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On Invariant Density Estimation for Ergodic Diffusion Processes

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Abstract

We present a review of several results concerning invariant density estimation by observations of ergodic diffusion process and some related problems. In every problem we propose a lower minimax bound on the risks of all estimators and then we construct an asymptotically efficient estimator.

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1 Introduction

Suppose that we observe a trajectory $X^T = \{X_t, 0 \leq t \leq T\}$ of the diffusion process

$$\mathrm{d}X_t = S(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \quad X_0, \quad t \geq 0 \tag{1}$$

where the trend coefficient $S(\cdot)$ is an unknown function and the diffusion coefficient $\sigma(\cdot)^2$ is a known positive function. We assume that these functions are such that the equation (1) has a unique weak solution (see, e.g. Durret (1996). Moreover, we suppose that the following conditions are fulfilled.

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Conditions $\mathscr{A}_0$:

$$\int_0^x \exp\left\{-2\int_0^y \frac{S(v)}{\sigma(v)^2}\,\mathrm{d}v\right\}\mathrm{d}y \to \pm\infty, \quad \text{as} \qquad x \to \pm\infty$$

and

$$G(S) = \int_{-\infty}^{\infty} \frac{1}{\sigma(x)^2}\exp\left\{2\int_0^x \frac{S(v)}{\sigma(v)^2}\,\mathrm{d}v\right\}\,\mathrm{d}x < \infty.$$

By these conditions the solution of equation (1) has ergodic properties, with the invariant density

$$f_S(x) = \frac{1}{G(S)\sigma(x)^2}\exp\left\{2\int_0^x \frac{S(v)}{\sigma(v)^2}\,\mathrm{d}v\right\}, \tag{2}$$

i.e., for any function $h(\cdot)$ such that $\mathbf{E}_S\,|h(\xi)| < \infty$ the *law of large numbers*

$$\frac{1}{T}\int_0^T h(X_t)\,\mathrm{d}t \longrightarrow \mathbf{E}_S\,h(\xi) \qquad\qquad \text{p.s.}$$

holds. We denote by ξ a random variable with density function $f_S(\cdot)$.

We consider the problem of estimation of the invariant density by observations X^T and study the properties of its estimators in the asymptotic of *large samples* $T \to \infty$. The initial value X_0 is supposed to have the same density function $f_S(\cdot)$, so the observed process is stationary and the observed value X_t is a random variable with density $f_S(\cdot)$. We recall that in i.i.d. case the density of one observation entirely defines the distribution of the whole sample, but in the case of continuous time stochastic processes the distribution of the observed trajectory is defined by all finite dimensional distributions. Hence the density of one observation does not identify the model. For an ergodic diffusion process this density nevertheless identifies the whole model, because the model is entirely defined by the trend (unknown) and diffusion (known) coefficients and having invariant density we can write the trend coefficient as

$$S(x) = \frac{\left(\sigma(x)^2 f_S(x)\right)'}{2 f_S(x)}. \tag{3}$$

The problem of invariant density estimation was considered by many authors (see, e.g., Nguen (1979), Delecroix (1980), Castellana and Leadbetter (1986), Bosq (1998), van Zanten (2001) *et al.*). In particular, Castellana and Leadbetter (1986) showed that for any stationary process with one and two dimensional densities $f(y)$, $f(\tau, y, z)$ under condition :

