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New aging properties of the Clayton-Oakes model based on multivariate dispersion*

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Abstract

In this work we present a recent definition of Multivariate Increasing Failure Rate (MIFR) based on the concept of multivariate dispersion. This new definition is an extension of the univariate characterization of IFR distributions under dispersive ordering of the residual lifetimes. We apply this definition to the Clayton-Oakes model. In particular, we provide several conditions to order in the multivariate dispersion sense the residual lifetimes of random vectors with a dependence structure given by the Clayton-Oakes survival copula. We illustrate our results with a graphical method.

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1. Introduction

We use the following notations throughout the paper. For every random variable or vector Z and an event A, let $[Z \mid A]$ denote a random variable or vector whose distribution is the conditional distribution of Z given A. For a random variable Z with distribution function F_Z we will denote by $\bar{F}_Z(t) = 1 - F_Z(t)$ the survival function and by $Q_Z(p) \equiv \inf\{x : F_Z(x) \geq p\}$ the quantile function. When we refer to $=_{st}$, we mean

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equality in law. For every matrix $A \in M_{n\times m}$ we denote by A^t the transpose matrix. We will denote in bold all entities concerned with more than one dimension. We will assume that all multivariate distribution functions are absolutely continuous functions.

The following univariate stochastic orders are common in Stochastic Order Theory. Let X and Y be two random variables with distribution functions F and G. The random variable X is said to be smaller than Y in the univariate dispersive ordering, denoted by $X \leq_{disp} Y$, if $Q_X(q) - Q_X(p) \leq Q_Y(q) - Q_Y(p)$ for all $0 < p \leq q < 1$. In other words, if any pair of quantiles of Y are more widely separated than the corresponding of X. Let us consider now $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors in $\mathbb{R}^n$. The random vector $\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ in the usual stochastic ordering, denoted by $\mathbf{X} \leq_{st} \mathbf{Y}$, if $\mathbb{E}(h(\mathbf{X})) \leq \mathbb{E}(h(\mathbf{Y}))$ for any increasing function $h : \mathbb{R}^n \mapsto \mathbb{R}$ for which the expectations exist. Note that if X and Y are two random variables, $n = 1$, then $X \leq_{st} Y$ if and only if $F_X(t) \geq F_Y(t)$ for every t. Roughly speaking, $\mathbf{X} \leq_{st} \mathbf{Y}$ if $\mathbf{X}$ is less likely than $\mathbf{Y}$ to take on large values. For more details about these stochastic orders the reader may see Shaked and Shanthikumar (2007).

Univariate notions of aging constitute a well established core of reliability theory. We focus on the definition of the IFR notion. Let T be a nonnegative random variable which represents the lifetime of a unit or system. For a survival time t such that $\bar{F}_T(t) > 0$, the conditional residual lifetime distribution is given by $T_t = [T - t \mid T > t]$. Then the random variable T (or its distribution) is said to be IFR [increasing failure rate] if the survival function of the residual lifetime is decreasing when t increases that is,

$$\Pr\{T_t > h\} = \frac{\bar{F}_T(t+h)}{\bar{F}_T(t)} \text{ is decreasing in } 0 < t < \infty \text{ for all } h \geq 0. \tag{1}$$

If the density function $f_T(t)$ exists, a straightforward computation leads to the following characterization:

$$T \text{ is IFR} \quad \Leftrightarrow \quad r_T(t) = \frac{f_T(t)}{\bar{F}_T(t)} \text{ is increasing in } t \geq 0. \tag{2}$$

The function $r_T(t)$ given in (2) is the well known concept of failure or hazard rate, and can be interpreted as the “probability” of instant failure for a unit or a system with survival time t. Therefore the IFR notion means that the probability of instant failure or death is increasing in the survival time, see Barlow and Proschan (1975) for more details.

The IFR univariate definition has a clear interpretation and provides the basis for many useful results, which apply when dealing with the analysis of a single unit or of several units with stochastically independent lifetimes. It is worth to mention that most of the units that are alive at time t will inexorably have the IFR aging property when the time passes. Among other results, the IFR aging class can be characterized by dispersive comparisons of residual lifetimes. It holds that

$$T \text{ is IFR} \quad \Leftrightarrow \quad T_{t'} \leq_{disp} T_t, \tag{3}$$

whenever $0 \leq t \leq t'$. We can find (3) in Belzunce, Candel and Ruiz (1996) and Pellerey and Shaked (1997). A more detailed explanation for these topics can be found in Arias-Nicolás *et al.* (2009) and Belzunce and Shaked (2007). Note that expression (3) reflects the effect of the time over the dispersion of the residual lifetimes.

We also note that the definition of the DFR [Decreasing Failure Rate] aging class follows by replacing decreasing by increasing in (1), increasing by decreasing in (2) and reversing the inequality in (3).

On the other hand, multivariate IFR notions are rather controversial. In fact, starting from the univariate definition, several types of multivariate extensions can be defined. Harris (1970), Brindley and Thompson (1972), Basu (1971), Marshall (1975), Block (1977a) and (1977b), Johnson and Kotz (1975), Savits (1985), Arjas (1981) and Shaked and Shanthikumar (1991) yield different point of view which are useful in different contexts.

