha), \overline{PV}(\alpha)\right] =$$

$$= \left[\sum_{j=1}^{n}\left[F_j - r_{F_j}(1-\alpha)\right]\left(1+i+r_i(1-\alpha)\right)^{-t_j}, \sum_{j=1}^{n}\left[F_j + r_{F_j}(1-\alpha)\right]\left(1+i-r_i(1-\alpha)\right)^{-t_j}\right]$$

$$(12)$$

So, $\widetilde{PV}$ does not maintain the STFN shape. In this sense, authors like Jiménez and Rivas (1998) or Grzegorzewski and Pasternak-Winiarska (2014) consider relevant using triangular approximations since this kind of FNs have a more intuitive interpretation and likewise they are easier to handle in subsequent quantitative analysis. So, bearing (4) and (11) in mind, we can approximate the fuzzy present value as $\widetilde{PV} \approx (PV(F_C, i), r_{PV})$ where:

$$PV(F_C, i) = \sum_{j=1}^{n} F_C (1+i)^{-t_j} \tag{13a}$$

$$r_{PV} = PV(r_F, i) + PV(F_C, i) \frac{D(F_C, i)}{1+i} r_i \tag{13b}$$

The above result is very appealing from a financial perspective. The most reliable value of the approximation for $\widetilde{PV}$ is obtained by evaluating the present value in the most possible values of cash flows and interest rates. The uncertainty of the present value, r_{PV} , comes from two sources: the uncertainty of cash flows, r_F, and the spread of the interest rate. The contribution of cash flow spreads in the uncertainty of the approximation of $\widetilde{PV}$ is its own present value with the most reliable interest rate. Likewise, the uncertainty that comes from the variability of interest rate is calculated with the Macaulay duration, as it is very usual in fixed income analysis (see for example Fabozzi and Fong (1994, p. 45)).

Let us to evaluate in an example the closeness of the proposed STFN approximation to (12). Table 1 shows the α-cuts of the present value of the cash flows $\tilde{F}_1 = (1000,50)$, $\tilde{F}_2 = (1500,75)$, $\tilde{F}_3 = (2000,100)$ and $\tilde{F}_4 = (2500,100)$, whose maturities are 1, 2, 3 and 4 years, respectively. The interest rate is $\tilde{\iota} = (0.02, \ 0.005)$. We define the errors:

$$\underline{err}(\alpha) = \frac{\left| \underline{PV}(\alpha) - \underline{PV}'(\alpha) \right|}{\underline{PV}(\alpha)}$$

$$\overline{err}(\alpha) = \frac{\left| \overline{PV}(\alpha) - \overline{PV}'(\alpha) \right|}{\overline{PV}(\alpha)}$$

being $\underline{PV}(\alpha), \overline{PV}(\alpha)$ the extremes of the α-cuts obtained with (12) and $\underline{PV}'(\alpha)$ and $\overline{PV}'(\alpha)$ these extremes in the STFN approximation of $\widetilde{PV}$ in (13). Notice that the error increases when α decreases because the Taylor expansion is developed from $\alpha = 1$ to lower levels $(0 \leq \alpha < 1)$. In any case, they are never greater than 0.1% which suggests that the proposed approximation is quite accurate[2] in this case.

Table 1. Comparison of the α-cuts of the fuzzy present value and its STFN approximation

α	$\widetilde{PV}$		$(PV(F,i), r_{PV})$			
	$\underline{PV}(\alpha)$	$\overline{PV}(\alpha)$	$\underline{PV}'(\alpha)$	$\overline{PV}'(\alpha)$	$\underline{err}(\alpha)$	$\overline{err}(\alpha)$
1	6616.40	6616.40	6616.40	6616.40	0.00%	0.00%
0,75	6516.80	6716.64	6516.49	6716.32	0.00%	0.00%
0,5	6417.84	6817.52	6416.57	6816.24	0.02%	0.02%
0,25	6319.50	6919.04	6316.65	6916.15	0.05%	0.04%
0	6221.79	7021.22	6216.74	7016.07	0.08%	0.07%

Now, let us symbolize the present value of a stream of unitary cash flows as:

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[2] If the value of the parameters changes, the value of the errors may vary but usually may be not great. For example, when the variability of the interest rate increases, the duration performs worse the sensibility of the present value and so, the errors of the proposed STFN approximation are greater. In our example, if interest rate spread respect to the 2% is 1% instead 0.5% (i.e. $\tilde{\iota} = (0.02, \ 0.01)$), we will find that $\underline{err}(0) = 0.19\%$ and $\overline{err}(0) = 0.17\%$, i.e., they are still very small.

$$PV(i) = \sum_{j=1}^{n}(1+i)^{-t_j} \tag{13c}$$

and its duration as:

$$D(i) = \frac{\sum_{j=1}^{n} t_j(1+i)^{-t_j}}{PV(i)} \tag{13d}$$

Table 2 shows the expressions of the present value and the Macaulay duration for unitary financial and life contingencies structures. Notice that Li and Panjer (1994) generalize the concept of duration under several yield rate regimes in life insurance pricing with deterministic mortality. Furthermore, whole life contracts are included by considering $n = \varpi - x + 1$, with ϖ the maximum attainable age in the mortality table, and non-deferred liabilities are also evaluated by taking $m = 0$.

Table 2. Present value and duration of several unitary financial and life contingency structures

	Present value of unitary payments	Macaulay duration of the unitary payments
Discount factor	$v^t = (1+i)^{-t}$	$D(v^t) = t$
n-term annuity-immediate	$a_{\overline{n}|} = \dfrac{1 - v^n}{i}$	$D(a_{\overline{n}|}) = \dfrac{\left(1+\frac{1}{i}-n\right) a_{\overline{n}|} - \frac{n}{i}}{a_{\overline{n}|}}$
Deferred m years n-term annuity-due	$_{m|}\ddot{a}_{\overline{n}|} = \displaystyle\sum_{t=m}^{m+n-1}(1+i)^{-t} = a_{\overline{n}|}v^{m-1}$	$D\left(_{m|}\ddot{a}_{\overline{n}|}\right) = \dfrac{\left[\left(m+\frac{1}{i}-n\right) a_{\overline{n}|} - \frac{n}{i}\right] v^{m-1}}{_{m|}\ddot{a}_{\overline{n}|i}}$
n-year pure endowment for a person aged x	$A_{\substack{x:\overline{n}| \\ 1}} = v^n\, _np_x$	$D\left(A_{\substack{x:\overline{n}| \\ 1}}\right) = n$
n-year life insurance deferred m years for a person aged x	$_{m|}A_{\substack{x:\overline{n}| \\ 1}} = \displaystyle\sum_{t=m+1}^{m+n} v^t\, _{t-1|}q_x$	$D\left(_{m|}A_{\substack{x:\overline{n}| \\ 1}}\right) = \dfrac{\sum_{t=m+1}^{m+n} tv^t\, _{t-1|}q_x}{_{m|}A_{\substack{x:\overline{n}| \\ 1}}}$
n-year endowment insurance deferred m years for a person aged x	$_{m|}A_{x:\overline{n}|} = A_{\substack{x:\overline{m+n}| \\ 1}} + _{m|}A_{\substack{x:\overline{n}| \\ 1}}$	$D\left(_{m|}A_{x:\overline{n}|}\right) = \dfrac{(m+n)v^{m+n}\, _{m+n}p_x + \sum_{t=m+1}^{m+n} tv^t\, _{t-1|}q_x}{_{m|}A_{x:\overline{n}|}}$
Deferred m years n-term life annuity-due for a person aged x	$_{m|}\ddot{a}_{x:\overline{n}|} = \displaystyle\sum_{t=m}^{m+n-1} v^t\, _tp_x$	$D\left(_{m|}\ddot{a}_{x:\overline{n}|}\right) = \dfrac{\sum_{t=m}^{nm+n-1} tv^t\, _tp_x}{_{m|}\ddot{a}_{x:\overline{n}|}}$

Notes: * $_np_x$ is the probability that the insured, aged x, survives at age $x + n$.