$\mathcal{CL}$. The functions $f(y)$, $f(\tau, y, z), \tau > 0$ *are continuous at point x and*

$$|f(\tau, y, z) - f(y)f(z)| \le \psi(\tau) \in \mathcal{L}_1(\mathbf{R}_+), \tag{4}$$

this density $f(y)$ can be estimated with the *parametric rate* $\sqrt{T}$, i.e., let $\hat{f}_T(x)$ be a kernel type estimator

$$\hat{f}_T(x) = \frac{1}{T\varphi_T}\int_0^T K\left(\frac{X_t - x}{\varphi_T}\right) \mathrm{d}t \tag{5}$$

where $\varphi_T \to 0, T\varphi_T \to \infty$ and the kernel $K(\cdot)$ satisfies the usual properties, then

$$\lim_{T\to\infty} T\,\mathbf{E}\left(\hat{f}_T(x) - f(x)\right)^2 = A(x).$$

Here

$$A(x) = 2\int_0^\infty \left[f(\tau, x, x) - f(x)^2\right] \mathrm{d}\tau.$$

The condition $\mathcal{CL}$ can be verified for ergodic diffusion processes (see Veretennikov (1999) for sufficient conditions). Note that this verification requires much more regularity from the coefficients $S(\cdot)$ and $\sigma(\cdot)$, than we really need in this estimation problem.

In this work we propose several asymptotically efficient estimators of the density function without supposing that the condition $\mathcal{CL}$ is fulfilled.

2 Lower bound

We start with the minimax lower bound on the risks of all estimators. This bound was established for a wide class of loss functions (see Kutoyants (1997b), (1998)) but for simplicity of exposition we consider quadratic loss functions only.

Fix some $S_*(\cdot)$ and and $\delta > 0$ and introduce the set

$$V_\delta = \{S(\cdot)\ :\sup_{x\in\mathbf{R}}|S(x) - S_*(x)| \leq \delta\}.$$

The role of Fisher information in our problem plays the quantity

$$\mathrm{I}_f(S, x) = \left\{4\, f_S(x)^2\, \mathbf{E}_S\left(\frac{\chi_{\{\xi>x\}} - F_S(\xi)}{\sigma(\xi) f_S(\xi)}\right)^2\right\}^{-1}$$

where $F_S(\cdot)$ is the distribution function of the invariant law. Hence $F_S(\xi)$ is a uniform $[0, 1]$ random variable. We have the following result.

Theorem 1 *Let* $\sup_{S\in V_\delta} G(S) < \infty$, *and* $\mathrm{I}_f(S_*, x) > 0$, *then for all estimators* $\bar{f}_T(x)$

$$\underline{\lim}_{\delta\to 0}\, \underline{\lim}_{T\to\infty}\, \sup_{S(\cdot)\in V_\delta} T\,\mathbf{E}_S\left(\bar{f}_T(x) - f_S(x)\right)^2 \geq \mathrm{I}_f(S_*, x)^{-1}.$$

As usual in this type of problem (see Ibragimov and Khasminskii (1981), Chapter 4) the proof is based on the estimate

$$\sup_{S(\cdot)\in V_\delta} \mathbf{E}_S \left(\bar{f}_T(x) - f_S(x)\right)^2 \geq \sup_{\vartheta\in\Theta_\delta} \mathbf{E}_\vartheta \left(\bar{f}_T(x) - f_\vartheta(x)\right)^2$$

where the parametric sub-model corresponds to the trend coefficient $S(\vartheta, x) = S_*(x) + (\vartheta - \vartheta_0)\,\psi(x)\,\sigma(x)^2 \in V_\delta$, with the function $\psi(\cdot)$ from the class

$$\mathcal{K} = \left\{\psi(\cdot):\ \mathbf{E}_{S_*}\int_\xi^x \psi(v)\,\mathrm{d}v = \left(2\,f_{S_*}(x)\right)^{-1}\right\}.$$

For this parametric family with Fisher information I_ψ we obtain a Hajek-Le Cam minimax bound and then choose the *least favourable family* (with minimal Fisher information) as follows

$$\inf_{\psi\in\mathcal{K}} \mathrm{I}_\psi = \mathrm{I}_f(S_*, x).$$

Note that for $\psi(\cdot) \in \mathcal{K}$ we have $f_\vartheta(x) = \vartheta + o(1)$ as $\delta \to 0$. The details of the proof can be found in Kutoyants (1997d), (1998), (2003).