Arias-Nicolás *et al.* (2009) present a new concept of MIFR [Multivariate Increasing Failure Rate] based on a natural generalization of (3) via the multivariate dispersion order defined in Fernández-Ponce and Suárez-Llorens (2003), denoted by disp-MIFR. They study the main properties of this new multivariate aging concept and apply it to some well known families of multivariate distributions. They also study the relationships with the other multivariate extensions. The main purpose of this paper is the study of this new notion in the context of Clayton-Oakes model. The paper is organized as follows. In Section 2, we recall the definition of multivariate dispersion order and provide a new property which relates dispersion and copula. In Section 3, we consider the multivariate aging notion defined by Arias-Nicolás *et al.* (2009) in the context of the Clayton-Oakes model. In Section 4, we provide a graphical tool in order to clarify the exposition.

2. The multivariate dispersion order

Several attempts have been made in the literature to extend the univariate dispersion order to the multivariate case. Important contributions have been made by Oja (1983) and Giovagnoly and Wynn (1995). These authors define multivariate dispersion orders through the existence of a multivariate function k which stochastically maps a random vector $\mathbf{X}$ to another random vector $\mathbf{Y}$, i.e., $\mathbf{Y} =_{st} k(\mathbf{X})$. Shaked and Shantikumar (2007) summarize two multivariate dispersion concepts based on a particular transformation by means of the standard construction, viz., the multivariate dispersion orders defined in Shaked and Shanthikumar (1998) and Fernández-Ponce and Suárez-Llorens (2003). Recently Belzunce, Ruiz and Suárez-Llorens (2008) consider another multivariate dispersion order also based on the standard construction and study the relationship with the other definitions. We recall here the multivariate dispersion order defined in Fernández-

Ponce and Suárez-Llorens (2003). We want to emphasize that this order has desirable properties when we compare two random vectors with the same dependence structure, i.e., with the same copula.

Let $\mathbf{X}$ be a random vector and let $\mathbf{u} = (u_1, \ldots, u_n)$ in $[0,1]^n$. The standard construction for $\mathbf{X}$, denoted by

$$\hat{\mathbf{x}}(\mathbf{u}) = (\hat{x}_1(u_1), \hat{x}_2(u_1,u_2), \ldots, \hat{x}_n(u_1,\ldots,u_n)),$$

is defined as follows in terms of the univariate quantile function Q

$$\begin{aligned}
&\hat{x}_1(u_1) = Q_{X_1}(u_1)\\
&\hat{x}_i(u_1,\ldots,u_i) = Q_{\left[X_i \mid \bigcap_{j=1}^{i-1} X_j = \hat{x}_j(u_1,\ldots,u_j)\right]}(u_i), \text{ for } i = 2,\ldots,n.
\end{aligned}$$

This well known construction is widely used in simulation theory and plays the role of the quantile function in the multivariate case. It is well known that $\hat{\mathbf{x}}(\mathbf{U}) =_{st} \mathbf{X}$ where $\mathbf{U}$ is a random vector with n independent uniform components in $[0,1]$.

Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors in $\mathbb{R}^n$. We say that $\mathbf{X}$ is less than $\mathbf{Y}$ in the multivariate dispersion order, denoted by $\mathbf{X} \leq_{disp} \mathbf{Y}$, if

$$\| \hat{\mathbf{x}}(\mathbf{v}) - \hat{\mathbf{x}}(\mathbf{u}) \|_2 \quad \leq \quad \| \hat{\mathbf{y}}(\mathbf{v}) - \hat{\mathbf{y}}(\mathbf{u}) \|_2,$$

for all $\mathbf{u}$ and $\mathbf{v}$ in $(0,1)^n$, where $\|\cdot\|_2$ means the Euclidean norm.

Fernández-Ponce and Suárez-Llorens (2003) showed that the $\leq_{disp}$ order is equivalent to verifying whether the multivariate function $\Phi = (\Phi_1, \ldots, \Phi_n)$, defined as

$$\begin{aligned}
&\Phi_1(x_1) = Q_{Y_1}(F_{X_1}(x_1))\\
&\Phi_i(x_1,\cdots,x_i) = Q_{\left[Y_i \mid \bigcap_{j=1}^{i-1} Y_j = \Phi_j(x_1,\cdots,x_j)\right]}\left(F_{\left[X_i \mid \bigcap_{j=1}^{i-1} X_j = x_j\right]}(x_i)\right), \text{ for } i = 2,\ldots,n,
\end{aligned} \tag{4}$$

which satisfies that $\mathbf{Y} =_{st} \Phi(\mathbf{X})$, is an expansion function. Recall from Giovagnoly and Wynn (1995) that a function $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is called an *expansion* if

$$\|\Phi(\mathbf{x}_2) - \Phi(\mathbf{x}_1)\|_2 \geq \|\mathbf{x}_2 - \mathbf{x}_1\|_2 \quad \text{for all } \mathbf{x}_2 \text{ and } \mathbf{x}_1 \text{ in } \mathbb{R}^n,$$

or equivalently if $J_\Phi(\mathbf{x})^t J_\Phi(\mathbf{x}) - I_n$ is non-negative for all $\mathbf{x} \in \mathbb{R}^n$, where $J_\Phi(\mathbf{x})$ and I_n denote the Jacobian and the identity matrix, respectively.

Fernández-Ponce and Suárez-Llorens (2003), Arias-Nicolás *et al.* (2005), Belzunce *et al.* (2008) and Arias-Nicolás *et al.* (2009) provide many properties of the $\leq_{disp}$ order and study the relationship with other well known multivariate dispersion concepts. For the purpose of our study, we are interested in recalling the relationship between the multivariate dispersion order and the notion of a copula.

