This is a post-print of an article published in Social Choice and Welfare. The final authenticated version is available online at: https://doi.org/10.1007/s00355-013-0789-0

Metadata of the article that will be visualized in OnlineAlone

Electronic supplementary material	Below is the link to the electronic supplementary material. **MOESM1:** Supplementary material 1 (f 173 KB).

Soc Choice Welf
DOI 10.1007/s00355-013-0789-0

ORIGINAL PAPER

A new approach for bounding awards in bankruptcy problems

José-Manuel Giménez-Gómez · **M. Carmen Marco-Gil**

Received: 7 March 2011 / Accepted: 21 December 2013

Abstract The solution for the *contested garment problem*, proposed in the Babylonic Talmud, suggests that each agent should receive at least some part of the resources whenever the claim exceeds the available amount. In this context, we propose a new method to define lower bounds on awards, an idea that has underlied the theoretical analysis of bankruptcy problems from its beginning (O'Neill, Math Soc Sci 2:345–371, 1982) to present day (Dominguez and Thomson, Econ Theory 28:283–307, 2006). Specifically, starting from the fact that a society establishes its own set of *commonly accepted equity principles*, our proposal ensures to each agent the smallest amount she gets according to all the *admissible* rules. We analyze its recursive application for different sets of equity principles.

1 Introduction

A bankruptcy problem is a situation where a group of agents claim more of a perfectly divisible resource (the endowment) than what is available. In this context, a rule prescribes how to share the endowment among the agents, according to the profile of claims. A natural question arises: Should each agent have a guaranteed level of awards when dividing the endowment?

Electronic supplementary material The online version of this article (doi:10.1007/s00355-013-0789-0) contains supplementary material, which is available to authorized users.

J.-M. Giménez-Gómez
Departament d'Economia and CREIP, Universitat Rovira i Virgili, Av. Universitat 1, 43204 Reus, Spain

M. C. Marco-Gil (✉)
Department of Economics, Universidad Politécnica de Cartagena, C/ Real, 3, 30201 Cartagena, Spain
e-mail: carmen.marco@upct.es

Journal: 355 Article No.: 0789 TYPESET DISK LE CP Disp.:2014/1/28 Pages: 23 Layout: Small-X

The axiomatic and game theory approaches have been used for the normative analysis of bankruptcy problems, whose main goal is to identify rules by means of appealing properties. Following this line, the establishment of lower bounds on awards has been found reasonable by many authors. In fact, the formal definition of a rule already includes both upper and lower bounds on awards by requiring that no agent receives more than her claim and less than zero. O'Neill (1982) provides another lower bound on awards called *respect of minimal rights,* which requires that each agent receives at least what is left once the other agents have been fully compensated, or zero if this amount is negative. Herrero and Villar (2001, 2002) introduce the following two properties that bound awards. *Sustainability* says that if we truncate all claims at an agent i's claim and there is enough to honor all claims, then agent i's award should be equal to her claim. *Exemption* demands that agent i not be rationed when equal division provides her more than she claims. Moulin (2002) defines a new restriction on awards, called *lower bound:* each agent receives at least the amount corresponding to the egalitarian division except those who demand less, in which case their claims are met in full. Moreno-Ternero and Villar (2004) present a weaker notion of Moulin's *lower bound*, named *securement*, which says that each agent should obtain at least the n-th part of her claim truncated at the endowment. Finally, Dominguez (2012) proposes the *min lower bound* which modifies *securement* by replacing each agent's claim by the smallest one.

Apart from *respect of minimal rights*, a property that is implied by the definition of a rule, the other proposed lower bounds on awards have been justified by their own reasonability or appeal. In this paper, we propose a new definition along the line of O'Neill's proposal. Specifically, we define the agent's *P-right* as the smallest amount recommended to her by all the rules satisfying a set of 'basic' requirements. This set of commonly accepted principles is formed by those properties that a specific society decides to apply in the resolution of bankruptcy problems. Then, we define the associated bound on awards, *respect of P-rights*, by demanding that each agent should receive at least her *P-right*.

In general, the aggregate guaranteed amount by means of our *P-rights* will not exhaust the endowment. That is why we propose and analyze its recursive application, called the *recursive P-rights process*. Once each agent receives her *P-right* in the original problem, it is revised accordingly in order to define the residual problem. Then, each agent receives her *P-right* in this residual problem, and so on. The idea of recursion is not new. Indeed, it has already been used in the context of bankruptcy problems by Alcalde et al. (2005), in order to generalize a proposal by Ibn Ezra, and by Dominguez and Thomson (2006), whose starting point is Moreno-Ternero and Villar's concept of boundedness. Dominguez (2012) also studies the behavior of the recursive application of a generic bound.

We first apply our methodology to the singleton P_1, whose only element is *order preservation*. We find that the *recursive P-rights process* leads to the *Constrained Equal Losses* rule. This result could be written as follows: 'For each bankruptcy problem, in the set of *admissible* rules according to P_1, the recursive application of the *P-rights* leads to the rule that provides greater awards to the agents with the larger claims'. Then, we analyze the generalization of this statement. With this aim we consider a new set of socially accepted principles, P_2, by adding to *order preservation*

Journal: 355 Article No.: 0789 TYPESET DISK LE CP Disp.:2014/1/28 Pages: 23 Layout: Small-X

the requirement of *resource monotonicity* and the *midpoint property*. In this case we demonstrate that the above statement is true, but only for two-agent problems in which the *Dual of Constrained Egalitarian* rule is obtained. Moreover, we conclude that the *recursive P-rights process* for n-agent problems presents important shortcomings. Specifically, we provide a three-agent problem for P_2 in which the resulting rule defined by this process does not satisfy the equity principles on which it is based. Finally, we note that it is possible, even for two-agent problems, that the *recursive P-rights process* singles out a rule that is not the most generous to the largest claimant.

The paper is organized as follows. Section 2 presents the model. Section 3 proposes our new approach for bounding awards and its recursive application. By using the previous ideas for P_1, Sect. 4 provides a new basis for the *Constrained Equal Losses* rule. Section 5 considers our process for other sets of equity principles and shows that, in general, it can not be extended to more than two agents. Section 6 summarizes our conclusions. All the proofs are relegated to the appendices.

2 Preliminaries

We consider a group of agents, $N = \{1, \ldots, i, \ldots, n\}$, having claims on a resource. A bankruptcy problem is a situation where the sum of the agents' claims is equal to or greater than the amount available. Each agent $i \in N$ has a claim c_i on the endowment, E, a perfectly divisible good. Formally,

Definition 1 A bankruptcy problem, or simply a **problem**, is a vector $(E, c) \in \mathbb{R}_{++} \times \mathbb{R}^n_+$ such that $E \leq \sum_{i \in N} c_i$.

Hence, when the claims add up to more than the endowment, this should be rationed among agents.

Let $\mathscr{B}$ denote the set of all problems; given $(E, c) \in \mathscr{B}$, C denotes the sum of the agents' claims, $C = \sum_{i \in N} c_i$; L the total loss to distribute among the agents, $L = C - E$. Let $\mathscr{B}_0$ be the set of problems in which claims are increasingly ordered, that is problems with $c_i \leq c_j$ for $i < j$.

A rule associates within each problem a distribution of the endowment among the agents.

Definition 2 A **rule** is a function, $\varphi : \mathscr{B} \rightarrow \mathbb{R}^n_+$, such that for each $(E, c) \in \mathscr{B}$,

(a) $\sum_{i \in N} \varphi_i(E, c) = E$ (*efficiency*) and
(b) $0 \leq \varphi_i(E, c) \leq c_i$ for each $i \in N$ (*non-negativity* and *claim-boundedness*).

Next are rules that will be used in the following sections, emphasizing their dual relations.

The *Constrained Equal Awards* rule (Maimonides, 12th century, among others) recommends equal awards to all agents subject to no one receiving more than her claim.

Journal: 355 Article No.: 0789 TYPESET DISK LE CP Disp.:2014/1/28 Pages: 23 Layout: Small-X

Constrained Equal Awards rule, *CEA*: for each $(E, c) \in \mathscr{B}$ and each $i \in N$, $CEA_i(E, c) \equiv \min\{c_i, \mu\}$, where μ is chosen so that $\sum_{i \in N} \min\{c_i, \mu\} = E$.

Piniles' rule (Piniles 1861) provides, for each problem $(E, c) \in \mathscr{B}$, the awards that the *Constrained Equal Awards* rule recommends for $(E, c/2)$ when the endowment is less than the half-sum of the claims. Otherwise, each agent first receives her half-claim, then the *Constrained Equal Awards* rule is re-applied to the residual problem $(E - C/2, c/2)$.

Piniles' rule, *Pin*: for each $(E, c) \in \mathscr{B}$ and each $i \in N$,

$$Pin_i(E, c) \equiv \begin{cases} CEA_i(E, c/2) & \text{if } E \leq C/2 \\ c_i/2 + CEA_i(E - C/2, c/2) & \text{if } E \geq C/2 \end{cases}$$

The next rule, introduced by Chun et al. (2001), is inspired by the *Uniform* rule (Sprumont 1991), a solution to the problem of fair division when preferences are single-peaked. It makes the minimal adjustment in the formula for the *Uniform* rule, taking the half-claims as peaks and guaranteeing that awards are ordered in same way as claims.

