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Variance reduction technique for calculating value at risk in fixed income portfolios

Pilar Abad* and Sonia Benito**

Abstract

Financial institutions and regulators increasingly use Value at Risk (VaR) as a standard measure for market risk. Thus, a growing amount of innovative VaR methodologies is being developed by researchers in order to improve the performance of traditional techniques. A variance-covariance approach for fixed income portfolios requires an estimate of the variance-covariance matrix of the interest rates that determine its value. We propose an innovative methodology to simplify the calculation of this matrix. Specifically, we assume the underlying interest rates parameterization found in the model proposed by Nelson and Siegel (1987) to estimate the yield curve. As this paper shows, our VaR calculating methodology provides a more accurate measure of risk compared to other parametric methods.

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Keywords: Value at Risk, Fixed Income Portfolios, Market risk, Nelson and Siegel model

1. Introduction

One of the most important tasks financial institutions face is evaluating their market risk exposure. This risk is a consequence of changes in the market prices of the assets in their portfolios. A possible way to measure this risk is to evaluate likely losses taking place by means of market price changes, which is what Value at Risk (VaR) methodology does. This methodology has been extensively used in recent times and it has become a basic market risk management tool for financial institutions and regulators.

The VaR of a portfolio is a statistical measure which tells us the maximum amount that an investor may lose over a given time horizon and a probability. Although VaR is

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a simple concept, its calculation is not trivial. Formally, VaR(α%) is the percentile α of the probability distribution of changes in the value of a portfolio, i.e.: the value for which α% of the values lie to the left on the distribution. Consequentially, in order to calculate VaR, we first must estimate the probability distribution of the changes in the value of the portfolio. Several methods have been developed to estimate VaR of a portfolio. Among them, parametric methods or variance-covariance approaches, historical and Monte Carlo Simulations were initially proposed[1]. The literature on VaR has focused on two main directions: proposals for methodological innovations which aim to overcome limitations in some of the VaR methods and performance comparisons of VaR methods.

Along the first strand of literature, shortcomings in VaR methods have stimulated development of new methodologies. For example, in the case of the variance-covariance approach: distributions different from the Normal one have been considered [see, e.g., Mittnik *et al.* (2002), Kamdem (2005), Aas and Haff (2006) or Miller and Liu (2006)]; non-parametric distributions have been introduced [see, e.g., Cai (2002), Cakici and Foster (2003), Fan and Gu (2003), or Albanese *et al.* (2004)], or the extreme value theory has been applied to calculate the percentile of the tail distribution [see, e.g., McNeil and Frey (2000), or Brooks *et al.* (2005)]. Furthermore, in a parametric models framework the application of switching volatility models has been proposed [see, e.g., Billio and Pelizzon (2000) or Li and Lin (2004)] and variance reduction techniques which simplify calculations of the variance-covariance matrix needed to compute VaR under the parametric method, [see, e.g., Christiansen (1999), Alexander (2001) or Cabedo and Moya (2003)][2].

The results found in the existing literature regarding relative performances from different VaR models are somewhat inconclusive. No one model is better than others. Recent works include a wider range of methods (historical, Monte Carlo simulation, parametric methods including non-parametric distribution, and the extreme value theory), such as Bao *et al.* (2006), Consigli (2002) and Daníelsson (2002). They show that parametric models provide satisfactory results in stable periods but they are less satisfactory in periods of high volatility. Some further evidence in favour of parametric methods is provided in: Sarma *et al.* (2003) by comparing historical simulation with parametric methods; Daníelsson and Vries (2000) by including the extreme value theory in their analysis and Chong (2004) who uses parametric methods to estimate VaR and compares the Normal distribution against a Student-t distribution to find that VaR performs better under a Normal distribution.

Although consensus on the most accurate model to estimate VaR has not been reached, parametric methods are the most popular in financial practice, as indicated by many authors such as Chong (2004) or Sarma *et al.* (2003). Therefore, this study

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1. Linsmeier and Pearson (2000) discuss the advantages and disadvantages of the three methods for computing VaR.

2. Other studies also propose variance reduction techniques for estimating VaR, which are used in the Monte Carlo Simulation method, e.g. Glasserman *et al.* (2000).

departs from a parametric or variance-covariance method and proposes a variance reduction technique for estimating VaR. Unlike Christiansen (1999), Alexander (2001) and Cabedo and Moya (2003), our proposal uses the parameterization of interest rates that underlies the model of Nelson and Siegel (1987) to estimate the yield curve.

The parametric approach, based on the assumption that changes in a portfolio's value follow a known distribution, only needs a priori calculation of the conditional variance from changes in the value of the portfolio. However, computing this variance is not a trivial exercise as a variance-covariance matrix for the portfolio assets needs to be estimated. Two types of problems are then involved: (1) a dimensionality problem and (2) a viability problem. The former is related to the large dimension of the matrix which complicates estimation. This is a more sensitive problem for fixed income portfolios where their value depends on a large number of interest rates with different maturities. The later problem stems from the complex task of estimating conditional covariances when sophisticated models such as multivariate GARCH models are used. The estimation of such models is very costly in terms of computation. These type of problems are usually overcome through use of multivariate analysis [Christiansen (1999), Alexander (2001) or Cabedo and Moya (2003)], which are based on the assumption that there are common factors in the volatility of the interest rates and that these same factors explain the changes in the temporal structure of interest rates (TSIR). Under these two assumptions, it turns out possible from a theoretical point of view to obtain, through a multivariate technique and at a low calculation cost, the variance-covariance matrix from a vector of interest rates.

This paper proposes an alternative method of estimating the variance-covariance matrix of interest rates at a low computational cost. No specific assumptions need to be stated to apply our technique. We depart from Nelson and Siegel (1987) model, initially developed to estimate the TSIR. This model gives an expression for interest rates as a function of four parameters. Therefore, we can obtain the interest rates variance-covariance matrix by calculating variances for only four variables – the principal components of the changes in the four parameters. Financial institutions and banks routinely compute the parameters we need from Nelson and Siegel's model for purposes other than VaR related. Consequently, estimated parameters are thus readily available as inputs to be used in a VaR estimation and do not represent an additional computational burden. This fact is an obvious advantage from our approach.

This paper is organized as follows. In section 2 we present our methodological proposal to estimate the variance-covariance matrix for a large vector of interest rates and at a low computational cost. The next three sections evaluate the proposed method for a Spanish market data sample. In section 3 we describe the data we use briefly before applying the method proposed in order to obtain the variance-covariance matrix of a vector of interest rates. In section 4 we evaluate the proposed methodology to calculate VaR for fixed income portfolios so that we can compare the results with those obtained from standard methods of calculation. Finally, section 5 presents the main conclusions from the paper.

2. A parametric model for estimating risk

In this section we present a methodology to calculate the variance-covariance matrix for a large vector of interest rates at a low computational cost. In order to do so we start with the model proposed by Nelson and Siegel (1987), originally designed to estimate the yield curve.

