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Positivity theorem for a general manifold

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Abstract

We give a generalization in the non-compact case to various positivity theorems obtained by Malliavin Calculus in the compact case.

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1 Introduction

For background about differential geometry, the reader can see the appendix.

Let us consider $X_0, X_1, \ldots, X_m$ $m + 1$ smooth vector fields on R^d. Let us consider B_t^i m independent Brownian motions. We consider the equation in Stratonovitch sense on R^d:

$$dx_t(x) = X_0(x_t(x))dt + \sum_{i>0} X_i(x_t(x))dB_t^i \tag{1}$$

issued of $x \in R^d$. If we perform a change of coordinates through a diffeomorphism of R^d, the vector fields are transformed according this change of coordinates, and since in Itô-Stratonovitch Calculus, the Itô formula is the traditional one, equation (1) has a meaning independent of the system of coordinates chosen. This means that we can look at (1) on a manifold. On R^d, we can consider the quadratic form

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$g(x)^{-1} \; : \; \xi \to \sum_{i>0} < X_i(x), \xi >^2$. This quadratic form depends smoothly on x. We say that we are in an elliptic situation if this quadratic form is non-degenerated. We can introduce the measure on R^d $d\mu(x) = detg(x)^{1/2}dx$ where dx is the Lebesgue measure on R^d. $d\mu$ is transformed intrinsically under a change of coordinates. We say that R^d endowed with the family of quadratic forms $g(x)$ is a Riemannian manifold endowed with the Riemannian measure $d\mu$. These considerations lead to the definition of a general (curved) Riemannian manifold. If X is a vector field on R^d, we can define its divergence with respect of the measure $d\mu$ by the following integration by parts formula:

$$\int_{R^d} Xfd\mu = \int_{R^d} fdivXd\mu \tag{2}$$

If f is a function on R^d, the differential $df(x)$ can be assimilated via the non-degenerated quadratic form $g(x)$ to a vector field $gradf$ (We suppose that we are in an elliptic situation). The Laplace-Beltrami operator is therefore $\Delta = div\, grad$. Moreover, (1) generates a Markov process whose generator is $L = X_0 + 1/2 \sum_{i>1} X_i^2$. We can find a drift $\tilde{X}_0$ such that

$$-L = \tilde{X}_0 + \Delta \tag{3}$$

All these considerations are invariant under a change of coordinates on R^d, and apply to a general manifold.

Let us consider a general Riemannian manifold M, endowed with its Riemannian measure dy. On the space $C_K^\infty(M)$ of smooth functions f with compact support on M, there is a canonical second order operator Δ called the Laplace-Beltrami operator. Δ is symmetric positive:

$$\int_M g(y)\Delta f(y)dy = \int_M f(y)\Delta g(y)dy \tag{4}$$

By analytical methods (see Gilkey(1995) for instance), we can solve the parabolic differential equation:

$$\frac{\partial}{\partial t}u_t = -\Delta u \,;\; u_0 = f \tag{5}$$

where f is smooth with compact support. We call $u_t = \exp[-t\Delta]f$. By classical analytical technics (see Gilkey, 1995), the semi-group has smooth heat-kernels:

$$u_t(x) = \int_M p_t(x,y)f(y)dy \tag{6}$$

where $(t, x, y) \to p_t(x, y)$ is smooth from $R^{+*} \times M \times M$ into R^{+*}. The fact that $p_t(x, y) > 0$ can be proved by analytical methods based upon the maximum principle.

Let us suppose that M is compact. By using a finite cover by small balls and some suitable partition of unity, we can write Δ under Hoermander's form:

$$-\Delta = X_0 + 1/2 \sum X_i^2 \tag{7}$$

for some suitable smooth vector fields on M (this decomposition is still true over each relatively compact open subset of M in the non compact case). Reciprocally, we can consider an Elliptic Hoermander's type operator L on M: the vector fields X_i $i \neq 0$ span in all points the tangent space of M if $L = X_0 + 1/2 \sum_{i>0} X_i^2$. In such a case, we can introduce a metric on M and a drift $\tilde{X}_0$ such that

