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Average monotonic cooperative games with nontransferable utility

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Abstract

A non-negative transferable utility (TU) game is *average* monotonic if there exists a non-negative allocation according to which the relative worth is not decreasing when enlarging the coalition. We generalize this definition to the nontransferable utility (NTU) case. It is shown that an average monotonic NTU game shares several properties with an average monotonic TU game. In particular it has a special core element and there exists a population monotonic allocation scheme. We show that an NTU bankruptcy game is average monotonic with respect to the claims vector.

Keywords: nontransferable utility; average monotonicity; core; population monotonicity
JEL classification: C71

1. Introduction

Izquierdo and Rafels (2001) define *average monotonic* cooperative games with transferable utility that allow to model multilateral interactive decision problems in economic situations with increasing average profits in which side payments are possible. For instance, consider a group of investors such that each of them has an amount of money to invest, and a bank offering a yield that depends increasingly on the total amount of the deposited money. Then, if the investors can combine their resources and invest them in the bank, there are incentives to form a coalition, since increasing investments generate an increasing interest rate (see Izquierdo, 1996, and Izquierdo and Rafels, 2001, for further details.)

In this context an arbitrary coalition of decision makers may form and select a feasible alternative that creates a profit for each of its members. The mentioned authors now assume that the arising aggregate profit may be redistributed in an arbitrary way to its members, i.e., it is assumed that side payments are possible. Therefore, the game that suitably models such a decision problem is a cooperative *transferable utility* game, a TU game.

If, on the other hand, side payments are not possible (they may be prohibited or physically impossible), then such a situation may be modeled as a cooperative *non transferable utility* game, an NTU game. It should be noted that a TU game may be regarded as a special NTU game.

Based on the definition of a average monotonic cooperative TU game given by Izquierdo and Rafels (2001), we define the average monotonic cooperative NTU game. Specifically, we generalize the definition of average monotonicity to the NTU case, showing that a TU game is average monotonic if and only if its corresponding NTU game is average monotonic. We show that an average monotonic NTU game has some properties in common with an average TU game. In face, it turns out that, as for an average monotonic TU game, the "proportional distribution" is a remarkable core element of an average monotonic NTU game as well.

Furthermore, we show that the allocation scheme that assigns to each coalition its proportional distribution does not decrease the payoffs of the players of a coalition when they form a larger coalition. That is, the extension to all coalitions of the proportional distribution is a *population monotonic allocation scheme* in the sense of (Moulin, 1990).

Finally, we prove that every NTU *bankruptcy game* in the sense of Orshan et al. (2003) is average monotonic with respect to the claims vector.

The paper is organized as follows. In Section 2 we formally present some basics about TU and NTU games. In Section 3 we present the definition of an average monotonic NTU game with respect to (w.r.t.) a vector α and show that each subgame of such a game is average monotonic w.r.t. the restricted vector of α and that the corresponding proportional distribution is in the core of the game. In Section 4 we show that the extension of the proportional distribution to all coalitions is a population monotonic allocation schemes. Finally, Section 5 is devoted to show that NTU bankruptcy games are average monotonic w.r.t. the claims vectors, and Section 6 conclude the paper.

2. Notation and basic definitions

We start with some notations. Throughout, let N be a finite nonempty set of elements called *players*. A *coalition* S is a nonempty subset of N and we denote by s the number of players in S. By $\mathcal{N}$ we denote the set of all coalitions in N, i.e., $\mathcal{N} = 2^N \backslash \{\emptyset\}$. The elements of $\mathbb{R}^N$ will be identified with n-dimensional vectors whose coordinates are indexed by the members of N. For each coalition N denote by 0_N the zero-vector of $\mathbb{R}^N$. Further, if $x \in \mathbb{R}^N$, and $S \in \mathcal{N}$ is a coalition, we write x_S for the restriction of x to S, i.e. $x_S := (x_i)_{i \in S} \in \mathbb{R}^S$ and $x(S) = \sum_{i \in S} x_i$.