This lower bound allows us define the asymptotically efficient estimator ϑ_T^* by the following equality:

$$\lim_{\delta\to 0}\lim_{T\to\infty}\sup_{S(\cdot)\in V_\delta} T\,\mathbf{E}_S\left(f_T^*(x) - f_S(x)\right)^2 = \mathrm{I}_f(S_*, x)^{-1}.$$

3 Asymptotically efficient estimators

We consider below three type of estimators: *local time, unbiased* and *kernel type*.

Local time estimator

Recall that local time $\Lambda_T(x)$ of the diffusion process (1) is defined by the formula

$$\Lambda_T(x) = \lim_{\varepsilon\downarrow 0}\frac{\mathrm{meas}\{t:\ |X_t - x| \leq \varepsilon,\ 0 \leq t \leq T\}}{4\,\varepsilon}$$

and it admits the representation (see Karatzas and Shreve (1991))

$$2\Lambda_T(x) = |X_T - x| - |X_0 - x| + \int_0^T \mathrm{sgn}\,(x - X_t)\ \mathrm{d}X_t$$

Local time estimator of the density is defined by the equality

$$f_T^{\circ}(x) = \frac{2\,\Lambda_T(x)}{\sigma(x)^2\,T}.$$

We study its asymptotic behavior under the following conditions:

$\mathcal{U}$. *The law of large numbers*

$$\mathbf{P}_S - \lim_{T\to\infty} \frac{4 f_S(x)^2}{T} \int_0^T \left(\frac{\chi_{\{X_t>x\}} - F_S(X_t)}{\sigma(X_t)\, f_S(X_t)}\right)^2 \mathrm{d}t = \mathrm{I}_f(S,x)^{-1}$$

is uniform on $S(\cdot) \in V_\delta$.

$\mathcal{B}$.

$$\sup_{S(\cdot)\in V_\delta} \left(\mathbf{E}_S \left|\frac{\chi_{\{\xi>x\}} - F_S(\xi)}{\sigma(\xi)\, f_S(\xi)}\right|^2 + \mathbf{E}_S \left|\int_0^{\xi} \frac{\chi_{\{v>x\}} - F_S(v)}{\sigma(v)^2 f_S(v)} \mathrm{d}v\right|^2\right) < \infty.$$

Theorem 2 *Let the conditions* $\mathcal{U},\mathcal{B}$ *be fulfilled and* $\mathrm{I}_f(S,x)$ *be continuous on* V_δ*, then the estimator* $f_T^{\circ}(x)$ *is unbiased:* $\mathbf{E}_S\, f_T^{\circ}(x) = f_S(x)$*, asymptotically normal:*

$$\mathcal{L}_S\left\{\sqrt{T}\left(f_T^{\circ}(x) - f_S(x)\right)\right\} \Rightarrow \mathcal{N}\left(0, \mathrm{I}_f(S,x)^{-1}\right)$$

and is asymptotically efficient.

The proof follows directly from the representation

$$\begin{aligned}\sqrt{T}\left(f_T^{\circ}(x) - f_S(x)\right) &= \frac{2\, f(x)}{\sqrt{T}} \int_{X_0}^{X_T} \frac{\chi_{\{v>x\}} - F(v)}{\sigma(v)^2 f(v)}\, \mathrm{d}v \\ &\quad - \frac{2\, f(x)}{\sqrt{T}} \int_0^T \frac{\chi_{\{X_t>x\}} - F(X_t)}{\sigma(X_t)\, f(X_t)}\, \mathrm{d}W_t \end{aligned} \qquad (6)$$

and the central limit theorem for stochastic integrals.