The Nelson and Siegel formulation specifies a parsimonious representation of the forward rate function given by:

$$\varphi_m^t = \beta_0 + \beta_1 e^{\left(-\frac{m}{\tau}\right)} + \beta_2 \frac{m}{\tau} e^{\left(-\frac{m}{\tau}\right)} \tag{1}$$

This expression allows us to accommodate various functional features such as level, slope sign or curve shape in relation to four parameters (β_0, β_1, β_2, τ).

Bearing in mind the fact that the spot interest rate at maturity m can be expressed as the sum of the instantaneous forward interest rates from 0 up to m, that is, by integrating the expression that defines the instantaneous forward rate:

$$r_t(m) = \int_0^m \varphi_u^t d_u \tag{2}$$

we obtain the following expression for the spot interest rate at maturity m:

$$r_t(m) = \beta_0 - \beta_1 \frac{\tau}{m} e^{\left(-\frac{m}{\tau}\right)} + \beta_1 \frac{\tau}{m} + \beta_2 \frac{\tau}{m} - \beta_2 e^{\left(-\frac{m}{\tau}\right)} - \beta_2 \frac{\tau}{m} e^{\left(\frac{-m}{\tau}\right)} \tag{3}$$

Equation (3) shows that spot interest rates are a function of only four parameters. In accordance with this function, changes in these parameters are the variables that determine changes in the interest rates. By using a linear approximation we can estimate the change in the zero-coupon interest rate at maturity m from the following expression:

$$dr_t(m) \approx \left[\frac{\partial r_t(m)}{\partial \beta_0}, \frac{\partial r_t(m)}{\partial \beta_1}, \frac{\partial r_t(m)}{\partial \beta_2}, \frac{\partial r_t(m)}{\partial \tau}\right] \begin{bmatrix} d\beta_0, t \\ d\beta_1, t \\ d\beta_2, t \\ d\tau_t \end{bmatrix} \tag{4}$$

In a multivariate context, the changes in the vector of interest rates that make up the TSIR can be expressed by generalizing equation (4) in the following way:

$$dr_t = G_t d\beta_t + \varepsilon_t \tag{5}$$

where

$$dr_t = [dr_t(1), dr_t(2), \ldots, dr_t(k)], \qquad d\beta_t' = [d\beta_{0,t}, d\beta_{1,t}, d\beta_{2,t}, d\tau_t],$$

$$G_t = \begin{bmatrix} \frac{\partial r_t(1)}{\partial \beta_0} & \frac{\partial r_t(1)}{\partial \beta_1} & \frac{\partial r_t(1)}{\partial \beta_2} & \frac{\partial r_t(1)}{\partial \tau} \\ \frac{\partial r_t(2)}{\partial \beta_0} & \frac{\partial r_t(2)}{\partial \beta_1} & \frac{\partial r_t(2)}{\partial \beta_2} & \frac{\partial r_t(2)}{\partial \tau} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial r_t(k)}{\partial \beta_0} & \frac{\partial r_t(k)}{\partial \beta_1} & \frac{\partial r_t(k)}{\partial \beta_2} & \frac{\partial r_t(k)}{\partial \tau} \end{bmatrix}$$

and ε_t is the errors vector.

From expression (5) we can calculate the variance-covariance matrix of a vector of changes in the k interest rates using the following expression:

$$var(dr_t) = G_t \Psi_t G_t' + var(\varepsilon_t) \tag{6}$$

where:

$$\Psi_t = \begin{bmatrix} var(\beta_{0,t}) & cov(\beta_{0,t}\,\beta_{1,t}) & cov(\beta_{0,t}\,\beta_{2,t}) & cov(\beta_{0,t}\,\tau_t) \\ & var(\beta_{1,t}) & cov(\beta_{1,t}\,\beta_{2,t}) & cov(\beta_{1,t}\,\tau_t) \\ & & var(\beta_{2,t}) & cov(\beta_{2,t}\,\tau_t) \\ & & & var(\tau_t) \end{bmatrix}$$

At this point, it is worth noting that we have greatly simplified the dimension of the variance-covariance matrix we need to estimate. Instead of having to estimate $k(k+1)/2$ variances and covariances for a vector of k interest rates, we now only need to estimate 10 second order moments. Nevertheless, the problem associated with the difficulty of the covariances estimations persists.

However, by applying principal components to the vector of the changes in the parameters ($d\beta_t$), we can simplify the calculation of the variance-covariance matrix even further. Accordingly, the vector of changes in the parameters of Nelson and Siegel (1987) model can be expressed as:

$$d\beta_t = AF_t \tag{7}$$

$$F_t = \begin{bmatrix} f_{1,t} & f_{2,t} & f_{3,t} & f_{4,t} \end{bmatrix}$$

and

$$A = \begin{bmatrix} a^1_{\beta_0} & a^2_{\beta_0} & a^3_{\beta_0} & a^4_{\beta_0} \\ a^1_{\beta_1} & a^2_{\beta_1} & a^3_{\beta_1} & a^4_{\beta_1} \\ a^1_{\beta_2} & a^2_{\beta_2} & a^3_{\beta_2} & a^4_{\beta_2} \\ a^1_{\tau} & a^2_{\tau} & a^3_{\tau} & a^4_{\tau} \end{bmatrix}$$

where F_t is the principal components vector associated with the vector $d\beta_t$ and A is the constants matrix from the eigenvectors associated with each of the four eigenvalues for the variance-covariance matrix of changes in the parameters from Nelson and Siegel model ($d\beta_t$).

Substituting equation (7) into equation (5) and given that each principal component is orthogonal to the rest, we can express the interest rates variance-covariance matrix as follows:

$$var(dr_t) = G_t^* \Omega_t G_t^{*'} + var(\varepsilon_t) \tag{8}$$

where:

$$\Omega_t = \begin{bmatrix} var(f_{1,t}) & 0 & 0 & 0 \\ 0 & var(f_{2,t}) & 0 & 0 \\ 0 & 0 & var(f_{3,t}) & 0 \\ 0 & 0 & 0 & var(f_{4,t}) \end{bmatrix}$$

and $G_t^* \approx G_t \times A$

Ignoring $var(\varepsilon_t)$, let us approximate:

$$var(dr_t) \approx G_t^* \Omega_t G_t^{*'} \tag{9}$$

Therefore, equation (9) provides us an alternative method to estimate the variance-covariance matrix of changes in a k interest rates vector by using the four principal components estimation for changes in the parameters of Nelson and Siegel (1987) model. In this way, the dimensionality problem associated with the calculation of the covariance has finally been solved.

Note that $var(dr_t)$ will be positive semi-definite, but it may not be strictly positive definite unless $\varepsilon_t = 0$. Although Ω_t is positive definite because it is a diagonal matrix with positive elements, nothing guarantees that $G_t^* \ \Omega_t G_t^{*'}$ will be positive definite when $\varepsilon_t \neq 0$. If the covariance matrix is based on (9), we should ensure strictly positive definiteness through checking eigenvalues. However, it is reasonable to expect that approximation (9) will give a strict positive definite variance-covariance matrix if representation (5) is done with a high degree of accuracy.

In the following sections we evaluate this method, to calculate both the variance matrix of a vector of interest rates and VaR for fixed income portfolios.

