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Stress-strength reliability of Weibull distribution based on progressively censored samples

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Abstract

Based on progressively Type-II censored samples, this paper deals with inference for the stress-strength reliability $R = P(Y < X)$ when X and Y are two independent Weibull distributions with different scale parameters, but having the same shape parameter. The maximum likelihood estimator, and the approximate maximum likelihood estimator of R are obtained. Different confidence intervals are presented. The Bayes estimator of R and the corresponding credible interval using the Gibbs sampling technique are also proposed. Further, we consider the estimation of R when the same shape parameter is known. The results for exponential and Rayleigh distributions can be obtained as special cases with different scale parameters. Analysis of a real data set as well a Monte Carlo simulation have been presented for illustrative purposes.

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1. Introduction

The Weibull distribution is one of the most widely used distributions in the reliability and survival studies. The two-parameter Weibull distribution denoted by $W(\alpha,\theta)$ has the probability density function (pdf)

$$f(x,\alpha,\theta) = \frac{\alpha}{\theta} x^{\alpha-1} e^{-\frac{x^\alpha}{\theta}}, \quad x > 0,\ \alpha,\ \theta > 0, \tag{1}$$

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and the cumulative distribution function (cdf)

$$F(x,\alpha,\theta) = 1 - e^{-\frac{x^\alpha}{\theta}}, \qquad x > 0,\ \alpha,\ \theta > 0. \tag{2}$$

Here α is the shape parameter and θ is the scale parameter.

It has been used very effectively for analyzing lifetime data, particularly when the data are censored. Among various censoring schemes, the Type II progressive censoring scheme has become very popular one in the last decade. It can be described as follows: Consider N units are placed under a study and only $n(< N)$ units are completely observed until failure. At the time of the first failure (the first stage), r_1 of the $N-1$ surviving units are randomly withdrawn (censored intentionally) from the experiment. At the time of the second failure (the second stage), r_2 of the $N-2-r_1$ surviving units are withdrawn and so on. Finally, at the time of the nth failure (the nth stage), all the remaining $r_n = N-n-r_1-\cdots-r_{n-1}$ surviving units are withdrawn. We will refer to this as progressive Type-II right censoring with scheme $(r_1, r_2, \ldots, r_n)$. It is clear that this scheme includes the conventional Type-II right censoring scheme (when $r_1 = r_2 = \cdots = r_{n-1} = 0$ and $r_n = N-n$) and complete sampling scheme (when $N = n$ and $r_1 = r_2 = \cdots = r_n = 0$). For further details on progressively censoring and relevant references, the reader may refer to the book by Balakrishnan and Aggarwala (2000).

In the stress-strength modelling, $R = P(Y < X)$ is a measure of component reliability when it is subjected to random stress Y and has strength X. For a particular situation, consider Y as the pressure of a chamber generated by ignition of a solid propellant and X as the strength of the chamber. Then R represents the probability of successful firing of the engine. In this context, R can be considered as a measure of system performance and it is naturally arise in electrical and electronic systems. It may be mentioned that R is of greater interest than just reliability since it provides a general measure of the difference between two populations and has applications in many area. For example, if X is the response for a control group, and Y refers to a treatment group, R is a measure of the effect of the treatment. Also, it may be mentioned that R equals the area under the receiver operating characteristic (ROC) curve for diagnostic test or bio-markers with continuous outcome (see Bamber (1975)). The ROC curve is widely used, in biological, medical and health service research, to evaluate the ability of diagnostic tests or bio-markers to distinguish between two groups of subjects, usually non-diseased and diseased subjects. For more application of R, see Kotz et al. (2003). Many authors have studied the stress-strength parameter R. Among them, Ahmad et al. (1997), Awad et al. (1981), Kundu and Gupta (2005, 2006), Adimari and Chiogna (2006), Baklizi (2008), Raqab et al. (2008) and Rezaei et al. (2010).

Based on complete X-sample and Y-sample, Kundu and Gupta (2006) considered the estimation of $R = P(Y < X)$ when $X \sim W(\alpha, \theta_1)$ and $Y \sim W(\alpha, \theta_2)$ are two independent Weibull distributions with different scale parameters, but having the same shape parameter. In this paper, we extend their results for the case when the samples are progressively Type-II censored. The layout of this paper is as follows: In Section 2, we derive the maximum likelihood estimator (MLE) of R. It is observed that the MLE can be obtained using an iterative procedure. We further propose an approximate MLE (AMLE) of R, which can be obtained explicitly. Different confidence intervals (C.I.'s) are presented in Section 3. A Bayes estimator of R and the corresponding credible interval using the Gibbs sampling technique have been proposed in Section 4. Analysis of a real data set as well a Monte Carlo simulation based comparison of the proposed methods are performed in Section 5. Finally, we conclude the paper in Section 6.

2. Maximum likelihood estimator of *R*

Let $X \sim W(\alpha,\theta_1)$ and $Y \sim W(\alpha,\theta_2)$ be independent random variables. Then it can be easily seen that

$$R = P(Y < X) = \frac{\theta_1}{\theta_1+\theta_2}. \tag{3}$$

Our interest is in estimating R based on progressive Type-II censored data on both variables. To derive the MLE of R, first we obtain the MLE's of α, θ_1 and θ_2. Suppose $\mathbf{X} = (X_{1:N}, X_{2:N}, \ldots, X_{n:N})$ is a progressively Type-II censored sample from $W(\alpha,\theta_1)$ with censored scheme $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ and $\mathbf{Y} = (Y_{1:M}, Y_{2:M}, \ldots, Y_{m:M})$ is a progressively Type-II censored sample from $W(\alpha,\theta_2)$ with censored scheme $\mathbf{r}' = (r'_1, r'_2, \ldots, r'_m)$. For notation simplicity, we will write $(X_1, X_2, \ldots, X_n)$ for $(X_{1:N}, X_{2:N}, \ldots, X_{n:N})$ and $(Y_1, Y_2, \ldots, Y_m)$ for $(Y_{1:M}, Y_{2:M}, \ldots, Y_{m:M})$. Therefore, the likelihood function of α, θ_1 and θ_2 is given (see Balakrishnan and Aggarwala (2000)) by