Unbiased estimator

Let us introduce the estimator of the density function

$$f_T^*(x) = \frac{1}{T} \int_0^T R_x(X_t)\, \mathrm{d}X_t + \frac{1}{T} \int_0^T N_x(X_t)\, \mathrm{d}t$$

where $h(\cdot) \in C'(\mathbf{R})$ and

$$R_x(y) = \frac{2\chi_{\{y<x\}} h(y)}{\sigma(x)^2\, h(x)}, \qquad N_x(y) = \frac{\chi_{\{y<x\}} h'(y)\sigma(y)^2}{\sigma(x)^2\, h(x)}.$$

Then it is easy to see that $\mathbf{E}_S\, f_T^*(x) = f_S(x)$ and

$$\mathcal{L}_S\left\{\sqrt{T}\left(f_T^*(x) - f_S(x)\right)\right\} \Rightarrow \mathcal{N}\left(0, \mathrm{I}_f\,(S,x)^{-1}\right).$$

The detailed proof (with conditions like $\mathcal{U}, \mathcal{B}$) can be found in Kutoyants (1998), (2003). It is based on the representation

$$\sqrt{T}\left(f_T^*(x) - f_S(x)\right) = \sqrt{T}\left(f_T^\circ(x) - f_S(x)\right) + o(1).$$

In particular, if $\sigma(x) \equiv 1$ and $h(x) = x^3$, then for $x \neq 0$

$$f_T^*(x) = \frac{2}{Tx^3}\int_0^T \chi_{\{X_t<x\}}\, X_t^3\, \mathrm{d}X_t + \frac{3}{Tx^3}\int_0^T \chi_{\{X_t<x\}}\, X_t^2\, \mathrm{d}t$$

is unbiased and asymptotically efficient estimator of the density.

Kernel type estimator

Let us introduce the *kernel type estimator*

$$\hat{f}_T(x) = \frac{1}{\sqrt{T}}\int_0^T K\left(\sqrt{T}(X_t - x)\right)\mathrm{d}t$$

where the kernel $K(\cdot)$ is a bounded function with compact support $[A, B]$ and

$$\int_A^B K(u)\,\mathrm{d}u = 1, \qquad \int_A^B u\, K(u)\,\mathrm{d}u = 0.$$

To study this estimator we need to suppose that the function $f_S(x)$ is continuously differentiable (the function $S(x)$ is continuous and $\sigma(x)^2$ is continuously differentiable). Then we obtain the representation

$$\sqrt{T}\left(\hat{f}_T(x) - f(x)\right) = \sqrt{T}\,(f_T^\circ(x) - f(x)) + o(1)$$

and the asymptotic normality of this estimator

$$\mathcal{L}_S\left\{\sqrt{T}\left(\hat{f}_T(x) - f_S(x)\right)\right\} \Rightarrow \mathcal{N}\left(0, \mathrm{I}_f\,(S,x)^{-1}\right)$$

follows from the asymptotic normality of the local time estimator. The detailed proof with exact conditions can be found in Kutoyants (1998), (2003). Using similar arguments we obtain its asymptotic efficiency. It is clear that $A(x) = \mathrm{I}_f(S,x)^{-1}$. The consistent estimation of the quantity $\mathrm{I}_f(S,x)^{-1}$ is proposed in Dehay and Kutoyants (2004).

4 Semiparametric estimation

Let us consider the problem of parameter

$$\vartheta_S = \mathbf{E}_S R(\xi)\, S(\xi) + \mathbf{E}_S N(\xi)$$

estimation by observations (1) (with unknown $S(\cdot)$ and known $\sigma(\cdot)^2$). Here $R(\cdot)$ and $N(\cdot)$ are known functions. We will see later that the problem of invariant density estimation is a particular case of this problem.

Introduce the Fisher information

$$\mathrm{I}_\vartheta(S) = \left\{ \mathbf{E}_S \left(\frac{R(\xi)\,\sigma(\xi)^2\, f_S(\xi) + 2M_S(\xi)}{\sigma(\xi)\, f_S(\xi)} \right)^2 \right\}^{-1}$$

where

$$M_S(y) = \mathbf{E}\left(\left[F_S(y) - \chi_{\{\xi<y\}} \right] \left[R(\xi)\, S(\xi) + N(\xi) \right] \right).$$

The first result is the lower bound similar to that of Theorem 1.