** $_{n|}q_x$ is the probability that the insured, aged x, dies within the nth year.

Let us define the fuzzy unitary payment by means of a FN of "approximately 1 monetary unit (m.u.)", which will be symbolized as $\tilde{u}$. If it is a STFN, then $\tilde{u} = (1, r_u)$ where $r_u \leq 1$. Of course, we can generate any other fuzzy positive quantity by using any scalar $F \geq 0$, $\tilde{F} = F\tilde{u}$. In our opinion, fuzzy monetary amounts may be suitable for modeling situations in which they are linked to the evolution of an economic index or when we are evaluating not only claimed quantities, but also future expenses associated with them that are not known exactly (e.g. settlement costs). In this situation, fuzzy unitary cash-flows cover the

case where the cash flows are "approximately constant". In any case, all the results can be extended to fuzzy variable cash-flows.

For a fuzzy interest rate $\tilde{\iota} = (i, r_i)$, the present value of the stream of fuzzy unitary quantities is the FN $\widetilde{PV}$ and using the STFN approximation $\widetilde{PV} \approx (PV(i), r_{PV})$, where, considering (13):

$$PV(i) = \sum_{j=1}^{n}(1 + i)^{-t_j} \tag{14a}$$

$$r_{PV} = r_u PV(i) + PV(i)\frac{D(i)}{1+i}r_i = PV(i)\left(r_u + \frac{D(i)}{1+i}r_i\right) \tag{14b}$$

Next, we approximate with STFN several classical financial and actuarial structures with fuzzy interest rates and payments. To do so, we use the results in Table 2, (4) and (14). Notice that the four last expressions contained in the first column of Table 2 can be obtained from classical approach to life contingencies valuation, which Wolthuis and van Hoek (1986, p. 218) name as classical deterministic model for life contingencies. So, in these expressions each possible outcome of the insured contingency and its probability is replaced for its expected value. This is the approach used in Lemaire (1990), Ostaszewski (1993) and Betzuen *et al.* (1997). In this way, evaluating life insurances and annuity contracts is reduced to evaluating (10) with fuzzy parameters as we have just shown above. The results are not STFNs, though.

- For a fuzzy unitary payment $\tilde{u} = (1, r_u)$, where $r_u \leq 1$, and for an interest rate $\tilde{\iota} = (i, r_i)$ the discount factor, is:

$$\tilde{v}^t = \tilde{u}(1 + \tilde{\iota})^{-t}$$

and by using the STFN approximation $\tilde{v}^t \approx (v^t, r_{v^t})$

$$v^t = (1 + i)^{-t}$$

$$r_{v^t} = v^t\left(r_u + \frac{t}{1 + i}r_i\right)$$

- In the case of a deferred m years n-term annuity-due the fuzzy present value ${}_{m|}\widetilde{\ddot{a}}_{\overline{n}|}$ is:

$${}_{m|}\widetilde{\ddot{a}}_{\overline{n}|} = \tilde{u}\sum_{t=m}^{m+n-1}(1 + \tilde{\iota})^{-t}$$

and it can be approximated by ${}_{m|}\widetilde{\ddot{a}}_{\overline{n}|} \approx \left({}_{m|}\ddot{a}_{\overline{n}|}, r_{{}_{m|}\ddot{a}_{\overline{n}|}}\right)$, where, by using (14):

$${}_{m|}\ddot{a}_{\overline{n}|} = \frac{1 - v^n}{i}v^{m-1}$$

$$r_{{}_{m|}\ddot{a}_{\overline{n}|}} = {}_{m|}\ddot{a}_{\overline{n}|}\left(r_u + \frac{D\left({}_{m|}\ddot{a}_{\overline{n}|}\right)}{1 + i}r_i\right)$$

- The fuzzy mathematical expectation of the present value of an n-year pure endowment for a person aged x is:

$$\tilde{A}_{\underset{x:\bar{n}|}{1}} = \tilde{v}^n \, {}_np_x$$

and $\tilde{A}_{\underset{x:\bar{n}|}{1}} \approx \left(A_{\underset{x:\bar{n}|}{1}}, r_{A_{\underset{x:\bar{n}|}{1}}} \right)$, being:

$$A_{\underset{x:\bar{n}|}{1}} = v^n \, {}_np_x \tag{15a}$$

$$r_{A_{\underset{x:\bar{n}|}{1}}} = A_{\underset{x:\bar{n}|}{1}} \left(r_u + \frac{n}{1+i} r_i \right) \tag{15b}$$

- In the case of an n-year life insurance deferred m years for a person aged x and fuzzy unitary payments, the fuzzy mathematical expectation value is:

$$_{m|}\tilde{A}_{\underset{x:\bar{n}|}{1}} = \tilde{u} \sum_{t=m+1}^{m+n} \tilde{v}^t \, {}_{t-1|}q_x$$

So, $_{m|}\tilde{A}_{\underset{x:\bar{n}|}{1}} \approx \left({}_{m|}A_{\underset{x:\bar{n}|}{1}}, r_{{}_{m|}A_{\underset{x:\bar{n}|}{1}}} \right)$, where:

$$_{m|}A_{\underset{x:\bar{n}|}{1}} = \sum_{t=m+1}^{m+n} v^t \, {}_{t-1|}q_x \tag{16a}$$

$$r_{{}_{m|}A_{\underset{x:\bar{n}|}{1}}} = {}_{m|}A_{\underset{x:\bar{n}|}{1}} \left(r_u + \frac{D\left({}_{m|}A_{\underset{x:\bar{n}|}{1}} \right)}{1+i} r_i \right) \tag{16b}$$