3. Estimating the variance-covariance matrix

3.1. The data

With the purpose of examining the method proposed in this paper, we estimate a daily term structure of interest rates using the actual mean for daily prices of Treasury transactions. The original data set consists of daily observations from actual transactions in all bonds traded on the Spanish government debt market. The database for bonds traded on the secondary market of Treasury debt covers the period from January, 1st 2002 to December, 31th 2004. We use this daily database to estimate the daily term structure of interest rates. We fit Nelson and Siegel's (1987) exponential model for the estimation of the yield curve and minimise price errors weighted by duration. We work with daily data for interest rates at 1, 2,. . . , 15 year maturities.

3.2. The results

In this section we examine this new approach to variance-covariance matrix estimation. The first section begins by comparing the changes in estimated and observed interest rates. Changes in interest rates are modelled by equation (5) so that we can then compare them with observed ones.

We then proceed to estimate the variance-covariance matrix of a vector of 10 interest rates, using the methodology proposed in the previous section. We compare these estimations (Indirect Estimation) with those obtained through some common univariate procedures (Direct Estimation).

In both cases, direct and indirect estimation, we need a method for estimating variances and covariance. For the indirect estimation case, the estimation method gives us the variances of the four principal components of changes in the parameters in Nelson and Siegel model. Indeed, this enables us to obtain the interest rates variance-covariance matrix from equation (9).

We use two alternative measures of volatility to estimate the variance-covariance matrixes of interest rates changes and principal components variance: exponentially weighted moving average (EWMA) and Generalized Autoregressive Conditional Heteroskedasticity models (GARCH)[3].

(1) Under the first alternative, the variance-covariance matrix is estimated with RiskMetrics methodology as developed by J. P. Morgan (1995). RiskMetrics uses the so called exponentially weighted moving average (EWMA) method. Accordingly, the estimator for the variance is:

3. GARCH models are standard in finance (see Ferenstein and Gasowski, 2004).

$$var(dx_t) = (1-\lambda)\sum_{j=0}^{N-1}\lambda^j(dx_{t-j}-\overline{dx})^2 \tag{10}$$

and the estimator for the covariance is:

$$cov(dx_t dy_t) = (1-\lambda)\sum_{j=0}^{N-1}\lambda^j(dx_{t-j}-\overline{dx})(dy_{t-j}-\overline{dy}) \tag{11}$$

J.P. Morgan uses the EWMA method to estimate VaR in their portfolios. For $\lambda = 0.94$ with $N = 20$, on a widely diversified international portfolio RiskMetrics produces the best back-testing results. Subsequently, we use both of these values in the paper.

Therefore, we obtain direct estimations of the interest rates variance-covariance matrix (D_EWMA) from equations (10) and (11) where x_t and y_t are interest rates at different maturities. For the case of indirect estimation of the variance-covariance matrix (I_EWMA), we use equation (10) to calculate the principal components variances (where x_t are now these principal components). Equation (9) gives us then the relevant matrix.

(2) The EWMA methodology currently used for RiskMetrics^{TM} data is quite acceptable for calculating VaR measures. Alternatively, some authors suggest using variance-covariance matrices obtained from multivariate GARCH. Nevertheless, the large variance-covariance matrices used in VaR calculations could never be estimated directly by implementing a full multivariate GARCH model due to insurmountable, computational complexity. For this reason we only compute variances of interest rates changes with univariate GARCH models and avoid computation of the covariance[4].

Given that indirect estimation (I_GARCH) does not require the estimation of covariance, we estimate the principal components conditional variance from changes in Nelson and Siegel model's parameters by using univariate GARCH models.

In sub-section two, we compare alternative estimations for the variance-covariance matrix as described above. Comparisons are then summarised in Table 1.

Table 1: *Type of variance-covariance matrix estimation.*

		Type of variance models	
		EWMA	**GARCH**
Type of estimation	**Direct Estimation**	D-EWMA	D-GARCH*
	Indirect Estimation	I-EWMA	I-GARCH

* We have not estimated multivariate GARCH model because of the computational complexity are insurmountable, so that only present the result of the variances which have been estimated using univariate GARCH models. All GARCH models are Exponential GARCH (EGARCH) model (see Nelson, 1991).

4. We use the most suitable model for each series. All of them are Exponential General Autoregressive Conditional Heteroskedastic (EGARCH) model (Nelson, 1991).

Note that estimating the variance-covariance matrix with the methodology proposed in this study (indirect estimation) involves a minimum calculation cost, since it is only necessary to estimate the variance of four variables (the principal components of daily changes in the parameters of the Nelson and Siegel model).

3.2.1. Comparing interest rates changes

Firstly, we have evaluated the capacity of the model that we propose here to estimate daily changes in an interest rates vector. We need to compare observed interest rates with their estimations from equation (5). In Figure 1 we show a scatter diagram relating observed changes with estimated changes in 1-year interest rates, the graph shows that they are closely related regardless of the maturity.

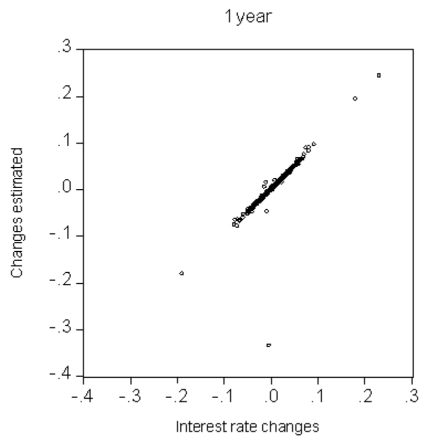


Figure 1: *Comparing the changes of 1-year interest rate observed with the estimated changes (equation (5)).*

In Table 2 we report some descriptive statistics for the interest rate estimation errors. The average error is less than a half basic point for all maturities i.e quite small. In relative terms, this error represents approximately 0.5% from the interest rates average. It is also worth noting that the average error and the standard deviation are very similar in all maturities therefore the model appears to be accurate for all maturities.

Table 2: *Estimation errors in interest rates. Descriptive statistics.*

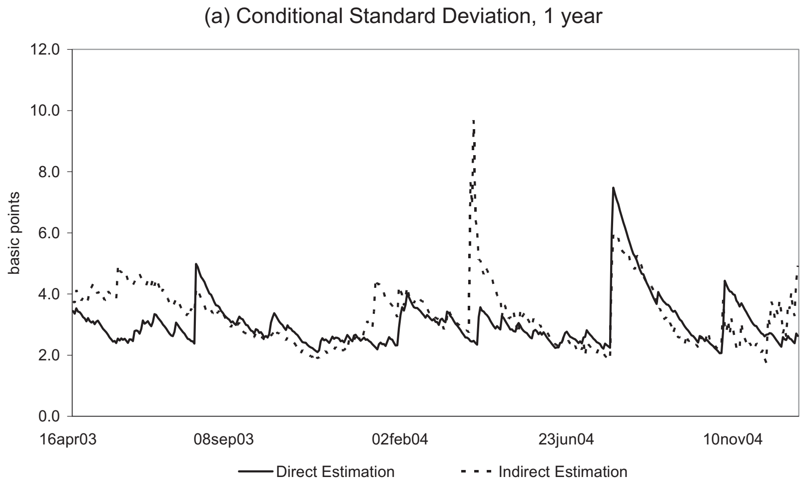
	1-year	2-year	3-year	4-year	5-year	6-year	7-year	8-year	9-year	10-year
Mean[(a)]	0.2	0.3	0.4	0.5	0.4	0.4	0.4	0.3	0.3	0.3
Standard deviation	1.5	3.2	2.7	2.1	1.8	1.8	1.7	1.6	1.4	1.3
Maximum error	32.9	66.0	45.4	32.8	34.5	33.2	30.4	27.1	23.8	20.9
Minimum error	−4.1	−1.5	−0.1	−0.1	−0.2	−4.0	−7.8	−9.2	−9.4	−9.4

Note: Sample period is from 1/1/2002 to 12/31/2004 (510 observations). The errors are the difference between the observed interest rates and their estimations from equation (5). The errors (and all statistics) are expressed in basic points. [(a)] The average error is calculated in absolute value.