$$L(\alpha,\theta_1,\theta_2) = \left[c_1 \prod_{i=1}^{n} f(x_i)[1-F(x_i)]^{r_i}\right] \times \left[c_2 \prod_{j=1}^{m} f(y_j)[1-F(y_j)]^{r'_j}\right], \tag{4}$$

where

$$c_1 = N(N-1-r_1)(N-2-r_1-r_2)\cdots(N-n+1-r_1\cdots-r_{n-1}),$$
$$c_2 = M(M-1-r'_1)(M-2-r'_1-r'_2)\cdots(M-m+1-r'_1\cdots-r'_{m-1}).$$

Upon using (1) and (2), we immediately have the likelihood function of the observed data as follows:

$$\begin{aligned} L(data|\alpha,\theta_1,\theta_2) &= c_1 c_2 \alpha^{n+m} \theta_1^{-n} \theta_2^{-m} \prod_{i=1}^{n} x_i^{\alpha-1} \prod_{j=1}^{m} y_j^{\alpha-1} \\ &\quad \times \exp\left\{-\frac{1}{\theta_1}\sum_{i=1}^{n}(r_i+1)x_i^{\alpha} - \frac{1}{\theta_2}\sum_{j=1}^{m}(r'_j+1)y_j^{\alpha}\right\}. \end{aligned} \tag{5}$$

From (5), the log-likelihood function is

$$l(\alpha,\theta_1,\theta_2) \propto (n+m)\ln\alpha - n\ln(\theta_1) - m\ln(\theta_2) + (\alpha-1)$$
$$\times \left[\sum_{i=1}^{n}\ln(x_i) + \sum_{j=1}^{m}\ln(y_j)\right] - \frac{1}{\theta_1}\sum_{i=1}^{n}(r_i+1)x_i^{\alpha} - \frac{1}{\theta_2}\sum_{j=1}^{m}(r'_j+1)y_j^{\alpha}.$$

The MLEs of α, θ_1 and θ_2, say $\widehat{\alpha}$, $\widehat{\theta}_1$ and $\widehat{\theta}_2$ respectively, can be obtained as the solution of

$$\frac{\partial l}{\partial \alpha} = \frac{n+m}{\alpha} + \left[\sum_{i=1}^{n}\ln(x_i) + \sum_{j=1}^{m}\ln(y_j)\right] - \frac{1}{\theta_1}\sum_{i=1}^{n}(r_i+1)x_i^{\alpha}\ln(x_i)$$
$$- \frac{1}{\theta_2}\sum_{j=1}^{m}(r'_j+1)y_j^{\alpha}\ln(y_j) = 0, \tag{6}$$

$$\frac{\partial l}{\partial \theta_1} = -\frac{n}{\theta_1} + \frac{1}{\theta_1^2}\sum_{i=1}^{n}(r_i+1)x_i^{\alpha} = 0, \tag{7}$$

$$\frac{\partial l}{\partial \theta_2} = -\frac{m}{\theta_2} + \frac{1}{\theta_2^2}\sum_{j=1}^{m}(r'_j+1)y_j^{\alpha} = 0. \tag{8}$$

From (7) and (8), we obtain

$$\widehat{\theta}_1(\alpha) = \frac{1}{n}\sum_{i=1}^{n}(r_i+1)x_i^{\alpha}, \text{ and } \widehat{\theta}_2(\alpha) = \frac{1}{m}\sum_{j=1}^{m}(r'_j+1)y_j^{\alpha}. \tag{9}$$

Substituting the expressions of $\widehat{\theta}_1(\alpha)$ and $\widehat{\theta}_2(\alpha)$ into (6), $\widehat{\alpha}$ can be obtained as a fixed point solution of the following equation:

$$k(\alpha) = \alpha, \tag{10}$$

where

$$k(\alpha) = \frac{n+m}{\frac{n\sum_{i=1}^{n}(r_i+1)x_i^{\alpha}\ln(x_i)}{\sum_{i=1}^{n}(r_i+1)x_i^{\alpha}} + \frac{m\sum_{j=1}^{m}(r'_j+1)y_j^{\alpha}\ln(y_j)}{\sum_{j=1}^{m}(r'_j+1)y_j^{\alpha}} - \left[\sum_{i=1}^{n}\ln(x_i) + \sum_{j=1}^{m}\ln(y_j)\right]}.$$

A simple iterative procedure $k(\alpha^{(j)}) = \alpha^{(j+1)}$ where $\alpha^{(j)}$ is the j-th iterate, can be used to find the solution of (10). Once we obtain $\widehat{\alpha}_{ML}$, the MLE of θ_1 and θ_2, can be deduced from (9) as $\widehat{\theta}_{1ML} = \widehat{\theta}_1(\widehat{\alpha}_{ML})$ and $\widehat{\theta}_{2ML} = \widehat{\theta}_2(\widehat{\alpha}_{ML})$. Therefore, we compute the MLE of R as

$$\widehat{R}_{ML} = \frac{\frac{1}{n}\sum_{i=1}^{n}(r_i+1)x_i^{\widehat{\alpha}_{ML}}}{\frac{1}{n}\sum_{i=1}^{n}(r_i+1)x_i^{\widehat{\alpha}_{ML}} + \frac{1}{m}\sum_{j=1}^{m}(r'_j+1)y_j^{\widehat{\alpha}_{ML}}}. \tag{11}$$