$$-L = \tilde{X}_0 + \Delta \tag{8}$$

where Δ is the Laplace-Beltrami operator associated to the Riemannian metric. We can solve the linear parabolic equation:

$$\frac{\partial}{\partial t} u_t = L u_t \ ; \ u_0 = f \tag{9}$$

We get a semi-group $\exp[tL]$ and we can show that there exists a smooth in (x, y) strictly positive heat-kernel such that:

$$\exp[tL] f(x) = \int_M p_t(x, y) f(y) dy \tag{10}$$

The proofs of these results are based upon elliptic theory and the maximum principle (Gilkey, 1995). The natural geometrical object associated with these operators is the Riemannian metric.

There are other distances, called Carnot-Caratheodory distances, which are associated with non-integrable systems of subspaces of the tangent space of the manifold. It is a natural object in Sub-Riemannian geometry. The big difference is the following: if the Riemannian distance is Lipschitz, a Sub-Riemannian distance is in general only Hoelder. Hoermander type operators are associated to Sub-Riemannian geometry. They are of the shape $L = 1/2 \sum X_i^2$. If in all x, the Lie algebra spanned by the X_i is equal to the tangent space of M (Strong Hoermander's hypothesis), the semi-group $\exp[tL]$ has a smooth **strictly positive** heat-kernel $p_t(x, y)$ (Hoermander, 1967). We are motivated by an extension of this theorem to Hoermander's type operator with drift $L = X_0 + 1/2 \sum X_i^2$.

Let us consider a general manifold M and some smooth vector fields $X_i, i = 0, \ldots, m$. In x, let us introduce the Lie ideal spanned by the vector fields X_i, X_0 excluded in the Lie algebra spanned by all the vector fields X_i, X_0 included. It is constructed as follows: we consider the space $F_0(x)$ spanned by the vector fields $X_i, i \neq 0$ in x. We define inductively $F_n(x)$ the linear space spanned by the Lie Brackets between an element of $F_{n-1}(x)$ and a vector X_i $i = 0, \ldots, m$. We suppose that in x, $\cup F_n(x) = T_x(M)$. This Hypothesis is called weak Hoermander's hypothesis in x. Let us consider a Hoermander's type operator $L = X_0 + 1/2 \sum_{i>0} X_i^2$. Hoermander's theorem (Hoermander, 1967) states that the semi-group generated by L has a smooth density $p_t(x, y)$ if the weak Hoermander hypothesis is checked in all x. We want to know when $p_1(x, y) > 0$. For that, we consider the control equation:

$$dx_t(h) = X_0(x_t(h))dt + \sum_{i>0} X_i(x_t(h))h_t^i dt \tag{11}$$

starting from x, where h_t^i belongs to $L^2([0,1])$. In R^d, when Hoermander's condition is satisfied in x and when the vector fields X_i are bounded with bounded derivatives of all orders, Ben Arous and Léandre (1991) have given the following criterion: $p_1(x,y) > 0$ if and only if there exists a h such that $x_1(h) = y$ and such that $h' \to x_1(h')$ is a submersion in h.

This last condition is called Bismut condition (Bismut, 1984).

The boundedness assumption in this theorem can be seen as a compactness assumption. Let us namely compactify R^n by adding a point at the infinity. We get the sphere S^n. The vector fields X_i bounded with bounded derivatives can be extended into smooth vector fields over S^n equal to 0 at the infinity.

Tools used by Ben Arous and Léandre were Malliavin Calculus. This theorem was generalized by Léandre (1990) for jump processes. An abstract version for diffusions was given by Aida, Kusuoka and Stroock (1993). Bally and Pardoux (1998) have given a version of this theorem for the case of a stochastic heat equation. A. Millet and M. Sanz Solé(1997) have given a positivity theorem for the case of a stochastic wave equation. Fournier (2001) has generalized the theorem of Léandre (1990) for the case of a non-linear jump process associated to the Boltzmann equation. Léandre (2003a) has studied the case of a delay equation on a manifold.