These results imply that when estimating changes in zero-coupon interest rates using equation (5) the error is virtually non-existent. In what follows, we evaluate the differences in the estimation of the variance-covariance matrix under various alternatives.

3.2.2. Comparing the estimations of variance-covariance matrix

In Figure 2 we show the conditional variances for the 1-year interest rate. We apply the exponentially weighted moving average method for direct and indirect: D_EWMA versus I_EWMA. Furthermore, in Figure 3 we show the direct estimation of the conditional variances for the interest rates using the GARCH (D_GARCH) models as well as an indirect estimation (I_GARCH). The variances estimated using the method proposed in this paper are very similar to the direct estimates for most of the maturities.



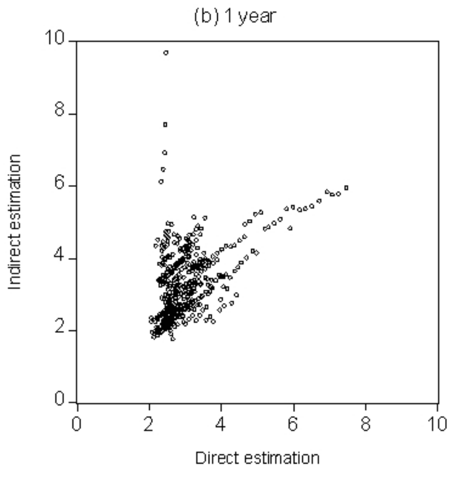


Figure 2: *Comparing the variance of changes of 1-year interest rate: Direct and indirect estimation using exponentially weighted moving average model.*

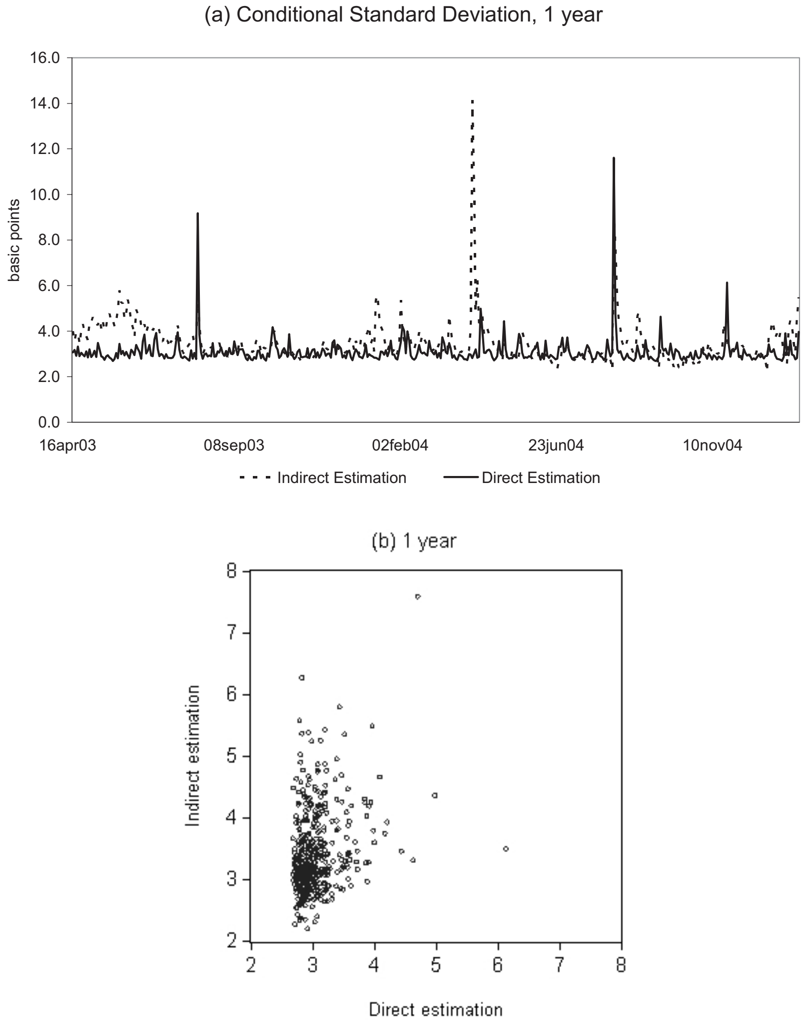


Figure 3: *Comparing the variance of changes of 1-year interest rate: Direct and indirect estimation using GARCH model.*

The descriptive statistics for the standard deviations differences estimated with both procedures are reported in Table 3. We compare the direct and indirect estimation methods using an EWMA model in panel (a), and using a GARCH model in panel (b). Panel (a) shows that the absolute value of the average differences for the EWMA specification, oscillates between 0.7 and 1.4 basic points. These average differences represent between 20% and 40% of the average of the estimated series. Panel (b) in Table 3 also shows that the average difference in absolute value for EGARCH specification is quite small. These differences are smaller than those of panel (a) taken as a percentage of the estimated conditional variance series. We can note that for both comparisons the range of differences for each pair of estimates is much wider for the 6-, 7- and 8-year interest rate than for the other maturities.

Table 3: *Differences in the estimation of the standard deviation on interest rates: direct vs. indirect method. Descriptive statistics.*

	1-year	2-year	3-year	4-year	5-year	6-year	7-year	8-year	9-year	10-year
	Panel (a): Comparing D_EWMA vs. I_EWMA									
Mean(a)	0.7	0.9	1.1	1.2	1.2	1.4	1.4	1.3	1.1	0.9
Standard deviation	0.9	1.7	2.0	1.8	1.7	1.8	1.9	1.8	1.5	1.2
Maximum error	1.6	4.4	6.6	3.9	1.4	1.3	1.2	1.1	1.0	0.8
Minimum error	−7.2	−19.7	−17.4	−12.9	−10.0	−8.6	−10.0	−10.0	−7.8	−8.0
	Panel (b): Comparing D_GARCH vs. I_GARCH									
Mean(a)	0.6	0.6	0.7	0.9	1.1	1.2	1.3	1.2	0.9	0.7
Standard deviation	0.9	1.2	1.2	1.3	1.5	1.6	1.7	1.7	1.6	1.5
Maximum error	3.6	4.8	3.9	2.3	1.4	1.1	0.9	0.7	0.4	0.4
Minimum error	−11.3	−18.0	−14.1	−12.0	−18.5	−22.5	−25.8	−27.0	−25.2	−23.4

Note: Sample period is from 1/1/2002 to 12/31/2004 (510 observations). I_EWMA indirect estimation (equation (9)) and D_EWMA direct estimation. RiskMetrics methodology (EWMA). I_GARCH: indirect estimation (equation (9)) and D_GARCH direct estimation. Conditional autoregressive volatility models (GARCH). (a) The average of the differences has been calculated in absolute value. Differences are measured in base points.

