Boundary powerful k-alliances in graphs*

I.G. Yero[†] and J. A. Rodríguez-Velázquez[‡]
Department of Computer Engineering and Mathematics
Rovira i Virgili University
Av. Països Catalans 26, 43007 Tarragona, Spain

Abstract

A global boundary defensive k-alliance in a graph $G = (V, E)$ is a dominating set S of vertices of G with the property that every vertex in S has k more neighbors in S than it has outside of S. A global boundary offensive k-alliance in a graph G is a set S of vertices of G with the property that every vertex in $V - S$ has k more neighbors in S than it has outside of S. We define a global boundary powerful k-alliance as a set S of vertices of G, which is both global boundary defensive k-alliance and global boundary offensive $(k + 2)$-alliance. In this paper we study mathematical properties of boundary powerful k-alliances. In particular, we obtain several bounds (closed formulas for the case of regular graphs) on the cardinality of every global boundary powerful k-alliance. In addition, we consider the case in which the vertex set of a graph G can be partitioned into two boundary powerful k-alliances, showing that, in such a case, $k = -1$ and, if G is δ-regular, its algebraic connectivity is equal to $\delta + 1$.

Keywords: Alliances in graphs, powerful alliances, domination, algebraic connectivity, Laplacian spectral radius.

AMS Subject Classification numbers: 05C69; 05A20; 05C50

1 Introduction

The mathematical properties of alliances in graphs were first studied by P. Kristiansen, S. M. Hedetniemi and S. T. Hedetniemi [12]. They proposed different types of alliances: namely, defensive [9, 10, 12, 13, 14, 17], offensive [4, 6, 13, 18] and dual or powerful alliances [1, 2]. The complexity of computing the minimum

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[†]ismael.gonzalez@urv.cat

[‡]juanalberto.rodriguez@urv.cat

cardinality of alliances is addressed in [3, 5, 6, 10, 11]. For instance, a *defensive alliance* of a graph G is a set S of vertices of G with the property that every vertex in S has at most one more neighbor outside of S than it has in S. An *offensive alliance* of a graph G is a set S of vertices of G with the property that every vertex in the boundary of S has at least one more neighbor in S than it has outside of S. The combination of these two kind of alliances is called *powerful alliance*, hence a powerful alliance is a set S of vertices of G, which is both, defensive and offensive. A generalization of defensive alliances was presented by K. H. Shafique and R. D. Dutton in [15, 16] where they define defensive k-alliance as a set S of vertices of G with the property that every vertex in S has at least k more neighbors in S than it has outside of S. In this paper, we study the mathematical properties of a particular case of k-alliances that we call boundary k-alliances. A *global boundary defensive k-alliance* in a graph $G = (V, E)$ is a dominating set S of vertices of G with the property that every vertex in S has k more neighbors in S than it has outside of S [19]. Analogously, a *global boundary offensive k-alliance* in a graph G is a dominating set S of vertices of G with the property that every vertex in $V - S$ has k more neighbors in S than it has outside of S. In this paper we focus our attention on sets of vertices of G that are both global boundary defensive k-alliances and global boundary offensive $(k + 2)$-alliances. We call these types of alliances *global boundary powerful k-alliances*.

We begin by stating the terminology used. Throughout this article, $G = (V, E)$ denotes a simple graph of order $|V| = n$ and size $|E| = m$. We denote two adjacent vertices u and v by $u \sim v$. For a nonempty set $X \subseteq V$, and a vertex $v \in V$, $N_X(v)$ denotes the set of neighbors v has in X: $N_X(v) := \{u \in X : u \sim v\}$ and the degree of v in X is denoted by $\delta_X(v) = |N_X(v)|$. We denote the degree of a vertex $v_i \in V$ by $\delta(v_i)$ (or by δ_i for short) and the degree sequence of G by $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$. The subgraph induced by $S \subset V$ is denoted by $\langle S \rangle$ and the complement of the set S in V is denoted by $\bar{S}$. Recall that a set $S \subset V$ is a *dominating set* in G if for every vertex $u \in \bar{S}$, $\delta_S(u) > 0$ (every vertex in $\bar{S}$ is adjacent to at least one vertex in S).

With the above notation we define boundary defensive (offensive) k-alliance as follows [19].