By using the mollifier in Malliavin sense introduced by Jones and Léandre (1997) and Léandre (1994), our goal is to remove the boundedness assumption in the theorem of Ben Arous and Léandre, and to generalize it to a general manifold M not necessarily complete. Our theorem is the following:

Main Theorem. *Let us suppose that in all points x of the manifold M, the weak Hoermander's hypothesis is checked. Then $p_1(x,y) > 0$ if and only if there exists an h such that $x_1(h) = y$ and such that $h' \to x_1(h')$ is a submersion in h.*

We refer for more details on Malliavin Calculus to the review of Meyer (1984), to the surveys of Léandre (1988), Léandre (1990), Kusuoka (1992) and Watanabe (1992) for the application of Malliavin Calculus to heat kernels in the compact or the bounded case. In the first part, we give a proof of the main theorem. In the second part, we give some extensions to other processes than diffusions.

2 Proof of the main theorem

Let us show that the condition is sufficient.

Let us introduce the solution of the stochastic differential equation in Stratonovitch sense, where B_t^i are some independent Brownian motions:

$$dx_t(x) = X_0(x_t(x))dt + \sum_{i>0} X_i(x_t(x))dB_t^i \tag{12}$$

starting from x. Let us introduce the exit time τ of the manifold. If f is a smooth function on M, we have classically (see Ikeda-Watanabe (1981), Nualart (1995)):

$$\int p_1(x,y)f(y)dy = E[f(x_1(x))1_{\tau>1}] \tag{13}$$

where $p_t(x,y)$ is the heat-kernel associated to the heat semi-group associated to the Hoermander's type operator $L = X_0 + 1/2\sum_{i>0} X_i^2$. In general, we cannot apply Malliavin Calculus to the diffusion $x_t(x)$. In order to be able to apply Malliavin Calculus, we introduce the mollifiers of Jones-Léandre (1997) and Léandre (1994). We consider a smooth function d from M into R^+, equal to 0 only in x and which tends to ∞ when y tends to infinity, the one compactification point of M. We consider a smooth function over $]-k,k[$ $(k \in R^+)$, equal to 1 over $[-k/2, k/2]$ and which behaves as $\frac{1}{(k-y)^r}$ when $y \to k_-$. Outside $]-k,k[$, this function, called $g_k(y)$ is equal to ∞. We suppose that $g_k \geq 1$.

We choose a big integer r. We choose a smooth function from $[1,\infty[$ into $[0,1]$, with compact support, equal to 1 in 1 and which decreases.

The mollifier functional of Jones-Léandre (1997) is

$$F_k = h(\int_0^1 g_k(d(x_s(x)))ds) \tag{14}$$

Lemma. *F_k belongs to all the Sobolev spaces in the sense of Malliavin Calculus if r is big enough, and is equal to 1 if $\sup_s d(x_s(x)) \leq k/2$, is smaller than 1 if $\sup_s d(x_s(x)) > k/2$ and is equal to 0 almost surely if $\sup_s d(x_s(x)) \geq k$. Moreover, $F_k \geq 0$.*

Proof of the Lemma. The support property of F_k comes from the fact that the paths of the diffusion $s \to x_s(x)$ are in fact almost surely Hoelder with a Hoelder exponent strictly smaller than $1/2$, instead of being only continuous.

Let us show that F_k belongs to all the Sobolev spaces.

Let us introduce some smooth vector fields X_i^k which are equal to X_i for $d \leq k$ and which are equal to 0 if $d \geq k+1$. We consider the stochastic differential equation in Stratonovitch sense starting from x:

$$dx_t^k(x) = X_0^k(x_t^k(x))dt + \sum_{i>0} X_i^k(x_t^k(x))dB_t^i \tag{15}$$

Since we consider a Stratonovitch equation, its solution is the limit in all the L^p of the solution of the random ordinary differential equation got when we replace the Stratonovich differential dB_t^i by the random ordinary differential of the polygonal approximation of the leading Brownian motion. It is called Wong-Zakai approximation (Ikeda-Watanabe (1981)). This explains, as we will see later, that the rules of computations with this equation are formally the same as for the solution of an ordinary differential equation, unlike an Itô equation. We put