Constrained Egalitarian rule, *CE*: for each $(E, c) \in \mathscr{B}$ and each $i \in N$,

$$CE_i(E, c) \equiv \begin{cases} CEA_i(E, c/2) & \text{if } E \leq C/2 \\ \max\{c_i/2, \min\{c_i, \delta\}\} & \text{if } E \geq C/2 \end{cases}$$

where δ is chosen so that $\sum_{i \in N} CE_i(E, c) = E$.

Given a rule φ, its dual distributes what is missing in the same way that φ divides what is available (Aumann and Maschler 1985).

The **dual** of φ, denoted by φ^d, is defined by setting for each $(E, c) \in \mathscr{B}$ and each $i \in N$, $\varphi_i^d(E, c) = c_i - \varphi_i(L, c)$.

It is straightforward to check that the duality operator is well defined, since for each $(E, c) \in \mathscr{B}$, $(L, c) \in \mathscr{B}$ and if φ satisfies *efficiency*, *non-negativity* and *claim-boundedness*, so does φ^d.

The next rule, discussed by Maimonides (Aumann and Maschler 1985), is the dual of the *Constrained Equal Awards* rule (Herrero 2003). Specifically, it chooses the awards vector at which all agents incur equal losses, subject to no one receiving a negative amount.

Constrained Equal Losses rule, *CEL*: for each $(E, c) \in \mathscr{B}$ and each $i \in N$, $CEL_i(E, c) \equiv \max\{0, c_i - \mu\}$, where μ is chosen so that $\sum_{i \in N} \max\{0, c_i - \mu\} = E$.

The *Dual of Piniles'* rule selects, for each problem $(E, c) \in \mathcal{B}$ the awards vector that the *Constrained Equal Losses* rule recommends for $(E, c/2)$ when the endowment is less than the half-sum of the claims. Otherwise, each agent first receives her half-claim, then the *Constrained Equal Losses* rule is re-applied to the residual problem $(E - C/2, c/2)$.

Journal: 355 Article No.: 0789 TYPESET DISK LE CP Disp.:2014/1/28 Pages: 23 Layout: Small-X

Dual of Piniles' rule, *DPin*: for each $(E, c) \in \mathscr{B}$ and each $i \in N$,

$$DPin_i(E,c) \equiv \begin{cases} c_i/2 - \min\{c_i/2, \lambda\} & \text{if } E \leq C/2 \\ c_i/2 + (c_i/2 - \min\{c_i/2, \lambda\}) & \text{if } E \geq C/2 \end{cases}$$

where λ is such that $\sum_{i \in N} DPin_i(E, c) = E$.

The *Dual of Constrained Egalitarian* rule gives the half-claims a central role. It makes the minimal adjustment in the formula for the *Uniform* rule, taking the half-claims as peaks and guaranteeing that losses are ordered in same way as claims.

Dual of Constrained Egalitarian rule, *DCE*: for each $(E, c) \in \mathscr{B}$ and each $i \in N$,

$$DCE_i(E,c) \equiv \begin{cases} c_i - \max\{c_i/2, \min\{c_i, \delta\}\} & \text{if } E \leq C/2 \\ c_i - \min\{c_i/2, \delta\} & \text{if } E \geq C/2 \end{cases}$$

where δ is chosen so that $\sum_{i \in N} DCE_i(E, c) = E$.

3 A new approach: bounding awards from equity principles

The lower bound of awards proposed by O'Neill (1982), called *respect of minimal rights*, requires that each agent receives at least what is left of the endowment after the other agents have been fully compensated, or zero if this amount is negative.

Respect of minimal rights: for each $(E, c) \in \mathscr{B}$ and each $i \in N$, $\varphi_i(E, c) \geq m_i(E, c) = \max\{E - \sum_{j \neq i} c_j, 0\}$.

This bound on awards is a consequence of *efficiency*, *non-negativity* and *claim-boundedness* together (Thomson 2003), the three conditions imposed by a rule (see Definition 2).[1]

Following this line we introduce a new method for bounding awards based on a set of principles that are commonly accepted by a society. We propose the following extension of a problem.

Definition 3 **A problem with legitimate principles** is a triplet (E, c, P_t), where $(E, c) \in \mathscr{B}$ and P_t is a set of properties on which a particular society has agreed.

Let P be the set of all subsets of properties of rules. Each $P_t \in P$ represents a specific society which will always apply such principles for solving its problems. Finally, let $\mathscr{B}_P$ be the set of all *problems with legitimate principles*.

This modelling becomes really interesting if it is applied to some specific types of problems, since the more available information we have, the easier it is to agree on these principles. For example, let $\mathscr{B}_P^T \subset \mathscr{B}_P$, the *problems with legitimate principles* that represent the collection of a given amount of taxes in a community. In this case,

[1] For each $i \in N$, if $m_i(E, c) > 0$ and $\varphi_i(E, c) < m_i(E, c)$ either $\sum_{i \in N} \varphi_i(E, c) < E$, contradicting *efficiency*, or there is $j \neq i$ such that $\varphi_j(E, c) > c_j$, contradicting *claim-boundedness*. Otherwise, that is, if $m_i(E, c) = 0$, by *non-negativity* $\varphi_i(E, c) \geq m_i(E, c)$.

Journal: 355 Article No.: 0789 TYPESET DISK LE CP Disp.:2014/1/28 Pages: 23 Layout: Small-X

progressivity (see Thomson 2003) may be commonly accepted. However, this property may not be reasonable in other situations.

For each *problem with legitimate principles*, a society will consider as socially *admissible* any rule that satisfies the properties in P_t.

Definition 4 A socially *admissible* rule, or simply an **admissible rule**, is a function, $\varphi : \mathscr{B}_P \to \mathbb{R}^n_+$, such that its application in $\mathscr{B}$, $\varphi : \mathscr{B} \to \mathbb{R}^n_+$, is a rule satisfying all properties in P_t.

Let Φ denote the set of all rules and let $\Phi(P_t)$ be the subset of rules satisfying P_t.

Taking extended problems as a starting point, we propose a new lower bound on awards based on the application of the ordinary meaning of a guarantee. This bound, called *P-rights*, provides each agent the smallest amount recommended to her by all *admissible* rules. Formally,

Definition 5 Given (E, c, P_t) in $\mathscr{B}_P$, the **P-right** of each $i \in N$, s_i, is

$$s_i(E, c, P_t) = \inf_{\varphi \in \Phi(P_t)} \{\varphi_i(E, c)\}.$$

Now, we say that a rule *respects P-rights* if, for each $P_t \in P$, each $(E, c) \in \mathscr{B}$ and each $i \in N$, $\varphi_i(E, c) \geq s_i(E, c, P_t)$.

Note that if P_t is the empty set, the *P-rights* correspond with the concept of *minimal rights*.

As in general, the sum of the agents' *P-rights* of a *problem with legitimate principles* does not exhaust the endowment, a requirement of composition from the profile of these bounds arises in a natural way. It says that the awards vector of each problem should be equivalently obtainable either directly or by means of the following process. First, assigning to each agent her lower bound on awards. Second, adjusting claims down by these amounts. And third, applying the rule to divide the remainder. The following definition applies this idea to our bound on awards.

Definition 6 Given $P_t \in P$, a rule φ satisfies **P-rights first** if for each $(E, c) \in \mathscr{B}$ and each $i \in N$, $\varphi_i(E, c) = s_i(E, c, P_t) + \varphi_i(E - \sum_{i \in N} s_i(E, c, P_t), c - s(E, c, P_t))$.

Although many of the proposed lower bounds on awards are respected by most of the rules, composition from these lower bounds is quite demanding. For instance, *respect of minimal rights* is satisfied by any rule, but none of the *Proportional, Constrained Equal Awards* or *Minimal Overlap* rules satisfy *minimal rights first* (Thomson 2003). Let us note that this kind of composition is equivalent to apply a recursive method from a lower bound on awards. In fact, this process has been used to generate new rules. The rule proposed by Dominguez and Thomson (2006) results from applying such a procedure to the *securement* lower bound. Similarly, we define the recursive application of our *P-rights*, which we call the *recursive P-rights process*.

Definition 7 For each $m \in \mathbb{N}$, the **recursive P-rights process** at the m-th step, RS^m, associates for each $(E, c, P_t) \in \mathscr{B}_P$ and each $i \in N$,

Journal: 355 Article No.: 0789 TYPESET DISK LE CP Disp.:2014/1/28 Pages: 23 Layout: Small-X

$$\left[RS^m(E,c,P_t)\right]_i = s_i(E^m, c^m, P_t),$$
$$\text{where } (E^1, c^1) \equiv (E, c) \text{ and for } m \geq 2,$$
$$(E^m, c^m) \equiv (E^{m-1} - \sum_{i\in N} s_i(E^{m-1}, c^{m-1}, P_t), \quad c^{m-1} - s(E^{m-1}, c^{m-1}, P_t)).$$

According to this process, at the first step each agent receives her *P-right* in the original problem. At the second step, we define a residual problem in which the endowment is what remains and the claims are adjusted down by the amounts just given. Then, each agent receives her *P-right* in this residual problem, and so on. In general, it cannot be ensured that the sum of the amounts that agents receive in each step exhausts the endowment. If this occurs, we define the *Recursive P-rights* rule.