- In the case of an n-year endowment insurance deferred m years for a person aged x, we have $_{m|}A_{x:\bar{n}|} = A_{\underset{x:\overline{m+n}|}{1}} + {}_{m|}A_{\underset{x:\bar{n}|}{1}}$. So we have only to sum two SFTNs to obtain the approximation of $_{m|}\tilde{A}_{x:\bar{n}|} \approx \left({}_{m|}A_{x:\bar{n}|}, r_{{}_{m|}A_{x:\bar{n}|}} \right)$, with:

$$_{m|}A_{x:\bar{n}|} = A_{\underset{x:\overline{m+n}|}{1}} + {}_{m|}A_{\underset{x:\bar{n}|}{1}} \tag{17a}$$

$$r_{{}_{m|}A_{x:\bar{n}|}} = A_{\underset{x:\overline{m+n}|}{1}} \left(r_u + \frac{m+n}{1+i} r_i \right) + {}_{m|}A_{\underset{x:\bar{n}|}{1}} \left(r_u + \frac{D\left({}_{m|}A_{\underset{x:\bar{n}|}{1}} \right)}{1+i} r_i \right) =$$

$$= {}_{m|}A_{x:\bar{n}|} \left(r_u + \frac{D\left({}_{m|}A_{x:\bar{n}|} \right)}{1+i} r_i \right) \tag{17b}$$

- Finally, regarding a deferred m years n-term life annuity-due for a person aged x, the fuzzy mathematical expectation value is:

$$_{m|}\tilde{\ddot{a}}_{x:\overline{n}|} = \sum_{t=m}^{m+n-1} \tilde{v}^t\ {}_tp_x$$

And so, $_{m|}\tilde{\ddot{a}}_{x:\overline{n}|} \approx \left(_{m|}\ddot{a}_{x:\overline{n}|}, r_{_{m|}\ddot{a}_{x:\overline{n}|}} \right)$ where:

$$_{m|}\ddot{a}_{x:\overline{n}|} = \sum_{t=m}^{m+n-1} v^t\ {}_tp_x \qquad (18a)$$

$$r_{_{m|}\ddot{a}_{x:\overline{n}|}} = {}_{m|}\ddot{a}_{x:\overline{n}|}\left(r_u + \frac{D\left(_{m|}\ddot{a}_{x:\overline{n}|}\right)}{1+i}r_i \right) \qquad (18b)$$

Whereas Table 3a shows the approximated fuzzy mathematical expectation for several types of life contingencies, Table 3b shows its defuzzified value with the concept of expected value of a FN contained in (2).

Table 3a. Approximated fuzzy mathematical expectation of the present value of several life contingencies

	$F \cdot {}_{m|}\tilde{A}_{\overset{1}{x:\overline{n}|}}$			$F \cdot {}_{m|}\tilde{A}_{x:\overline{n}|}$			$F \cdot {}_{m|}\tilde{\ddot{a}}_{x:\overline{n}|}$		
x	$F \cdot {}_{m|}A_{\overset{1}{x:\bar{n}|}}$	$F \cdot r_{_{m|}A_{\overset{1}{x:\bar{n}|}}}$	$D\left(_{m|}A_{\overset{1}{x:\bar{n}|}}\right)$	$F \cdot {}_{m|}A_{x:\overline{n}|}$	$F \cdot r_{_{m|}A_{x:\overline{n}|}}$	$D\left(_{m|}A_{x:\overline{n}|}\right)$	$F \cdot {}_{m|}\ddot{a}_{x:\overline{n}|}$	$F \cdot r_{_{m|}\ddot{a}_{x:\overline{n}|}}$	$D\left(_{m|}\ddot{a}_{x:\overline{n}|}\right)$
25	326.01	94.09	54.79	820.61	56.57	9.98	34374.28	4646.69	24.47
35	394.87	95.57	45.29	820.88	56.52	9.97	30863.17	3710.70	21.41
45	476.24	93.68	36.05	821.94	56.31	9.90	26713.92	2789.71	18.19
55	567.82	87.90	27.50	824.22	55.87	9.75	22042.86	1939.60	14.83
65	668.34	77.70	19.64	828.74	55.00	9.46	16917.07	1203.92	11.38
75	775.85	62.71	12.41	843.43	52.16	8.54	11433.78	624.45	7.97
85	872.96	46.06	6.68	884.77	44.08	6.08	6482.19	259.88	4.94
95	933.82	34.18	3.39	934.47	34.13	3.37	3367.70	100.54	2.71

Notes: * $\tilde{\imath} = (0.02,\ 0.005)$, $\tilde{u} = (1, 0.02)$ and F=1000 m.u. The life contingency probabilities are taken from the mortality tables of the Spanish population for both males and females in the year 2010 included in the Human Mortality Database (http://www.mortality.org), and obtained as explained in Wilmoth *et al.* (2007).

** In all cases, life contingencies are not deferred, i.e. $m = 0$ and both life insurance and annuities are whole life, i.e. $n = \varpi - x + 1$. On the other hand, for the endowment insurance we have considered $n = 10$.

Table 3b. Expected value for the fuzzy mathematical expectation of the present value of life contingencies in Table 3a

	$EV\left(1000 \cdot {}_{m|}\tilde{A}_{\overset{1}{x:\overline{n}|}};\lambda\right)$			$EV\left(1000 \cdot {}_{m|}\tilde{A}_{x:\overline{n}|};\lambda\right)$			$EV\left(1000 \cdot {}_{m|}\tilde{\ddot{a}}_{x:\overline{n}|};\lambda\right)$		
$x \backslash \lambda$	0	0.5	1	0	0.5	1	0	0.5	1
25	278.97	326.01	373.06	792.33	820.61	848.90	32050.94	34374.28	36697.63
35	347.09	394.87	442.66	792.62	820.88	849.14	29007.82	30863.17	32718.52
45	429.40	476.24	523.08	793.79	821.94	850.10	25319.07	26713.92	28108.78
55	523.87	567.82	611.77	796.29	824.22	852.16	21073.06	22042.86	23012.66
65	629.49	668.34	707.19	801.24	828.74	856.24	16315.11	16917.07	17519.03

75	744.50	775.85	807.21	817.35	843.43	869.51	11121.56	11433.78	11746.01
85	849.93	872.96	895.99	862.73	884.77	906.81	6352.25	6482.19	6612.13
95	916.73	933.82	950.91	917.41	934.47	951.54	3317.43	3367.70	3417.97

4. PRICING LIFE CONTINGENCIES WITH FUZZY RANDOM VARIABLES

In the previous section we calculated the present value of several life contingency structures by "fuzzifying" the classical deterministic model of life contingencies. This approach has the disadvantage that all the information providing a complete statistical description of the present values of life contingent liabilities is lost. To avoid this, we propose adapting the stochastic approach to life insurance and annuities to the case of fuzzy unitary cash flows and fuzzy discount interest rate by using the concept of FRV. Our developments here are based on Andrés-Sánchez and González-Vila (2012, 2014). In these papers, the authors present a general formulation for the fair value and risk of individual contracts by means of FRVs. However, they do not assume any shape for fuzzy interest rates *a priori* and, consequently no closed expressions for these magnitudes are developed. Likewise, crisp unitary amounts are considered.