$$\tilde{F}_k = h(\int_0^1 g_k(d(x_s^k(x)))ds) \tag{16}$$

We get clearly $\tilde{F}_k = F_k$. The interest to use the diffusion $x_t^k(x)$ instead of the initial diffusion is that we can apply Malliavin Calculus to it. Let us recall quickly how we proceed (see Meyer (1984) for a detailed exposition). Since the vector fields X_i^k have compact support, we can exhibit a smooth version of $x \to x_t(x)$ (See Ikeda-Watanabe(1981) and Meyer (1981)). We put

$$\phi_t^k(x) = \frac{\partial}{\partial x} x_t^k(x) \tag{17}$$

which is the solution of the linear equation in Stratonovitch sense:

$$d\phi_t^k(x) = \frac{\partial}{\partial x} X_0^k(x_t^k(x))\phi_t^k(x)dt + \sum_{i>0} \frac{\partial}{\partial x} X_i^k(x_t^k(x))\phi_t^k(x)dB_t^i \tag{18}$$

If we perturb dB_t^i into $dB_t^i + \lambda h_t^i dt$, we get by Ikeda-Watanabe (1981) and Meyer (1981) a smooth version of the solution $x_t^k(\lambda, x)$. Moreover , $\frac{\partial}{\partial\lambda} x_t^k(0, x)$ is solution of the stochastic differential equation with second member which is deduced from the first one by taking formally the derivative of the equation of $x_t^k(\lambda, x)$. These formal considerations are justified because the vector fields have compact supports (see Ikeda-Watanabe (1981) and Meyer (1981)). We get, in Stratonovitch sense:

$$\begin{aligned} d\frac{\partial}{\partial\lambda} x_t^k(0, x) &= \frac{\partial}{\partial x} X_0^k(x_t^k(x)) \frac{\partial}{\partial\lambda} x_t^k(0, x)dt \\ &+ \sum_{i>0} \frac{\partial}{\partial x} X_i^k(x_t^k(x)) \frac{\partial}{\partial\lambda} x_t^k(0, x)dB_t^i + \sum_{i>0} X_i^k(x_t^k(x))h_t^i dt \end{aligned} \tag{19}$$

Since we consider Stratonovitch differential, we can solve (19) by the method of variation of constant. We get:

$$\frac{\partial}{\partial\lambda} x_t^k(0, x) = D_h x_t^k(x) = \phi_t^k(x) \int_0^t (\phi_s^k(x))^{-1} X_i^k(x_t^k(x))h_t^i dt \tag{20}$$

Therefore the random kernel of $Dx_t^k(x)$ is given by

$$Dx_t^k(x)(s) = \phi_t^k(x)(\phi_s^k(x))^{-1}X_i^k(x_s^k(x))$$

for $s \leq t$. Since the vector fields have compact supports, $\phi_t^k(x)$ as well as its inverse are bounded in all L^p for finite p. So the kernel of $Dx_t^k(x)$ are bounded in all the L^p (see Meyer (1984)).

Moreover, the path $t \to x_t^k(x)$ is Hoelder with Hoelder exponent strictly smaller than $1/2$. By Kolmogorov lemma (see Meyer (1981)), the Hoelder norm of the diffusion $t \to x_t^k(x), t \leq 1$ belongs to all the L^p. We deduce that for r big enough (see Jones-Léandre (1997) (2.14))

$$P\{\sup_t \frac{1}{\left(k - d(x_t^k(x)\right)^+} > \frac{1}{\epsilon}; \int_0^1 \frac{dt}{(k - d(x_t^k(x)))^{+r}} < C\} < C(p)\epsilon^p \tag{21}$$

for all p.