Here the maximum likelihood approach does not give an explicit estimator for α and hence for R, based on a progressively Type-II censored sample. Now, we approximate the likelihood equation analogously to Kundu and Gupta (2006). It is based on the fact that if the random variable X has $W(\alpha,\theta)$, then $V = \ln X$, has the extreme value distribution with pdf as

$$f(v;\mu,\sigma) = \frac{1}{\sigma} e^{\frac{v-\mu}{\sigma} - e^{\frac{v-\mu}{\sigma}}}, \quad -\infty < v < \infty, \tag{12}$$

where $\mu = \frac{1}{\alpha}\ln\theta$ and $\sigma = 1/\alpha$. The density function (12) is known as the density function of an extreme value distribution, with location, and scale parameters as μ and σ respectively. The standard extreme value distribution has the pdf and cdf as

$$g(v) = e^{v-e^v}, \qquad G(v) = 1 - e^{-e^v}.$$

Suppose $X_1 < X_2 < \cdots < X_n$ is a progressively Type-II censored sample from $W(\alpha,\theta_1)$ with censored scheme $(r_1, r_2, \ldots, r_n)$ and $Y_1 < Y_2 < \cdots < Y_m$ is a progressively Type-II censored sample from $W(\alpha,\theta_2)$ with censored scheme $(r'_1, r'_2, \ldots, r'_m)$. Furthermore, we use the following notation for this subsection. $T_i = \ln X_i$, $Z_i = \frac{T_i-\mu_1}{\sigma}$, $i = 1,\ldots,n$ and $S_j = \ln Y_j$, $W_j = \frac{S_j-\mu_2}{\sigma}$, $j = 1,\ldots,m$, where $\mu_1 = \frac{1}{\alpha}\ln\theta_1$, $\mu_2 = \frac{1}{\alpha}\ln\theta_2$ and $\sigma = \frac{1}{\alpha}$. The log-likelihood function of the observed data $T_1,\ldots,T_n$ and $S_1,\ldots,S_m$ is

$$\begin{aligned} l^*(\mu_1,\mu_2,\sigma) \propto & -(n+m)\ln\sigma + \sum_{i=1}^{n}\ln(g(z_i)) + \sum_{i=1}^{n} r_i \ln(1-G(z_i)) \\ & + \sum_{j=1}^{m}\ln(g(w_j)) + \sum_{j=1}^{m} r'_j \ln(1-G(w_j)). \end{aligned} \tag{13}$$

Differentiating (13) with respect to μ_1, μ_2 and σ, we obtain the likelihood equations as

$$\frac{\partial l^*}{\partial\mu_1} = -\frac{1}{\sigma}\sum_{i=1}^{n}\frac{g'(z_i)}{g(z_i)} + \frac{1}{\sigma}\sum_{i=1}^{n} r_i \frac{g(z_i)}{1-G(z_i)} = 0, \tag{14}$$

$$\frac{\partial l^*}{\partial\mu_2} = -\frac{1}{\sigma}\sum_{j=1}^{m}\frac{g'(w_j)}{g(w_j)} + \frac{1}{\sigma}\sum_{j=1}^{m} r'_j \frac{g(w_j)}{1-G(w_j)} = 0, \tag{15}$$

$$\begin{aligned} \frac{\partial l^*}{\partial\sigma} = & -\frac{n+m}{\sigma} - \frac{1}{\sigma}\sum_{i=1}^{n} z_i \frac{g'(z_i)}{g(z_i)} + \frac{1}{\sigma}\sum_{i=1}^{n} r_i z_i \frac{g(z_i)}{1-G(z_i)} \\ & - \frac{1}{\sigma}\sum_{j=1}^{m} w_j \frac{g'(w_j)}{g(w_j)} + \frac{1}{\sigma}\sum_{j=1}^{m} r'_j w_j \frac{g(w_j)}{1-G(w_j)} = 0. \end{aligned} \tag{16}$$

We approximate the terms $p(z_i) = \frac{g'(z_i)}{g(z_i)}$ and $q(z_i) = \frac{g(z_i)}{1-G(z_i)}$ by expanding them in a Taylor series around $\nu_i = E(Z_i)$. Further, we also approximate the terms $\bar{p}(w_j) = \frac{g'(w_j)}{g(w_j)}$, and $\bar{q}(w_j) = \frac{g(w_j)}{1-G(w_j)}$ by expanding them in a Taylor series around $\bar{\nu}_j = E(W_j)$. It is known that $Z_i \stackrel{d}{=} G^{-1}(U_i)$, where U_i is the i-th progressively Type-II censored order statistic from the uniform $U(0,1)$ distribution. Therefore,

$$\nu_i = E(Z_i) \approx G^{-1}(\eta_i),$$

where $\eta_i = E(U_i)$. From Balakrishnan and Aggarwala (2000),

$$\eta_i = 1 - \prod_{k=n-i+1}^{n} \frac{k + R_{n-k+1} + \cdots + R_n}{k + 1 + R_{n-k+1} + \cdots + R_n}, \quad i = 1, \ldots, n.$$