In the case that the interest rate and the unitary cash flows are STFN, any kind of life contingency policy produces for the insurer a FRV *present value of life contingencies*, $\widetilde{\mathbf{Z}}$, whose realizations can be approximated by STFNs. Following the developments in the previous sections, we can write $\widetilde{\mathbf{Z}} \approx (\mathbf{Z}, \mathbf{r_Z})$. Furthermore, we denote the jth outcome of $\widetilde{\mathbf{Z}}$ as $\widetilde{PV}_j(i) = \left(PV_j(i), r_{PV_j(i)} \right)$, where $PV_j(i)$ denotes the present value of 1 m.u. or a unitary annuity whereas $r_{PV_j(i)}$ is, from (14):

$$r_{PV_j(i)} = PV_j(i) \left(r_u + \frac{D_j(i)}{1+i} r_i \right)$$

Thus, we can find the mathematical expectation of the FRV present value of life contingencies as:

$$E\big(\widetilde{\mathbf{Z}}\big) = \big(E(\mathbf{Z}), E(\mathbf{r_Z}) \big)$$

where:

$$E(\mathbf{Z}) = \sum_j PV_j(i) p_j$$

$$E(\mathbf{r_Z}) = \sum_j r_{PV_j(i)} \, p_j = \sum_j PV_j(i) \left(r_u + \frac{D_j(i)}{1+i} r_i \right) p_j \tag{19}$$

and p_j is the probability of the jth outcome.

For the variance of $\widetilde{\mathbf{Z}}$, taking into account (9), we can obtain:

$$V\big(\widetilde{\mathbf{Z}}\big) = V(\mathbf{Z}) + \tfrac{1}{3} V(\mathbf{r_Z})$$

being:

$$V(\mathbf{Z}) = E(\mathbf{Z}^2) - \left(E(\mathbf{Z})\right)^2 = \sum_j \left(PV_j(i)\right)^2 p_j - \left(\sum_j PV_j(i)p_j\right)^2$$

(20a)

and:

$$V(\mathbf{r_Z}) = E((\mathbf{r_Z})^2) - \left(E(\mathbf{r_Z})\right)^2 =$$

$$= \sum_j \left[PV_j(i)\left(r_u + \frac{D_j(i)}{1+i}r_i\right)\right]^2 p_j - \left(\sum_j PV_j(i)\left(r_u + \frac{D_j(i)}{1+i}r_i\right)p_j\right)^2$$

(20b)

Notice that using a STFN approximation to outcomes simplifies the treatment of the FRV "present value of life contingencies" initially proposed in Andrés-Sánchez and González-Vila (2012, 2014). Indeed, under their approach, to obtain numerical results, it is necessary to discretize the membership level α (in the abovementioned papers the exampes are developed for $\alpha = 1, 0.75, 0.5, 0.25, 0$) to obtain for each insured event a lower and upper present value by using (3) and (10). In such a way, $\widetilde{Z}$ will finally be characterized, for each of the predefined values of α, by an infima and suprema conventional RV. So, under the hypothesis about fuzzy amounts and interest rates assumed in our paper, and this discretization of α, $\widetilde{Z}$ will be characterized by 10 RV[3]. However, with our introduced STFN approach the number of RV that have to be used to determine $\widetilde{Z}$ is reduced to only two: a mode and a spread.

In a life insurance contract, the randomness derives from the maturity of the payment of the insured amounts. So, we will symbolize as $\mathbf{T}$ the discrete RV "payment maturity". If there is no insured money for some death ages, then the outcome of $\mathbf{T}$ is ∞ years. So, in any contract of this kind, the FRV $\widetilde{Z}$ can be expressed as:

$$\widetilde{\mathbf{Z}} = \tilde{v}^T$$

(21a)

In terms of a STFN approximation (21a) can be written as:

$$\widetilde{\mathbf{Z}} \approx (\mathbf{Z}, \mathbf{r_Z}) = \left(v^T, v^T\left(r_u + \frac{T}{1+i}r_i\right)\right)$$

(21b)

Similarly, in a life annuity the randomness derives from the number of terms to be paid by the insurer. We symbolize as $\mathbf{N}$ the discrete RV "number of terms" for a life annuity-due. In this case we can express $\widetilde{Z}$ as:

$$\widetilde{\mathbf{Z}} = \sum_{j=1}^{N} \tilde{v}^{j-1}$$

(22a)

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[3] In the case of more discrete values of the membership grade α, the number of RV would be greater. In general, if the membership level α is discretized by n values, $\widetilde{Z}$ will be characterized with $2n$ RV.

That under the STFN approximation of this paper can be written as:

$$\widetilde{\mathbf{Z}} \approx (\mathbf{Z}, \boldsymbol{r_Z}) = \left({}_{m|}\ddot{a}_{\overline{N}|}, {}_{m|}\ddot{a}_{\overline{N}|} \left(r_u + \frac{D\left({}_{m|}\ddot{a}_{\overline{N}|} \right)}{1+i} r_i \right) \right)$$

(22b)

From this general setting, below we study some actuarial structures by applying the stochastic approach to life insurance and annuities to the case of fuzzy unitary cash flows and fuzzy discount interest rate.

- Let us consider an n-year pure endowment for a person aged x. In this case, it is supposed that the liability is $\tilde{u}$ m.u. if the insured survives n years and no payment otherwise. From the mortality tables we can deduce the RV "payment maturity" $\boldsymbol{T} = \{n, \infty\}$ with probabilities[4] $\{ {}_np_x, {}_nq_x \}$. A fuzzy random approach to this kind of insurance for any shape of fuzzy interest rates can be found in Andrés-Sánchez and González-Vila (2014). In our particular case, where the outcomes are approximated with STFNs, the FRV present value of the pure endowment, which we symbolize as $\widetilde{A}_{x:\bar{n}|}^{\,1}$, can be represented as the pair $\widetilde{A}_{x:\bar{n}|}^{\,1} \approx \left(A_{x:\bar{n}|}^{\,1}, r_{A_{x:\bar{n}|}^{\,1}} \right)$ where, from (21b):

$$\left(A_{x:\bar{n}|}^{\,1}, r_{A_{x:\bar{n}|}^{\,1}} \right) = \begin{cases} 0 & \text{with probability} \quad {}_nq_x \\ \left(v^n, v^n \left(r_u + \frac{n}{1+i} r_i \right) \right) & \text{with probability} \quad {}_np_x \end{cases}$$

(23)

So, $E\left(\widetilde{A}_{x:\bar{n}|}^{\,1} \right) = \left(E\left(A_{x:\bar{n}|}^{\,1} \right), E\left(r_{A_{x:\bar{n}|}^{\,1}} \right) \right) = \tilde{A}_{x:\bar{n}|}^{\,1} \approx \left(A_{x:\bar{n}|}^{\,1}, r_{A_{x:\bar{n}|}^{\,1}} \right)$, i.e. (15a) and (15b).