Definition 8 The **Recursive P-rights** rule, φ^R, associates for each $(E, c, P_t) \in \mathscr{B}_P$ and each $i \in N$, $\varphi_i^R(E, c, P_t) = \sum_{m=1}^{\infty} \left[RS^m(E, c, P_t)\right]_i$, whenever

$$\sum_{i\in N}\left(\sum_{m=1}^{\infty}\left[RS^m(E,c,P_t)\right]_i\right) = E.$$

Note that the *Recursive P-rigths* rule satisfies *non-negativity* and *claim-boundedness* by construction. The next result, which is a specification to our context of a theorem provided by Dominguez (2012) for the recursive process based on a generic lower bound on awards, shows that whenever the *P-rights* provide a positive amount to some agent at each step, *efficiency* is met.[2]

Theorem (Dominguez 2012) *For each* $(E, c, P_t) \in \mathscr{B}_P$, $\sum_{i\in N}\left(\sum_{m=1}^{\infty}\left[RS^m(E, c, P_t)\right]_i\right) = E$ *whenever for each* $m \in \mathbb{N}$ *there is* $i \in N$ *such that* $s_i(E^m, c^m, P_t) > 0$.

At this point we should mention some contributions that have certain features in common with our approach. In the context of Nash's bargaining model, Damme (1986) uses the research on Nash equilibria of a non-cooperative game which is induced by a mechanism of successive concessions. Specifically, the agents' strategies are the choice of a rule among a set of reasonable ones. The first variants of van Damme's work for bargaining and bankruptcy problems were introduced by Chun (1984, 1989). From them, other mechanisms, related to ours, have been proposed. The unanimous concessions mechanism, provided by Marco et al. (1995) and modified by Herrero (2003) for its application to bankruptcy, is close to our *recursive P-rights process*, but the starting point and analysis of the two are quite different. Also for bargaining problems, Thomson (2012) introduces and studies the concept of *closedness under recursion* of a family of solutions, which means that the solution defined through the process is not only well-defined, but also belongs to the family of solutions considered. This idea, although in a different framework, is close to our definition of *admissible* rule, but the process he uses has no relation to ours.

[2] In this sense, the recursive application of the *minimal rights* does not satisfy *efficiency*, since from the second step on, each agent receives nothing.

Journal: 355 Article No.: 0789 TYPESET DISK LE CP Disp.:2014/1/28 Pages: 23 Layout: Small-X

4 A minimal requirement of fairness

Next, we apply our method to the singleton P_1 whose only element is *order preservation*,

$$P_1 \equiv \{order\ preservation\}.$$

This property was introduced by Aumann and Maschler (1985). In fact, in our setting, where there are neither absolute nor relative priority classes, *order preservation* has been understood as a minimal requirement of fairness by many authors. It requires that if agent i's claim is at least as large as agent j's claim, she should receive at least as much as agent j does; furthermore, agent i's loss should be at least as large as agent j's loss.

Order preservation: for each $(E, c) \in \mathscr{B}$ and each $i, j \in N$ such that $c_i \geq c_j$, $\varphi_i(E, c) \geq \varphi_j(E, c)$ and $c_i - \varphi_i(E, c) \geq c_j - \varphi_j(E, c)$.

Lemma 5 in Appendix 2 shows that the P_1-*rights* for agents 1 and n are given by the *Constrained Equal Losses* and the *Constrained Equal Awards* rules, respectively. As a direct consequence of this result, for two-agent problems, these two rules mark out the area of all the *admissible* rules in P_1. However, as shown in the next example, this fact cannot be generalized for problems with more than two agents.

Example 1 Let $N = \{1, 2, 3\}$ and $(E, c) = (49, (18, 27, 40)) \in \mathscr{B}$. Thus, $CEA\,(E, c) = \left(16\frac{1}{3}, 16\frac{1}{3}, 16\frac{1}{3}\right)$ and $CEL\,(E, c) = (6, 15, 28)$. By Lemma 5 in Appendix 2, $s_1(E, c, P_1) = 6$ and $s_3(E, c, P_1) = 16\frac{1}{3}$. Moreover, $T\,(E, c) = \left(9, 13\frac{1}{2}, 26\frac{1}{2}\right)$, where T denotes the *Talmud* rule.[3] This rule satisfies *order preservation* and $T_2\,(E, c) = 13\frac{1}{2} < CEL_2\,(E, c) < CEA_2\,(E, c)$. Therefore, for agent 2 neither of the amounts provided by both *CEL* and *CEA*, is the smallest one she can get according to P_1.

The next result shows the *Recursive P-rights* rule for P_1.

Theorem 1 *For each $(E, c, P_1) \in \mathscr{B}_P$, the Recursive P-rights rule is the Constrained Equal Losses rule, that is, $\varphi^R(E, c, P_1) = CEL(E, c)$.*

Proof See Appendix 2. □

To conclude this section, let us note that we have proved that the *Recursive P-rights* rule for P_1 leads to the *admissible* rule which favors the largest claimant.

5 Other sets of legitimate principles

In this section, we consider other possible choices of '*commonly accepted equity principles*'. First, we propose P_2 obtained by adding to P_1 *resource monotonicity* and *midpoint property*,

[3] The *Talmud* rule (Aumann and Maschler 1985) assigns the awards that the *Constrained Equal Awards* rule recommends for $(E, c/2)$, when the endowment is less than the half-sum of the claims. Otherwise, each agent receives her half-claim plus the amount provided by the *Constrained Equal Losses* rule when it is applied to the residual problem (E-C/2, c/2).

Journal: 355 Article No.: 0789 TYPESET DISK LE CP Disp.:2014/1/28 Pages: 23 Layout: Small-X

$$P_2 \equiv \{order\ preservation,\ resource\,monotonicity,\ midpoint property\}.$$

Resource monotonicity Curiel et al. (1987), Young (1988) and others says that if the endowment increases, then all individuals should get at least what they received initially. This property has been widely accepted. Moreover, no rule violating this property has been proposed.

Resource monotonicity: for each $(E, c) \in \mathscr{B}$ and each $E' \in \mathbb{R}_+$ such that $C \geq E' > E$, $\varphi_i(E', c) \geq \varphi_i(E, c)$, for each $i \in N$.

Midpoint property (Chun et al. 2001) requires that if the endowment is equal to the sum of the half-claims, then all agents should receive their half-claim. In this situation both gains and losses are equal. Thus, this property treats the problem of dividing awards or losses equally, but only in a very special case. In the words of Aumann and Maschler (1985), 'it is socially unjust for different creditors to be on opposite sides of the halfway point, $C/2$'.

Midpoint property: for each $(E, c) \in \mathscr{B}$ and each $i \in N$, if $E = C/2$, then $\varphi_i(E, c) = c_i/2$.

From Lemma 6 in Appendix 3 we obtain that the *Constrained Egalitarian* and the *Dual of Constrained Egalitarian* rules mark out the area of all the *admissible* rules satisfying properties in P_2 for two-agent problems. Next, we show that the *Recursive P-rights* rule for P_2 leads to the *Dual of Constrained Egalitarian* rule, but only for two-agent problems. Besides this, we demonstrate that the *recursive P-rights process* for P_2 presents important shortcomings for n-agent problems. Specifically, this process provides a rule that is not *admissible* since it does not satisfy one of the equity principles upon which society initially agreed to found its decisions.

Theorem 2 *For each two-agent problem with legitimate principles in $\mathscr{B}_P$ with $P = P_2$, the Recursive P-rights rule is the Dual of Constrained Egalitarian rule, that is, $\varphi^R(E, c, P_2) = DCE(E, c)$.*

Proof See Appendix 3. □

Proposition 1 *The Recursive P-rights rule for P_2 does not satisfy resource monotonicity for n-agent problems with $n > 2$.*

Proof See Appendix 4. □

All of our previous results can be summarized by the following two statements:

(a) For each two-agent problem, the *P-rights* recursive application of the two sets of properties considered up to now, leads to the *admissible* rule which favors the largest claimant.
(b) If a society agree on the set of principles P_2, the *recursive P-rights process* cannot be applied for n-agent problems.

Next, we note that for other reasonable legitimate principles, on the one hand, statement (a) cannot be extended and, on the other hand, statement (b) can also be applied.

Journal: 355 Article No.: 0789 TYPESET DISK LE CP Disp.:2014/1/28 Pages: 23 Layout: Small-X

Let us consider the set of *commonly accepted equity principles* P_3, obtained from P_2 by substituting *order preservation* for a strengthened version, *super-modularity*,

$$P_3 \equiv \{super\text{-}modularity,\ resourcemonotonicity,\ midpoint\ property\}.$$

Super-modularity Dagan et al. (1997) demands that, when the endowment increases, if agent i's claim is at least as large as agent j's claim, agent i's share of the increment should be at least as large as agent j's. Apart from the *Constrained Egalitarian* rule and its dual, all of the rules that have been introduced in the literature satisfy *super-modularity*.

Super-modularity: for each $(E, c) \in \mathscr{B}$, each $E' \in \mathbb{R}_+$ and each $i, j \in N$ such that $C \geq E' > E$ and $c_i \geq c_j$, $\varphi_i(E', c) - \varphi_i(E, c) \geq \varphi_j(E', c) - \varphi_j(E, c)$.

In this context, the next results are obtained.

Remark 1 For each two-agent *problem with legitimate principles* in $\mathscr{B}_P$ with $P = P_3$, the *Recursive P-rights* rule is *admissible*, but neither the *Dual of Piniles*' rule nor *Piniles*' rule coincides with it, that is, $DPin(E, c) \neq \varphi^R(E, c, P_3) \neq Pin(E, c)$.

Remark 2 The *Recursive P-rights* rule for P_3 does not satisfy *super-modularity* for n-agent problems with $n > 2$.