Since, $G^{-1}(u) = \ln(-\ln(1-u))$, we can approximate ν_i by $\ln(-\ln(1-\eta_i))$. Similarly, we approximate $\bar{\nu}_j$ by $\ln(-\ln(1-\eta_j))$. Now, upon expanding the function $p(z_i)$, $\bar{p}(w_j)$, $q(z_i)$ and $\bar{q}(w_j)$ keeping only the first two terms, we get

$$p(z_i) \approx \alpha_i + \beta_i z_i, \qquad \bar{p}(w_j) \approx \bar{\alpha}_j + \bar{\beta}_j w_j,$$
$$q(z_i) \approx \gamma_i + \delta_i z_i, \qquad \bar{q}(w_j) \approx \bar{\gamma}_j + \bar{\delta}_j w_j,$$

where

$$\begin{aligned}
\alpha_i &= p(\nu_i) - \nu_i p'(\nu_i) = 1 + (\nu_i - 1)e^{\nu_i}, & \beta_i &= p'(\nu_i) = -e^{\nu_i},\\
\bar{\alpha}_j &= \bar{p}(\bar{\nu}_j) - \bar{\nu}_j \bar{p}'(\bar{\nu}_j) = 1 + (\bar{\nu}_j - 1)e^{\bar{\nu}_j}, & \bar{\beta}_j &= \bar{p}'(\bar{\nu}_j) = -e^{\bar{\nu}_j},\\
\gamma_i &= q(\nu_i) - \nu_i q'(\nu_i) = (1 - \nu_i)e^{\nu_i}, & \delta_i &= q'(\nu_i) = e^{\nu_i},\\
\bar{\gamma}_j &= \bar{q}(\bar{\nu}_j) - \bar{\nu}_j \bar{q}'(\bar{\nu}_j) = (1 - \bar{\nu}_j)e^{\bar{\nu}_j}, & \bar{\delta}_j &= \bar{q}'(\bar{\nu}_j) = e^{\bar{\nu}_j}.
\end{aligned}$$

Therefore, (14), (15), and (16) can be approximated respectively as

$$\frac{\partial l^*}{\partial \mu_1} = -\frac{1}{\sigma} \sum_{i=1}^{n} [(\alpha_i - r_i\gamma_i) + z_i(\beta_i - r_i\delta_i)] = 0, \tag{17}$$

$$\frac{\partial l^*}{\partial \mu_2} = -\frac{1}{\sigma} \sum_{j=1}^{m} \left[(\bar{\alpha}_j - r'_j\bar{\gamma}_j) + w_j(\bar{\beta}_j - r'_j\bar{\delta}_j)\right] = 0, \tag{18}$$

$$\begin{aligned}
\frac{\partial l^*}{\partial \sigma} = &-\frac{n+m}{\sigma} - \frac{1}{\sigma} \sum_{i=1}^{n} \left[z_i(\alpha_i - r_i\gamma_i) + z_i^2(\beta_i - r_i\delta_i)\right]\\
&- \frac{1}{\sigma} \sum_{j=1}^{m} \left[w_j(\bar{\alpha}_j - r'_j\bar{\gamma}_j) + w_j^2(\bar{\beta}_j - r'_j\bar{\delta}_j)\right] = 0.
\end{aligned} \tag{19}$$

If we denote $\tilde{\mu}_1$, $\tilde{\mu}_2$ and $\tilde{\sigma}_1$ as the solutions of (17), (18) and (19) respectively, then observe that

$$\tilde{\mu}_1 = A_1 + B_1\tilde{\sigma}, \;\; \tilde{\mu}_2 = A_2 + B_2\tilde{\sigma}, \;\text{ and }\; \tilde{\sigma} = \frac{-D + \sqrt{D^2 - 4(n+m)E}}{2(n+m)},$$

where

$$A_1 = \frac{\sum_{i=1}^{n} t_i(\beta_i - r_i\delta_i)}{\sum_{i=1}^{n}(\beta_i - r_i\delta_i)}, \qquad B_1 = \frac{\sum_{i=1}^{n}(\alpha_i - r_i\gamma_i)}{\sum_{i=1}^{n}(\beta_i - r_i\delta_i)},$$

$$A_2 = \frac{\sum_{j=1}^{m} s_j(\bar{\beta}_j - r'_j\bar{\delta}_j)}{\sum_{j=1}^{m}(\bar{\beta}_j - r'_j\bar{\delta}_j)}, \qquad B_2 = \frac{\sum_{j=1}^{m}(\bar{\alpha}_j - r'_j\bar{\gamma}_j)}{\sum_{j=1}^{m}(\bar{\beta}_j - r'_j\bar{\delta}_i)},$$

$$\begin{aligned} D &= \sum_{i=1}^{n} t_i(\alpha_i - r_i\gamma_i) - 3A_1\sum_{i=1}^{n}(\alpha_i - r_i\gamma_i) + 2A_1B_1\sum_{i=1}^{n}(\beta_i - r_i\delta_i) \\ &\quad + \sum_{j=1}^{m} s_j(\bar{\alpha}_j - r'_j\bar{\gamma}_j) - 3A_2\sum_{j=1}^{m}(\bar{\alpha}_j - r'_j\bar{\gamma}_j) + 2A_2B_2\sum_{j=1}^{m}(\bar{\beta}_j - r'_j\bar{\delta}_j), \\ E &= \sum_{i=1}^{n} t_i^2(\beta_i - r_i\delta_i) - A_1\sum_{i=1}^{n} t_i(\beta_i - r_i\delta_i) \\ &\quad + \sum_{j=1}^{m} s_j^2(\bar{\beta}_j - r'_j\bar{\delta}_j) - A_2\sum_{j=1}^{m} s_j(\bar{\beta}_j - r'_j\bar{\delta}_j). \end{aligned}$$

Once $\tilde{\sigma}$ is obtained, $\tilde{\mu}_1$ and $\tilde{\mu}_2$ can be obtained immediately. Therefore, the approximate MLE of R is given by

$$\tilde{R} = \frac{\tilde{\theta}_1}{\tilde{\theta}_1 + \tilde{\theta}_2},$$

where

$$\tilde{\theta}_1 = \exp\left(\frac{1}{\tilde{\sigma}}(A_1 + B_1\tilde{\sigma})\right), \;\text{ and }\; \tilde{\theta}_2 = \exp\left(\frac{1}{\tilde{\sigma}}(A_2 + B_2\tilde{\sigma})\right).$$

3. C.I.'s of R

Based on asymptotic behavior of R, we present an asymptotic C.I. of R. We further, propose two C.I.'s based on the non-parametric bootstrap method.