A *TU game* (on N) is a pair (N, v) such that $v : 2^N \to \mathbb{R}$, the *coalition function* (also called *characteristic function*), satisfies $v(\emptyset) = 0$. For $S \in \mathcal{N}$ and $x, y \in \mathbb{R}^S$, we write $x > y$ if $x_i > y_i$ for each $i \in S$, $x \geqslant y$ means $x_i \geqslant y_i$ for each $i \in S$ and $x \gneqq y$ means $x_i \geqslant y_i$ for all $i \in S$ and $x \neq y$. Let $\mathbb{R}^S_+ := \{x \in \mathbb{R}^S | x \geqslant 0\}$ and $\mathbb{R}^S_{++} := \{x \in \mathbb{R}^S | x > 0\}$.

We say that the set $A \subseteq \mathbb{R}^S$, where $S \in \mathcal{N}$ is *comprehensive* if $x \in A$ and $y \leqslant x$ imply $y \in A$.

Using this notation we define a cooperative game without transferable utility, an NTU game, as follows.

Definition 2.1. *An NTU game (on N) is a pair (N, V) such that V is a mapping that assigns to each coalition $S \in \mathcal{N}$ a subset $V(S)$ of $\mathbb{R}^S$ of attainable payoff vectors satisfying the following conditions:*

(i) $V(S)$ is nonempty, closed and comprehensive for all $S \in \mathcal{N}$,

(ii) $V(S) \cap \mathbb{R}^S_+$ is nonempty and bounded for all $S \in \mathcal{N}$.

It is also assumed that $V(\emptyset) = \emptyset$.

A TU game (N, v) can be considered as an NTU game in the following natural way. Indeed, let (N, V_v) be the NTU game defined by $V_v(S) = \{x \in \mathbb{R}^S \mid x(S) \leqslant v(S)\}$ for all $S \in \mathcal{N}$. Then we say that (N, V_v) is the NTU game *corresponding to* the TU game (N, v).

Let (N, V) be an NTU game. For every $i \in N$ let

$$v_i = \max\{x_i \mid x_{\{i\}} \in V(\{i\})\}.$$

We often identify (N, V) with its characteristic function V. The intended interpretation is that $x \in V(S)$ if cooperation within the coalition S allows to create the utility allocation x for the members of S. In order to simplify the notation we will write $V(i)$ instead of $V(\{i\})$.

The *core* of (N, V), $\mathcal{C}(N, V)$, is the set of all vectors $x \in V(N)$ such that, for each coalition S and each allocation $y \in V(S)$, there exists $i \in S$ such that $x_i \geqslant y_i$. Note that core of a TU game (N, v) coincides with the core of its corresponding NTU game (N, V_v). The well-known Bondareva-Shapley theorem shows that the core of a TU game is nonempty if and only if the TU game is balanced (Bondareva, 1963; Shapley, 1967). Say that a TU game is *totally balanced* if each of its subgames – a subgame of (N, v) on a coalition S has S as set of players, and its characteristic function is the original characteristic function restricted to the subsets of S– is balanced.

For all $S, T \in \mathcal{N}$ with $S \subseteq T$ and all $X \subseteq \mathbb{R}^T$, we denote by X_S the projection of X on $\mathbb{R}^S$, i.e., $X_S = \{x_S \mid x \in X\}$.

For all $S \in \mathcal{N}$ and $X \subseteq \mathbb{R}^S$ we denote by $P(X)$ the *Pareto frontier* of X, i.e., $P(X) = \{x \in X \mid \nexists y \in X \text{ such that } y \gneqq x\}$. The *weak Pareto frontier of* X is the set $WP(X) = \{x \in X \mid \nexists y \in V(S) \text{ such that } y > x\}$. Note that for an NTU game (N, V) and a coalition $S \in \mathcal{N}$, $WP((V)(S)) = \partial V(S)$, where ∂ means boundary.

For each $S \in \mathcal{N}$, by slightly abusing notation, we denote by (S, V_S) its *subgame* on S. That is, the set of players is S and $V_S(T) = V(T)$ for any $T \subseteq S$.

Next, we recall some properties of NTU games that we use.

An NTU game (N, V) is *superadditive* if, for all $S, T \in \mathcal{N}$ such that $S \cap T = \emptyset$, $V(S) \times V(T) \subseteq V(S \cup T)$. Moreover, (N, V) is *weakly* superadditive if the foregoing condition is just requested in the case that $T = \{i\}$ is a singleton. Note that a TU game satisfies (weak) superadditivity if and only if its corresponding NTU game does.