Now, let us obtain the variance $V\left(\widetilde{A}_{x:\bar{n}|}^{\,1} \right)$. Following (9) and (20) it can be obtained as:

$$V\left(\widetilde{A}_{x:\bar{n}|}^{\,1} \right) = V\left(A_{x:\bar{n}|}^{\,1} \right) + \frac{1}{3} V\left(r_{A_{x:\bar{n}|}^{\,1}} \right)$$

It is easy to show that:

$$V\left(A_{x:\bar{n}|}^{\,1} \right) = (v^n)^2 \, {}_np_x - \left(A_{x:\bar{n}|}^{\,1} \right)^2 = v^{2n} \, {}_np_x \, {}_nq_x$$

$$V\left(r_{A_{x:\bar{n}|}^{\,1}} \right) = \left(r_u + \frac{n}{1+i} r_i \right)^2 v^{2n} \, {}_np_x - \left(r_{A_{x:\bar{n}|}^{\,1}} \right)^2 = \left(r_u + \frac{n}{1+i} r_i \right)^2 v^{2n} \, {}_np_x \, {}_nq_x$$

and so:

---

[4] In this kind of insurances, the insured person does not receive any amount if he dies within the first n years of the contract. It is equivalent to consider the maturity $t \to \infty$, since in this case the present value of any amount is also 0.

$$V\left(\widetilde{A}_{\underset{x:\overline{n}|}{1}}\right) = \left[1 + \frac{1}{3}\left(r_u + \frac{n}{1+i}r_i\right)^2\right]v^{2n}\,{}_np_x\,{}_nq_x$$

- Now, let us take an n-year life insurance, deferred m years, for a person aged x. Of course, if it were a whole life insurance policy, n would be $\varpi - x + 1$. The insured party aged x will receive the STFN $\tilde{u}$ m.u. at the end of the year of his death, if he dies between the ages $x + m$ and $x + m + n$ and otherwise he does not receive anything. So, the RV "maturity of the benefits", T, has as its outcomes $\{m+1, m+2, \dots, m+n, \infty\}$ with probabilities $\{{}_{m|}q_x, {}_{m+1|}q_x, \dots, {}_{m+n-1|}q_x, {}_{m+n}p_x + {}_mq_x\}$. The FRV present value of the life insurance, ${}_{m|}\widetilde{A}_{\underset{x:\overline{n}|}{1}}$, developed for a more general shape of discount rates in Andrés-Sánchez and González-Vila (2014), can be represented with our STFN approximation as

$${}_{m|}\widetilde{A}_{\underset{x:\overline{n}|}{1}} \approx \left({}_{m|}A_{\underset{x:\overline{n}|}{1}}, r_{{}_{m|}A_{\underset{x:\overline{n}|}{1}}}\right) \text{ where, from (21b):}$$

$$\left({}_{m|}A_{\underset{x:\overline{n}|}{1}}, r_{{}_{m|}A_{\underset{x:\overline{n}|}{1}}}\right) =$$

$$= \begin{cases} 0 & \text{with probability} & {}_{m+n}p_x + {}_mq_x \\ \left(v^t, v^t\left(r_u + \frac{t}{1+i}r_i\right)\right) & \text{with probability} & {}_{t-1|}q_x, t = m+1, m+2, \dots, m+n \end{cases}$$

(24)

So, $E\left({}_{m|}\widetilde{A}_{\underset{x:\overline{n}|}{1}}\right) = \left(E\left({}_{m|}A_{\underset{x:\overline{n}|}{1}}\right), E\left(r_{{}_{m|}A_{\underset{x:\overline{n}|}{1}}}\right)\right) = {}_{m|}\tilde{A}_{\underset{x:\overline{n}|}{1}} \approx \left({}_{m|}A_{\underset{x:\overline{n}|}{1}}, r_{{}_{m|}A_{\underset{x:\overline{n}|}{1}}}\right)$ i.e. (16a) and (16b).

Considering (9), it turns out that:

$$V\left({}_{m|}\widetilde{A}_{\underset{x:\overline{n}|}{1}}\right) = V\left({}_{m|}A_{\underset{x:\overline{n}|}{1}}\right) + \frac{1}{3}V\left(r_{{}_{m|}A_{\underset{x:\overline{n}|}{1}}}\right)$$

and from the definition of ${}_{m|}\widetilde{A}_{\underset{x:\overline{n}|}{1}}$, (20) and Table 2 we find:

$$V\left({}_{m|}A_{\underset{x:\overline{n}|}{1}}\right) = \sum_{t=m+1}^{m+n} v^{2t}\,{}_{t-1|}q_x - \left({}_{m|}A_{\underset{x:\overline{n}|}{1}}\right)^2$$

$$V\left(r_{{}_{m|}A_{\underset{x:\overline{n}|}{1}}}\right) = \sum_{t=m+1}^{m+n} v^{2t}\left(r_u + \frac{t}{1+i}r_i\right)^2\,{}_{t-1|}q_x - \left(r_{{}_{m|}A_{\underset{x:\overline{n}|}{1}}}\right)^2$$

- For an n-year endowment insurance, deferred m years, linked to a person aged x years, the outcomes of the RV "maturity of the benefits", T, are $\{\infty, m+1, m+2, \dots, m+n-1, m+n\}$ with probabilities $\{{}_mq_x, {}_{m|}q_x, {}_{m+1|}q_x, \dots, {}_{m+n-1|}q_x, {}_{m+n}p_x\}$. As all the benefits are considered as STFN, the fuzzy RV present value of liabilities, ${}_{m|}\widetilde{A}_{x:\overline{n}|}$, that was exposed

for any kind of fuzzy interest rate in Andrés-Sánchez and González Vila (2014), can be approximated, following our STFN, by $\left({}_{m|}\boldsymbol{A}_{x:\overline{n}|}, \boldsymbol{r}_{{}_{m|}A_{x:\overline{n}|}}\right)$ where, bearing in mind (21b):

$$\left({}_{m|}\boldsymbol{A}_{x:\overline{n}|}, \boldsymbol{r}_{{}_{m|}A_{x:\overline{n}|}}\right) =$$

$$= \begin{cases} 0 & \text{with probability} & {}_{m}q_x \\ \left(v^t, v^t\left(r_u + \dfrac{t}{1+i}r_i\right)\right) & \text{with probability} & {}_{t-1|}q_x, t = m+1, m+2, \dots, m+n \\ \left(v^{m+n}, v^{m+n}\left(r_u + \dfrac{m+n}{1+i}r_i\right)\right) & \text{with probability} & {}_{m+n}p_x \end{cases}$$

$$(25)$$

In this case $E\left({}_{m|}\widetilde{\boldsymbol{A}}_{x:\overline{n}|}\right) = \left(E\left({}_{m|}\boldsymbol{A}_{x:\overline{n}|}\right), E\left(\boldsymbol{r}_{{}_{m|}A_{x:\overline{n}|}}\right)\right) = {}_{m|}\widetilde{A}_{x:\overline{n}|}$ so, considering (17), $E\left({}_{m|}\boldsymbol{A}_{x:\overline{n}|}\right) \approx {}_{m|}A_{x:\overline{n}|}$ and $E\left(\boldsymbol{r}_{{}_{m|}A_{x:\overline{n}|}}\right) \approx r_{{}_{m|}A_{x:\overline{n}|}}$.