A copula is a function that links univariate marginals to their multivariate distribution. Copulas were introduced in the context of probabilistic metric spaces, but the copula method for understanding multivariate distributions has a relatively short history in the statistics literature. In fact, most of the statistical applications have arisen in the last ten years. Given an n-dimensional distribution $\mathbf{F}(x_1,\ldots,x_n)$, with marginals $F_1,\ldots,F_n$, there exists an n-dimensional distribution function C, with marginals uniformly distributed over the interval $[0,1]$, such that

$$\mathbf{F}(x_1,\ldots,x_n) = C(F_1(x_1),\ldots,F_n(x_n)),$$

for all $(x_1,\ldots,x_n) \in \mathbb{R}^n$. Moreover, this copula representation is unique if the margins are continuous. As we can see, this fact allows to separate the marginal feature and the dependence structure which is represented by the copula. For more details about the notion of copula see Nelsen (1999).

From the above, a natural question arises about comparing in dispersion two random vectors with the same dependence structure. Belzunce *et al.* (2008) provide some results concerning both copula and dispersion. The following result can be found in Arias-Nicolás *et al.* (2005). Let $\mathbf{X} = (X_1,\ldots,X_n)$ and $\mathbf{Y} = (Y_1,\ldots,Y_n)$ be n-dimensional random vectors with the same copula. Then

$$\mathbf{X} \leq_{disp} \mathbf{Y} \text{ if and only if } X_i \leq_{disp} Y_i, \text{ for all } i = 1,\ldots,n. \tag{5}$$

Note that in case of a common copula we only have to be care about the comparison of the marginal distributions. Next we provide a new result concerning the multivariate dispersion ordering that will be used later on.

Theorem 1 *Let* $\mathbf{X} = (X_1,\ldots,X_n)$ *and* $\mathbf{Y} = (Y_1,\ldots,Y_n))$ *be two random vector sharing the same copula. If the marginal distributions* X_i *and* Y_i *have the same finite left endpoint for* $i = 1,\ldots,n$. *If* $\mathbf{X} \leq_{disp} \mathbf{Y}$ *then*

$$\mathbf{X} \leq_{st} \mathbf{Y}$$

$$\left[(X_i,\ldots,X_n)\left|\bigcap_{j=1}^{i-1} X_j = \hat{x}_j(u_1,\ldots,u_j)\right.\right] \leq_{st} \left[(Y_i,\ldots,Y_n)\left|\bigcap_{j=1}^{i-1} Y_j = \hat{y}_j(u_1,\ldots,u_j)\right.\right], \tag{6}$$

for $i = 2,\ldots,n$ *and* $\mathbf{u} \in [0,1]^n$.

Proof For random vectors sharing the same copula, Arias-Nicolás *et al.* (2005) showed that the function Φ, defined in (4), can be expressed as

$$\begin{aligned}\Phi_i(x_1,\ldots,x_i) &= Q_{[Y_i|\bigcap_{j=1}^{i-1} Y_j=\Phi_j(x_1,\cdots,x_j)]}(F_{[X_i|\bigcap_{j=1}^{i-1} X_j=x_j]}(x_i)), [0.2cm] \\ &= Q_{Y_i}(F_{X_i}(x_i)).\end{aligned}$$

for $i = 1, \ldots, n$. On the other hand, by construction the function Φ maps the standard construction of $\mathbf{X}$ to the corresponding one of $\mathbf{Y}$, see Fernández-Ponce and Suárez-Llorens (2003), then it is clear that

$$F_{X_i}(\hat{x}_i(u_1, \ldots, u_i)) = F_{Y_i}(\hat{y}_i(u_1, \ldots, u_i)), \text{ for } i = 1, \ldots, n. \tag{7}$$

By hypothesis assumption and using (5), $X_i \leq_{disp} Y_i$ holds for $i = 1, \ldots, n$. It is well known that the univariate dispersive order implies the stochastic order when we compare distribution functions having the same left endpoint in their supports, see Shaked and Shantikumar (2007). Hence we obtain that $X_i \leq_{st} Y_i$, for $i = 1, \ldots, n$, which trivially implies that $Q_{X_i}(u) \leq Q_{Y_i}(u)$ for all $u \in [0, 1]$. From the expression (7), it easily holds that $\hat{x}_i(u_1, \ldots, u_i)$ and $\hat{y}_i(u_1, \ldots, u_i)$ represents the same univariate quantile for the marginal distributions X_i and Y_i, respectively. Therefore

$$\hat{\mathbf{x}}_i(u_1, \ldots, u_i) \leq \hat{\mathbf{y}}_i(u_1, \ldots, u_i), \text{ for } i = 1, \ldots, n.$$