We now relax weak superadditivity further and say that an NTU game (N, V) is *weakly* superadditive* if $\bigtimes_{i \in S} V(i) \subseteq V(S)$ for all $S \in \mathcal{N}$. Hence, subgames of (weakly*) superadditive games are (weakly*) superadditive.

An NTU game is *monotonic* (Hart and Mas-Colell, 1996) if for all coalitions $S, T \in \mathcal{N}$ with $S \subseteq T$ and all $x \in V(S)$, there exists $y \in V(T)$ with $y_S \geqslant x$ and $y_{T \setminus S} \geqslant 0$, i.e., $V(S) \times \{0_{T \setminus S}\} \subseteq V(T)$. Note that the foregoing definition of monotonicity expands the classical definition of monotonicity for TU games, i.e., a TU game is monotonic if and only if its corresponding NTU game is.

According to Otten et al. (1998) an NTU game (N, V) is *weakly monotonic* if for all coalitions $S, T \in \mathcal{N}$ with $S \subseteq T$ and all $x \in V(S)$, there exists an $y \in V(T)$ with $y_S \geqslant x$, i.e., $V(S) \subseteq \{y_S \mid y \in V(T)\}$.

Note that a monotonic NTU game is weakly monotonic as well as an NTU game that corresponds to an arbitrary TU game, but there are weakly monotonic NTU games that are not monotonic (e.g., all NTU game that correspond to non-monotonic TU games).

Additionally, in general, we do not require an NTU game (N, V) to be convex-valued, i.e., for a coalition S, $V(S)$ is not required to be convex, unless explicitly stated.

3. Average monotonic games with non transferable utility

Izquierdo and Rafels (2001) introduce and study average monotonic TU games. We now recall the corresponding definition. The TU game (N, v) is *average monotonic* w.r.t. $\alpha \in \mathbb{R}^N_+ \backslash \{0_N\}$ if $v(S) \geqslant 0$ for all $S \in \mathcal{N}$ and v *does not decrease in average w.r.t.* α*, i.e., for all* $S, T \in \mathcal{N}$ *with* $S \subseteq T$, $\alpha(T)v(S) \leqslant \alpha(S)v(T)$. Say that the NTU game (N, v) is *average monotonic* if there exists $\alpha \in \mathbb{R}^N_+ \backslash \{0_N\}$ such that (N, V) is average monotonic w.r.t. α.

It is known that an average monotonic TU game is monotonic, superadditive, totally balanced, and has a population monotonic allocation scheme.

We expand the definition of average monotonicity to NTU games in a natural way and show that average monotonic NTU games still have a nonempty core and a population monotonic allocation scheme.

Remark 3.1. Let (N, v) be a TU game.

(1) Let (N, v) be average monotonic w.r.t. some $\alpha \in \mathbb{R}^N_+ \backslash \{0_N\}$ and let $S \in N$ be such that $\alpha_S = 0_S$. As $\alpha(N) > 0$ and $\alpha(N)v(S) \leqslant \alpha(S)v(N) = 0$, $v(S) = 0$.

(2) If $v(S) = 0$ for all $S \subseteq \mathcal{N}$, then (N, v) is average monotonic w.r.t. each $\alpha \in \mathbb{R}^N \backslash \{0_N\}$.

(3) By the first two remarks, (N, v) is average monotonic if and only if $v(S) \geqslant 0$ for all $S \in \mathcal{N}$ and there exists $\alpha \in \mathbb{R}^N_+$ such that, for all $S, T \in \mathcal{N}$ and $S \subseteq T$, $\alpha(T)v(S) \leqslant \alpha(S)v(T)$ and, if $\alpha_S = 0_S$, then $v(S) = 0$. In this case we, again, say that (N, v) is average monotonic w.r.t. α.

Lemma 3.2. *The TU game (N, v) is average monotonic w.r.t. $\alpha \in \mathbb{R}^N_+$ if and only if the following three conditions hold:*

$$S \in \mathcal{N} \quad \Rightarrow \quad v(S) \geqslant 0 \tag{3.1}$$

$$R \in \mathcal{N} \text{ and } \alpha_R = 0_R \quad \Rightarrow \quad v(R) = 0 \tag{3.2}$$

$$S, T \in \mathcal{N}, q > 0, S \subseteq T, \alpha_S \neq 0_S, \alpha(S)q \leqslant v(S) \quad \Rightarrow \quad \alpha(T)q \leqslant v(T) \tag{3.3}$$

The proof can be deduced from Theorem 3.1 of Izquierdo and Rafels (2001) and it is added for completeness reasons.