Theorem 3 *Let* $\sup_{S\in V_\delta} G(S) < \infty$ *and* $\mathrm{I}_\vartheta(S_*) > 0$, *then for all estimators* $\bar\vartheta_T$

$$\varliminf_{\delta\to 0} \varliminf_{T\to\infty} \sup_{S(\cdot)\in V_\delta} T\, \mathbf{E}_S \left(\bar\vartheta_T - \vartheta_S\right)^2 \geq \mathrm{I}_\vartheta(S_*)^{-1}.$$

The proof can be found in Kutoyants (1997c) and (2003).

By direct calculation we verify that the *empirical estimator*

$$\tilde\vartheta_T = \frac{1}{T} \int_0^T R(X_t)\,\mathrm{d}X_t + \frac{1}{T}\int_0^T N(X_t)\,\mathrm{d}t$$

(under moments conditions) is asymptotically normal:

$$\mathcal{L}_S\left\{ \sqrt{T}\left(\tilde\vartheta_T - \vartheta_S\right)\right\} \Rightarrow \mathcal{N}\left(0, \mathrm{I}_\vartheta(S)^{-1}\right).$$

and asymptotically efficient:

$$\lim_{\delta\to 0}\lim_{T\to\infty}\sup_{S(\cdot)\in V_\delta} T\,\mathbf{E}_S\left(\tilde\vartheta_T-\vartheta_S\right)^2=\mathrm{I}_\vartheta(S_*)^{-1}.$$

Let us consider three different choices of the functions $R(\cdot)$ and $N(\cdot)$.

- **Distribution function** estimation. Let us put $R(y)=0$ and $N(y)=\chi_{\{y<x\}}$, then $\vartheta_S=F_S(x)$. Hence the empirical distribution function

  $$\tilde\vartheta_T=\hat F_T(x)=\frac{1}{T}\int_0^T\chi_{\{X_t<x\}}\,\mathrm{d}t$$

  is asymptotically efficient estimator of the invariant distribution function Kutoyants (1997a).

- **Density** estimation. Let us put $R(y)=\dfrac{\mathrm{sgn}(x-y)}{\sigma(x)^2}$ and $N(y)=0$, then $\vartheta_S=f_S(x)$ and

  $$\tilde\vartheta_T=\overline{f_T}(x)=\frac{1}{T\sigma(x)^2}\int_0^T\mathrm{sgn}(x-X_t)\ \mathrm{d}X_t.$$

  Therefore, $\overline{f_T}(x)$ is an asymptotically efficient estimator of the density.

- **Moments** estimation. Let us put $R(y)=0$ and $N(y)=y^k$, then $\vartheta_S=\mathbf{E}_S\xi^k$ and the empirical moment

  $$\tilde\vartheta_T=\frac{1}{T}\int_0^T X_t^k\,\mathrm{d}t$$

  is asymptotically efficient estimator of the moments of ergodic diffusion process.

5 Integral type risk

Let us consider integral type quadratic risk

$$\mathcal{R}\left(\overline{f_T},f_S\right)=\mathbf{E}_S\int\left(\overline{f_T}(x)-f_S(x)\right)^2\mathrm{d}x$$

and denote by

$$\mathcal{R}_f(S)=\int\mathrm{I}_f(S_*,x)^{-1}\,\mathrm{d}x$$

the limit value of this risk for local time estimator.