We now compare directly estimated covariances with those obtained with the procedure suggested in this paper. As above mentioned, given the extreme complexity of the GARCH multivariate model estimations, the direct estimation of the covariances is only approached with EWMA models.

Figure 4 shows estimated covariances for 3 and 1-year interest rates and for both procedures: D_EWMA versus I_EWMA. As it can be checked, estimated covariances behave similarly, although it should be noted that in most maturities there are greater differences than for the variances. In Table 4 we report some of the descriptive statistics of the estimated covariances. The average difference in absolute value is very small, between 0.0004 and 0.0014. However, this represents about 40% of the average estimated covariance.

Table 4: *Differences in the estimation of covariances of interest rates: direct vs. indirect method. Descriptive statistics.*

Comparing D_EWMA vs. I_EWMA

	1-year			**3-year**		**5-year**
	3-year	**5-year**	**10-year**	**5-year**	**10-year**	**10-year**
Mean(a)	0.0004	0.0005	0.0005	0.0013	0.0013	0.0014
Standard deviation	0.001	0.001	0.001	0.004	0.003	0.003
Maximum error	0.002	0.004	0.003	0.008	0.001	0.001
Minimum error	−0.022	−0.014	−0.003	−0.042	−0.018	−0.021

Note: Sample period is from 1/1/2002 to 12/31/2004 (510 observations). I_EWMA indirect estimation (equation (8)) and D_EWMA direct estimation. RiskMetrics methodology (EWMA). (a) The average difference is calculated in absolute value.

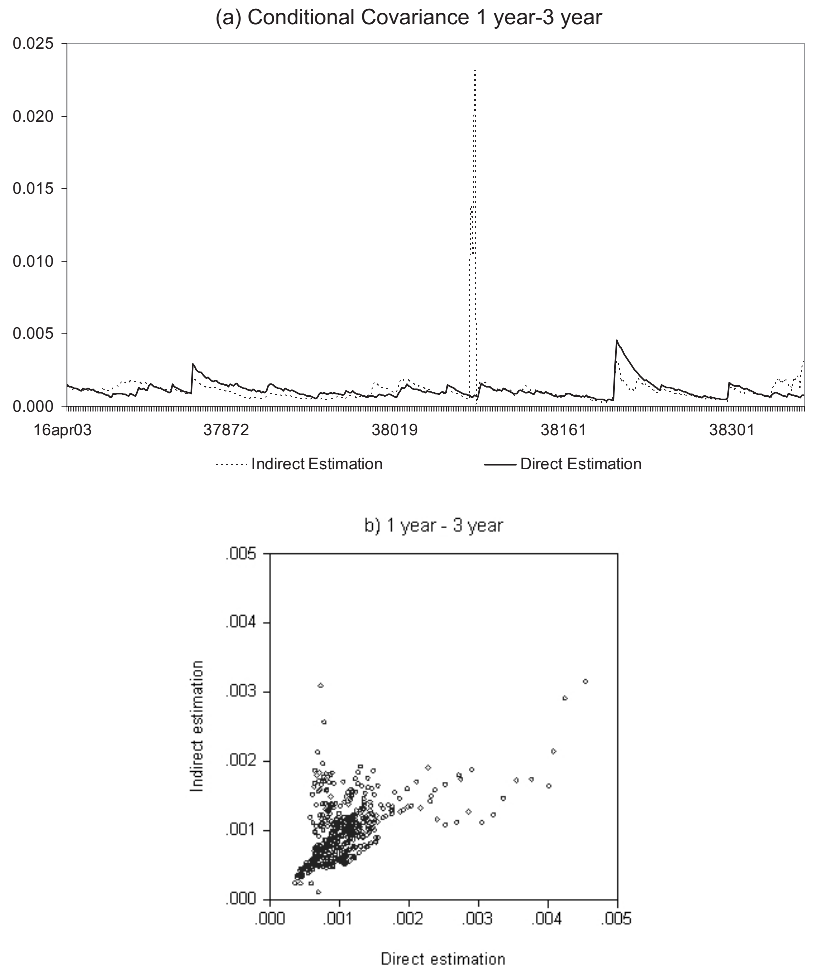


Figure 4: *Comparing the covariance between 3-year and 1-year interest rates: Direct and indirect estimation using exponentially weighted moving average (EWMA) model.*

In order to summarize the section we can conclude that we have shown how the procedure proposed in this paper to estimate the variance-covariance matrix of a large interest rates vector generates quite satisfactory results. In the following section we evaluate whether these small differences are important for risk management. Consequently, we apply the proposed methodology VaR calculation in several fixed income portfolios.

4. Estimating value at risk

In this section we evaluate the utility of our methodological proposal for risk management in fixed income portfolios. Thus, we create a parametric measure of VaR as an indicator of the risk of a given portfolio.

4.1. Value at risk

The VaR of a portfolio is a measure of the maximum loss that the portfolio may suffer over a given time horizon and with a given probability. Formally, the VaR measure is defined as the lower limit of the confidence interval of one tail:

$$\Pr\left[\Delta V_t(\tau) < VaR_t\right] = \alpha \quad (12)$$

where α is the level of confidence and $\Delta V_t(\tau)$is the change in the value of the portfolio over the time horizon τ.

The methods based on the parametric or variance-covariance approaches depart from the assumption that changes in the value of a portfolio follow a Normal distribution. Assuming that the average change is zero, the VaR for one day of portfolio j is obtained as:

$$VaR_{j,t}(\alpha\%) = \sigma_{t,dV_j} k_{\alpha\%} \quad (13)$$

where $k_{\alpha\%}$is the α percentile of the Standard Normal distribution, and the parameter that needs to be estimated is the standard deviation conditional of the value of portfolio j (σ_{t,dV_j}).

In a fixed income asset portfolio, duration can be used to obtain the variance of the value of portfolio j from the interest rates variance as shown in Jorion (2000):

$$\sigma^2_{t,dV_j} = D_{j,t} \Sigma_t D'_{j,t} \quad (14)$$

where Σ_t is the variance-covariance matrix of the interest rates and $D_{j,t}$ is the vector of the duration of portfolio j. This vector represents the sensitivity of the value of the portfolio to changes in the interest rates that determine its value.