Proof. For the only if part, suppose that (N, v) is average monotonic w.r.t. α. Remark 3.1 (3) directly implies (3.1) and (3.2). In order to show (3.3), let S, T, q satisfy the required conditions. Then $q \leqslant \frac{v(S)}{\alpha(S)}$ and, hence, $\alpha(T)q \leqslant \frac{\alpha(T)v(S)}{\alpha(S)} \leqslant \frac{\alpha(S)v(T)}{\alpha(S)} = v(T)$, where the last inequality is due to average monotonicity.

For the if part suppose that (3.1) – (3.3) are satisfied. In view of Remark 3.1 (3), it remains to show that, if $\alpha_S \neq 0$ and $S, T \in \mathcal{F}(N)$ satisfy $S \subseteq T$, then $\alpha(T)v(S) \leqslant \alpha(S)v(T)$. To this end let $q = \frac{v(S)}{\alpha(S)}$. By (3.3), $\alpha(T)q \leqslant v(T)$, hence $\alpha(T)v(S) = \alpha(T)\alpha(S)q \leqslant \alpha(S)v(T)$. ■

The foregoing observations motivate the following definition.

Definition 3.3. *The NTU game* (N, V) *is* average monotonic *if there exists* $\alpha \in \mathbb{R}^N$ *such that the following three conditions hold:*

$$S \in \mathcal{N} \quad \Rightarrow \quad 0_S \in V(S) \tag{3.4}$$

$$R \in \mathcal{N} \text{ and } \alpha_R = 0_R \quad \Rightarrow \quad 0_R \in \partial V(R) \tag{3.5}$$

$$S, T \in \mathcal{N}, q > 0, S \subseteq T, \alpha_S \neq 0_S, \alpha_S q \in V(S) \quad \Rightarrow \quad \alpha_T q \in V(T) \tag{3.6}$$

In this case we say that (N, V) *is average monotonic* w.r.t. α.

Note that the foregoing definition generalizes the definition of average monotonicity for TU games. Indeed, in view of Lemma 3.2, a TU game (N, v) is average monotonic if and only if its corresponding NTU game (N, V_v) is average monotonic.

Lemma 3.4. *An average monotonic NTU game is weakly* superadditive.*

Proof. Let (N, V) be an NTU game and let $\alpha \in \mathbb{R}^N_+$. Assume that (N, V) is average monotonic w.r.t. α and let $S \in \mathcal{N}$. If $\alpha_S = 0_S$, then $V(i) = -\mathbb{R}_+$ for all $i \in S$ and $0_S \in \partial V(S)$ so that $\times_{i \in S} V(i) \subseteq V(S)$. Hence, we may assume that $\alpha_S \neq 0_S$. For each $i \in S$ such that $\alpha_i \neq 0$, put $q_i^* = \frac{v_i}{\alpha_i}$. Then $V(i) = \{q_i \alpha_i \mid q_i \leqslant q_i^*\}$. By average monotonicity w.r.t. α, $q_i \alpha_S \in V(S)$ for all $q_i \leqslant q_i^*$. Let $q^* = \max\{q_i^* \mid i \in S, \alpha_i > 0\}$. Then $q \alpha_S \in V(S)$ for all $q \leqslant q^*$. As $v_j = 0$ for all $j \in S$ with $\alpha_j = 0$, $\times_{i \in S} V(i) \subseteq V(S)$. ■

As shown in the following example, in contrast with TU games, if an NTU game is average monotonic, then it need neither be monotonic nor weakly superadditive.