Definition 1. *A set $S \subseteq V$ is a boundary defensive k-alliance in G, $k \in \{-\delta_1, \ldots, \delta_1\}$, if*

$$\delta_S(v) = \delta_{\bar{S}}(v) + k, \quad \forall v \in S. \tag{1}$$

Definition 2. *A set $S \subseteq V$ is a boundary offensive k-alliance in G, $k \in \{2 - \delta_1, \ldots, \delta_1\}$, if*

$$\delta_S(v) = \delta_{\bar{S}}(v) + k, \quad \forall v \in \cup_{u \in S} N_{\bar{S}}(u). \tag{2}$$

2 Boundary powerful k-alliances

Definition 3. *A set $S \subseteq V$ is a (global) boundary powerful k-alliance in $G = (V, E)$, $k \in \{2 - \delta_1, \dots, \delta_1\}$, if S is both (global) boundary defensive k-alliance and (global) boundary offensive $(k + 2)$-alliance.*

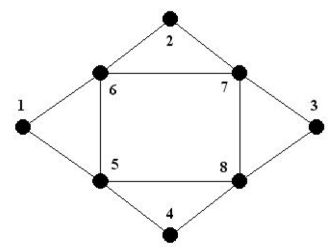

Figure 1: $S_1 = \{1, 2, 3, 4\}$ is a boundary powerful (-2)-alliance and $S_2 = \{5, 6, 7, 8\}$ is a boundary powerful (0)-alliance.

It was shown in [19] that the cardinality of a boundary defensive k-alliance S is bounded by

$$\left\lceil \frac{\delta_n + k + 2}{2} \right\rceil \leq |S| \leq \left\lfloor \frac{2n - \delta_n + k}{2} \right\rfloor.$$

Analogously, in the case of a boundary offensive k-alliance S we have:

$$\left\lceil \frac{\delta_n + k}{2} \right\rceil \leq |S| \leq \left\lfloor \frac{2n - \delta_n + k - 2}{2} \right\rfloor.$$

Thus, the following result follows.

Remark 1. *If S is a boundary powerful k-alliance in a graph, then*

$$\left\lceil \frac{\delta_n + k + 2}{2} \right\rceil \leq |S| \leq \left\lfloor \frac{2n - \delta_n + k}{2} \right\rfloor.$$

Corollary 2. *If S is a boundary powerful k-alliance in a complete graph $G = K_n$, then $|S| = \left\lceil \frac{n+k+1}{2} \right\rceil$.*

Theorem 3. *If S is a global boundary powerful k-alliance in a graph, then*

$$\left\lceil \frac{2m + n(k + 2)}{2\delta_1 + 2} \right\rceil \leq |S| \leq \left\lfloor \frac{2m + n(k + 2)}{2\delta_n + 2} \right\rfloor.$$

Proof. As S is a global boundary powerful k-alliance in G, then, for every $v \in S$, $\delta(v) = 2\delta_{\bar{S}}(v) + k$, and for every $v \in \bar{S}$, $\delta(v) = 2\delta_S(v) - (k+2)$. Hence,

$$\sum_{v \in S} \delta(v) = 2 \sum_{v \in S} \delta_{\bar{S}}(v) + k|S|,$$

and

$$\sum_{v \in \bar{S}} \delta(v) = 2 \sum_{v \in \bar{S}} \delta_S(v) - (k+2)(n - |S|).$$

Therefore, as $\sum_{v \in S} \delta_{\bar{S}}(v) = \sum_{v \in \bar{S}} \delta_S(v)$,

$$2m = 4 \sum_{v \in S} \delta_{\bar{S}}(v) + |S|(2k+2) - n(k+2). \tag{3}$$

Moreover,

$$\frac{\delta_n - k}{2} \leq \delta_{\bar{S}}(v) \leq \frac{\delta_1 - k}{2}, \quad \forall\, v \in S. \tag{4}$$

Now, by using the above inequalities in equation (3) we obtain the bounds on $|S|$. □

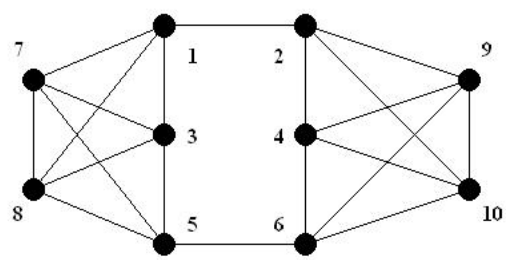

Figure 2: $S = \{1,2,3,4,5,6\}$ is a global boundary powerful (0)-alliance.

Since for any δ-regular graph, $m = \frac{\delta n}{2}$, the above theorem gives a closed formula for the cardinality of any global boundary powerful k-alliance.