From the mentioned fact that $\hat{\mathbf{x}}(\mathbf{U}) =_{st} \mathbf{X}$ and $\hat{\mathbf{y}}(\mathbf{U}) =_{st} \mathbf{Y}$ where $\mathbf{U}$ is a random vector with n independent uniform components in $[0, 1]$ and using Theorem 6.B.1 in Shaked and Shanthikumar (2007) we obtain that $\mathbf{X} \leq_{st} \mathbf{Y}$. Taking in account that $(\hat{x}_i(u_1, \ldots, u_i), \ldots, \hat{x}_n(u_1, \ldots, u_n))$ represents the standard construction evaluated at $(u_i, \ldots, u_n)$ for the conditional random vector

$$\left[(X_i, \ldots, X_n) \middle| \bigcap_{j=1}^{i-1} X_j = \hat{x}_j(u_1, \ldots, u_j) \right],$$

the rest of the proof follows directly with an equivalent argument. □

3. The Disp-MIFR notion in the context of Clayton-Oakes model

Arias-Nicolás *et al.* (2009) generalize condition (3) via the multivariate dispersion ordering. Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a nonnegative random vector with an absolutely continuous distribution function which represent the lifetimes of n individuals in some system. Given a vector $\mathbf{t} = (t_1, \ldots, t_n)$ on $[0, \infty)^n$, the residual lifetime of $\mathbf{T}$ conditional on the observed survival data $\mathbf{t}$ is given by $\mathbf{T_t} = [\mathbf{T} - \mathbf{t} \mid \mathbf{T} > \mathbf{t}]$. Note that in the general case the residual lifetime of $\mathbf{T}$ takes into account different ages for the individuals. In this paper we restrict our study to a particular survival data, $\mathbf{t} = (t, \ldots, t)$, where all individuals have the same age. The following definition can be found in Arias-Nicolás *et al.* (2009).

Definition 1 *Let* $\mathbf{T} = (T_1, \ldots, T_n)$ *be a non-negative absolutely continuous random vector and let* $\mathbf{t} = (t, \ldots, t)$ *and* $\mathbf{t}' = (t', \ldots, t')$ *be two observations of survival data* *such that* $0 \leq t \leq t'$. *We will say that* $\mathbf{T}$ *is disp3-MIFR (disp3-MDFR) if*

$$\mathbf{T}'_{\mathbf{t}} = [\mathbf{T} - \mathbf{t}' \mid \mathbf{T} > \mathbf{t}'] \quad \leq_{disp} (\geq_{disp}) \quad \mathbf{T}_{\mathbf{t}} = [\mathbf{T} - \mathbf{t} \mid \mathbf{T} > \mathbf{t}]. \tag{8}$$

It is worth to mention that Arias-Nicolás *et al.* (2009) also studied the disp-MIFR and disp2-MIFR definitions for $\mathbf{t} = (t_1, \ldots, t_n)$, $\mathbf{t}' = (t'_1, \ldots, t'_n)$ and $\mathbf{t} = (t_1, \ldots, t_n)$, $\mathbf{t}' = (t_1 + t, \ldots, t_n + t)$, respectively. As we have mentioned, disp3-MIFR could be appropriated in situations where all individuals have the same age. For instance, the well known problems for twins or left-eye and right-eye.

In many types of applications, the dependence structure of a random vector $\mathbf{T}$ is given by the Clayton-Oakes survival copula:

$$\bar{C}(u_1, \ldots, u_n) = \left(\sum_{i=1}^{n} u_i^{1-\theta} - (n-1) \right)^{\frac{1}{1-\theta}}, \tag{9}$$

where, $\theta > 1$. A survival copula is a copula which yields the value of the joint survival function in terms of the values of the marginal survival functions. A multivariate distribution function $\mathbf{F}$ has the above survival copula if

$$\bar{\mathbf{F}}(x_1, \ldots, x_n) = \bar{C}(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n))$$

holds for all $\mathbf{x}$ in $\mathbb{R}^n$. Two multivariate distribution functions have the same survival copula if and only if they have the same copula, for more details see Nelsen (1999).

The family given by (9) has been widely studied in the biostatistics literature. Cook and Johnson (1981) used it to model hydro geochemical data, it is used to generalize the multivariate Pareto distribution and has been used in survival analysis, where it is generally referred to as the gamma frailty model, see Clayton (1978). In epidemiological and actuarial studies there is strong empirical evidence that supports the dependence of mortality on pairs of individuals. This type of copula is useful not only for detecting dependency but also for fitting multivariate data. In the literature we can find examples in medicine, see Sun, Wang and Sun (2006), Bogaerts and Lessafre (2008a) and (2008b) or hidrology, see De Michele *et al.* (2005) and Genest and Favre (2007).

One of the reasons why Clayton-Oakes survival copula becomes so important in biostatistics is the truncation-invariance property. If the random vector $\mathbf{T}$ has a Clayton-Oakes survival copula, then the residual lifetime $\mathbf{T}_{\mathbf{t}}$ has also the same copula. This property characterizes the Clayton-Oakes survival copula, see Sungur (1999) and (2002), Oakes (2005), Charpentier and Juri (2006) and Ahamadi Javid (2008).

From the truncation-invariance property and using (5), if $\mathbf{T}$ has a Clayton-Oakes survival copula, then $\mathbf{T}$ is disp3-MIFR (disp3-MDFR) if and only if

$$[T_i - t' \mid \mathbf{T} \geq \mathbf{t}'] \leq_{disp} (\geq_{disp}) [T_i - t \mid \mathbf{T} \geq \mathbf{t}], \tag{10}$$

for all $\mathbf{t} = (t,\ldots,t)$ and $\mathbf{t}' = (t',\ldots,t')$ such that $0 \le t \le t'$, $i = 1,\ldots,n$. This fact was pointed out in Proposition 8 in Arias-Nicolás *et al.* (2009) for the general definition disp-MIFR. The next proposition presents a result for the stochastic comparison of the residual lifetimes.