Introduce the condition

$$\sup_{S(\cdot)\in V_\delta}\left\{\int \mathbf{E}_S\left|\frac{\chi_{\{\xi>x\}}-F_S(\xi)}{\sigma(\xi)\,f_S(\xi)}\right|^2 \mathrm{d}x+ \right.$$

$$\left. +\int \mathbf{E}_S\left|\int_0^{\xi}\frac{\chi_{\{v>x\}}-F_S(v)}{\sigma(v)^2 f_S(v)}\mathrm{d}v\right|^2 \mathrm{d}x\right\}<\infty. \qquad (7)$$

Theorem 4 *Let the condition* (7) *be fulfilled, then*

$$\lim_{\delta\to 0}\lim_{T\to\infty}\inf_{\bar f_T}\sup_{S(\cdot)\in V_\delta} T\,\mathcal{R}\left(\bar f_T, f_S\right)=\mathcal{R}_f\left(S_*\right).$$

Note that this theorem contains two results. The first one is the lower bound for all estimators

$$\underline{\lim}_{\delta\to 0}\,\underline{\lim}_{T\to\infty}\sup_{S(\cdot)\in V_\delta} T\,\mathcal{R}\left(\bar f_T, f_S\right)\geq \mathcal{R}_f\left(S_*\right). \qquad (8)$$

The upper bound we obtain with the help of local time estimator. Slight modification of the conditions allows obtain the same limit for the risks of unbiased and kernel type estimators. Therefore, all these estimators are asymptotically efficient in the sense of the bound (8). The proofs can be found in Kutoyants (2003). Note that Negri (2001) establishes the asymptotic efficiency of the local time estimator for the loss function with uniform metric, i.e., for $\mathbf{E}_S\,\ell\left(\sup_x \sqrt{T}\left|\bar f_T(x)-f_S(x)\right|\right)$.

6 Second order efficiency

Having so many asymptotically efficient estimators we seek now the second order efficient one. Let us study the quantity $\left(T\mathcal{R}\left(\bar f_T, f_S\right)-\mathcal{R}_f(S)\right)$. Note that for LTE

$$T^{\frac12}\left(T\mathcal{R}\left(f_T^\circ, f_S\right)-\mathcal{R}_f(S)\right)\to\ Q\neq 0.$$

It can be shown that if the function $S(\cdot)$ is $k-1$ times differentiable then for certain kernel type estimators $\hat f_T(\cdot)$

$$T^{\frac{1}{2k-1}}\left(T\mathcal{R}\left(\hat f_T, f_S\right)-\mathcal{R}_f(S)\right)\to\ -P<0.$$

To answer the question why the rate $T^{\frac{1}{2k-1}}$ is better than $T^{\frac12}$ we write the last expression as

$$T\mathcal{R}\left(\hat f_T, f_S\right)=\mathcal{R}_f(S)-P\,T^{-\frac{1}{2k-1}}\left(1+o(1)\right)$$

and it is clear now that the slower rate is better. Now we can compare the different estimators by their constants P and the estimator with biggest P will be the best (second order efficient). Therefore, as before we have two problems: the first one is to obtain a lower bound on the risks of all estimators (to find the biggest P) and the second is to construct an estimator which attains this bound. The similar problem of second order efficient estimation was consider by Golubev and Levit (1996) and our proof follows the main steps of their work. Note that this result is in the spirit of Pinsker's approach (see Pinsker (1980)).

For simplicity of exposition we put $\sigma(\cdot) = 1$.

Theorem 5 (Dalalyan, Kutoyants (2003)) *Suppose that the function $S(\cdot)$ is $(k-1)$-times differentiable ($k > 1$), satisfies the condition*

$$\lim_{|x|\to\infty} \operatorname{sgn}(x)\, S(x) < 0,$$

where $S_*(x) = -x$ and belongs to the set

$$\Sigma_k = \left\{ S(\cdot) : \int \left[f_S^{(k)}(x) - f_{S_*}^{(k)}(x) \right]^2 \mathrm{d}x \le R \right\}.$$

Then for all estimators $\bar{f}_T(\cdot)$

$$\underline{\lim}_{T\to\infty} \sup_{S(\cdot)\in\Sigma_k} T^{\frac{1}{2k-1}} \left[T\mathcal{R}\left(\bar{f}_T, f_S\right) - \mathcal{R}_f(S) \right] \ge -\hat{\Pi}(k,R)$$

where

$$\hat{\Pi}(k,R) = 2(2k-1)\left(\frac{4k}{\pi(k-1)(2k-1)} \right)^{\frac{2k}{2k-1}} R^{-\frac{1}{2k-1}}.$$

The proof can also be found in Kutoyants (2003).