The proofs of the previous remarks are constructive. To prove Remark 1, several structures of a generic bankruptcy problem are considered regarding the values of the endowment and the agents' claims. Then, knowing the *P-rights* for P_3 (by means of an analogous result to Lemma 6) allows applying the *recursive P-rights process* to each structure and obtaining the *Recursive P-rights* rule for P_3. The development of this proof reveals that the magnitude of the endowment with respect to the sum of the half-claims can change for different steps of the *recursive P-rights process* and this fact prevents that a similar result to Theorem 2 can be reached. The proof of Remark 2 is similar to that of Proposition 1. Starting from defining a rule that recommends the smallest amount for agent 2 among all the *admissible* rules for P_3, some steps of the *recursive P-rights process* are computed for two particular problems to contradict that the *Recursive P-rights* rule for P_3 satisfies *super-modularity*.[4]

This analysis warns of the dangers that may involve the composition of the puzzle with 'a priori' suitable pieces: 'reasonable' principles and recursion. Unfortunately, we have ascertained that it does not always provide *admissible* rules.

6 Conclusions

We would like to remark that our approach can be rewritten for losses by using the idea of duality. Because all the considered properties are *self-dual*, P_1, P_2 and P_3 will be the same when focusing on losses.[5] Moreover, let us note that given a set of *self-dual*

[4] The proofs of Remarks 1 and 2 are available from the authors under request.

[5] *Self-duality* requires invariance regarding the perspective from which the problem is derived, that is, dividing 'what is available' or 'what is missing'. Formally, two properties, $\mathcal{P}$ and $\mathcal{P}'$, are **dual** if whenever a rule, φ, satisfies $\mathcal{P}$, its dual, φ^d, satisfies $\mathcal{P}'$. A property, $\mathcal{P}$, is **self-dual** when it coincides with its dual.

Journal: 355 Article No.: 0789 TYPESET DISK LE CP Disp.:2014/1/28 Pages: 23 Layout: Small-X

properties on which a particular society has agreed, P_t, a rule, φ, is *admissible* if and only if its dual, φ^d, is also *admissible*. Specifically, by considering (L, c, P_t) for each (E, c, P_t) we have that

$$s_i(L, c, P_t) = \inf_{\varphi \in \Phi(P_t)} \{\varphi_i(L, c)\} = \inf_{\varphi \in \Phi(P_t)} \{c_i - \varphi_i^d(E, c)\} = c_i - \sup_{\varphi \in \Phi(P_t)} \{\varphi_i(E, c)\}.$$

Thus, our process applied to losses is equivalent to the following. First, determine the agents' upper bound on awards by searching for the supremum of what they are assigned among all the *admissible* rules in P_t. Then, revise each agent's claim by her upper bound and if the sum of the revised claims is greater than the endowment, follow the recursive process until the sum of the revised claims is equal to the endowment.

Therefore, if for each $(E, c) \in \mathscr{B}$ we consider its associated distribution of losses, that is the problem (L, c), the recursive application of the P-*rights* leads to: (i) the *Constrained Equal Awards* for $P = P_1$; (ii) the *Constrained Egalitarian* rule for two-agent problems when $P = P_2$; (iii) a new *admissible* rule for two-agent problems if $P = P_3$; (iv) inadmissible rules for n-agent problems when $n > 2$ for both P_2 and P_3.

In addition, let us note that none of our results requires the use of many of the axioms proposed in the literature on the theoretical analysis of bankruptcy problems. Nevertheless, it can be straightforwardly checked that all of them (Theorems 1 and 2, Proposition 1 and Remarks 1 and 2) remain the same if we add to the considered sets of legitimate principles (P_1, P_2, P_3) some standard properties (such as *continuity*, *claims monotonicity* or *homogeneity*, among others) that are satisfied by those *admissible* rules that play a central role (*CEA* and *CEL* in Theorem 1; *CE* and *DCE* in Theorem 2 and Proposition 1; and Pin and $DPin$ in Remarks 1 and 2).[6] This fact has conditioned the analysis of the generalization of the conclusion reached for P_1. Specifically, the other legitimate principles sets have been established in looking for a trade-off between reasonability of properties and inadmissibility of the *Constrained Equal Losses* rule, for P_2, and inadmissibility of both the *Constrained Equal Losses* and the *Dual of Constrained Egalitarian* rules, for P_3.

To sum up, this paper offers the understanding of old bankruptcy rules from a different perspective and uncovers some shortcomings with the application of our recursive process. In this line, the following issues are open: (i) the analysis of conditions on legitimate principles sets to guarantee that such principles are upheld when applying our approach, and (ii) the search for new procedures that ensure compatibility with socially accepted equity principles.

Acknowledgments The authors are deeply grateful to Francisco Javier Martínez de Albéniz, Montserrat Ferré, Josep Enric Peris, Begoña Subiza, William Thomson, an anonymous referee and the associated editor of this journal, who really helped us improve our initial draft. For their suggestions we also thank seminar participants at both Universities Barcelona and Rovira i Virgili. The usual caveats apply. Financial support from Universitat Rovira i Virgili, Banco Santander and Generalitat de Catalunya under project 2011LINE-06 and the Barcelona GSE are gratefully acknowledged.

[6] Formal definitions of these properties can be found in Thomson (2003).

Appendix 1: General claims

We present three claims which are used in the proofs of Appendices 2 and 3. Henceforth, $m \in \mathbb{N}$ denotes the m-th step of the *recursive P-rights process* (see Definition 7).

First, for any *problem with legitimate principles*, the total loss to distribute is the same at every step of the *recursive P-rights process.*

Claim 1 *For each $(E, c, P_t) \in \mathscr{B}_P$ and each $m \in \mathbb{N}$, $L^m = L$.*

Proof Let $(E, c, P_t) \in \mathscr{B}_P$ and $m \in \mathbb{N}$. Then,

$$\begin{aligned} L^m = C^m - E^m &= \sum_{i \in N} \left(c_i - \sum_{k=1}^{m-1} s_i(E^k, c^k, P_t) \right) - \left(E - \sum_{i \in N} \sum_{k=1}^{m-1} s_i(E^k, c^k, P_t) \right) \\ &= C - E = L. \end{aligned}$$

□

Second, for each $P \in \{P_1, P_2, P_3\}$, the order of the agents' claims remains the same along the *recursive P-rights process*.

Claim 2 *For each $(E, c, P_t) \in \mathscr{B}_P$ with $P_t \in \{P_1, P_2, P_3\}$ and each $i, j \in N$, if $c_i^m \leq c_j^m$, then $c_i^{m+1} \leq c_j^{m+1}$.*

Proof Let $(E, c, P_t) \in \mathscr{B}_P$ with $P_t \in \{P_1, P_2, P_3\}$, $i, j \in N$ such that $c_i^m \leq c_j^m$ and φ^*, φ' belonging to $\Phi(P_t)$.

Since, for each $P_t \in \{P_1, P_2, P_3\}$, all the *admissible* rules satisfy *order preservation*, for each $\varphi \in \Phi(P_t)$, $c_i^m - \varphi_i(E^m, c^m) \leq c_j^m - \varphi_j(E^m, c^m)$ so that,

(a) If $s_i^m(E, c, P_t) = \varphi_i^*(E^m, c^m)$ and $s_j^m(E, c, P_t) = \varphi_j^*(E^m, c^m)$, by *order preservation*, $c_i^m - s_i^m(E^m, c^m, P_t) \leq c_j^m - s_j^m(E^m, c^m, P_t)$. Therefore, $c_i^{m+1} \leq c_j^{m+1}$.

(b) If $s_i^m(E, c, P_t) = \varphi_i^*(E^m, c^m)$ and $s_j^m(E, c, P_t) = \varphi_j'(E^m, c^m)$, by Definition 5, $\varphi_j'(E^m, c^m) \leq \varphi_j^*(E^m, c^m)$, so that, $c_i^m - \varphi_i^*(E^m, c^m) \leq c_j^m - \varphi_j^*(E^m, c^m) \leq c_j^m - \varphi_j'(E^m, c^m)$. Therefore, $c_i^{m+1} \leq c_j^{m+1}$. □

Third, for each $P_t \in \{P_1, P_2, P_3\}$, the sum of the amounts that agents are assigned by the *recursive P-rights process* is the entire endowment.

Claim 3 *For each $(E, c, P_t) \in \mathscr{B}_P$ with $P_t \in \{P_1, P_2, P_3\}$, $\sum_{i \in N} \left(\sum_{m=1}^{\infty} \left[RS^m (E, c, P_t) \right]_i \right) = E$.*

Proof Given that for each $P_t \in \{P_1, P_2, P_3\}$ the *P-rights* always provide a positive amount to certain agents in each step, *efficiency* of the *recursive P-rights process* straightforwardly comes from Theorem 1 in Dominguez (2012), that we have particularized within our context in Sect. 3. □

Appendix 2: Proof of Theorem 1

We assume throughout this Appendix, without loss of generality, that $(E, c) \in \mathscr{B}_0$.[7]

The proof is based on five lemmas. Before presenting them, we note the following two facts.