Proposition 1 *Let* $\mathbf{T} = (T_1,\ldots,T_n)$ *be a non-negative absolutely continuous random vector having a Clayton-Oakes survival copula and let* $\mathbf{t} = (t,\ldots,t)$ *and* $\mathbf{t}' = (t',\ldots,t')$ *such that* $0 \le t \le t'$. *If* $\mathbf{T}$ *is disp3-MIFR (disp-3MDFR) then*

$$\mathbf{T}_{\mathbf{t}'} \le_{st} (\ge_{st}) \mathbf{T}_{\mathbf{t}}$$

$$\left[(T_i - t',\ldots,T_n - t')\left|\bigcap_{j=1}^{i-1} T_j = t', \bigcap_{j=i}^{n} T_j > t'\right.\right] \le_{st} (\ge_{st}) \left[(T_i - t,\ldots,T_n - t)\left|\bigcap_{j=1}^{i-1} T_j = t, \bigcap_{j=i}^{n} T_j > t\right.\right],$$

for all $i = 2,\ldots,n$.

Proof From the truncation-invariance property of the Clayton-Oakes survival copula the random vectors $\mathbf{T}_{\mathbf{t}'}$ and $\mathbf{T}_{\mathbf{t}}$ share a common copula. Let us denote by $\hat{\mathbf{h}'}_{\mathbf{t}'}(\mathbf{u})$ and $\hat{\mathbf{h}}_{\mathbf{t}}(\mathbf{u})$ the standard constructions of $\mathbf{T}_{\mathbf{t}'}$ and $\mathbf{T}_{\mathbf{t}}$, respectively. The proof follows directly from Theorem 1. It is only necessary to note that $\hat{\mathbf{h}'}_{\mathbf{t}'}(0,\ldots,0) = \hat{\mathbf{h}}_{\mathbf{t}}(0,\ldots,0) = (0,\ldots,0)$ for all $\mathbf{t}$ and $\mathbf{t}'$. □

Proposition 1 implies that the disp3-MIFR (disp-3MDFR) property for a random vector with a Clayton-Oakes survival copula is a sufficient condition for the aging property studied in Mulero and Pellerey (2008) based on the stochastic order of the residual lifetimes.

As a common practice in Biostatistics we will consider now a random vector $\mathbf{T}$ having an exchangeable distribution function, i.e. symmetric permutation. Some examples of this last assumption can be found in clinical trials which involve randomizing clusters or groups of subjects or units into two or more treatment arms, see Manatunga and Chen (2000).

With those settings we provide the main result of the paper. We also need a technical result before about establishing the dispersive order among members of a parametric family of univariate probability distributions.

Theorem 2 (Saunders and Moran (1978)) *Let* X_a *be a univariate random variable with distribution function* F_a *for each* $a \in \mathbb{R}$ *such that:*

1. F_a *is supported on some interval* $(X_-^{(a)}, X_+^{(a)}) \subseteq (-\infty, +\infty)$,
2. F_a *has density* f_a *which does not vanish on any subinterval of* $(X_-^{(a)}, X_+^{(a)})$, *and*
3. *the derivative of* F_a *with respect to* a *exists and is denoted by* $\frac{d}{da}F_a(x)$.

Then,

$$X_a \ge_{disp} X_{a^*} \text{ for } a, a^* \in \mathbb{R}, a > a^*, \tag{11}$$

if and only if

$$\frac{\frac{d}{da}F_a(x)}{f_a(x)} \text{ is decreasing in } x. \tag{12}$$

Remark 1 Although Saunders and Moran (1978) did not mention this explicitly, it is immediate to observe, just considering the parameter $a' = 1/a$, that Theorem 2 is also valid replacing simultaneously $\leq_{disp}$ for $\geq_{disp}$ in (11) and increasing for decreasing in (12).

Theorem 3 (The main result) *Let* $\mathbf{T}$ *be a non-negative absolutely continuous random vector having a Clayton-Oakes survival copula. If* $\mathbf{T}$ *has an exchangeable distribution with margins having a common distribution* F_T*, then* $\mathbf{T}$ *is disp3-MIFR (disp3-MDFR) if and only if the function* $\phi(s)$ *defined by*

$$\phi(s) = \frac{n-(n-1)\bar{F}_T(t+s)^{\theta-1}}{r_T(t+s)} \tag{13}$$

is decreasing (increasing) in s.

Proof Without lack of generality, we will prove the result just for disp3-MIFR. Due to the fact that $\mathbf{T}$ has an exchangeable distribution with a Clayton-Oakes survival copula and using (10), $\mathbf{T}$ is disp3-MIFR if and only if

$$[T_1 - t \mid \mathbf{T} \geq \mathbf{t}] \leq_{disp} [T_1 - t' \mid \mathbf{T} \geq \mathbf{t}'], \tag{14}$$

for all $\mathbf{t} = (t,\ldots,t)$ and $\mathbf{t}' = (t',\ldots,t')$ such that $0 \leq t \leq t'$.

Note that the first component of the residual lifetime $\mathbf{T_t}$, denoted by $[\mathbf{T_t}]_1 \equiv [T_1 - t \mid \mathbf{T} \geq \mathbf{t}]$, can be considered a parametric class of univariate probability distributions depending on parameter t. Then using Theorem 2 and Remark 1, it is clear that inequality (14) holds if and only if the function

$$\frac{\frac{d}{dt}F_{[\mathbf{T_t}]_1}(s)}{f_{[\mathbf{T_t}]_1}(s)} \tag{15}$$

is increasing in s.