Let us introduce a subdivision of $\mathbf{R}$ on intervals $I_m = [a_m - \delta_T, a_m + \delta_T]$, where $a_m = 2m\delta_T, m = 0, \pm1, \pm2, \ldots$ and $\delta_T \to 0$. The asymptotically second order efficient estimator (for $x \in I_m$) can be written as

$$\hat{f}_T(x) = \frac{1}{2T\delta_T} \int_0^T \sum_{l=-\hat{\nu}_T}^{\hat{\nu}_T} \left(1 - \left| \frac{l}{\hat{\nu}_T} \right|^{k_T} \right) \cos\left(\frac{\pi l (x - X_t)}{\delta_T} \right) \chi_{\{X_t\in I_m\}} \mathrm{d}t$$

or

$$\hat{f}_T(x) = \sum_{l=-\hat{\nu}_T}^{\hat{\nu}_T} \left(1 - \left| \frac{l}{\hat{\nu}_T} \right|^{k_T} \right) \frac{1}{2\delta_T} \int_{a_m-\delta_T}^{a_m+\delta_T} \cos\left(\frac{\pi l (x - y)}{\delta_T} \right) f_T^{\circ}(y)\, \mathrm{d}y$$

Here $k_T = k + \mu_T, \mu_T = 1/\sqrt{\log T} \to 0$,

$$\hat{\nu}_T = \delta_T \left(\frac{8k\pi^{2(k-1)}}{RT(k-1)(2k-1)} \right)^{-\frac{1}{2k-1}} \to \infty$$

Theorem 6 (Dalalyan, Kutoyants (2004)) *Let the conditions of Theorem 5 be fulfilled, then*

$$\lim_{T\to\infty} \sup_{S(\cdot)\in\Sigma_k} T^{\frac{1}{2k-1}} \left[T\mathcal{R}\left(\hat{f}_T, f_S\right) - \mathcal{R}_f(S) \right] = -\hat{\Pi}(k, R).$$

The proof can also be found in Kutoyants (2003).

7 Trend estimation

Let us consider the problem of trend coefficient estimation. As before the observed process (1) is ergodic diffusion with unknown $S(\cdot)$ and known diffusion coefficient, which we put (for simplicity of exposition) to be equal 1. The problem of trend estimation was studied by several authors (see, e.g., Banon (1978), Galtchouk and Pergamenshchikov (2001)). Therefore we observe a trajectory $X^T = \{X_t, 0 \le t \le T\}$ of the solution of the stochastic differential equation

$$\mathrm{d}X_t = S(X_t)\,\mathrm{d}t + \mathrm{d}W_t, \quad X_0, \quad 0 \le t \le T.$$

The trend coefficient can be written with the help of invariant density $f_S(\cdot)$ as

$$S(x) = \frac{f_S'(x)}{2\, f_S'(x)}. \tag{9}$$

Hence for for estimation of $S(x)$ we can use the estimators of density and its derivative. The error of the estimators we measure with the help of the following risk

$$\mathcal{R}\left(\bar{S}_T, S\right) = \mathbf{E}_S \int \left(\bar{S}_T(x) - S(x)\right)^2 f_S(x)^2\, \mathrm{d}x.$$

The conditions of the regularity are similar to that of the Section 5.

Conditions $\mathcal{S}_\delta$.