Corollary 4. *If S is a global boundary powerful k-alliance in a δ-regular graph, then*

$$|S| = \left\lceil \frac{n(\delta + k + 2)}{2(\delta + 1)} \right\rceil.$$

The Euler formula states that for a connected planar graph of order n, size m and f faces, $m = n + f - 2$. Hence, we obtain the following corollary of Theorem 3.

Corollary 5. *Let G be a planar connected graph with f faces. If S is a global boundary powerful k-alliance in G, then*

$$\left\lceil \frac{n(k+4)+2f-4}{2\delta_1+2} \right\rceil \leq |S| \leq \left\lfloor \frac{n(k+4)+2f-4}{2\delta_n+2} \right\rfloor$$

and, if G is δ-regular,

$$|S| = \frac{n(k+4)+2f-4}{2(\delta+1)}.$$

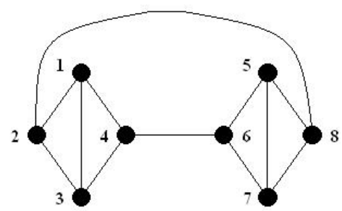

Figure 3: A planar and connected (3-regular) graph with $f = 6$ faces where $S = \{1, 2, 4, 5, 6, 8\}$ is a global boundary powerful (1)-alliance.

Theorem 6. *If S is a global boundary powerful k-alliance in a graph, then*

$$\left\lceil \frac{n(2\delta_n+k+2)-2m}{2\delta_n+2} \right\rceil \leq |S| \leq \left\lfloor \frac{n(2\delta_1+k+2)-2m}{2\delta_1+2} \right\rfloor.$$

Proof. As S is a global boundary offensive $(k+2)$-alliance in G, then

$$\frac{\delta_n+k+2}{2} \leq \delta_S(v) \leq \frac{\delta_1+k+2}{2}, \quad \forall\, v \in \bar{S}. \tag{5}$$

Now, as $\sum_{v\in S} \delta_{\bar{S}}(v) = \sum_{v\in \bar{S}} \delta_S(v)$, by using (5), in equation (3), we obtain both bounds on $|S|$. $\square$

Notice that the above theorem leads to Corollary 4.

Theorem 7. *Let $G = (V, E)$ be a graph and let $S \subset V$. Let c be the number of edges of G with one endpoint in S and the other endpoint outside of S. If S is a global boundary powerful k-alliance in G, with $k \neq -1$, then*

$$|S| = \frac{2(m+n-2c)+nk}{2(k+1)}.$$

Proof. Let $m\langle S\rangle$ be the size of $\langle S\rangle$. As S is a boundary defensive k-alliance in G,

$$2m\langle S\rangle = \sum_{v\in S}\delta_S(v) = \sum_{v\in S}\delta_{\bar{S}}(v) + k|S| = c + k|S|.$$

Moreover, as S is a global boundary offensive $(k+2)$-alliance in G,

$$c = \sum_{v\in\bar{S}}\delta_S(v) = \sum_{v\in\bar{S}}\delta_{\bar{S}}(v) + (n-|S|)(k+2) = 2m\langle\bar{S}\rangle + (n-|S|)(k+2).$$

Now, as $m = m\langle S\rangle + m\langle\bar{S}\rangle + c$, we obtain the value of $|S|$. $\square$

Corollary 8. *Let $G=(V,E)$ be a δ-regular graph and let $S\subset V$. Let c be the number of edges of G with one endpoint in S and the other endpoint outside of S. If S is a global boundary powerful k-alliance in G, with $k\neq -1$, then*

(i) $|S| = \frac{n(\delta+k+2)-4c}{2k+2}$,

(ii) $c = \frac{n(\delta^2+2\delta-k^2-2k)}{4(\delta+1)}$.

Proof. (i) is trivial and (ii) is a direct consequence of Corollary 4 and Theorem 7. $\square$

3 Partitioning a graph into two global boundary powerful k-alliances

Remark 9. *Let $G=(V,E)$ be a graph.*

(i) *$S\subset V$ is a global boundary powerful (-1)-alliance in G, if and only if, $\bar{S}$ is a global boundary powerful (-1)-alliance in G.*

(ii) *If G can be partitioned into two global boundary powerful k-alliances, then $k=-1$.*

Proof. If S is a global boundary powerful k-alliance in G, then

$$\delta_S(v) = \delta_{\bar{S}}(v) + k, \quad \forall\, v\in S \tag{6}$$

and

$$\delta_S(v) = \delta_{\bar{S}}(v) + k + 2, \quad \forall\, v\in \bar{S}. \tag{7}$$