3.1. Asymptotic C.I. of R

Let us first start with obtaining the Fisher information matrix of $\theta = (\alpha, \theta_1, \theta_2)$. If $X_{1:N} < X_{2:N} < \cdots < X_{n:N}$ is a progressively Type-II censored sample from the $W(\alpha, \theta)$ distribution with censored scheme $\mathbf{r} = (r_1, r_2, \ldots, r_n)$. Then $Z_{1:N} < Z_{2:N} < \cdots < Z_{n:N}$, where $Z_{i:N} = \frac{X^{\alpha}_{i:N}}{\theta}$ $(i = 1, \ldots, n)$ is a progressively Type-II censored sample from the standard Exponential distribution with censored scheme $\mathbf{r} = (r_1, r_2, \ldots, r_n)$. Hence (see Balakrishnan and Aggarwala (2000), p. 19.)

$$E[(X_{i:N})^{\alpha}] = \theta E(Z_{i:N}) = \theta \mu_i,$$

where

$$\mu_i = \sum_{k=1}^{i} \frac{1}{N - \sum_{s=0}^{k-1} r_s - k + 1}, \quad i = 1, \ldots, n.$$

The pdf of $X_{i:N}$ (see, for example, Kamps and Cramer (2001)) is

$$f_{X_{i:N}}(x) = c_{i-1} \sum_{k=1}^{i} a_{k,i} \frac{\alpha}{\theta} x^{\alpha-1} e^{-\gamma_k \frac{x^{\alpha}}{\theta}},\ x > 0,$$

where

$$\gamma_k = N - k + 1 + \sum_{s=k}^{n} r_s,\ \ c_{i-1} = \prod_{s=1}^{i} \gamma_s,\ \text{ and }\ a_{k,i} = \prod_{\substack{s=1\\ s\neq k}}^{i} \frac{1}{\gamma_s - \gamma_i}.$$

The Fisher information matrix of $\theta = (\alpha, \theta_1, \theta_2)$ (cf. Kundu and Gupta (2005, 2006)) is obtained to be

$$I(\theta) = -\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 l}{\partial \alpha \partial \theta_1}\right) & E\left(\frac{\partial^2 l}{\partial \alpha \partial \theta_2}\right) \\ E\left(\frac{\partial^2 l}{\partial \theta_1 \partial \alpha}\right) & E\left(\frac{\partial^2 l}{\partial \theta_1^2}\right) & E\left(\frac{\partial^2 l}{\partial \theta_1 \partial \theta_2}\right) \\ E\left(\frac{\partial^2 l}{\partial \theta_2 \partial \alpha}\right) & E\left(\frac{\partial^2 l}{\partial \theta_2 \partial \theta_1}\right) & E\left(\frac{\partial^2 l}{\partial \theta_2^2}\right) \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}$$

where

$$\begin{aligned} -I_{11} &= -\frac{n+m}{\alpha^2} - \frac{1}{\theta_1} \sum_{i=1}^{n} \left[(r_i+1) E[X_i^{\alpha} (\ln(X_i))^2]\right] - \frac{1}{\theta_2} \sum_{j=1}^{m} \left[(r'_j+1) E[Y_j^{\alpha} (\ln(Y_j))^2]\right] \\ &= -\frac{1}{\alpha^2} \sum_{i=1}^{n} \left[(r_i+1) c_{i-1} \sum_{k=1}^{i} \frac{a_{k,i}}{\gamma_k^2} \left[\Gamma''(2) + 2\ln(\frac{\theta_1}{\gamma_k})\Gamma'(2) + (\ln(\frac{\theta_1}{\gamma_k}))^2 \Gamma(2)\right]\right] \\ &\quad - \frac{1}{\alpha^2} \sum_{j=1}^{m} \left[(r'_j+1) c'_{j-1} \sum_{k=1}^{j} \frac{a'_{k,j}}{\gamma'^2_k} \left[\Gamma''(2) + 2\ln(\frac{\theta_2}{\gamma'_k})\Gamma'(2) + (\ln(\frac{\theta_2}{\gamma'_k}))^2 \Gamma(2)\right]\right] - \frac{n+m}{\alpha^2}, \end{aligned}$$

$$-I_{22} = \frac{n}{\theta_1^2} - \frac{2}{\theta_1^3}\sum_{i=1}^{n}(r_i+1)E[X_i^{\alpha}] = \frac{1}{\theta_1^2}\left[n - 2\sum_{i=1}^{n}(r_i+1)\mu_i\right],$$

$$-I_{33} = \frac{m}{\theta_2^2} - \frac{2}{\theta_2^3}\sum_{j=1}^{m}(r'_j+1)E[Y_j^{\alpha}] = \frac{1}{\theta_2^2}\left[m - 2\sum_{j=1}^{m}(r'_j+1)\mu'_j\right],$$

$$\begin{aligned}-I_{12} &= \frac{1}{\theta_1^2}\sum_{i=1}^{n}(r_i+1)E[X_i^{\alpha}\ln(X_i)] \\ &= \frac{1}{\alpha\theta_1}\sum_{i=1}^{n}(r_i+1)c_{i-1}\sum_{k=1}^{i}\frac{a_{k,i}}{\gamma_k^2}\left[\Gamma'(2)+\ln(\frac{\theta_1}{\gamma_k})\Gamma(2)\right] = -I_{21}. \\ -I_{13} &= \frac{1}{\theta_2^2}\sum_{j=1}^{m}(r'_j+1)E[Y_j^{\alpha}\ln(Y_j)] \\ &= \frac{1}{\alpha\theta_2}\sum_{j=1}^{m}(r'_j+1)c'_{j-1}\sum_{k=1}^{j}\frac{a'_{k,j}}{\gamma_k'^2}\left[\Gamma'(2)+\ln(\frac{\theta_2}{\gamma'_k})\Gamma(2)\right] = -I_{31},\end{aligned}$$

$$I_{23} = I_{32} = 0,$$

where

$$\mu'_j = \sum_{k=1}^{j}\frac{1}{M - \sum_{s=0}^{k-1} r'_s - k + 1} \qquad \gamma'_k = M - k + 1 + \sum_{s=k}^{m} r'_s$$

$$c'_{j-1} = \prod_{s=1}^{j}\gamma'_s \qquad a'_{k,j} = \prod_{\substack{s=1\\ s\neq k}}^{j}\frac{1}{\gamma'_s - \gamma'_j}.$$

From the asymptotic properties of the MLE's and the fact that the two-parameter Weibull distribution satisfies all the regularity conditions (cf. Bain (1978)), we state the following theorem.