The kernel of the first derivative of $\tilde{F}_k$ is not 0 only when $\sup d(x_t^k(x)) \leq k$. It is given by

$$h'\left(\int_0^1 g_k(d(x_t^k(x))dt\right)\int_0^1 g_k'(d(x_t^k(x)))d'(x_t^k(x))Dx_t^k(x)(s)dt \tag{22}$$

It remains to use the inequality

$$\begin{aligned} &|\int_0^1 g_k'(d(x_t^k(x)))d'(x_t^k(x))Dx_t^k(x)(s)dt| \\ &\leq \left(\int_0^1 (g_k'(d(x_t^k(x))\right)^2 dt)^{1/2}\left(\int_0^1 (d'(x_t^k(x))Dx_t^k(s)(s))^2 dt\right)^{1/2} \end{aligned} \tag{23}$$

and to use (21) in order to deduce that $D\tilde{F}_k(s)$ is bounded in all the L^p. The same holds for the derivatives of higher order of $\tilde{F}_k$. □

We introduce the auxiliary measure μ_k:

$$\mu_k : f \to E[F_k f(x_1(x))] \tag{24}$$

To the measure μ_k, we can apply Malliavin Calculus. Namely, $\mu_k[f] = E[\tilde{F}_k f(x_1^k(x))]$. In particular μ_k has a density q_k smaller than $p_1(x, y)$. In particular, if there exists a h such that $x_1(h) = y$ and $h' \to x_1(h')$ is a submersion in h, we can find k large enough such that $q_k(y) > 0$, by the positivity theorem of Ben Arous and Léndre (1991) in the compact case with the extra-condition that $\tilde{F}_k$ has to be strictly positive. This shows that the condition is sufficient.

In order to show that the condition is necessary, we remark that if $p_1(x, y) > 0$ in y, $q_k(y)$ is still strictly positive for k large enough , because for k enough large, for ϵ small

$$|E[(1_{\tau>1} - F_k)f(x_1(x))]| \leq \epsilon||f||_\infty \tag{25}$$

where $||f||_\infty$ denotes the uniform norm of f.

Therefore, it is enough to apply the Ben Arous-Léandre result in the other sense.

Remark: Let us suppose that Hoermander's condition is satisfied only in x. We can suppose that h is decreasing and that g_k decreases to 1, such that F_k increases to $1_{\tau>1}$. By Malliavin Calculus, μ_k has a density q_k, which increases. Let us consider the function $f = 1_A$ for a set A of measure 0 for the Lebesgue measure over M. We have:

$$\mu_k[f] = 0 \tag{26}$$

But

$$\mu_k[f] = E[F_k f(x_1(x))] = 0 \tag{27}$$

anf $F_k f(x_1(x))$ increases and tends to $1_{\tau>1} f(x_1(x))$, which is in L^1. We deduce that

$$E[1_{\tau>1} f(x_1(x))] = 0 \tag{28}$$

This means that the the law of $x_1(x)$ has a density without to suppose that Hoermander's hypothesis is satisfied in all points.

Remark: The localization procedure given in this work is a localization procedure of all the paths between 0 and $x_1(x)$, when we cannot apply Malliavin Calculus to all the diffusions $x_t(x)$. It is different of various localization procedures, developped in Léandre (1988) for instance, in order to get some estimates of hypoelliptic heat-kernels in small time, which were used when we can apply the machinery of the Malliavin Calculus to all the diffusion $x_t(x)$: namely, in Léandre (1988), we consider vector fields X_i on R^n with bounded derivatives of all orders in order to apply Malliavin Calculus. This allows to get a rough estimate of the heat kernel. Nash inequality (Carlen-Kusuoka-Stroock (1987)) allows to get rough estimates of the heat kernel: in Léandre (2002) we mix the localization procedures developped in this part and the Nash inequality, in order to localize the estimates which were got previously by Malliavin Calculus (see Kusuoka (1992), Léandre (1988), Léandre (1990), Watanabe (1992)) under the restrictions of Malliavin Calculus, and to avoid the classical boundedness assumption of Malliavin Calculus.

Remark: Since the Laplace-Beltrami operator is an elliptic Hoermander's type operator on each locally compact open subset of the manifold, we can apply the previous localization method to show that the heat-kernel associated to the Laplace-Beltrami operator on a Riemannian manifold is strictly positive.

Remark: If the drift X_0 is identically equal to 0, this theorem recovers the fact that the heat kernel associated to the operator $1/2 \sum X_i^2$ under the strong Hoermander's hypothesis is strictly positive, by using the technics of Léandre (1988) Theorem II.1.