Fact 1 For each $(E, c) \in \mathscr{B}_0$ and each $i \in N$, $CEL_i(E, c) = \max\{0, c_i - \mu\}$, where μ is such that $\sum_{i \in N} \max\{0, c_i - \mu\} = E$.
Therefore, μ can be understood as the losses incurred by the agents who receive positive amounts by applying the *CEL* rule. A straightforward way to compute this rule, which will be useful later on, is as follows.
For each $(E, c) \in \mathscr{B}_0$ and each $i \in N$, the loss imposed on agent i by *CEL* is

$$\gamma_i = \min\{c_i, \alpha_i\},$$

where

$$\alpha_i = \left(L - \sum_{j<i} \gamma_j\right) / (n - i + 1).$$

Therefore, for each $i \in N$,

$$CEL_i(E, c) = c_i - \gamma_i.$$

Hereinafter, for each $m \in \mathbb{N}$, μ^m, α_i^m and γ_i^m will denote μ, α_i and γ_i solving $\sum_{i \in N} CEL_i(E^m, c^m) = E^m$, respectively.

Fact 2 By Fact 1 and Claim 1 we have:

(a) For each $(E, c) \in \mathscr{B}_0$ and each $i \in N$, if $\gamma_i = c_i$ then, for each $j < i$, $\gamma_j = c_j$.
(b) For each $(E, c) \in \mathscr{B}_0$ and each $i \in N$, if $\gamma_i = \alpha_i$ then, $\alpha_i = \mu$ and for each $j > i$, $\alpha_j = \alpha_i$. Therefore, $\gamma_i = \mu$.
(c) For each $m \in \mathbb{N}$ and each $i \in N$, α_i^m only depends on both the initial problem (E, c) and agent j's claim for each $j < i$.

Next, we provide the five lemmas on which Theorem 1 is based.

The first lemma says that the losses incurred by the agents who receive positive amounts by applying the *CEL* rule is the same at all steps of the *recursive P-rights process* for P_1.

Lemma 1 *For each* (E, c, P_1) *with* $(E, c) \in \mathscr{B}_0$ *and each* $m \in \mathbb{N}$, $\mu^{m+1} = \mu^m$.

Proof Let agent i be the first agent who receives a positive amount at step $m \in \mathbb{N}$ according to the *CEL* rule. That is, if $i = 1$, for each $k \in N$, $CEL_k(E^m, c^m) > 0$. Otherwise, (i) $CEL_i(E^m, c^m) > 0$ and (ii) for each $j < i$, $CEL_j(E^m, c^m) = 0$.

[7] If $(E, c) \in \mathscr{B} \backslash \mathscr{B}_0$, there is a permutation π such that $\pi(c)$ is increasingly ordered and we can compute $\varphi(E, c) = \pi^{-1}[\varphi(E, \pi(c))]$. Where a permutation is a bijection applying $\mathcal{N}$ to itself and, abusing notation, $\pi(c)$ will denote the claim vector obtained by applying permutation π to its components, i.e. the i-th component of $\pi(c)$ is c_j whenever $j = \pi(i)$. Similar considerations apply for $\pi[\varphi(E, c)]$.

Journal: 355 Article No.: 0789 TYPESET DISK LE CP Disp.:2014/1/28 Pages: 23 Layout: Small-X

By (i) and Fact 2, $c_i^m > \mu^m = \alpha_i^m$. Given (ii) and Definition 7 of the *recursive P-rights process* for P_1 at the m-th step, for each $j < i$, $c_j^{m+1} = c_j^m$. By Fact 2-(c), $\alpha_i^{m+1} = \alpha_i^m = \mu^m < c_i^m$. Furthermore,

$$
\begin{aligned}
c_i^{m+1} &= c_i^m - \min_{\varphi \in \Phi(P_1)} \{\varphi_i(E^m, c^m)\} \\
&\geq c_i^m - CEL_i(E^m, c^m) = c_i^m - \left(c_i^m - \mu^m\right) \\
&= \mu^m = \alpha_i^{m+1}.
\end{aligned}
$$

Therefore, by Claim 2 and Fact 2-(b), $\gamma_i^{m+1} = \alpha_i^{m+1} = \mu^{m+1}$. □

The second lemma states that if at some step $m \in \mathbb{N}$ the agent i 's *P-right* for P_1 is $CEL_i(E^m, c^m)$, then at each subsequent step, her *P-right* for P_1 is zero.

Lemma 2 *For each* $(E, c) \in \mathscr{B}_0$ *and each* $i \in N$, *if there is* $m \in \mathbb{N}$ *such that*

$$s_i(E^m, c^m, P_1) = CEL_i(E^m, c^m)$$

then, for each $h \in \mathbb{N}$

$$s_i(E^{m+h}, c^{m+h}, P_1) = 0.$$

Proof Let $(E, c) \in \mathscr{B}_0$, $i \in N$ and $m \in \mathbb{N}$ be such that

$$s_i(E^m, c^m, P_1) = CEL_i(E^m, c^m).$$

Since

$$
\begin{aligned}
&CEL_i(E^m, c^m) = c_i^m - \min\left\{c_i^m, \mu\right\}, \\
&\quad c_i{}^{m+1} = c_i{}^m - CEL_i(E^m, c^m) = c_i{}^m - \left(c_i^m - \min\left\{c_i^m, \mu\right\}\right) = \min\left\{c_i^m, \mu\right\}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
CEL_i(E^{m+1}, c^{m+1}) &= c_i^{m+1} - \min\left\{c_i^{m+1}, \mu\right\} \\
&= \min\left\{c_i^m, \mu\right\} - \min\left\{\min\left\{c_i^m, \mu\right\}, \mu\right\} \\
&= \min\left\{c_i^m, \mu\right\} - \min\left\{c_i^m, \mu\right\} = 0.
\end{aligned}
$$

Therefore,

$$s_i(E^{m+1}, c^{m+1}, P_1) = 0.$$

By Fact 1 we have that if at some step $k \in \mathbb{N}$ $CEL_i(E^k, c^k) = 0$, then $CEL_i(E^{k+h}, c^{k+h}) = 0$ for each $h \in \mathbb{N}$. Then, agent i's *P-right* for P_1 is, from step $m + 1$ on, zero. □

The next lemma establishes that, if agent i's *P-right* for P_1 is, at each step, a different amount from that provided by the *CEL* rule then, the total amount received by this agent is at most her award as calculated by the *CEL* rule applied to the initial problem.

Journal: 355 Article No.: 0789 TYPESET DISK LE CP Disp.:2014/1/28 Pages: 23 Layout: Small-X

Lemma 3 *For each* $(E, c) \in \mathscr{B}_0$ *and each* $i \in N$, *if for each* $m \in \mathbb{N}$

$$s_i(E^m, c^m, P_1) \neq CEL_i(E, c), \text{ then}$$

$$\varphi_i^R(E, c, P_1) = \sum_{k=1}^{\infty} s_i(E^k, c^k, P_1) \leq CEL_i(E, c).$$

Proof Let $(E, c) \in \mathscr{B}_0$ and $i \in N$. If for each $m \in \mathbb{N}$ $s_i(E^m, c^m, P_1) \neq CEL_i(E, c)$, by Definition 5 ,

$$s_i(E^m, c^m, P_1) < CEL_i(E^m, c^m) = c_i^m - \mu = c_i - \sum_{k=1}^{m-1} s_i(E^k, c^k, P_1) - \mu.$$

So that,

$$s_i(E^m, c^m, P_1) + \sum_{k=1}^{m-1} s_i(E^k, c^k, P_1) < c_i - \mu,$$

that is, for each $m \in \mathbb{N}$

$$\sum_{k=1}^{m} s_i(E^k, c^k, P_1) < CEL_i(E, c).$$

Therefore, the sequence $\{a_m\}_{m\in\mathbb{N}}$, where $a_m = \sum_{k=1}^{m} s_i(E^k, c^k, P_1)$ for each $m \in \mathbb{N}$, is bounded above. Since, by construction $\{a_m\}_{m\in\mathbb{N}}$ is monotonically increasing, by applying basic properties of sequences limit computation (see, for instance, Blume and Simon 1994) we have that $\lim_{m\to\infty} \sum_{k=1}^{m} s_i(E^k, c^k, P_1)$ exists and $\lim_{m\to\infty} \sum_{k=1}^{m} s_i(E^k, c^k, P_1) \leq CEL_i(E, c)$. □

The fourth lemma says that if at some step $m \in \mathbb{N}$, an agent's *P-right* for P_1 is the amount provided by the *CEL* rule for the problem (E^m, c^m), then the total amount received by this agent up to that step is given by the *CEL* rule applied to the initial problem.