To obtain the variance $V\left({}_{m|}\widetilde{\boldsymbol{A}}_{x:\overline{n}|}\right) = V\left({}_{m|}\boldsymbol{A}_{x:\overline{n}|}\right) + \frac{1}{3}V\left(\boldsymbol{r}_{{}_{m|}A_{x:\overline{n}|}}\right)$, we have to take into account that, from (20):

$$V\left({}_{m|}\boldsymbol{A}_{x:\overline{n}|}\right) = \sum_{t=m+1}^{m+n} v^{2t}\,{}_{t-1|}q_x + v^{2(m+n)}\,{}_{m+n}p_x - \left({}_{m|}A_{x:\overline{n}|}\right)^2$$

$$V\left(\boldsymbol{r}_{{}_{m|}A_{x:\overline{n}|}}\right) = \sum_{t=m+1}^{m+n} v^{2t}\left(r_u + \frac{t}{1+i}r_i\right)^2 {}_{t-1|}q_x + v^{2(m+n)}\left(r_u + \frac{m+n}{1+i}r_i\right)^2 {}_{m+n}p_x - \left(r_{{}_{m|}A_{x:\overline{n}|}}\right)^2$$

- Finally, regarding a deferred m years n-term life annuity-due for a person aged x the randomness will derive from the number of terms to be paid by the insurer. For this reason, the discrete RV "number of terms", $\boldsymbol{N}$, has to be considered. Its outcomes are $\{0,1,2,\dots,n-1,n\}$ where their probabilities are $\left\{{}_{m}q_x, {}_{m|}q_x, {}_{m+1|}q_x, \dots, {}_{m+n-2|}q_x, {}_{m+n-1}p_x\right\}$. For any shape of discount rates, a general present value formulation of the annuity can be found in de Andrés-Sánchez and González-Vila (2012) but, once again, if we consider that the outcomes of the FRV present value of the annuity are approximated by STFNs, this present value can be represented as ${}_{m|}\widetilde{\ddot{\boldsymbol{a}}}_{x:\overline{n}|} \approx \left({}_{m|}\ddot{\boldsymbol{a}}_{x:\overline{n}|}, \boldsymbol{r}_{{}_{m|}\ddot{a}_{x:\overline{n}|}}\right)$.

Therefore, following (22b):

$$\left({}_{m|}\ddot{\boldsymbol{a}}_{x:\overline{n}|}, \boldsymbol{r}_{{}_{m|}\ddot{a}_{x:\overline{n}|}}\right) =$$

$$= \begin{cases} 0 & \text{with probability} & {}_{m}q_x \\ \left({}_{m|}\ddot{a}_{\overline{j}|}, {}_{m|}\ddot{a}_{\overline{j}|}\left(r_u + \dfrac{D\left({}_{m|}\ddot{a}_{\overline{j}|}\right)}{1+i}r_i\right)\right) & \text{with probability} & {}_{m+j-1|}q_x, j = 1,2,\dots,n-1 \\ \left({}_{m|}\ddot{a}_{\overline{n}|}, {}_{m|}\ddot{a}_{\overline{n}|}\left(r_u + \dfrac{D\left({}_{m|}\ddot{a}_{\overline{n}|}\right)}{1+i}r_i\right)\right) & \text{with probability} & {}_{m+n-1}p_x \end{cases}$$

Furthermore, $E\left({}_{m|}\widetilde{\ddot{a}}_{x:\overline{n}|}\right) = {}_{m|}\widetilde{\ddot{a}}_{x:\overline{n}|} \approx \left({}_{m|}\ddot{a}_{x:\overline{n}|}, r_{{}_{m|}\ddot{a}_{x:\overline{n}|}}\right)$, where ${}_{m|}\ddot{a}_{x:\overline{n}|}$ and $r_{{}_{m|}\ddot{a}_{x:\overline{n}|}}$ can be obtained with the expressions contained in (18).

The determination of the variance requires considering the relation:

$$V\left({}_{m|}\widetilde{\ddot{a}}_{x:\overline{n}|}\right) = V\left({}_{m|}\ddot{a}_{x:\overline{n}|}\right) + \frac{1}{3}V\left(r_{{}_{m|}\ddot{a}_{x:\overline{n}|}}\right)$$

where:

$$V\left({}_{m|}\ddot{a}_{x:\overline{n}|}\right) = \sum_{j=1}^{n-1}\left({}_{m|}\ddot{a}_{\overline{j}|}\right)^2 {}_{m+j-1|}q_x + \left({}_{m|}\ddot{a}_{\overline{n}|}\right)^2 {}_{m+n-1}p_x - \left({}_{m|}\ddot{a}_{\overline{n}|}\right)^2$$

$$V\left(r_{{}_{m|}\ddot{a}_{x:\overline{n}|}}\right) = \sum_{j=1}^{n-1}\left({}_{m|}\ddot{a}_{\overline{j}|}\right)^2 \left(r_u + \frac{D\left({}_{m|}\ddot{a}_{\overline{j}|}\right)}{1+i}r_i\right)^2 {}_{m+j-1|}q_x +$$

$$+ \left({}_{m|}\ddot{a}_{\overline{n}|}\right)^2 \left(r_u + \frac{D\left({}_{m|}\ddot{a}_{\overline{n}|}\right)}{1+i}r_i\right)^2 {}_{m+n-1}p_x - \left(r_{{}_{m|}\ddot{a}_{x:\overline{n}|}}\right)^2$$

Table 4 shows the standard deviation of the present value of the same life contingencies as those presented in Table 3a.

Table 4. Standard deviation of the present value of life contingencies in Table 3a

	$1000 \cdot \widetilde{A}_{{}^1_{x:\overline{\varpi-x+1}|}}$			$1000 \cdot \widetilde{A}_{x:\overline{10}|}$			$1000 \cdot \widetilde{\ddot{a}}_{x:\overline{\varpi-x+1}|}$		
x	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
25	93.81	5.02	93.85	5.24	1.03	5.28	4782.24	1156.03	4828.59
35	106.55	6.79	106.62	7.31	1.43	7.36	5432.41	1111.38	5470.17
45	117.97	10.58	118.13	12.66	2.48	12.74	6016.22	1012.33	6044.55
55	120.98	14.05	121.25	19.70	3.86	19.83	6170.71	844.53	6189.95
65	113.97	16.11	114.35	28.31	5.55	28.49	5815.99	632.11	5827.43
75	98.85	16.48	99.31	43.01	8.45	43.29	5046.75	411.42	5052.34
85	73.01	13.83	73.44	55.32	11.02	55.69	3738.57	217.17	3740.67
95	46.29	8.94	46.58	42.40	8.72	42.70	2238.69	93.88	2239.35

Notes: * Life contingencies and technical basis are the same as in Table 3a. So, the mathematical expectation for the present value of liabilities is contained in this table.

** (1) stands for the standard deviation of the RV defined by the centers, (2) for the standard deviation of the RVs whose outcomes are the spreads and (3) is the standard deviation of the FRV present value.

5. PRICING PORTFOLIOS OF FUZZY UNITARY LIFE CONTINGENCIES

Given that the mathematical expectation and the variance of FRVs have similar properties to analogous indicators for conventional RVs, computing the mathematical expectation

and the variance for the whole portfolio, under the hypothesis of independence of mortality between policies, is straightforward. This hypothesis implies, following Wolthuis and van Hoek (1986, p.233), that the policies are considered stochastically independent; for instance, the portfolio does not contain more than one policy on a single life.