3 Extensions

The main novelty of the Malliavin Calculus with respect to its preliminary forms (See works of Hida, Elworthy, Albeverio, Fomin, Berezanskii..) is the following: it can be applied to diffusions, and can differentiate some functionals which are only almost surely defined. There are other examples of Wiener functionals, almost surely defined, where we can apply the Malliavin Calculus and where we can get some positivity theorems. We sketch the proof only.

Nualart-Sanz (1985) consider some smooth vector fields $X_i, i = 0, \dots, d$ on R^n with derivatives at each order bounded. They consider d independent Brownian sheets $B^i(s, t)$ $s \geq 0, t \geq 0$. Let us recall that it is a Gaussian process indexed by $R^+ \times R^+$ defined by:

$$E[B(s, t)B(s', t')] = (s \wedge s')(t \wedge t') \tag{29}$$

and

$$E[B(s, t)] = 0 \tag{3.2}$$

They consider the Cairoli equation (δ denotes the Itô integral):

$$x_{(s,t)} = x + \sum_{i>0} \int_{[0,s]\times[0,t]} X_i(x_{(u,v)})\delta B^i(u, v) + \int_{[0,s]\times[0,t]} X_0(x_{(u,v)})dudv \tag{30}$$

By Malliavin Calculus, Nualart-Sanz (1985) can show if $st > 0$, that $x_{(s,t)}$ has a law having a smooth density with respect of the Lebesgue measure on R^n if in all x, the vector fields X_i $i \neq 0$ span R^n (Nualart-Sanz (1985) study in fact a more degenerated situation). Millet-Sanz (1997) have shown that this density is strictly positive under this non degenerate assumption (They establish in fact under a more general assumption a necessary and sufficient condition for this density to be strictly positive).

By using the fact that the path $(s, t) \to x_{(s,t)}$ is Hoelder, we can remove the hypothesis that the derivative at each order of the vector fields are bounded. If we remove these hypothesis, the two-parameter diffusion can blow up. We introduce O the measurable set where $x_{(u,v)}$ does not blow up on $[0, s] \times [0, t]$. By using the technics of Part III, we can prove the following theorem:

Theorem III.1. *Let us suppose that the vector fields are smooth, and that in all x the vector fields X_i $i > 0$ span R^n. Let us consider the measure: $f \to E[1_O f(x_{(s,t)})]$. This* *measure is bounded below by a measure having a strictly positive smooth density with respect of the Lebesgue measure.*

We can restrict the Brownian sheet $B(t,x)$ to the set $R^+ \times [0,1]$ and study the Walsh equation:

$$\frac{\partial}{\partial t}x(t,x) = \frac{\partial^2}{\partial x^2}x(t,x) + \psi(x(t,x)) + \phi(x(t,x))\frac{\partial^2}{\partial t\partial x}B(t,x) \tag{31}$$

where ψ and ϕ are bounded smooth functions with bounded derivatives at each order and $\frac{\partial^2}{\partial t\partial x}B(t,x)$ is the formal white noise associated to $B(t,x)$. We refer to Walsh (1986) for a complete study of this stochastic heat equation. We consider the initial smooth condition $x(0,x) = x_0(x)$ and the Neumann boundary conditions:

$$\frac{\partial}{\partial x}x(t,0) = \frac{\partial}{\partial x}x(t,1) = 0 \tag{32}$$

We suppose that $\phi > 0$ in order to simplify the exposition.

Pardoux-Zhang (1993) have shown that under these conditions, we can apply the Malliavin Calculus to the solution $x_t(x)$ of (31). The final result of Bally-Pardoux (1998) is the following: let us consider $0 \le x_1 < x_2 < \cdots < x_d \le 1$. Under these assumptions the law of $x(t,x_1),\dots,x(t,x_d)$ has a strictly positive smooth density.