The conditional distribution $[\mathbf{T_t}]_1$ is given by the expression

$$F_{[\mathbf{T_t}]_1}(s) = 1 - \frac{\bar{\mathbf{F}}_\mathbf{T}(t+s,t,\ldots,t)}{\bar{\mathbf{F}}_\mathbf{T}(t,\ldots,t)},$$

where

$$\bar{\mathbf{F}}_\mathbf{T}(t_1,t_2,\ldots,t_n) = \left(\bar{F}_T(t_1)^{1-\theta} + \bar{F}_T(t_2)^{1-\theta} + \ldots + \bar{F}_T(t_n)^{1-\theta} - (n-1)\right)^{\frac{1}{1-\theta}}.$$

Therefore, a straightforward computation leads to

$$f_{[\mathbf{T_t}]_1}(s) = \frac{d}{ds}F_{[\mathbf{T_t}]_1}(s) = \left(\frac{\bar{\mathbf{F}}_{\mathbf{T}}(t+s,t,\ldots,t)}{\bar{F}_T(t+s)}\right)^{\theta} \frac{f_T(t+s)}{\bar{\mathbf{F}}_{\mathbf{T}}(t,\ldots,t)}.$$

Now if we take the partial derivative of $F_{[\mathbf{T_t}]_1}(s)$ with respect to the parameter t we obtain

$$\begin{aligned}\frac{d}{dt}F_{[\mathbf{T_t}]_1}(s) &= f_{[\mathbf{T_t}]_1}(s) + (n-1)\left(\frac{\bar{\mathbf{F}}_{\mathbf{T}}(t+s,t,\ldots,t)}{\bar{F}_T(t)}\right)^{\theta} \frac{f_T(t)}{\bar{\mathbf{F}}_{\mathbf{T}}(t,\ldots,t)} \\ &\quad -n\left(\frac{\bar{\mathbf{F}}_{\mathbf{T}}(t,\ldots,t)}{\bar{F}_T(t)}\right)^{\theta} \frac{f_T(t)\bar{\mathbf{F}}_{\mathbf{T}}(t+s,t,\ldots,t)}{\bar{\mathbf{F}}_{\mathbf{T}}(t,\ldots,t)^2}.\end{aligned}$$

Hence we have to study the expression

$$\begin{aligned}\frac{\frac{d}{dt}F_{[\mathbf{T_t}]_1}(s)}{f_{[\mathbf{T_t}]_1}(s)} &= 1+\left((n-1)-n\left(\frac{\bar{\mathbf{F}}_{\mathbf{T}}(t+s,t,\ldots,t)}{\bar{\mathbf{F}}_{\mathbf{T}}(t,\ldots,t)}\right)^{1-\theta}\right)\left(\frac{\bar{F}_T(t+s)}{\bar{F}_T(t)}\right)^{\theta}\frac{f_T(t)}{f_T(t+s)} \\ &= 1-\left(\frac{\bar{\mathbf{F}}_{\mathbf{T}}(t+s,\ldots,t+s)}{\bar{\mathbf{F}}_{\mathbf{T}}(t,\ldots,t)}\right)^{1-\theta}\left(\frac{\bar{F}_T(t+s)}{\bar{F}_T(t)}\right)^{\theta}\frac{f_T(t)}{f_T(t+s)}.\end{aligned}$$

Therefore it is clear that (15) is increasing in s, if and only if the function

$$\frac{\bar{\mathbf{F}}_{\mathbf{T}}(t+s,\ldots,t+s)^{1-\theta}\bar{F}_T(t+s)^{\theta}}{f_T(t+s)} = \frac{n-(n-1)\bar{F}_T(t+s)^{\theta-1}}{f_T(t+s)/\bar{F}_T(t+s)}$$

is decreasing in s. □

Bassan and Spizzichino (2005) pointed out the importance of studying the relations among univariate aging, multivariate aging and dependence structure for multivariate lifetimes. Note that Theorem 3 relates the new concept of MIFR-disp aging with the survival function and the hazard rate function of the margins for a particular dependence structure. We emphasize in those relations in the following results.

Corollary 1 *Let* **T** *be a non-negative absolutely continuous random vector having a Clayton-Oakes survival copula. If* **T** *has an exchangeable distribution with margins having a common distribution* F_T *and a non-increasing hazard rate function, then* **T** *is disp3-MDFR.*

Proof From Theorem 3 we only have to prove that the function $\phi(s)$ given by the expression (13) is increasing in s. From the hypothesis assumption for margins, the function $r_T(t+s)$ is non-increasing in s. The proof is immediate just noting that $n-(n-1)\bar{F}(t+s)^{\theta-1}$ is always increasing in s. □

Corollary 2 *Let* $\mathbf{T}$ *be a non-negative absolutely continuous random vector having a Clayton-Oakes survival copula with* $\theta \leq \frac{n}{n-1}$. *If* $\mathbf{T}$ *has an exchangeable distribution with margins having a common convex distribution* F_T*, then* $\mathbf{T}$ *is disp3-MIFR.*

Proof From Theorem 3 we only have to prove that the function $\phi(s)$ given by the expression (13) is decreasing in s. If we take the logarithm of $\phi(s)$ we obtain

$$\log(\phi(s)) = \log(n-(n-1)\bar{F}_T(t+s)^{\theta-1}) + \log \bar{F}_T(t+s) - \log f_T(t+s).$$