$\mathcal{S}_1$. *The function* $S(\cdot)$ *has polynomial majorant and*

$$\overline{\lim_{|x|\to\infty}} \operatorname{sgn}(x)\ S(x) < 0.$$

$\mathcal{S}_2$. *The function* $S(\cdot) \in C^k(\mathbf{R})$ *with some* $k \geq 1$ *and belongs to the set*

$$\Sigma_\delta = \left\{ S(\cdot) \in V_\delta \; : \; \int_{\mathbf{R}} \left[f_S^{(k+1)}(x) - f_{S_*}^{(k+1)}(x) \right]^2 \, \mathrm{d}x \leq 4\,R \right\}.$$

$\mathcal{S}_3$. *The Fourier transform* $\varphi_*(\cdot)$ *of the function* $f'_{S_*}(\cdot)$ *is such that*

$$\int_{\mathbf{R}} |\lambda|^{2k+\tau} \left|\varphi_*(\lambda)\right|^2 \mathrm{d}\lambda < \infty$$

with some positive constant τ.

Let us put

$$\Pi(k,R) = (2k+1) \left(\frac{k}{\pi(k+1)(2k+1)} \right)^{\frac{2k}{2k+1}} R^{\frac{1}{2k+1}}$$

The first result is the minimax lower bound.

Theorem 7 (Dalalyan, Kutoyants (2002)) *Let the conditions* $\mathcal{S}_\delta$ *be fulfilled. Then for any estimator* $\bar{S}_T(\cdot)$ *we have*

$$\varliminf_{T \to \infty} \inf_{\bar{S}_T} \sup_{S(\cdot) \in \Sigma_\delta} T^{\frac{2k}{2k+1}} \mathcal{R}\left(\bar{S}_T, S \right) \geq \Pi(k,R).$$

According to (9) we introduce the estimator

$$\hat{S}_T(x) = \frac{\hat{\vartheta}_T(x)}{2\; f_T^\circ(x) + \varepsilon_T\; e^{-l_T|x|}}$$

where $f_T^\circ(x)$ is the local-time estimator of the density, $\varepsilon_T = T^{-(1-\kappa)/2}$, $l_T = [\ln T]^{-1}$, the constant $\kappa < 1/(2k+1)$ and $\hat{\vartheta}_T(x)$ is the asymptotically efficient estimator of the derivative $f'_S(x)$ Dalalyan, Kutoyants (2003):

$$\hat{\vartheta}_T(x) = \frac{2\nu_T}{T} \int_0^T K^*\left(\nu_T (x - X_t) \right) \mathrm{d}X_t$$

where the kernel

$$K^*(x) = \frac{1}{\pi} \int_0^1 (1 - u^{k+\mu_T}) \; \cos(ux) \, \mathrm{d}u$$

and

$$\nu_T = \left(\frac{\pi\; R\; (k+1)(2k+1)}{4k} \right)^{\frac{1}{2k+1}} \; T^{\frac{1}{2k+1}}.$$

Here $\mu_T = (\log T)^{-1/2}$.

Theorem 8 (Dalalyan, Kutoyants (2002)) *Let the conditions $\mathcal{S}_\delta$ be fulfilled then*

$$\lim_{T\to\infty}\sup_{S(\cdot)\in\Sigma_\delta} T^{\frac{2k}{2k+1}}\mathcal{R}\left(\hat{S}_T,S\right)=\Pi\left(k,R\right).$$

If the values $k\geq 2$ and $R>0$ are unknown then it is possible to construct an adaptive estimator $\tilde{S}_T(\cdot)$, which has the same asymptotic properties as $\hat{S}_T(\cdot)$.

Theorem 9 (Dalalyan (2003)) *Let the conditions $\mathcal{S}_\delta$ be fulfilled then*

$$\lim_{T\to\infty}\sup_{S(\cdot)\in\Sigma_\delta} T^{\frac{2k}{2k+1}}\mathcal{R}\left(\tilde{S}_T,S\right)=\Pi\left(k,R\right).$$

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