Lemma 4 *For each* $(E, c) \in \mathscr{B}_0$ *and each* $i \in N$, *if there is* $m^* \in \mathbb{N}$, $m^* > 1$, *such that* $s_i(E^{m^*}, c^{m^*}, P_1) = CEL_i(E^{m^*}, c^{m^*})$ *and* $s_i(E^{m^*-1}, c^{m^*-1}, P_1) \neq CEL_i(E^{m^*-1}, c^{m^*-1})$ *then,*

$$\sum_{k=1}^{m^*} s_i(E^k, c^k, P_1) = CEL_i(E, c).$$

Proof Let $(E, c) \in \mathscr{B}_0$, $i \in N$ and $m^* \in \mathbb{N}$, $m^* > 1$ be such that $s_i(E^{m^*}, c^{m^*}, P_1) = CEL_i(E^{m^*}, c^{m^*})$ and $s_i(E^{m^*-1}, c^{m^*-1}, P_1) \neq CEL_i(E^{m^*-1}, c^{m^*-1})$. Since $\varphi_i(E^{m^*-1}, c^{m^*-1}) < CEL_i(E^{m^*-1}, c^{m^*-1})$, $CEL_i(E^{m^*-1}, c^{m^*-1}) > 0$. Therefore, $c_i^{m^*-1} > \mu$

Journal: 355 Article No.: 0789 TYPESET DISK LE CP Disp.:2014/1/28 Pages: 23 Layout: Small-X

and by Lemma 1, $c_i^{m^*} \geq \mu$. Then, at step m^*, agent i has received

$$\sum_{k=1}^{m^*} s_i(E^k, c^k, P_1) = \sum_{k=1}^{m^*-1} s_i(E^k, c^k, P_1) + CEL_i(E^{m^*}, c^{m^*})$$

$$= \sum_{k=1}^{m^*-1} s_i(E^k, c^k, P_1) + \left[c_i^{m^*} - \min\left\{c_i^{m^*}, \mu\right\}\right]$$

$$= \sum_{k=1}^{m^*-1} s_i(E^k, c^k, P_1)$$

$$+ \left[\left(c_i - \sum_{k=1}^{m^*-1} s_i(E^k, c^k, P_1)\right) - \min\left\{c_i^{m^*}, \mu\right\}\right]$$

$$= c_i - \min\left\{c_i^{m^*}, \mu\right\} = c_i - \mu.$$

Therefore,

$$\sum_{k=1}^{m^*} s_i(E^k, c^k, P_1) = CEL_i(E, c).$$

□

The last lemma shows that the *P-rights* for agents 1 and n, when considering P_1, correspond to the *CEL* and *CEA* rules, respectively.

Lemma 5 *For each* $(E, c, P_1) \in \mathscr{B}_P$ *with* $(E, c) \in \mathscr{B}_0$, $s_1(E, c, P_1) = CEL_1(E, c)$ *and* $s_n(E, c, P_1) = CEA_n(E, c)$.

Proof Let (E, c, P_1) with $(E, c) \in \mathscr{B}_0$. First, we show that $s_1(E, c, P_1) = CEL_1(E, c)$. There are two cases.

- $CEL_1(E, c) = 0$. By *non-negativity*, $s_1(E, c, P_1) = CEL_1(E, c)$.
- $CEL_1(E, c) > 0$. By the definition of the *CEL* rule, $c_1 - CEL_1(E, c) = c_j - CEL_j(E, c)$ for each $j \neq 1$. Let us suppose that there is $\varphi \in \Phi(P_1)$ such that $\varphi_1(E, c) < CEL_1(E, c)$. By *efficiency*, $\varphi_j(E, c) > CEL_j(E, c)$ for some $j \neq 1$. Then, $c_1 - \varphi_1(E, c) > c_j - \varphi_j(E, c)$, contradicting *order preservation*. Therefore, $s_1(E, c, P_1) = CEL_1(E, c)$.

Second, it can be similarly obtained that $s_n(E, c, P_1) = CEA_n(E, c)$. □

Proof of Theorem 1 Let $(E, c) \in \mathscr{B}_0$. There are two cases.

Case a: All claims are equal. Then, by definition of *P-rights* for P_1, each agent receives the same amount and the entire endowment is distributed at the first step. Therefore, $\varphi^R(E, c, P_1) = CEL(E, c)$.

Case b: There are at least two agents whose claims differ. Let $S = \{r \in N| s_r(E^m, c^m, P_1) = CEL_r(E^m, c^m)$ at some step $m \in \mathbb{N}\}$ and $T = N \setminus S$. By Lemma 5,

Journal: 355 Article No.: 0789 TYPESET DISK LE CP Disp.:2014/1/28 Pages: 23 Layout: Small-X

$s_1(E, c, P_1) = CEL_1(E, c)$. Furthermore, by Lemmas 2 and 4, for each agent $r \in S$, we have that $\varphi_r^R(E, c, P_1) = CEL_r(E, c)$. Moreover, for each agent $l \in T$, by Lemma 3, $\varphi_l^R(E, c, P_1) \leq CEL_l(E, c)$. Then, since $\varphi^R(E, c, P_1)$ exhausts the endowment, by Claim 3, $\varphi^R(E, c, P_1) = CEL(E, c)$. □

Appendix 3: Proof of Theorem 2

We assume throughout this Appendix, without loss of generality, that $(E, c) \in \mathscr{B}_0$ (see Footnote 7). Next we present a lemma and a fact, which the proof of Theorem 2 is based on.

The lemma shows that the *P-rights* for agents 1 and n, when considering P_2, correspond to the *DCE* and *CE* rules, respectively.

Lemma 6 *For each* $(E, c, P_2) \in \mathscr{B}_P$ *with* $(E, c) \in \mathscr{B}_0$, $s_1(E, c, P_2) = DCE_1(E, c)$ *and* $s_n(E, c, P_2) = CE_n(E, c)$.

Proof First, we show that $s_1(E, c, P_2) = DCE_1(E, c)$. Let (E, c, P_2) with $(E, c) \in \mathscr{B}_0$. If $E = C/2$, by the *midpoint property*, $s_1(E, c, P_2) = DCE_1(E, c)$. Next, we consider the rest of the possibilities.

Case a: $E < C/2$. There are four subcases.

- $DCE_1(E, c) = 0$. By *non-negativity*, $s_1(E, c, P_2) = DCE_1(E, c)$.
- $DCE_1(E, c) > 0$ and $DCE_j(E, c) = c_j/2$ for each $j \neq 1$. Let us suppose that there is $\varphi \in \Phi(P_2)$ such that $\varphi_1(E, c) < DCE_1(E, c)$. By *efficiency*, $\varphi_j(E, c) > c_j/2$ for some $j \neq 1$. By the *midpoint property*, $\varphi(C/2, c) = c/2$. Then, $\varphi_j(E, c) > \varphi_j(C/2, c)$, contradicting *resource monotonicity*. Therefore, $s_1(E, c, P_2) = DCE_1(E, c)$.
- $DCE_1(E, c) > 0$ and $DCE_j(E, c) \neq c_j/2$ for each $j \neq 1$. By the definition of the *DCE* rule, $c_1 - DCE_1(E, c) = c_j - DCE_j(E, c)$ for each $j \neq 1$. Let us suppose that there is $\varphi \in \Phi(P_2)$ such that $\varphi_1(E, c) < DCE_1(E, c)$. By *efficiency*, $\varphi_j(E, c) > DCE_j(E, c)$ for some $j \neq 1$. Then, $c_1 - \varphi_1(E, c) > c_j - \varphi_j(E, c)$, contradicting *order preservation*. Therefore, $s_1(E, c, P_2) = DCE_1(E, c)$.
- $DCE_1(E, c) > 0$ and there are S, T, $\varnothing \neq S \subset N\backslash\{1\}$, and $\varnothing \neq T \subset N\backslash\{1\}$ such that for each $l \in S$, $DCE_l(E, c) \neq c_l/2$, and for each $k \in T$, $DCE_k(E, c) = c_k/2$. By the definition of the *DCE* rule, $c_1 - DCE_1(E, c) = c_j - DCE_j(E, c)$ for each $j \in S$. Let us suppose that there is $\varphi \in \Phi(P_2)$ such that $\varphi_1(E, c) < DCE_1(E, c)$. By *efficiency*, $\varphi_j(E, c) > DCE_j(E, c)$ for some $j \neq 1$. Then, if $j \in S$, $c_1 - \varphi_1(E, c) > c_j - \varphi_j(E, c)$, contradicting *order preservation*. If $j \in T$, by the *midpoint property*, $\varphi(C/2, c) = c/2$. Then, $\varphi_j(E, c) > \varphi_j(C/2, c)$, contradicting *resource monotonicity*. Therefore, $s_1(E, c, P_2) = DCE_1(E, c)$.

Case b: $E > C/2$. There are two subcases.

- $DCE_1(E, c) = c_1/2$. Let us suppose that there is $\varphi \in \Phi(P_2)$ such that $\varphi_1(E, c) < c_1/2$. By the *midpoint property*, $\varphi(C/2, c) = c/2$. Then, $\varphi_1(E, c) < \varphi_1(C/2, c)$, contradicting *resource monotonicity*. Therefore, $s_1(E, c, P_2) = DCE_1(E, c)$.

Journal: 355 Article No.: 0789 TYPESET DISK LE CP Disp.:2014/1/28 Pages: 23 Layout: Small-X

- $DCE_1(E,c) > c_1/2$. By the definition of the *DCE* rule, $c_1 - DCE_1(E,c) = c_j - DCE_j(E,c)$, for each $j \in N\backslash\{1\}$. Let us suppose that there is $\varphi \in \Phi(P_2)$ such that $\varphi_1(E,c) < DCE_1(E,c)$. By *efficiency*, $\varphi_j(E,c) > DCE_j(E,c)$ for some $j \neq 1$. Then, $c_1 - \varphi_1(E,c) > c_j - \varphi_j(E,c)$, contradicting *order preservation*. Therefore, $s_1(E,c,P_2) = DCE_1(E,c)$.

Second, it can be similarly obtained that $s_n(E,c,P_2) = CE_n(E,c)$. □

The following fact provides two conditions that will be used in the proof of Theorem 2.