If F_T is convex it is clear that $-\log f_T(t+s)$ is decreasing. Now, if we take the derivative of the first and second term of $\log(\phi(s))$ with respect to s, we obtain that

$$\begin{aligned}
\frac{d}{ds}\left(\log\left(n-(n-1)\bar{F}(t+s)^{\theta-1}\right)+\log \bar{F}(t+s)\right) &\leq 0 \Longleftrightarrow \\
f_T(t+s)\frac{\theta(n-1)\bar{F}_T(t+s)^{\theta-1}-n}{(n-(n-1)\bar{F}_T(t+s)^{\theta-1})\bar{F}_T(t+s)} &\leq 0 \Longleftrightarrow \\
\theta(n-1)\bar{F}_T(t+s)^{\theta-1}-n &\leq 0 \Longleftrightarrow \\
(n-1)\,(\theta-1)\,\bar{F}(t+s,\ldots,t+s)^{\theta-1} &\leq 1. \qquad (16)
\end{aligned}$$

The proof concludes just observing that $\theta \leq \frac{n}{n-1}$ is a sufficient condition for inequality (16). $\square$

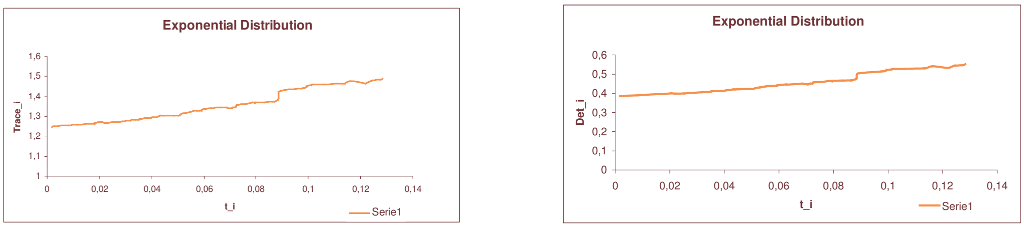
4. A graphical example

In this section we only provide a graphical tool which can help to evaluate the disp3-MIFR (disp3-DMFR) notion from a practical point of view. Arias-Nicolás *et al.* (2009) pointed out the $\leq_{disp}$ order preserves many classical multivariate dispersion measures given in the literature. In particular, if $\mathbf{X} \leq_{disp} \mathbf{Y}$ then $\text{trace}[\text{Cov}(\mathbf{X}))] \leq \text{trace}[\text{Cov}(\mathbf{Y}))]$ and $\det[\text{Cov}(\mathbf{X}))] \leq \det[\text{Cov}(\mathbf{Y}))]$, where Cov($\mathbf{X}$) means the variance-covariance matrix of $\mathbf{X}$ and the same for $\mathbf{Y}$. Both dispersion measures based on the trace and the determinant of the variance-covariance matrix are well known in the literature and easy to estimate. The first one is known as the Total Variance, and the second one as the Wilk's Generalized Variance. Based on these properties authors provide a graphical tool to evaluate the dispersion of the multivariate residual lifetimes. Let $\mathbf{T} = (T_1, \cdots, T_n)$ be a random vector and let t be a real number in $[0,\infty)$. We denote by f_1 and f_2 the following real functions:

$$f_1 : [0,\infty) \mapsto [0,\infty),\; f_1(t) = \text{trace}\left(\text{Cov}\left(\left[T_1-t,\ldots,T_n-t|\bigcap_{i=1}^{n} T_i > t\right]\right)\right)$$

$$f_2 : [0,\infty) \mapsto [0,\infty),\; f_2(t) = \det\left(\text{Cov}\left(\left[T_1-t,\ldots,T_n-t|\bigcap_{i=1}^{n} T_i > t\right]\right)\right)$$

From the above discussion, it is clear that if $\mathbf{T}$ is disp3-MIFR (disp3-DMFR), then the functions f_1 and f_2 are decreasing (increasing) when times increases. In practice, the functions f_1 and f_2 can be easily estimated from the well-known non-parametric estimator of the variance-covariance matrix based on the empirical distribution. From a practical point of view we can use the non-parametric estimation of these functions to detect aging properties. We illustrate this method with two simulated examples in the bivariate case.



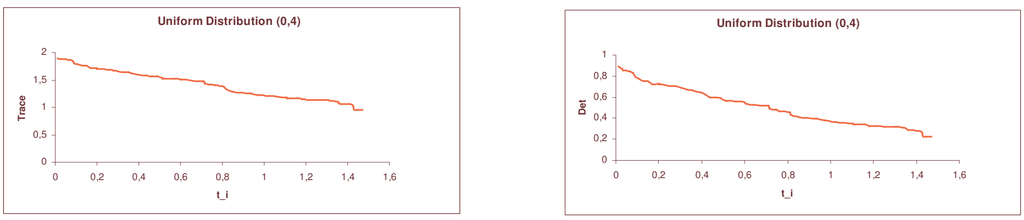
(a) Estimation of f_1 (b) Estimation of f_2

Figure 1: *For* $\mathbf{T}=(T_1,T_2)$ *where* $T_1 =_{st} T_2 =_{st} \mathrm{Exp}(0,5)$