But we remark that $(t,x) \to x(t,x)$ is almost surely Hoelder, if $x \to x_0(x)$ is smooth (see Walsh (1986)). Let us introduce the measurable set O where the solution $x(s,x)$ does not blow-up on $[0,t]\times[0,1]$. We can get by using the technics of the third part:

Theorem III.2. *Let us suppose that ϕ and ψ are smooth and that $\phi > 0$. Let us consider $0 \le x_1 < x_2 \cdots < x_d \le 1$. Let us consider the measure over R^d:*

$$f \to E[1_O f(x(t,x_1),\dots,x(t,x_d))] \tag{33}$$

This measure is bounded below by a measure having a strictly positive density on R^d.

The last studied extension is the case of a delay equation on a manifold M. Let us consider a compact Riemannian manifold. If $t \to x_t$ is a semi-martingale on M, we can define the parallel transport from t to t' on the tangent bundle of M endowed with the Levi-Civita connection (See appendix) $\tau_{t',t}$ for $t < t'$. Let us consider some smooth vector fields X_i on M and d independent Brownian motions B^i.

Léandre-Mohammed (2001) have introduced and studied the following delay equation on a manifold in Statonovitch sense:

$$dx_t = \tau_{t,t-\delta}\sum X_i(x_{t-\delta})dB_t^i \tag{34}$$

with initial condition on $[-\delta, 0]$ equal to the finite energy path $s \to \gamma_s$ defined on $[-\delta, 0]$. The parallel transport considered is the stochastic parallel transport associated to the solution.

Let us suppose that the vector fields X_i span in all points the tangent space. Under these assumptions, Léandre (2003a) has shown that if $t > 0$ the law of x_t has a strictly smooth density with respect of the Riemannian measure, by using the Malliavin Calculus. We remark that $t \to x_t$ is Hoelder. This will allow us to remove the compactness hypothesis on M. If M is not compact, $s \to x_s$ can blow up on $[0, t]$. Let us introduce the measurable set O where $s \to x_s$ does not blow-up on $[0, t]$. By using the technics of the part III, we get the following theorem:

Theorem III.3. *Let us suppose that the smooth vector fields X_i on the non-compact manifold M span in all points the tangent space of M. Let us introduce the measure $f \to E[1_O f(x_t)]$. This measure is bounded below by a measure having a smooth strictly positive density on M.*

4 Appendix: a brief review about stochastic differential geometry

We refer to Elworthy (1982), Emery (1989) and Ikeda-Watanabe (1981) for an extensive study of the material of this part.

Let us recall that a smooth manifold of finite dimension M is locally homeomorphic to an open subset of R^n and that the transition function between different local charts are smooth. We can define the algebra $C^\infty(M)$ of smooth functionals over it. Let (Ω, F_s, P) be a filtered probability space. A continuous semi martingale x_s with values in M is a process such that, by definition, $f(x_s)$ is a semi-martingale with values in R for any smooth functions f.

A vector field X is an operator on $C^\infty(M)$ such that

$$X(fg) = gX(f) + fX(g) \tag{35}$$

We check that $XY - YX$ is still a vector field called the Lie bracket $[X, Y]$ of the two vector fields X and Y. Let $X_i, i = 0, \dots, d$ some smooth vector fields with compact support: $X_i f$ is equal to zero if the support of f does not intersect the support of X_i. Let B_t^i some independent Brownian motions over R. We introduce the solution of the Stratonovitch differential equation:

$$dx_t(x) = X_0(x_t(x))dt + \sum_{i>0} X_i(x_t(x))dB_t^i\,;\; x_0(x) = x \tag{36}$$

This means that for all smooth functions f, the process $x_t(x)$ has to satisfy:

$$f(x_t(x)) = f(x) + \int_0^t X_0 f(x_s(x))ds + \sum_{i>0} \int_0^t X_i f(x_s(x))dB_s^i \tag{37}$$

This differential equation has a unique solution which is a semi-martingale. We can extend this notion to the case where the vector have no-compact supports, if we take care that the solution of (37) can have an blowing-up time $\tau(x)$.