If we symbolize the present value of the jth liability of the portfolio as $\widetilde{Z}_j = \left(Z_j, r_{Z_j} \right)$, the present value of a portfolio made up of J contracts is a FRV $\tilde{L}$ where:

$$\tilde{L} = \sum_{j=1}^{J} \widetilde{Z}_j$$

And so, $\tilde{L} = (L, r_L)$ with:

$$(L, r_L) = \left(\sum_{j=1}^{J} Z_j, \sum_{j=1}^{J} r_{Z_j} \right)$$

The rules for the expectation of RVs are identical to those for FRV. So, for $E(\tilde{L}) = \left(E(L), E(r_L) \right)$ we immediately derive:

$$E(L) = \sum_{j=1}^{J} E(Z_j)$$

$$E(r_L) = \sum_{j=1}^{J} E\left(r_{Z_j} \right)$$

Likewise, given that we are dealing with independent RVs, to obtain $V(\tilde{L})$ we have to compute:

$$V(L) = \sum_{j=1}^{J} V(Z_j)$$

$$V(r_L) = \sum_{j=1}^{J} V\left(r_{Z_j} \right)$$

And so, $V(\tilde{L}) = V(L) + \frac{1}{3} V(r_L)$.

On the other hand, under our hypothesis that the cost of claims and discount rates are STFN, we can extend the results in Andrés-Sánchez and González-Vila (2012, 2014) to compute the quantiles of the present value of liabilities which are commonly used to measure the risk of the life contingencies insurance portfolio (see, for example, Alegre and Claramunt (1995)).

If the group of policies can be divided into large homogeneous sets of liabilities, a suitable way to approach $\widetilde{Z}$ is by using a fuzzy normal RV with a fuzzy mean $E(\widetilde{Z}) =$

$\left(E(\boldsymbol{Z}), E(\boldsymbol{r_Z})\right)$ and a crisp standard deviation $SD(\widetilde{\boldsymbol{Z}})$. Fuzzy normal RVs have been used in the financial context for portfolio selection (Inuiguchi and Ramik (2000)) and to obtain value-at-risk in a fuzzy environment (Zmeskal (2005)). In an actuarial context, Shapiro (2009) suggests their use in practical applications.

When the set of liabilities cannot be divided into broad groups of homogeneous policies, Andrés-Sánchez and González-Vila (2012, 2014) propose using statistical simulation to obtain empirically the FRV "present value of the portfolio's life contingencies". In these papers, the authors adapt the schema proposed in Pitacco (1986) to the fact that the outcomes are FNs. The steps that have to be taken to obtain the percentiles of the present value are presented below.

Step 1 Simulate S times for each of the J policies the RV "payment maturity" ($\boldsymbol{T}$) in insurance contracts and "number of terms" ($\boldsymbol{N}$) in annuity policies.

Step 2. By applying (23), (24), (25) or (26) to the sth simulation of the jth life contingency contract, we obtain a STFN that quantifies the fuzzy present value of this policy in the sth simulation. Notice that the use of the STFN approximation reduces notably the calculations that have to be implemented to obtain simulated fuzzy present values respect to Andrés-Sánchez and González-Vila (2012, 2014). In fact, in our paper, the quantification of the present value that derives from the sth simulation for the jth policy only requires the calculation of one crisp present value (that is the mode of the fuzzy present value) and a variability margin (the spread of this FN) closely linked to the Macaulay duration of this present value. On the other hand, in those papers, given that the present value calculations are based on the α-cuts of discount rates, if n values of α were considered, the fuzzy present value of the sth simulation for the jth policy would require calculating $2n$ values. So, the reduction in the computations is obvious.

Step 3. The sth simulation of the whole portfolio of life contingencies is obtained as a STFN by summing the present value of the J policies for that sth simulation. So, we have an approximation to the FRV present value of the portfolio, $\widetilde{\boldsymbol{L}}$, where each outcome has the same probability $1/S$.

Step 4. From the α-cut representation of the approximation to $\widetilde{\boldsymbol{L}}$ we can obtain the bounds of their percentiles by using (6a) and (6b).

Tables 5a and 5b describe a small set of non-deferred whole life insurances and show the 99th and 95th percentiles of the present value of liabilities for segregated and aggregate subsets of contracts. Likewise, in Tables 6a and 6b we develop an analogous numerical application for a set of life annuities. Our examples have been performed with Microsoft Excel 2000® and we have done S=5000 simulations for the value of life contingencies portfolios. Likewise, to measure the simplification of our STFN approach, we have also developed the same examples with the method proposed in Andrés-Sánchez and González-Vila (2012, 2014) taking $\alpha = 1, 0.75, 0.5, 0.25$ and 0. Regarding the run time, we have found that it corresponds, essentially, to the simulation of $\boldsymbol{T}$ or $\boldsymbol{N}$ (step 1). So, we have not found any relevant difference between the two calculation methods of the present value of liabilities (steps 2 and 3). However, this does not follow in the weight of the files that contain the calculations. The weight of the Excel files made with our STFN approach is

less than half of those made with the developments in Andrés-Sánchez and González-Vila (2012, 2014).

Table 5a. Portfolio of whole life insurances $1000 \cdot \tilde{A}_{1 \atop x:\overline{\varpi-x+1}|}$ with several ages and contracts

Age	Number of contracts	$\tilde{L}$		
		$E(L)$	$E(r_L)$	$SD(\tilde{L})$
55	5	2839.09	439.52	271.12
65	10	6683.37	776.95	361.61
75	5	3879.23	313.53	222.06
Whole portfolio	20	13401.70	1530.00	503.57

Note: * Whole life insurances and technical basis are the same as those considered in Tables 3a and 4.

Table 5b. 99th and 95th percentiles of portfolio and sub-portfolios of whole life insurances in Table 5a

	Contract: $1000 \cdot \tilde{A}_{1 \atop 55:\overline{\varpi-55+1}|}$ Number of contracts: 5				Contract: $1000 \cdot \tilde{A}_{1 \atop 65:\overline{\varpi-65+1}|}$ Number of contracts: 10			
α	$\underline{Q^{0.99}(\tilde{L})}_\alpha$	$\overline{Q^{0.99}(\tilde{L})}_\alpha$	$\underline{Q^{0.95}(\tilde{L})}_\alpha$	$\overline{Q^{0.95}(\tilde{L})}_\alpha$	$\underline{Q^{0.99}(\tilde{L})}_\alpha$	$\overline{Q^{0.99}(\tilde{L})}_\alpha$	$\underline{Q^{0.95}(\tilde{L})}_\alpha$	$\overline{Q^{0.95}(\tilde{L})}_\alpha$
1	3577.59	3577.59	3323.50	3323.50	7637.49	7637.49	7347.51	7347.51
0.75	3492.02	3662.01	3229.71	3417.30	7476.46	7798.52	7175.04	7519.98
0.5	3406.46	3745.89	3134.78	3511.10	7315.47	7959.54	7002.58	7692.58
0.25	3320.89	3829.76	3037.86	3604.79	7154.49	8120.57	6830.11	7865.34
0	3237.87	3913.64	2940.93	3698.41	6993.52	8281.59	6655.65	8038.11