Let $\mathbf{T}=(T_1,T_2)$ be a bivariate random vector having a Clayton-Oakes survival copula. Let us consider an i.i.d. sample of size n of $\mathbf{T}$ denoted by (t_{1j},t_{2j}), $j=1,\ldots,n$, where simulation is done using the well known algorithm proposed by Marshall and Olkin (1988), i.e. we first generate a bivariate sample (u_{1j},u_{2j}), $j=1,\ldots,n$, from the Clayton-Oakes copula for a particular parameter θ and secondly we consider $t_{1j}=F_{T_1}^{-1}(u_{1j})$ and $t_{2j}=F_{T_2}^{-1}(u_{2j})$, $j=1,\ldots,n$, where T_i, $i=1,2$, represent the marginal distributions. Observe that for exchangeable vectors we will consider identical marginal distributions, $T_1 =_{st} T_2 =_{st} T$. From the data, it is easy to observe that the sets

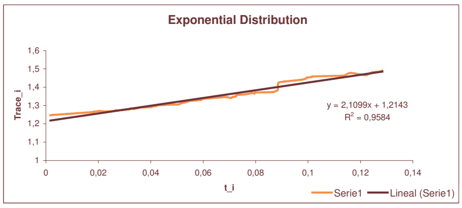
$$\left\{p_j=(a_j,\mathrm{trace}(\hat{\mathrm{Cov}}([T_1-a_j,T_2-a_j \mid T_1>a_j,T_2>a_j,]))),for\ \ j=1,\ldots,m\right\}, \quad (17)$$

$$\left\{q_j=(a_j,\det(\hat{\mathrm{Cov}}([T_1-a_j,T_2-a_j \mid T_1>a_j,T_2>a_j,]))),for\ \ j=1,\ldots,m\right\} \quad (18)$$

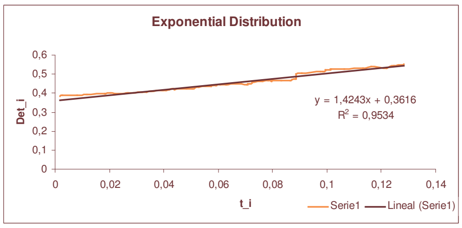


(c) Estimation of f_1 (d) Estimation of f_2

Figure 2: *For* $\mathbf{T}=(T_1,T_2)$ *where* $T_1 =_{st} T_2 =_{st} U(0,4)$

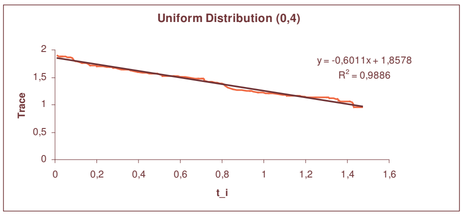


(e) Testing of tendency of f_1

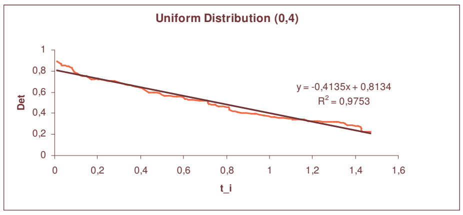


(f) Testing of tendency of f_2

Figure 3: *Fitting a regression model for* $\mathbf{T} = (T_1, T_2)$ *where* $T_1 =_{st} T_2 =_{st} \mathrm{Exp}(0,5)$



(g) Testing of tendency of f_1



(h) Testing of tendency of f_2

Figure 4: *Fitting a regression model for* $\mathbf{T} = (T_1, T_2)$ *where* $T_1 =_{st} T_2 =_{st} U(0,4)$

provide a non-parametric estimation of the graph of f_1 and f_2 by a family of m points, $m < n$. Note that $\hat{\mathrm{Cov}}$ represents the non-parametric estimator of the variance-covariance matrix based on the empirical distribution and a_j, $j = 1, \ldots m$ are univariate sample values included in the support of T.

From the expression (17) and (18), we have simulated two estimations based on exponential and uniform marginal distributions for $n = 100$. Figure 1 represents the non-parametric estimation of f_1 and f_2 for a bivariate vector $\mathbf{T}$ having a Clayton-Oakes survival copula with parameter $\theta = 1.7$ and identical components that are exponentially distributed with mean 2 and analogously for Figure 2 but considering $\theta = 1.5$ and identical components that are uniformly distributed on $(0,4)$. Observe that we have considered $m = 60$ to guarantee a good estimation of the variance-covariance matrix. It is well known that the exponential distribution has a constant hazard rate. Hence, using Corollary 1, it easily holds that the vector $\mathbf{T} = (T_1, T_2)$ with exponential margins is disp3-MDFR. On the other hand, the uniform distribution has a convex distribution function. Hence using Corollary 2, where $\theta = 1.5 < 2$, it is clear that the vector $\mathbf{T} = (T_1, T_2)$, having a Clayton-Oakes survival copula with parameter $\theta = 1.5$, with uniform margins is disp3-MIFR.

From a practical point view, these graphs are not difficult to compute and interpret. We can graphically see the dispersion of the residual lifetimes. If the graph decreases (increases) when the time increases, we could expect a behaviour less (more) dispersive when the time increases, which is closely related to the increase (decrease) of the capacity for predicting an imminent failure. To finalize we can also fit a classical regression model to evaluate the significance of the tendency of f_1 and f_2. Figures 3 and 4 show the result of fitting some classical regression models. Note that the R^2 coefficient is larger than 0.95 in all cases.

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