Let L be the operator $X_0 + 1/2 \sum_{i>0} X_i^2$. We can consider the semi-group $\exp[tL]$. It has the following stochastic representation:

$$\exp[tL]f(x) = E[1_{\tau(x)>t} f(x_t(x))] \tag{38}$$

We can consider a vector field as a section of a linear bundle $T(M)$ called the tangent bundle, by looking at a trivialization of M and patching together these trivializations modulo linear maps in the fiber related to the differential of the transition diffeomorphism between the local charts of M. A Riemannian metric is a strictly positive quadratic form over $T_x(M)$, which is intrinsic (it depends consistently upon the different change of trivialization of $T(M)$), and which depends smoothly on x. Let us write in local coordinates the metric $\sum g_{i,j}(x)dx \otimes dx^j$. We can see that the measure $det(g_{i,j}(x))^{-1/2} \prod dx_i$ is intrisically defined. This allows to define the Riemannian measure dx on M.

The application $X(x) \to Xf(x)$ defines a continuous form on the tangent space. By duality, we can write $Xf(x) =< X, grad f >_{T_x(M)}$. Moreover, we have some integration by parts formulas:

$$int_M Xf(x)dx = \int_M f(x)\, divX(x)dx \tag{39}$$

The Laplace-Beltrami operator is defined intrinsically by:

$$\Delta f = div\, grad\, f \tag{40}$$

We can write Δ in local coordinates. For that, let us recall that there is a unique differential operator ∇_Y for a vector field Y acting on the vector fields, which satisfies to the following requirements:

$$\nabla_Y(fX) = f\nabla_Y X + (Yf)X \tag{41}$$

$$\nabla_{\lambda Y+\lambda' Y'} X = \lambda \nabla_Y X + \lambda' \nabla_{Y'} X \tag{42}$$

$$X < Y, Z > =< \nabla_X Y, Z > + < Y, \nabla_X Z > \tag{43}$$

$$\nabla_X Y - \nabla_Y X = [X, Y] \tag{44}$$

(43) says that the connection ∇ is metric. (44) says that the connection is without torsion. In local coordinates, if $X = \sum \lambda_i \frac{\partial}{\partial x_i}$ and $Y = \sum \mu_j \frac{\partial}{\partial x_j}$, we have:

$$\nabla_X Y = \sum \lambda_i \frac{\partial}{\partial x_i} Y + \sum \Gamma_{j,k} \lambda_j \mu_k \frac{\partial}{\partial x_i} \tag{45}$$

The set of $\Gamma^i_{j,k}$, called the Christoffel-Symbols of the Levi-Civita connection, defines in local coordinates a 1-form A with values in the endomorphism of $T_x(M)$.

Let $(g^{i,j}) = (g_{i,j})^{-1}$. In local coordinates,

$$\Delta f = g^{i,j} \left(\frac{\partial^2}{\partial x_i \partial x_j} f(x) - \Gamma^k_{i,j} \frac{\partial}{\partial x_k} f(x) \right) \tag{46}$$

(We use Einstein summation convention).

The Laplace-Beltrami operator is locally an Hoermander's type operator. It generates a semi-group called the heat semi-group on the manifold.

If x_s is a semi-martingale with values in M, we can solve in local coordinates the linear equation:

$$d\tau_t = -A_{dx_t} \tau_t \tag{47}$$

Since, in Stratonovitch Calculus, the Itô formula is the traditional one:

$$f(x_t) = f(x) + \int_0^t < df(x_s), dx_s > \tag{48}$$

the local linear differential equations (47) patch together, and we get a global process which is an isometry from $T_{x_0}(M)$ to $T_{x_t}(M)$ called the stochastic parallel transport along the semi-martingale x_t.

This allows to get the construction of Eells-Elworthy-Malliavin of the Brownian motion starting from x on the Riemannian manifold M:

$$dx_s(x) = \tau_s dB_s \tag{49}$$

where $s \to \tau_s$ is the stochastic parallel transport for the Levi-Civita connection along the solution $x_s(x)$ and B_s a linear Brownian motion in $T_x(M)$. (49) has a unique solution up to an exit stopping time $\tau(x)$. We get:

$$\exp[-t/2\Delta] f(x) = E[1_{\tau(x)>t} f(x_t(x))] \tag{50}$$

Remark. The equation (49) can be extended in the degenerated case by using Langerock' connection, in order to get geometrical degenerated operators (Léandre (2004)).

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