Theorem 1. *For $n \to \infty$ and $m \to \infty$,*

$$\left(\sqrt{m}(\widehat{\alpha}-\alpha),\ \sqrt{n}(\widehat{\theta}_1-\theta_1),\ \sqrt{n}(\widehat{\theta}_2-\theta_2)\right) \xrightarrow{d} N_3(0, A^{-1}(\alpha,\theta_1,\theta_2))$$

where

$$A(\alpha,\theta_1,\theta_2) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix},$$

with

$$a_{11} = \frac{I_{11}}{m},\ a_{22} = \frac{I_{22}}{n},\ a_{33} = \frac{I_{33}}{n},\ a_{12} = a_{21} = \frac{I_{12}}{\sqrt{nm}},\ a_{13} = a_{31} = \frac{I_{13}}{\sqrt{nm}}.$$

Theorem 2. *For $n \to \infty$, $m \to \infty$, $\sqrt{m}(\widehat{R}-R) \to N(0,B)$, where*

$$B = \frac{1}{u(\theta_1+\theta_2)^4}\left[(a_{11}a_{22}-a_{12}^2)\theta_1^2 - 2a_{12}a_{13}\theta_1\theta_2 + (a_{11}a_{33}-a_{13}^2)\theta_2^2\right],$$

and $u = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22}$.

Proof. See the Appendix.

From Theorem 2, we construct the asymptotic C.I. of R. Using the MLE's of α, θ_1 and θ_2, B can be estimated. As a consequence of that, a $100(1-\gamma)\%$ asymptotic C.I. for R can be presented of the form,

$$(\widehat{R} - z_{1-\frac{\gamma}{2}}\frac{\sqrt{\widehat{B}}}{\sqrt{m}}, \widehat{R} + z_{1-\frac{\gamma}{2}}\frac{\sqrt{\widehat{B}}}{\sqrt{m}}),$$

where z_γ is 100γth percentile of $N(0,1)$. The C.I. of R by using the asymptotic distribution of the AMLE of R can be obtained similarly by replacing α, θ_1 and θ_2 in B by their respective AMLE's.
It is of interest to observe that when the shape parameter α is known, the MLE of R can be obtained explicitly as

$$\hat{R}_{ML} = \frac{S_1(\mathbf{x})}{S_1(\mathbf{x}) + \frac{n}{m}S_2(\mathbf{y})}, \tag{20}$$

where $S_1(\mathbf{x}) = \sum_{i=1}^{n}(r_i+1)x_i^\alpha$ and $S_2(\mathbf{y}) = \sum_{j=1}^{m}(r'_j+1)y_j^\alpha$. It is easily checked that $(2/\theta_1)S_1(\mathbf{X})$ and $(2/\theta_2)S_2(\mathbf{Y})$ have chi-square distribution with $2n$ and $2m$ degrees of freedom, respectively. Alternatively, we have

$$\widehat{R}_{ML} \stackrel{d}{=} \frac{1}{1+\frac{\theta_2}{\theta_1}W},$$

or

$$W \stackrel{d}{=} \frac{1-\widehat{R}_{ML}}{\widehat{R}_{ML}}.\frac{R}{1-R},$$

where W has an F distribution with $2m$ and $2n$ degrees of freedom. Then, a $100(1-\gamma)\%$ C.I. for R can be presented as

$$\left(\frac{1}{1+\left[\frac{1-\widehat{R}_{ML}}{\widehat{R}_{ML}}\right]F_{1-\frac{\gamma}{2},2n,2m}}, \frac{1}{1+\left[\frac{1-\widehat{R}_{ML}}{\widehat{R}_{ML}}\right]F_{\frac{\gamma}{2},2n,2m}}\right),$$

where $F_{\frac{\gamma}{2},2n,2m}$ and $F_{1-\frac{\gamma}{2},2n,2m}$ are the lower and upper $\frac{\gamma}{2}th$ percentile points of an F distribution with $2n$ and $2m$ degrees of freedom.

3.2. Bootstrap C.I.'s

It is evident that the C.I.'s based on the asymptotic results do not perform very well for small sample size. For this, we propose two C. I.'s based on the non-parametric bootstrap methods: (i) percentile bootstrap method (we call it Boot-p) based on the idea of Efron (1982), and (ii) bootstrap-t method (we refer to it as Boot-t) based on the idea of Hall (1988). We illustrate briefly how to estimate C.I.'s of R using both methods.