Example 3.5. Let (N, V) be the NTU game, where $N = \{1, 2, 3\}, V(1) = V(2) = V(3) = 0.5 - \mathbb{R}_+$, $V(\{1, 2\}) = \{(6, 1), (1, 6)\} - \mathbb{R}^2_+$, $V(\{1, 3\}) = V(\{2, 3\}) = \{(1, 1)\} - \mathbb{R}^2_+$, $V(N) = \{(4, 4, 4)\} - \mathbb{R}^3_+$. Let $\alpha = (1, 1, 1)$ and, for $S \in \mathcal{N}$, $q_S^* = \max\{q \in \mathbb{R}_+ \mid q \alpha_S \in V(S)\}$. Then $q_i^* = 0.5$ for all $i \in N$, $q_S^* = 1$ for all $S \in \mathcal{N}$ of cardinality 2, and $q_N^* = 4$. Hence, (N, v) is average monotonic w.r.t. α. As $(6, 1) \in V(\{1, 2\})$ and $(6, 1, 0) \notin V(N)$, (N, V) is not monotonic. As $0 \in V(3)$, we conclude that $V(\{1, 2\}) \times V(3) \nsubseteq V(N)$ *so that* (N, V) *is not weakly superadditive.*

For each TU game (N, v) that is average monotonic w.r.t. $\alpha \in \mathbb{R}^N_+$, we recall that Izquierdo and Rafels (2001) define the *proportional distribution* w.r.t. α, $p(v, \alpha)$, by $p(v, \alpha) = 0$ if $\alpha = 0_N$ and $p(v, \alpha) = \alpha \frac{v(N)}{\alpha(N)}$ if $\alpha \neq 0_N$.

We generalize the definition of the proportional distribution to NTU game. For each NTU game (N, V) that is average monotonic w.r.t. $\alpha \in \mathbb{R}^N_+$, we define $p(V, \alpha) \in \mathbb{R}^N$ as follows. If $\alpha = 0_N$, then $p(V, \alpha) = 0_N$. If $\alpha \neq 0_N$, then $q^* = \max\{q \geqslant 0 \mid \alpha q \in V(N)\}$ exists because $V(N) \cap \mathbb{R}^N_+ \neq \varnothing$ is assumed to be compact. In this case put $p(V, \alpha) = \alpha q^*$.

Remark 3.6. Let (N, V) be an average monotonic NTU game w.r.t. $\alpha \in \mathbb{R}^N_+$. Then $p(V, \alpha) \in C(N, V)$, and if the game corresponds to a TU game for some TU game (N, v), then $p(V, \alpha) = p(v, \alpha)$.

Every subgame of an average monotonic game is also an average monotonic game.

Lemma 3.7. *Let the NTU game (N, V) be average monotonic w.r.t. $\alpha \in \mathbb{R}^N_+$. Then, for any $R \in \mathcal{N}$, the subgame (R, V_R) is average monotonic w.r.t. α_R.*

Proof. Let $R \in \mathcal{N}$. It is straightforward to check that the three conditions (3.4), (3.5), and (3.6) are satisfied when replacing $\mathcal{N}$ by $\{S \subseteq R \mid S \neq \varnothing\}$. ∎

4. Population monotonic allocation schemes

The notion of population monotonic allocation scheme (PMAS) for a cooperative TU game (N, v) was introduced by Sprumont (1990), and extended, in a straightforward manner, to non transferable utility games by Moulin (1990), who investigated the monotonic core, i.e., the set of all PMAS of the game. From its definition we can directly deduce that a PMAS $x = (x^S)_{S \in \mathcal{N}}$ selects a core allocation $x^S \in C(S, V_S)$ of the subgame (S, V_S) in such a way that the payoff to a player cannot decrease when her coalition becomes larger. We now recall the formal definition of a population monotonic allocation scheme.

Definition 4.1. *A collection of vectors* $x = (x^S)_{S \in \mathcal{N}}$ *is a* population monotonic allocation scheme *(*PMAS*) of the NTU game* (N, V) *if and only if it satisfies the following conditions:*

(1) *For all* $S \in \mathcal{N}$*,* $x^S \in \mathbb{R}^S$ *and* $x^S \in \partial V(S)$*,*
(2) *For all* $S, T \in \mathcal{N}$ *with* $S \subseteq T$*,* $x^S \leqslant x_S^T$*.*

Proposition 4.2. *Let* (N, V) *be an NTU game and* $\alpha \in \mathbb{R}_+^N$*. If* (N, V) *is average monotonic w.r.t.* α*,* $(p(V_S, \alpha_S))_{S \in \mathcal{N}}$ *is a PMAS of* (N, V)*.*