In this section, VaR measures are calculated and compared. In the parametric approach, we use the estimations of the variance-covariance matrix as obtained in the previous section (see Table 1). Table 5 illustrates the four measures of VaR that we develop from the four variance-covariance models:

Table 5: *Type of VaR measures.*

	Type of variance-covariance matrix estimation	**Type o VaR measure**
Direct Estimation	D_EWMA	VaR_D_EWMA
	D_GARCH	VaR_D_GARCH*
Indirect Estimation	I_EWMA	VaR_I_EWMA
	I_GARCH	VaR_I_GARCH

* We did not compute VAR_D_GARCH because of the impossibility to estimate a multivariate GARCH model with 10 variables.

For the first VaR measure, VaR_D_EWMA, VaR is obtained by directly estimating Σ_t with an EWMA model. This is a popular approach to measuring market risk, used by JP Morgan (RiskMetricTM). The second VaR measure, VaR_D_GARCH, is also calculated by directly estimating the variance-covariance matrix, but using GARCH models to estimate the second order moments. Given that the large variance-covariance matrices used in VaR calculations could never be estimated directly using a full multivariate GARCH model, this VaR measure has not been calculated as computational complexity would be insurmountable.

Two final VaR measures are then computed by estimating the variance-covariance matrix of the interest rates following the procedure described in Section 2. We can estimate the variance-covariance matrix of interest rates indirectly, by substituting equation (9) into equation (14) to deduct a new expression for the variance of the changes in the value of the portfolio:

$$\sigma^2_{t,dV_j} = D_{j,t} G^* \Omega_t G^{*'} D^{'}_{j,t} = D^m_{j,t} \Omega_t D^{m'}_{j,t} \qquad (15)$$

In indirect estimation, $\Omega_t$is a diagonal matrix containing the conditional variance for the principal components of changes in the four parameters of Nelson and Siegel's model in its main diagonal. Also, $D^m_{j,t}$ is the modified vector of durations of portfolio j (with 1×4 dimension) which represents the sensitivity of the value of the portfolio to changes in the principal components of the four parameters in Nelson and Siegel model. In the VaR_I_EWMA, we use an EWMA model to estimate the variance of the principal components; and a suitable GARCH model to estimate these variances for the calculation of the VaR_I_GARCH measure.

4.2. The portfolios

In order to evaluate the procedure proposed in this paper for VaR calculation, we have considered 4 different portfolios made up of theoretical bonds with maturities at 3-, 5-, 10- and 15-years and constructed from real data from the Spanish debt market. The bond coupon is 3.0% in every portfolio. The period of analysis goes from April, 15th 2002 to December, 31st 2004, which allows us to perform 437 estimations of daily VaR for each portfolio.

In order to estimate the daily VaR we have assumed that the main portfolios features remain constant during the analysis period: the initial value of the portfolio, the maturity date and the coupon rate. In this case, results are comparable for the entire period of analysis since we avoid both the pull to par effect (the value of the bonds tends to par as the maturity date of the bond approaches) and the roll down effect (the volatility of the bond decreases over time).

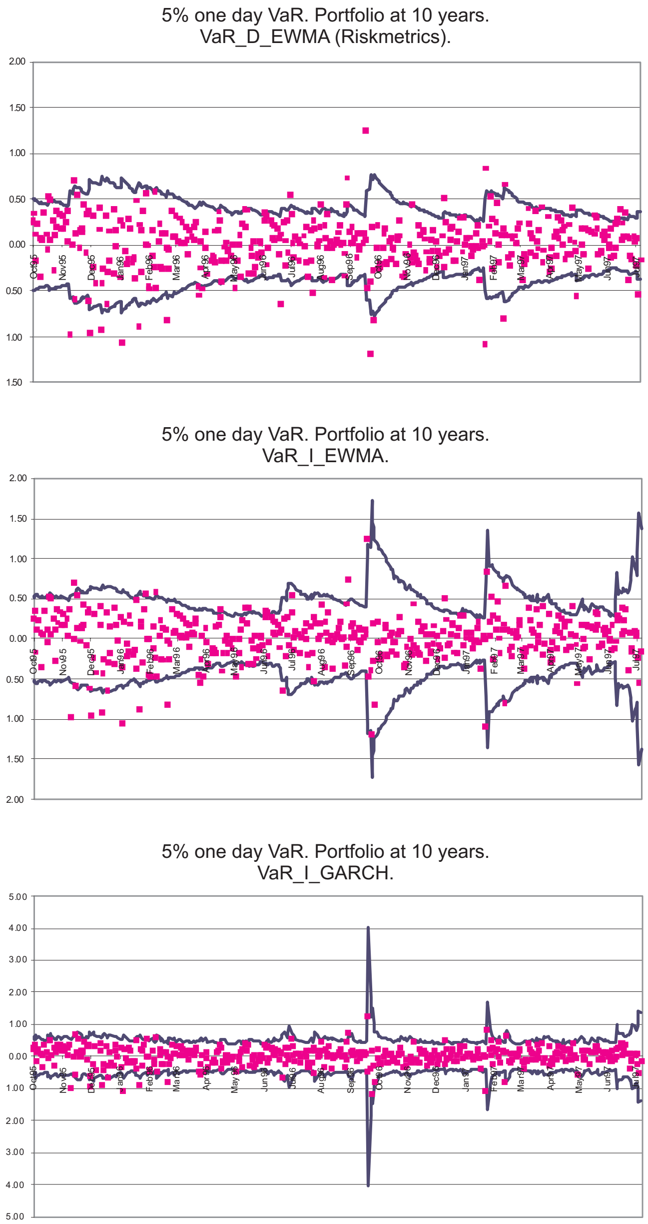


__Figure 5:__ The 5% one day VaR for a 10 year portfolio. Direct estimation using an exponentially weighted moving average model [VaR_D_EWMA(5%)], indirect estimation using an exponentially weighted moving average model [VaR_I_EWMA(5%)] and indirect estimation using a GARCH model [VaR_I_GARCH(5%)].

4.3. Comparing VaR measures

In this section VaR measures are compared. We calculate daily VaR at a 5%, 4%, 3%, 2% and 1% confidence level for all portfolios. First, before formally evaluating the precision of the VaR measures under comparison, we examine actual daily portfolio value changes (as implied by daily fluctuations in the zero-coupon interest rate) and compare them with the 5% VaR. In Figure 5 we show the actual change in a 10-year portfolio together with the VaR at 5% for the three VaR measures we consider: VaR_D_EWMA (Graph 1), VaR_I_EWMA (Graph 2) and VaR_I_GARCH (Graph 3). In Graph 1 and 2, we observe that the portfolio's value falls below VaR more often than in Graph 3. In all cases, the number of times the value of the portfolio falls below VaR is closer to its theoretical level. This is also a clear result for the other portfolios considered, but we will not show it due to space limitations. This preliminary analysis suggests that VaR estimations from both models, both directly and indirectly are very precise; however, a more rigorous evaluation of the precision of the estimations is required[5].