(i) Boot-p method

1. Generate a bootstrap sample of size n, $\{x_1^*,\ldots,x_n^*\}$ from $\{x_1,\ldots,x_n\}$, and generate a bootstrap sample of size m, $\{y_1^*,\ldots,y_m^*\}$ from $\{y_1,\ldots,y_m\}$. Based on $\{x_1^*,\ldots,x_n^*\}$ and $\{y_1^*,\ldots,y_m^*\}$, compute the bootstrap estimate of R say $\widehat{R}^*$ using (11).
2. Repeat 1 NBOOT times.
3. Let $H_1(x) = P(\widehat{R}^* \leq x)$ be the cumulative distribution function of $\widehat{R}^*$. Define $\widehat{R}_{Bp}(x) = H_1^{-1}(x)$ for a given x. The approximate $100(1-\gamma)\%$ confidence interval of R is given by

$$\left(\widehat{R}_{Bp}(\frac{\gamma}{2})\ ,\ \widehat{R}_{Bp}(1-\frac{\gamma}{2})\right)$$

(ii) Boot-t method

1. From the sample $\{x_1,\ldots,x_n\}$ and $\{y_1,\ldots,y_m\}$, compute $\widehat{R}$.
2. Same as in Boot-p method, first generate bootstrap sample $\{x_1^*,\ldots,x_n^*\}$, $\{y_1^*,\ldots,y_m^*\}$ and then compute $\widehat{R}^*$, the bootstrap estimate of R. Also, compute the statistic

$$T^* = \frac{\sqrt{m}(\widehat{R}^* - \widehat{R})}{\sqrt{Var(\widehat{R}^*)}}$$

   Compute $Var(\widehat{R}^*)$ using Theorem 2.
3. Repeat 1 and 2 NBOOT times.
4. Let $H_2(x) = P(T^* \leq x)$ be the cumulative distribution function of T^*. For a given x define $\widehat{R}_{Bt}(x) = \widehat{R} + H_2^{-1}(x)\sqrt{\frac{Var(\widehat{R})}{m}}$. The approximate $100(1-\gamma)\%$ C.I. of R is given by

$$\left(\widehat{R}_{Bt}(\frac{\gamma}{2})\ ,\ \widehat{R}_{Bt}(1-\frac{\gamma}{2})\right).$$

4. Bayes estimation of R

In this section, we obtain the Bayes estimation of R under assumption that the shape parameter α and scale parameters θ_1 and θ_2 are random variables. Following the approach of Berger and Sun (1993), it is assumed that the prior density of θ_j is the inverse gamma $IG(a_j,b_j)$, $j=1,2$ with density function

$$\pi_j(\theta_j)=\pi(\theta_j|a_j,b_j)=e^{-\frac{b_j}{\theta_j}}\frac{\theta_j^{-a_j-1}b_j^{a_j}}{\Gamma(a_j)},$$

and α has the gamma $G(a_3,b_3)$ with density function

$$\pi_3(\alpha)=\pi(\alpha|a_3,b_3)=e^{-b_3\alpha}\frac{\alpha^{a_3-1}b_3^{a_3}}{\Gamma(a_3)}.$$

Moreover, it is assumed that θ_1, θ_2 and α are independent. Therefore the joint posterior density of α, θ_1 and θ_2 given the data is

$$L(\alpha,\theta_1,\theta_2|\text{data})=\frac{L(\text{data}|\alpha,\theta_1,\theta_2)\ \pi(\alpha)\ \pi_1(\theta_1)\ \pi_2(\theta_2)}{\int_0^\infty\int_0^\infty\int_0^\infty L(\text{data}|\alpha,\theta_1,\theta_2)\pi_1(\theta_1)\ \pi_2(\theta_2)\ \pi(\alpha)\ d\alpha\ d\theta_1 d\theta_2}. \quad (21)$$

From (21), it is obvious that the form of the posterior density function will not be tractable and the computation of its respective Bayes estimate will not be analytically obtained. Consequently, we opt for stochastic simulation procedures, namely, the Gibbs and Metropolis samplers (Gilks et al., 1995) to generate samples from the posterior distributions and then compute the Bayes estimate of R and the corresponding credible interval of R. The posterior pdfs of θ_1 and θ_2 are as follows:

$$\theta_1|\alpha,\theta_2,\text{data}\sim IG\left(n+a_1,b_1+\sum_{i=1}^{n}(r_i+1)x_i^\alpha\right),$$

$$\theta_2|\alpha,\theta_1,\text{data}\sim IG\left(m+a_2,b_2+\sum_{j=1}^{m}(r'_j+1)y_j^\alpha\right),$$

and

$$f(\alpha|\theta_1,\theta_2,\text{data})\propto\alpha^{n+m+a_3-1}\prod_{i=1}^{n}x_i^{\alpha-1}\prod_{j=1}^{m}y_j^{\alpha-1}$$
$$\times\exp\left\{-b_3\alpha-\frac{1}{\theta_1}\sum_{i=1}^{n}(r_i+1)x_i^\alpha-\frac{1}{\theta_2}\sum_{j=1}^{m}(r'_j+1)y_j^\alpha\right\}.$$

The posterior pdf of α is not known, but the plot of its show that it is similar to normal distribution. So to generate random numbers from this distributions, we use the Metropolis-Hastings method with normal proposal distribution. Therefore the algorithm of Gibbs sampling is as follows:

1. Start with an initial guess $(\alpha^{(0)}, \theta_1^{(0)}, \theta_2^{(0)})$.
2. Set t = 1.
3. Using Metropolis-Hastings, generate $\alpha^{(t)}$ from $f(\alpha|\theta_1^{(t-1)}, \theta_2^{(t-1)}, \text{data})$ with the $N(\alpha^{(t-1)}, 1)$ proposal distribution.
4. Generate $\theta_1^{(t)}$ from $IG(n+a_1, b_1+\sum_{i=1}^{n}(r_i+1)x_i^{\alpha^{(t-1)}})$.
5. Generate $\theta_2^{(t)}$ from $IG(m+a_2, b_2+\sum_{j=1}^{m}(r'_j+1)y_j^{\alpha^{(t-1)}})$.
6. Compute $R^{(t)}$ from (3).
7. Set $t = t+1$.
8. Repeat steps 3-7, T times.