Proof. Assume that (N, v) is average monotonic w.r.t. α and let $S \in \mathcal{N}$. By Lemma 3.7, (S, V_S) average monotonic w.r.t. α_S. By the definition of the proportional solution, $p(S, V_S) = q_S^* \alpha_S$ for a unique $q_S^* \geqslant 0$ that satisfies $q_S^* = 0$ in the case that $\alpha_S = 0_S$. By Remark 3.6, $q_S^* \alpha_S \in C(S, v_S)$. Now, if $T \in \mathcal{N}$ satisfies $S \subseteq T$, then $q_T^* \geqslant q_S^*$ because either $\alpha_T = 0_T$ and, hence, $\alpha_S = 0_S$ or $\alpha_T \neq 0_T$ and, hence, $q_T^* \alpha_T \in C(T, V_T)$. In each case the proof is finished. ■

As a consequence of the last proposition and the fact that every population monotonic allocation scheme assigns to each coalition a core element of the corresponding subgame, for an average monotonic game w.r.t. α, $x^S = q_S^* \alpha$ is a core element of (S, V_S).

5. Bankruptcy games with nontransferable utility

Aumann and Maschler (1985) introduced, for each TU bankruptcy problem, it corresponding TU bankruptcy game. We recall the NTU extension of a bankruptcy problem and its corresponding NTU game as given by Orshan et al. (2003).

An NTU *bankruptcy problem* on a set N is a pair (E, c), where $E \subseteq \mathbb{R}^N$, the *estate*, is closed, comprehensive and $E \cap \mathbb{R}_+^N$ is nonempty and bounded. The vector of *claims*, $c = (c_i)_{i \in N}$ is such that $c \in \mathbb{R}_+^N \backslash E$.

The estate, E, is the set of all feasible utility vectors, and c_i the utility claimed by player $i \in N$.

The NTU *bankruptcy game* associated to an NTU bankruptcy problem (E, c) on N, is the NTU game $(N, V_{E,c})$ defined by

$$V_{E,c}(S) = \{x_S \in \mathbb{R}^S \mid (x_S, c_{N\setminus S}) \in E\} \cup -\mathbb{R}^S_+ \text{ for all } S \in \mathcal{N}.$$

Let (N, V) be an NTU bankruptcy game. Let $S \in \mathcal{N}$. It is straightforward to check that $V(S)$ is nonempty, closed, and comprehensive.

Finally, we show that NTU bankruptcy games are average monotonic games with respect to their claims vectors.

Proposition 5.1. *Let (E, c) be an NTU bankruptcy problem. Then the bankruptcy game $(N, V_{E,c})$ is average monotonic w.r.t. c.*

Proof. Let $V = V_{E,c}$. Let $S \in \mathcal{N}$. As $-\mathbb{R}^S_+ \subseteq V(S)$, $0_S \in V(S)$, i.e., (3.4). If $c_S = 0_S$, then $(0_S, c_{N\setminus S}) = c \notin E$, hence $0_S \in \partial V(S)$, i.e., (3.5). Now, assume that $c_S \neq 0_S$. Let $T \in \mathcal{N}$ such that $S \subseteq T$. Let $q > 0$ such that $c_S q \in V(S)$. Hence, $(c_S q, c_{N\setminus S}) \in E$. As $c \notin E$ and as E is comprehensive, $q < 1$. Again as E is comprehensive, $(c_T q, c_{N\setminus T}) \in E$. We conclude that $c_T q \in V(T)$, i.e., (3.6). ■

6. Final remarks

As mentioned, according to the Bondareva-Shapley theorem, a TU game has a nonempty core if and only if it is balanced. Moreover, according to Scarf (1967) a balanced NTU game has a non-empty core, but the converse may not hold. Billera (1970) generalized this result to π-balanced NTU games.

According to our definition, however, an average monotonic NTU game is not necessarily (π-)balanced, which can easily be shown by means of 3-person games (N, V) corresponding to certain average monotonic TU games except that $V(N)$ is the comprehensive hull of the proportional distribution.

A main result of Izquierdo and Rafels (2001) is that for an average monotonic TU game, the classical bargaining set (Aumann and Maschler, 1964; Davis and Maschler, 1967) and the Mas-Colell bargaining set (Mas-Colell, 1989) coincide with the core. Whether this result remains valid for average monotonic NTU games has still to be investigated.

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