We then compare VaR measures with the actual change in a portfolio value on day t+1, denoted as ΔV_{t+1}. When $\Delta V_{t+1} < VaR$, we have an exception. For testing purposes, we define the exception indicator variable as

$$I_{t+1} = \begin{cases} 1 \text{ if } \Delta V_{t+1} < VaR \\ 0 \text{ if } \Delta V_{t+1} \geq VaR \end{cases} \tag{16}$$

a) Testing the level

The most basic test of a VaR procedure is to see if the stated probability level is actually achieved. The mean of the exception indicator series is the level of achievement for the procedure. If we assume a constant probability for the exception, the number of exceptions follows the binomial distribution. Thus, it is possible to build up confidence intervals for the level of each VaR measure (see Kupiec, 1995).

Table 6 shows the level achieved and the 95% confidence interval for each of the 1-day VaR estimates. An * indicates the cases in which the level is out of the confidence interval, evidence obtained rejects the null hypothesis at the 5% confidence level. The number of exceptions is inside the interval confidence for the three measures and almost all portfolios considered. Therefore, VaR estimates (direct and indirect) seem to be good.

Only for three cases is the number of exceptions out of the confidence interval. Specifically, for VaR_I_GARCH measure for 4% and 5% confidence level in 5- and 10-year portfolios. The number of exceptions in these cases is much lower than the theoretical level, so it would seem this measure overestimates the risk of these portfolios.

5. The empirical assessment of VaR is not developed through analysing the mean squared error (Longford, 2008). Instead, we use the standard test of VaR.

Table 6: *Testing the Level.*

VaR measures	Number of exceptions 3-year	5-year	10-year	15-year	Confidence intervals at the 95% level
Var_D_EWMA (1%)	6	7	7	5	(1-10)
Var_D_EWMA (2%)	9	9	14	10	(5-17)
Var_D_EWMA (3%)	15	12	17	14	(8-23)
Var_D_EWMA (4%)	17	14	17	16	(12-30)
Var_D_EWMA (5%)	20	20	23	20	(17-36)
Var_I_EWMA (1%)	7	7	10	8	(1-10)
Var_I_EWMA (2%)	13	11	11	13	(5-17)
Var_I_EWMA (3%)	16	12	15	17	(8-23)
Var_I_EWMA (4%)	19	15	16	19	(12-30)
Var_I_EWMA (5%)	19	19	19	24	(17-36)
Var_I_GARCH (1%)	6	6	6	6	(1-10)
Var_I_GARCH (2%)	11	9	8	9	(5-17)
Var_I_GARCH (3%)	12	10	11	11	(8-23)
Var_I_GARCH (4%)	16	10*	13	13	(12-30)
Var_I_GARCH (5%)	19	13*	14*	19	(17-36)

Note: Sample period 4/15/2002 to 12/31/2004 (437 observations). Confidence intervals derived from the number of exceptions follows the binomial distribution (437, x%) for x = 1, 2, 3, 4 and 5. An * indicates the cases in which the number of exceptions is out of the confidence interval, so that, we obtain evidence to reject the null hypothesis at the 5% level type I error rate.

b) Testing consistency of level

We want the VaR level found to be the stated level on average, but we also want to find the stated level at all points in time. One approach to test the consistency of the level is the Ljung-Box portmanteau test (Ljung and Box, 1978) on the exception indicator variable of zeros and ones. When using Ljung-Box tests, there is a choice of the number of lags in which to look for autocorrelation. If the test uses only a few lags but autocorrelation occurs over a long time frame, the test will miss some of the autocorrelation. Conversely, should a large number of lags be used in the test when the autocorrelation is only in a few lags, then the test will not be as sensitive as if the number of lags in the test matched the autocorrelation.

Different lags have been used for each estimate in order to have a good picture of autocorrelation. Table 7 shows the Ljung-Box statistics at lags of 4 and 8. We only detect the existence of autocorrelation in the 10-year portfolio with the measures VaR_I_EWMA(3%) and (4%). In general, the results of the Ljung-Box comparison indicate that autocorrelation is not present. When we consider other lags not shown for space reasons, the result is very similar. We can also conclude from this test the VaR estimates are good.

Table 7: *Testing Consistency of Level.*

	3-year		5-year		10-year		15-year	
Lags	**4**	**8**	**4**	**8**	**4**	**8**	**4**	**8**
VaR_D_EWMA (1%)	0.35	0.71	0.38	0.64	0.38	0.64	0.24	0.49
	(0.987)	(1.000)	(0.984)	(1.000)	(0.984)	(1.000)	(0.993)	(1.000)
VaR_D_EWMA (2%)	0.79	5.18	0.67	1.17	2.14	3.65	0.98	2.00
	(0.940)	(0.739)	(0.955)	(0.997)	(0.711)	(0.887)	(0.913)	(0.981)
VaR_D_EWMA (3%)	1.97	4.13	1.26	3.73	2.25	4.17	2.19	4.24
	(0.740)	(0.845)	(0.869)	(0.881)	(0.690)	(0.842)	(0.700)	(0.835)
VaR_D_EWMA (4%)	2.14	3.86	1.76	3.67	2.25	4.17	5.29	7.98
	(0.709)	(0.870)	(0.780)	(0.886)	(0.690)	(0.842)	(0.259)	(0.435)
VaR_D_EWMA (5%)	1.88	3.61	1.90	3.63	5.60	7.92	3.85	7.93
	(0.758)	(0.890)	(0.754)	(0.889)	(0.231)	(0.441)	(0.427)	(0.440)
VaR_I_EWMA (1%)	0.47	0.97	0.47	0.97	0.98	4.49	5.68	6.33
	(0.976)	(0.998)	(0.976)	(0.998)	(0.913)	(0.811)	(0.225)	(0.611)
VaR_I_EWMA (2%)	2.30	4.65	2.90	5.82	2.89	5.81	2.29	4.04
	(0.680)	(0.795)	(0.576)	(0.667)	(0.577)	(0.668)	(0.683)	(0.853)
VaR_I_EWMA (3%)	5.31	7.64	2.52	5.08	14.52*	16.77*	4.58	7.64
	(0.257)	(0.469)	(0.641)	(0.749)	(0.006)	(0.033)	(0.333)	(0.470)
VaR_I_EWMA (4%)	8.19	10.18	6.32	8.45	12.08*	14.41	3.70	7.55
	(0.085)	(0.253)	(0.177)	(0.391)	(0.017)	(0.072)	(0.449)	(0.479)
VaR_I_EWMA (5%)	8.19	10.18	8.19	8.32	9.07	12.00	8.42	12.30
	(0.085)	(0.253)	(0.085)	(0.403)	(0.059)	(0.151)	(0.077)	(0.138)
VaR_I_GARCH (1%)	0.35	0.71	0.35	0.71	0.35	0.71	0.35	0.71
	(0.987)	(1.000)	(0.987)	(1.000)	(0.987)	(1.000)	(0.987)	(1.000)
VaR_I_GARCH (2%)	1.19	4.12	0.79	5.18	0.62	6.33	0.79	1.62
	(0.879)	(0.846)	(0.940)	(0.739)	(0.961)	(0.610)	(0.940)	(0.991)
VaR_I_GARCH (3%)	1.43	3.98	0.98	4.49	1.19	4.12	1.19	2.44
	(0.840)	(0.859)	(0.913)	(0.811)	(0.879)	(0.846)	(0.879)	(0.965)
VaR_I_GARCH (4%)	5.64	10.97	0.98	4.49	2.29	4.63	2.29	4.63
	(0.228)	(0.203)	(0.913)	(0.811)	(0.683)	(0.796)	(0.683)	(0.796)
VaR_I_GARCH (5%)	4.61	8.37	2.30	5.23	2.18	4.43	4.60	6.60
	(0.329)	(0.398)	(0.680)	(0.733)	(0.702)	(0.816)	(0.330)	(0.581)