(i) follows immediately from (6) and (7). If $\bar{S}$ is a global boundary powerful k-alliance in G, then

$$\delta_{\bar{S}}(v) = \delta_S(v) + k, \quad \forall\, v\in \bar{S} \tag{8}$$

and

$$\delta_{\bar{S}}(v) = \delta_S(v) + k + 2, \quad \forall\, v\in S. \tag{9}$$

Hence, by (6) and (9) (or by (7) and (8)), we obtain that $k=-1$. $\square$

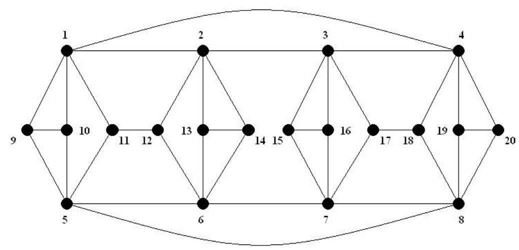

Figure 4: $S = \{1, ..., 8\}$ and $\bar{S}$ are global boundary powerful (-1)-alliances.

Theorem 10. *Let $G = (V, E)$ be a graph, if S is a global boundary powerful (-1)-alliance in G, then*

$$\left\lceil \frac{n(\delta_n + 1)}{\delta_1 + \delta_n + 2} \right\rceil \leq |S| \leq \left\lfloor \frac{n(\delta_1 + 1)}{\delta_1 + \delta_n + 2} \right\rfloor .$$

Proof. From (6) and (7) we have,

$$\sum_{v \in S} \delta_S(v) = \sum_{v \in S} \delta_{\bar{S}}(v) - |S| \tag{10}$$

and

$$\sum_{v \in \bar{S}} \delta_S(v) = \sum_{v \in \bar{S}} \delta_{\bar{S}}(v) + n - |S|. \tag{11}$$

Hence, as $\sum_{v \in S} \delta_{\bar{S}}(v) = \sum_{v \in \bar{S}} \delta_S(v)$,

$$\sum_{v \in S} \delta_S(v) = \sum_{v \in \bar{S}} \delta_{\bar{S}}(v) + n - 2|S|. \tag{12}$$

Thus, by (12) we obtain the following inequalities,

$$|S| \frac{\delta_1 - 1}{2} \geq (n - |S|) \frac{\delta_n - 1}{2} + n - 2|S| \tag{13}$$

and

$$|S| \frac{\delta_n - 1}{2} \leq (n - |S|) \frac{\delta_1 - 1}{2} + n - 2|S|. \tag{14}$$

Solving the above inequalities for $|S|$ we obtain the bounds. $\square$

Notice that the bounds obtained in the above theorem are attained, for instance, in the case of the graph in Figure 4.

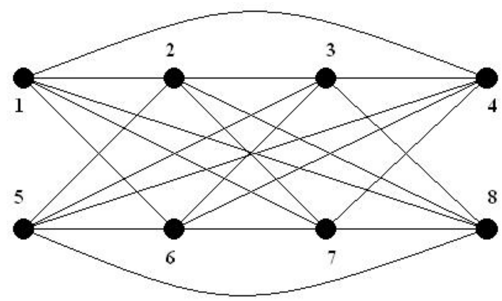

Figure 5: $S = \{1, 2, 3, 4\}$ and $\bar{S}$ are global boundary powerful (-1)-alliances.

Corollary 11. *If S is a global boundary powerful (-1)-alliance in a δ-regular graph, then $|S| = \frac{n}{2}$.*

Figure 5 shows an example of a 5-regular graph, which can be partitioned into two global boundary powerful (-1)-alliances.

Theorem 12. *Let $S \subset V$ be a global boundary powerful (-1)-alliance in a graph $G = (V, E)$ and let $M \subset E$ be a cut set with one endpoint in S and the other endpoint outside of S. Then $\left\lceil \frac{2m+n}{2\delta_1+2} \right\rceil \le |S| \le \left\lfloor \frac{2m+n}{2\delta_n+2} \right\rfloor$ and $|M| = \frac{2m+n}{4}$.*

Proof. As S is a global boundary defensive (-1)-alliance in G, for every $v \in S$, $\delta(v) = 2\delta_{\bar{S}}(v) - 1$, therefore,

$$\sum_{v \in S} \delta(v) = 2|M| - |S|.$$

Moreover, as $\bar{S}$ is a global boundary offensive 1-alliance in G, for every $v \in \bar{S}$, $\delta(v) = 2\delta_S(v) - 1$, therefore,

$$\sum_{v \in \bar{S}} \delta(v) = 2|M| - n + |S|.$$

Hence, $2m = 4|M| - n$. So, the value of $|M|$ follows. The bounds on $|S|$ are obtained from the above equation by using that, for every $v \in S$, $\frac{\delta_n+1}{2} \le \delta_{\bar{S}}(v) \le \frac{\delta_1+1}{2}$. $\square$

Notice that the above result leads to the Corollary 11.