	Contract: $1000 \cdot \tilde{A}_{1 \atop 75:\overline{\varpi-75+1}|}$ Number of contracts: 5				Whole portfolio			
α	$\underline{Q^{0.99}(\tilde{L})}_\alpha$	$\overline{Q^{0.99}(\tilde{L})}_\alpha$	$\underline{Q^{0.95}(\tilde{L})}_\alpha$	$\overline{Q^{0.95}(\tilde{L})}_\alpha$	$\underline{Q^{0.99}(\tilde{L})}_\alpha$	$\overline{Q^{0.99}(\tilde{L})}_\alpha$	$\underline{Q^{0.95}(\tilde{L})}_\alpha$	$\overline{Q^{0.95}(\tilde{L})}_\alpha$
1	4379.39	4379.39	4227.66	4227.66	14911.43	14911.43	14477.85	14477.85
0.75	4323.04	4436.62	4165.48	4292.05	14580.40	15242.46	14131.10	14824.65
0.5	4266.54	4493.85	4103.80	4356.44	14249.40	15573.49	13784.57	15173.82
0.25	4209.45	4551.09	4041.73	4420.77	13918.41	15904.52	13432.80	15525.92
0	4153.06	4608.32	3978.06	4482.84	13590.51	16235.54	13082.85	15877.63

Note: * Whole life insurances and technical basis are the same as those considered in Tables 3a and 4.

Table 6a. Portfolio of whole life annuities $1000 \cdot \tilde{a}_{x:\overline{\varpi-x+1}|}$ with several ages and contracts

Age	Number of contracts	$\tilde{L}$		
		$E(L)$	$E(r_L)$	$SD(\tilde{L})$
55	5	110214.28	9698.01	13841.14
65	10	169170.68	12039.18	18427.94
75	5	57168.89	3122.25	11297.37
Whole portfolio	20	57168.89	3122.25	25667.04

Table 6b. 99th and 95th percentiles of portfolio and sub-portfolios of whole life annuities in Table 6a

α	Contract: $1000 \cdot \tilde{\tilde{a}}_{55:\overline{\varpi-55+1}\vert}$ Number of contracts: 5				Contract: $1000 \cdot \tilde{\tilde{a}}_{65:\overline{\varpi-65+1}\vert}$ Number of contracts: 10			
	$\underline{Q^{0.99}(\tilde{L})}_{\alpha}$	$\overline{Q^{0.99}(\tilde{L})}_{\alpha}$	$\underline{Q^{0.95}(\tilde{L})}_{\alpha}$	$\overline{Q^{0.95}(\tilde{L})}_{\alpha}$	$\underline{Q^{0.99}(\tilde{L})}_{\alpha}$	$\overline{Q^{0.99}(\tilde{L})}_{\alpha}$	$\underline{Q^{0.95}(\tilde{L})}_{\alpha}$	$\overline{Q^{0.95}(\tilde{L})}_{\alpha}$
1	136297.47	136297.47	130500.05	130500.05	213630.32	213630.32	199509.08	199509.08
0.75	132853.68	139741.26	127345.77	133665.07	209360.34	217900.30	195620.09	203300.79
0.5	129409.88	143185.06	124191.48	136881.81	205090.36	222170.28	191769.42	207092.50
0.25	125966.09	146628.85	121037.20	140098.56	200820.38	226440.26	187962.84	210884.20
0	122522.30	150072.64	117882.91	143311.54	196550.40	230710.24	184156.64	214675.91

α	Contract: $1000 \cdot \tilde{\tilde{a}}_{75:\overline{\varpi-75+1}\vert}$ Number of contracts: 5				Whole portfolio			
	$\underline{Q^{0.99}(\tilde{L})}_{\alpha}$	$\overline{Q^{0.99}(\tilde{L})}_{\alpha}$	$\underline{Q^{0.95}(\tilde{L})}_{\alpha}$	$\overline{Q^{0.95}(\tilde{L})}_{\alpha}$	$\underline{Q^{0.99}(\tilde{L})}_{\alpha}$	$\overline{Q^{0.99}(\tilde{L})}_{\alpha}$	$\underline{Q^{0.95}(\tilde{L})}_{\alpha}$	$\overline{Q^{0.95}(\tilde{L})}_{\alpha}$
1	82582.50	82582.50	75146.83	75146.83	332116.81	332116.81	316496.67	316496.67
0.75	81276.75	83888.26	74014.93	76278.78	325290.95	338942.66	310049.54	322946.50
0.5	79970.99	85194.01	72884.71	77410.72	318465.10	345768.52	303651.51	329399.04
0.25	78665.57	86499.77	71754.97	78544.03	311639.24	352594.37	297136.33	335848.24
0	77360.20	87805.53	70625.22	79678.22	304716.27	359420.23	290618.21	342250.14

6. CONCLUSIONS

In this paper we have extended the results reported in Andrés-Sánchez and González-Vila (2012, 2014), which model life contingency pricing with fuzzy random variables (FRVs), in two ways. First, we allow not only the discount rate(s), but also the amounts to be paid out by the insurance companies, to be fuzzy. Thus, our schema can be used when part of the cost of an insurance policy is not known with precision (e.g. when insured amounts are indexed or when future expenses such as settlement costs need to be taken into account). Second, under the hypothesis that those parameters are symmetrical triangular fuzzy numbers (STFNs), we have obtained several indicators that enable us to obtain the fair (or expected) price of life contingencies, the reasonable variability of this mathematical expectation and the variability of the present cost of these life contingencies (i.e. the solvency cost).

Using the STFN approximation based on Dubois and Prade (1993, p. 131) to the present value with STFN parameters enable us to obtain interesting operational expressions that simplify the computational requirements of the results in Andrés-Sánchez and González-Vila (2012, 2014). In fact, the calculation of the fuzzy present value is reduced to obtain the modal value with the centers of interest rate and cash flows. Likewise, the variability of the present value is intuitively (from a financial perspective) evaluated by aggregating

the present value of cash flow spreads and the variability that comes from interest rate uncertainty, which is linearly approximated, as is usual in standard financial mathematics, with the Macaulay duration. The use of FRVs with STFN outcomes allows us to describe the present value of life contingencies using just two conventional random variables: the centers and the spreads (i.e. the uncertainty) of the present value of insured amounts. Thus, the calculation of such magnitudes as mathematical expectation, standard deviation, quantiles, etc. for a contract (and, consequently, for a portfolio of policies) is relatively easy and intuitive.

The results presented in this paper can, we believe, be readily extended to more general forms of fuzzy numbers. Moreover, in relation to life insurance policies, a slight generalization of our proposed developments can be derived from the consideration that the time when the insured amount is paid is not the end of the year of death, but any moment within that year and so, the payment maturity date is fuzzy. The latter may depend, for example, on the exact date of death, the delay before the heirs file their claim, or other factors. For example, if the insured dies in the rth year, the maturity payment may be the STFN $\left(r + \frac{1}{2}, \frac{1}{2}\right)$, i.e. it is considered that the payment of the insured amount can be made in any moment within the rth year, the most reliable date being at the midpoint of this year. Other natural ways in which our findings can be extended include the use of the fuzzy, non-flat, temporal structure of interest rates, as proposed in Ostaszewski (1993) and in Andrés-Sánchez and Terceño (2003), or the introduction of fuzzy uncertainty in the future lifetime, as suggested Shapiro (2013).

7. REFERENCES

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