Fact 3 Let $(E,c) \in \mathscr{B}_0$ be a two-agent problem. By Lemma 6, at each step $m \in \mathbb{N}$, $s_1(E^m, c^m, P_2) = DCE_1(E^m, c^m)$. Therefore, next inequality characterizes the fact that agent 1 is guaranteed nothing at each step $m \in \mathbb{N}$

$$s_1(E^m, c^m, P_2) = 0 \Leftrightarrow E^m \leq \min\left\{c_2^m - c_1^m, c_2^m/2\right\}. \tag{1}$$

Now, applying (1) to $m = 2$ and substituting, in terms of the problem at step $m-1$, the expressions of E^m and c_i^m for each $i \in N$, that is

$$E^m = E^{m-1} - s_1(E^{m-1}, c^{m-1}, P_2) - s_2(E^{m-1}, c^{m-1}, P_2)$$

and

$$c_i^m = c_i^{m-1} - s_i(E^{m-1}, c^{m-1}, P_2),$$

we have the next inequality,

$$s_1(E^2, c^2, P_2) = 0 \Leftrightarrow E \leq \min\begin{cases} c_2 - c_1 + 2s_1(E,c,P_2), \\ c_2/2 + s_2(E,c,P_2)/2 + s_1(E,c,P_2) \end{cases} \tag{2}$$

Proof of Theorem 2 Let $(E,c) \in \mathscr{B}_0$. By Lemma 6, at each step $m \in \mathbb{N}$, $s_1(E^m, c^m, P_2) = DCE(E^m, c^m)$. Given this, we show that agent 1's *P-right* for P_2 at each step $m \geq 2$, is zero, so agent 1's *Recursive P-rights* rule for P_2 is the *Dual of Constrained Egalitarian* rule. Then, since $\varphi^R(E,c,P_2)$ exhausts the endowment, given Claim 3, $\varphi^R(E,c,P_2) = DCE(E,c)$.

If $c_1 = c_2$, by the definition of the *Recursive P-rights* rule for P_2, each agent i receives the same amount at the initial step. If $c_1 \neq c_2$ and $E = (c_1 + c_2)/2$ by the *midpoint property*, each agent i receives her half-claim, $c_i/2$. Therefore, in both cases, at the initial step the endowment is exhausted and $\varphi^R(E,c,P_2) = DCE(E,c)$.

When $c_1 \neq c_2$ there are three cases.

Case 1: $s_1(E,c,P_2) = 0$.

By (1) for $m = 1$, $E \leq \min\{c_2 - c_1, c_2/2\}$. Now, in the following step (2) states that

$$s_1(E^2, c^2, P_2) = 0 \Leftrightarrow E \leq \min\{c_2 - c_1, c_2/2 + s_2(E,c,P_2)/2\},$$

which follows from

$$E \leq \min\{c_2 - c_1, c_2/2\}.$$

Given that $s_1(E^2, c^2, P_2) = 0$, if the previous reasoning is applied to (E^2, c^2, P_2), we obtain that $s_1(E^3, c^3, P_2) = 0$. Then, extending this argument henceforth we can conclude that $s_1(E^m, c^m, P_2) = 0$ at each step $m > 2$. So $\varphi_1^R(E, c, P_2) = 0$. Therefore, by Claim 3, $\varphi^R(E, c, P_2) = (0, E) = DCE(E, c)$.
In Cases 2 and 3, we show that at $m = 2$ agent 1's *P-right* for P_2 is zero. Case 1 can then be applied to the residual *problem with legitimate principles*, so from $m = 2$ on, $s_1(E^{m+h}, c^{m+h}, P_2) = 0$, for each $h \in \mathbb{N}$, and $\varphi_1^R(E, c, P_2) = s_1(E, c, P_2)$.

Case 2: $s_1(E, c, P_2) > 0$ and $c_2/2 \geq c_2 - c_1$. There are five subcases.
Subcase 2.1: $c_2 - c_1 \leq E \leq c_1$. Then, $s_1(E, c, P_2) = (E + c_1 - c_2)/2$ and $s_2(E, c, P_2) = E/2$. Now, substituting these expressions in (2),

$$s_1(E^2, c^2, P_2) = 0 \Leftrightarrow E \leq \min\{E, c_1/2 + 3E/4\} \Leftrightarrow E \leq 2c_1,$$

which is true, as in this region, $E \leq c_1$. Therefore,

$$\varphi^R(E, c, P_2) = ((E + c_1 - c_2)/2, (E - c_1 + c_2)/2) = DCE(E, c).$$

Subcase 2.2: $c_1 \leq E \leq (c_1 + c_2)/2$. Then, $s_1(E, c, P_2) = E - c_2/2$ and $s_2(E, c, P_2) = E - c_1/2$. Now, substituting these expressions in (2),

$$s_1(E^2, c^2, P_2) = 0 \Leftrightarrow E \leq \min\{2E - c_1, +3E/2 - c_1/4\} \Leftrightarrow E \geq c_1,$$

which is obviously fulfilled in this region. Therefore,

$$\varphi^R(E, c, P_2) = (E - c_2/2, c_2/2) = DCE(E, c).$$

Subcase 2.3: $(c_1+c_2)/2 \leq E \leq (c_1+c_2)/2+(c_2-c_1)/2 = c_2$. Then, $s_1(E, c, P_2) = c_1/2$ and $s_2(E, c, P_2) = c_2/2$. Again, by substituting these expressions in (2),

$$s_1(E^2, c^2, P_2) = 0 \Leftrightarrow E \leq \min\{c_2, 3c_2/4 + c_1/2\}.$$

On the one hand, $E \leq c_2$ is fulfilled since c_2 is the endowment-upper bound of this region. On the other hand, in Case 2 $c_2/2 \geq c_2 - c_1$ which implies $c_1/2 \geq c_2/4$ and $3c_2/4 + c_1/2 \geq c_2$ then, again by the endowment-upper bound of this region, $E \leq 3c_2/4 + c_1/2$ is true. Therefore,

$$\varphi^R(E, c, P_2) = (c_1/2, E - c_1/2) = DCE(E, c).$$

Subcase 2.4: $c_2 \leq E \leq 2c_1$. Then, $s_1(E, c, P_2) = (E+c_1-c_2)/2$ and $s_2(E, c, P_2) = E/2$. Now, substituting these expressions in (2),

$$s_1(E^2, c^2, P_2) = 0 \Leftrightarrow E \leq \min\{E, c_1/2 + 3E/4\} \Leftrightarrow E \leq 2c_1,$$

which is obviously fulfilled in this region. Therefore,

$$\varphi^R(E, c, P_2) = ((E + c_1 - c_2)/2, (E - c_1 + c_2)/2) = DCE(E, c).$$

Journal: 355 Article No.: 0789 TYPESET DISK LE CP Disp.:2014/1/28 Pages: 23 Layout: Small-X

Subcase 2.5: $2c_1 \leq E$. Then, $s_1(E, c, P_2) = (E + c_1 - c_2)/2$ and $s_2(E, c, P_2) = E - c_1$. Here, the substitution of these expressions in (2) does not imply any restriction, so that,

$$\varphi^R(E, c, P_2) = ((E + c_1 - c_2)/2, (E - c_1 + c_2)/2) = DCE(E, c).$$

Case 3: $s_1(E, c, P_2) > 0$ and $c_2/2 \leq c_2 - c_1$. There are four subcases.
Subcase 3.1: $c_2/2 \leq E \leq (c_1 + c_2)/2$. Then, $s_1(E, c, P_2) = E - c_2/2$ and $s_2(E, c, P_2) = E - c_1/2$. Now, substituting these expressions in (2),

$$s_1(E^2, c^2, P_2) = 0 \Leftrightarrow E \leq \min\{2E - c_1, +3E/2 - c_1/4\} \Leftrightarrow E \geq c_1,$$

inequality fulfilled as in this region $c_2/2 \leq c_2 - c_1$, implying $c_1 \leq c_2/2$. Therefore,

$$\varphi^R(E, c, P_2) = (E - c_2/2, c_2/2) = DCE(E, c).$$

Subcase 3.2: $(c_1 + c_2)/2 \leq E \leq c_1 + c_2/2$. Then $s_1(E, c, P_2) = c_1/2$ and $s_2(E, c, P_2) = c_2/2$. Now, substituting these expressions in (2),

$$s_1(E^2, c^2, P_2) = 0 \Leftrightarrow E \leq \min\{c_2, 3c_2/4 + c_1/2\}.$$

Both inequalities $E \leq c_2$ and $E \leq 3c_2/4 + c_1/2$ are satisfied as in this region $c_2/2 \leq c_2 - c_1$, which implies $c_1 \leq c_2/2$. Therefore,

$$\varphi^R(E, c, P_2) = (c_1/2, E - c_1/2) = DCE(E, c).$$

Subcase 3.3: $c_1 + c_2/2 \leq E \leq c_2$. Then, $s_1(E, c, P_2) = c_1/2$ and $s_2(E, c, P_2) = E - c_1$. Now, substituting these expressions in (2),

$$s_1(E^2, c^2, P_2) = 0 \Leftrightarrow E \leq \min\{c_2, c_2/2 + E/2\} \Leftrightarrow E \leq c_2,$$

which is the endowment-upper bound in this region. Therefore,

$$\varphi^R(E, c, P_2) = (c_1/2, E - c_1/2) = DCE(E, c).$$

Subcase 3.4: $c_2 \leq E$. Then, $s_1(E, c, P_2) = (E+c_1-c_2)/2$ and $s_2(E, c, P_2) = E-c_1$. Here, the substitution of these expressions in (2) does not imply any restriction, so that,

$$\varphi^R(E, c, P_2) = ((E + c_1 - c_2)/2, (E - c_1 + c_2)/2) = DCE(E, c).$$

□

Appendix 4: Proof of Proposition 1

Let us consider the rule $\varphi^* : \mathcal{B} \to \mathbb{R}^n_+$ which, without loss of generality, is defined for each $(E, c) \in \mathcal{B}_0$ as follows (see footnote 7).