It is well-known that the second smallest Laplacian eigenvalue of a graph, frequently called *algebraic connectivity*, is probably the most important information contained in the Laplacian spectrum. Also, the *Laplacian spectral radius* (the largest Laplacian eigenvalue of a graph) contains important information about the graph. These eigenvalues, are related to several important graph invariants and they impose reasonably good bounds on the values of several parameters of graphs which are very hard to compute.

The algebraic connectivity of G, μ, and the Laplacian spectral radius, μ_*, satisfy the following equalities shown by Fiedler [7],

$$\mu = 2n \min \left\{ \frac{\sum_{v_i \sim v_j} (w_i - w_j)^2}{\sum_{v_i \in V} \sum_{v_j \in V} (w_i - w_j)^2} : w \neq \alpha \mathbf{j} \text{ for } \alpha \in \mathbb{R} \right\} \tag{15}$$

and

$$\mu_* = 2n \max \left\{ \frac{\sum_{v_i \sim v_j} (w_i - w_j)^2}{\sum_{v_i \in V} \sum_{v_j \in V} (w_i - w_j)^2} : w \neq \alpha \mathbf{j} \text{ for } \alpha \in \mathbb{R} \right\}, \tag{16}$$

where $V = \{v_1, v_2, ..., v_n\}$, $\mathbf{j} = (1, 1, ..., 1)$ and $w \in \mathbb{R}^n$.

The following result shows the relationship between the algebraic connectivity (and the Laplacian spectral radius) of a graph and the cardinality of its global boundary powerful (-1)-alliances.

Theorem 13. *If $X \subset V$ is a global boundary powerful (-1)-alliance in $G = (V, E)$, then, without loss of generality,*

$$\frac{n}{2} + \left\lceil \sqrt{\frac{n^2(\mu - 1) - 2nm}{4\mu}} \right\rceil \leq |X| \leq \frac{n}{2} + \left\lfloor \sqrt{\frac{n^2(\mu_* - 1) - 2nm}{4\mu_*}} \right\rfloor$$

and

$$\frac{n}{2} - \left\lfloor \sqrt{\frac{n^2(\mu_* - 1) - 2nm}{4\mu_*}} \right\rfloor \leq |\bar{X}| \leq \frac{n}{2} - \left\lceil \sqrt{\frac{n^2(\mu - 1) - 2nm}{4\mu}} \right\rceil.$$

Proof. On one hand, by Theorem 12 we have,

$$\sum_{v \in X} \delta_{\bar{X}}(v) = \frac{2m + n}{4}. \tag{17}$$

On the other hand, by equations (15) and (16), taken $w \in \mathbb{R}^n$ defined as,

$$w_i = \begin{cases} 1 & \text{if} \quad v_i \in X; \\ 0 & \text{otherwise,} \end{cases} \tag{18}$$

we have

$$\mu \leq \frac{n \sum_{v \in X} \delta_{\bar{X}}(v)}{|X|(n - |X|)} \leq \mu_*. \tag{19}$$

Now, by using equation (17) in (19) we obtain both bounds on $|X|$. Moreover, as $|\bar{X}| = n - |X|$, the bounds on $|\bar{X}|$ follows. □

By Corollary 11 and the above theorem we obtain the following consequence.

Theorem 14. *Let $G = (V, E)$ be a δ-regular graph. If G contains a global boundary powerful (-1)-alliance, then the algebraic connectivity of G is $\mu = \delta + 1$.*

The above theorem gives a necessary condition for the existence global boundary powerful (-1)-alliances. Thus we obtain, for instance, that the Icosahedron does not contain global boundary powerful (-1)-alliances; because its algebraic connectivity is $\mu = 5 - \sqrt{5}$. Notice that the same occurs for the Petersen graph because, in this case, $\delta = 3$ and $\mu = 2$.

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