Note: Sample period 4/15/2002 to 12/31/2004. The Ljung-Box Q-statistics on the exception indicator variable and their *p*-values. The Q-statistic at lag 4(8) for the null hypothesis that there is no autocorrelation up to order 4(8). An * indicates that there is evidence to reject the null hypothesis at the 5% level type I error date.

c) The back-testing criterion

The back-testing criterion is used to evaluate the performance of VaR measures. The most popular back-testing measure for accuracy of the quantile estimator is the percentage of returns that falls below the quantile estimate which is denoted as $\hat{\alpha}$. For an accurate estimator of an α quantile, $\hat{\alpha}$ will be very close to $\alpha\%$. In order to determine the significance of α departure of from $\hat{\alpha}\%$, the following test statistic is used:

$$Z = (T\hat{\alpha} - T\alpha\%) / \sqrt{T\alpha\%(1-\alpha\%)} \longrightarrow N(0,1) \tag{17}$$

where T is the sample size.

Table 8: *The Back-testing Criterion.*

	% of exceptions			
	3-year	**5-year**	**10-year**	**15-year**
VaR_D_EWMA (1%)	1.37% [0.784]	1.60% [1.264]	1.60% [1.264]	1.14% [0.303]
VaR_D_EWMA (2%)	2.06% [0.089]	3.10% [1.644]	3.88%* [2.801]	4.26%* [3.380]
VaR_D_EWMA (3%)	3.43% [0.530]	2.75% [−0.311]	3.89% [1.091]	3.20% [0.250]
VaR_D_EWMA (4%)	3.89% [−0.117]	3.20% [−0.850]	3.89% [−0.117]	3.66% [−0.361]
VaR_D_EWMA (5%)	4.58% [−0.406]	4.58% [−0.406]	5.26% [0.252]	4.58% [−0.406]
VaR_I_EWMA (1%)	1.60% [1.264]	1.60% [1.264]	2.29%* [2.707]	1.83% [1.745]
VaR_I_EWMA (2%)	2.97% [1.456]	2.52% [0.772]	2.52% [0.772]	2.97% [1.456]
VaR_I_EWMA (3%)	3.66% [0.810]	2.75% [−0.311]	3.43% [0.530]	3.89% [1.091]
VaR_I_EWMA (4%)	4.35% [0.371]	3.43% [−0.605]	3.66% [−0.361]	4.35% [0.371]
VaR_I_EWMA (5%)	4.35% [−0.626]	4.35% [−0.626]	4.35% [−0.626]	5.49% [0.472]
VaR_I_GARCH (1%)	1.37% [0.784]	1.37% [0.784]	1.37% [0.784]	1.37% [0.784]
VaR_I_GARCH (2%)	2.52% [0.772]	2.06% [0.089]	1.83% [−0.253]	2.06% [0.089]
VaR_I_GARCH (3%)	2.75% [−0.311]	2.29% [−0.872]	2.52% [−0.592]	2.52% [−0.592]
VaR_I_GARCH (4%)	3.66% [−0.361]	2.29% [−1.826]	2.97% [−1.094]	2.97% [−1.094]
VaR_I_GARCH (5%)	4.35% [−0.626]	2.97% [−1.942]	3.20% [−1.723]	4.35% [−0.626]

Note: Sample period 4/15/2002 to 12/31/2004. Percentage of exceptions. In square brackets Back-testing Criterion: The Z statistic for determining the significance of departure for $\hat{\alpha} = x/T$ from $\alpha\%$. An * indicates that there is evidence to reject the null hypothesis at the 5% level type I error rate.

Table 8 presents the percentage of exception and, in square brackets, the *Z* statistic for VaR measures. For measures computed with EWMA (independently of the quantile considered) we reject the null hypothesis that the percentage of exceptions coincides with the corresponding quantile in three cases. More precisely, in two occasions with the VaR_D_EWMA measure and once with the VaR_I_EWMA measure. On the other hand, the null hypothesis is never rejected for the VaR_I_GARCH measure.

In summary, we can say that the VaR measures we obtain using the simplification proposed in this paper are, at least as good as those computed with RiskMetrics method (VaR_D_EWMA). Nevertheless, the advantage of the proposed method is a much lower computational cost to calculate VaR.

5. Conclusion

When we use the most commonly implemented parametric approach, we need to estimate the variance-covariance matrix of the portfolio assets. The variance-covariance matrix of prices of a bonds vector from a portfolio depends on the variance-covariance matrix of the interest rates that determine its value. The estimation of the interest rates matrix entails two types of practical problems: dimensionality (the number of variances and covariances to be estimated may be very large), and feasibility (the estimation of interest rates covariances using multivariate methods becomes unfeasible as the dimension increases).

The aim of this paper is to propose a method for calculating the variance-covariance matrix of a large set of interest rates with a low computational cost. The suggested methodology exploits the parameterization of the underlying interest rates proposed by Nelson and Siegel (1987) for estimating the term structure of interest rate (TSIR). Our method turns out to be useful for estimating Value at Risk (VaR), since it considerably simplifies the calculation of this measure.

We start with an explanatory model of interest rates: the Nelson and Siegel (1987) model originally developed to estimate the TSIR. This model provides a relationship to account for changes in interest rates as a function of changes in four parameters, using a linear approximation. Although this approximation reduces the dimension of the variance-covariance matrix, it still requires covariance to be estimated. In order to solve this problem, we propose applying principal components of the changes in the four parameters of the Nelson and Siegel model. Given orthogonality among principal components, the resulting diagonal variance-covariance matrix has a smaller dimension, i.e., all covariances are zero.

The procedure we propose in this paper has been tested using data from the Spanish debt market. The results obtained from applying our methodology are very satisfactory. On the one hand, the variances estimated with our procedure and those from a direct estimation are quite similar, regardless of the method used to estimate the volatility (exponentially weighted moving average or RiskMetrics methodology vs. Generalized

Autoregressive Conditional Heteroskedasticity models). As for VaR calculation, the estimations we obtain with this procedure are quite precise, independently of the method used to estimate the volatility.

An additional advantage of the proposed method is that it is not necessary to decompose the assets into cash-flow and subsequently assign cash to a series of vertexes (RiskMetrics cash flow mapping method). This stems from the fact that our method allows us to estimate the variances and covariances of a vector of interest rates at the same cost and independently from the dimension of the problem. It is unnecessary to reduce the TSIR to a small number of vertexes.

Finally, we should mention that the methodology proposed in this paper presupposes a small implementation cost for financial institutions, since the majority of them already use the Nelson and Siegel (1987) method to estimate TSIR or yield curve. Therefore, these institutions already have the information required to implement our procedure.

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