Case a: If $c_3 - c_2 \leq \frac{3}{16}c_1$ and $c_3 - c_2 \leq c_2 - c_1$,

$$\varphi^*(E,c) = \begin{cases} (0,0,E) & \text{if } 0 \leq E \leq c_3 - c_2 \\ \left(\frac{E-(c_3-c_2)}{3}, \frac{E-(c_3-c_2)}{3}, \frac{E+2(c_3-c_2)}{3}\right) & \text{if } c_3 - c_2 \leq E \leq 6(c_3-c_2) \\ \left(\frac{E}{2} - \frac{4}{3}(c_3-c_2), \frac{E}{2} - \frac{4}{3}(c_3-c_2), \frac{8}{3}(c_3-c_2)\right) & \text{if } 6(c_3-c_2) \leq E \leq 8(c_3-c_2) \\ \left(\frac{E}{3}, \frac{E}{3}, \frac{E}{3}\right) & \text{if } 8(c_3-c_2) \leq E \leq \frac{3}{2}c_1 \\ \left(\frac{c_1}{2}, \frac{c_1}{2}, E - c_1\right) & \text{if } \frac{3}{2}c_1 \leq E \leq \frac{3}{2}c_1 + c_3 - c_2 \\ \left(\frac{c_1}{2}, \frac{E-(c_3-c_2)}{2} - \frac{c_1}{4}, \frac{E+(c_3-c_2)}{2} - \frac{c_1}{4}\right) & \text{if } \frac{3}{2}c_1 + c_3 - c_2 \leq E \leq \frac{c_1}{2} + c_2 \\ \left(\frac{c_1}{2}, E - \frac{c_1+c_3}{2}, \frac{c_3}{2}\right) & \text{if } \frac{c_1}{2} + c_2 \leq E \leq \frac{C}{2} \\ CE(E,c) & \text{if } E \geq \frac{C}{2} \end{cases}$$

Case b: Otherwise, $\varphi^*(E, c) \equiv CE(E, c)$

Note that, it is easy to check that φ^* is an *admissible* rule for P_2 satisfying other standard properties such as *continuity*, *claims monotonicity* and *homogeneity* (see Footnote 6). Moreover, for each of the following problems in which we apply it, φ^* recommends the smallest amount for agent 2 among all the *admissible* rules for P_2. By Lemma 6, we know that for each $(E, c, P_2) \in B_P$, $s_1(E, c, P_2) = DCE_1(E, c)$ and $s_3(E, c, P_2) = CE_3(E, c)$. Taking into account these facts, next we compute some steps of the *recursive* P_2*-rights process* for the problem $(E, c) = \left(21, \left(5, 19\frac{1}{2}, 20\right)\right) \in \mathcal{B}$.

Step m = 1: $\left(E^1, c^1\right) = (21, (5, 19\frac{1}{2}, 20))$, $CE(E^1, c^1) = \left(2\frac{1}{2}, 9\frac{1}{4}, 9\frac{1}{4}\right)$, DCE $(E^1, c^1) = \left(1\frac{1}{4}, 9\frac{3}{4}, 10\right)$ and $\varphi^*(E^1, c^1) = \left(2\frac{1}{2}, 9, 9\frac{1}{2}\right)$. Then,

$$s(E^1, c^1, P_2) = \left(1\frac{1}{4}, 9, 9\frac{1}{4}\right).$$

Step m = 2: $\left(E^2, c^2\right) = \left(1\frac{1}{2}, \left(3\frac{3}{4}, 10\frac{1}{2}, 10\frac{3}{4}\right)\right)$, $CE(E^2, c^2) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, DCE $(E^2, c^2) = \left(0, \frac{1}{8}, \frac{7}{8}\right)$ and $\varphi^*(E^2, c^2) = \left(\frac{5}{12}, \frac{5}{12}, \frac{2}{3}\right)$. Then,

$$s(E^2, c^2, P_2) = \left(0, \frac{5}{12}, \frac{1}{2}\right).$$

Journal: 355 Article No.: 0789 TYPESET DISK LE CP Disp.:2014/1/28 Pages: 23 Layout: Small-X

Step m = 3: $\left(E^3, c^3\right) = \left(\frac{7}{12}, \left(3\frac{3}{4}, 10\frac{1}{12}, 10\frac{1}{4}\right)\right)$, $CE(E^3, c^3) = \left(\frac{7}{36}, \frac{7}{36}, \frac{7}{36}\right)$, DCE $(E^3, c^3) = \left(0, \frac{5}{24}, \frac{3}{8}\right)$ and $\varphi^*(E^3, c^3) = \left(\frac{5}{36}, \frac{5}{36}, \frac{11}{36}\right)$. Then,

$$s(E^3, c^3, P_2) = \left(0, \frac{5}{36}, \frac{7}{36}\right).$$

Step m = 4: $\left(E^4, c^4\right) = \left(\frac{1}{4}, \left(3\frac{3}{4}, 9\frac{17}{18}, 10\frac{1}{18}\right)\right)$, $CE(E^4, c^4) = \left(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right)$, DCE $(E^4, c^4) = \left(0, \frac{5}{72}, \frac{13}{72}\right)$ and $\varphi^*(E^4, c^4) = \left(\frac{5}{108}, \frac{5}{108}, \frac{17}{108}\right)$. Then,

$$s(E^4, c^4, P_2) = \left(0, \frac{5}{108}, \frac{1}{12}\right).$$

Therefore,

$$\begin{aligned}\varphi^R\left(21, \left(5, 19\frac{1}{2}, 20\right), P_2\right) &= \sum_{k=1}^{4} s(E^k, c^k, P_2) + \sum_{k=5}^{\infty} s(E^k, c^k, P_2)\\ &= \left(\frac{5}{4}, 9\frac{65}{108}, 10\frac{1}{36}\right) + \sum_{k=5}^{\infty} s(E^k, c^k, P_2).\end{aligned}$$

Now, let us consider the problem $(E', c) = \left(22\frac{1}{4}, \left(5, 19\frac{1}{2}, 20\right)\right)$. By the *midpoint property*,

$$\varphi^R\left(22\frac{1}{4}, \left(5, 19\frac{1}{2}, 20\right), P_2\right) = \left(2\frac{1}{2}, 9\frac{3}{4}, 10\right).$$

By Definition 5 we have that for each $m \in \mathbb{N}$ and each $i \in N$ $s_i(E^m, c^m, P_2) \geq 0$. Therefore, the two previous distributions contradict *resource monotonicity* as the highest agent receives less when the endowment increases.

References

Alcalde J, Marco MC, Silva JA (2005) Bankruptcy games and the Ibn Ezra's proposal. Econ Theory 26:103–114

Aumann R, Maschler M (1985) Game theoretic analysis of a bankruptcy problem from the Talmud. J Econ Theory 36:195–213

Blume L, Simon CP (1994) Mathematics for economics. Norton and Company Inc., London

Chun Y (1984) Note on "The Nash bargaining solution is optimal". University of Rochester, Mimeo, Rochester

Chun Y (1989) A non-cooperative justification for egalitarian surplus sharing. Math Soc Sci 17:245–261

Chun Y, Schummer J, Thomson W (2001) Constrained Egalitarianism: a new solution to bankruptcy problems. Seoul J Econ 14:269–297

Curiel I, Maschler M, Tijs SH (1987) Bankruptcy games. Z Oper Res 31:A143–A159

Dagan N, Serrano R, Volij O (1997) A non-cooperative view of consistent rules. Games Econ Behav 18:55–72

Dominguez D (2012) Lower bounds and recursive methods for the problem of adjudicating conflicting claim. Soc Choice Welf (forthcoming)

Dominguez D, Thomson W (2006) A new solution to the problem of adjudicating conflicting claims. Econ Theory 28:283–307
Herrero C (2003) Equal awards versus equal losses: duality in bankruptcy. In: Sertel MR, Koray S (eds) Advances in economic design. Springer, Berlin, pp 413–426
Herrero C, Villar A (2001) The three musketeers: four classical solutions to bankruptcy problems. Math Soc Sci 39:307–328
Herrero C, Villar A (2002) Sustainability in bankruptcy problems. TOP 10:261–273
Marco MC, Peris J, Subiza B (1995) A mechanism for meta-bargaining problems. A discussion WP-AD 98–17
Moreno-Ternero J, Villar A (2004) The Talmud rule and the securement of agents' awards. Math Soc Sci 47:245–257
Moulin H (2002) Axiomatic cost and surplus-sharing. In: Arrow K, Sen A, Suzumura K (eds) The handbook of social choice and welfare, vol 1. Elsevier, Amsterdam, pp 289–357
O'Neill B (1982) A problem of rights arbitration from the Talmud. Math Soc Sci 2:345–371
Piniles HM (1861) Drkah shel Torah. Forester, Viena
Sprumont Y (1991) The division problem with single-peaked preferences: a characterization of the uniform allocation rule. Econometrica 49:509–519
Thomson W (2003) Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. Math Soc Sci 45:249–297
Thomson W (2012) Bargaining theory: the axiomatic approach. Academic Press, New York
van Damme E (1986) The Nash bargaining solution is optimal. J Econ Theory 38:78–100
Young P (1988) Distributive justice in taxation. J Econ Theory 43:321–335