Now the approximate posterior mean, and posterior variance of R become

$$\widehat{E}(R|\text{data}) = \frac{1}{T}\sum_{t=1}^{T} R^{(t)},$$
$$\widehat{Var}(R|\text{data}) = \frac{1}{T}\sum_{t=1}^{T}\left(R^{(t)} - \widehat{E}(R|\text{data})\right)^2.$$

Based on T and R values, using the method proposed by Chen and Shao (1999), a $100(1-\gamma)\%$ HPD credible interval can be constructed as $\left(R_{[\frac{\gamma}{2}T]}, R_{[(1-\frac{\gamma}{2})T]}\right)$, where $R_{[\frac{\gamma}{2}T]}$ and $R_{[(1-\frac{\gamma}{2})T]}$ are the $[\frac{\gamma}{2}T]$-th smallest integer and the $[(1-\frac{\gamma}{2})T]$-th smallest integer of $\{R_t,\ t = 1,2,\ldots,T\}$, respectively.

Here we obtain the Bayes estimation of R under the assumptions that the scale parameters θ_1 and θ_2 are random variables and the shape parameter α is known. It is assumed that θ_1 and θ_2 have independent inverted Gamma priors with parameters (a_1, b_1) and (a_2, b_2), respectively. The posterior pdf's of θ_1 and θ_2 can be shown to be IGamma$(n+a_1, b_1+S_1(\mathbf{x}))$ and IGamma$(m+a_2, b_2+S_2(y))$ respectively. Since the priors θ_1 and θ_2 are independent, the posterior pdf of R becomes

$$f_R(z) = A\frac{z^{m+a_2-1}(1-z)^{n+a_1-1}}{[(b_1+S_1(\mathbf{x}))(1-z)+(b_2+S_2(\mathbf{y}))z]^{n+m+a_1+a_2}}, \qquad 0<z<1,$$

where $A = \frac{\Gamma(n+m+a_1+a_2)}{\Gamma(a_1+n)\Gamma(a_2+m)}[b_1+S_1(\mathbf{x})]^{n+a_1}[b_2+S_2(\mathbf{y})]^{m+a_2}$.

The Bayes estimate of R under the squares error loss function is the posterior mean

$$\widehat{R}_{BS} = \int_0^1 z f_R(z)dz. \tag{22}$$

Since the Bayes estimate of R under the squared erros loss can not be obtained analytically, we can obtain the approximate Bayes estimate of R by using the method of Lindley (1980). It can be shown that the approximate Bayes estimate of R, say $\widehat{R}_{ABS}$, under squared error loss function is

$$\widehat{R}_{ABS} = \tilde{R}\left[1 + \frac{\tilde{\theta}_2\tilde{R}^2}{\tilde{\theta}_1^2} \times \frac{(n+a_1-1)\tilde{\theta}_2 - (m+a_2-2)\tilde{\theta}_1}{(n+a_1-1)(m+b_2-1)}\right], \tag{23}$$

where $\tilde{\theta}_1 = \frac{n+a_1-1}{b_1+S_1(\mathbf{x})}$, $\tilde{\theta}_2 = \frac{m+a_2-1}{b_2+S_2(\mathbf{y})}$ and $\tilde{R} = \frac{\tilde{\theta}_1}{\tilde{\theta}_1+\tilde{\theta}_2}$.

The $100(1-\gamma)\%$ Bayesian interval for R is given by (L,U), where L and U are the lower and upper bounds, respectively, satisfying

$$P[R \le L|\text{data}] = \frac{\gamma}{2}, \quad \text{and} \quad P[R \le U|\text{data}] = 1 - \frac{\gamma}{2}.$$

By using some numerical integration methods, we can obtain Bayesian interval estimation of R.

5. Data analysis and comparison study

In this section, a Monte Carlo simulation study and a real data set are presented to illustrate all the estimation methods described in the preceding sections. All the computations are performed using Visual Maple (V12) package. The codes themselves are available from the authors.

5.1. Numerical comparison study

In this subsection, we present some results based on Monte Carlo simulations to compare the performance of the different methods for different censoring schemes, and for different parameter values. We compare the performances of the MLE, AMLE, and the Bayes estimates (with respect to the squared error loss function) in terms of biases, and mean squares errors (MSE). We also compare different C.I.'s, namely the C.I.'s obtained by using asymptotic distributions of the MLE and AMLE, bootstrap C.I.'s and the HPD credible intervals in terms of the average confidence lengths, and coverage percentages. We use different parameter values, different hyper parameters and different sampling schemes. We used three sets of parameter values $(\alpha = 0.5, \theta_1 = 1, \theta_2 = 1)$, $(\alpha = 1.5, \theta_1 = 1, \theta_2 = 1)$ and $(\alpha = 2.5, \theta_1 = 1, \theta_2 = 1)$ mainly to compare the MLEs and different Bayes estimators. For computing the Bayes estimators and HPD credible intervals, we assume 3 priors as follows:

Prior 1:	$a_j = 0,$	$b_j = 0,$	$j = 1,2,3,$
Prior 2:	$a_j = 1,$	$b_j = 2,$	$j = 1,2,3,$
Prior 3:	$a_j = 2,$	$b_j = 3,$	$j = 1,2,3.$

Prior 1 is the non-informative gamma prior for both the shape and scale parameters. Priors 2 and 3 are informative gamma priors. We also use three censoring schemes as given in Table 1.

Table 1: *Censoring schemes.*

	(n,N)	C. S.
r_1	(10, 30)	(0,0,0,0,0,0,0,0,0,20)
r_2	(10, 30)	(20,0,0,0,0,0,0,0,0,0)
r_3	(10, 30)	(2,2,2,2,2,2,2,2,2,2)

For different parameter values, different censoring schemes and different priors, we report the average biases, and MSE of the MLE, AMLE, and Bayes estimates of R over 1000 replications. The results are reported in Table 2. In our simulation experiments for both the bootstrap methods, we have computed the confidence intervals based on 250 re-sampling. The Bayes estimates and the corresponding credible intervals are based on 1000 sampling, namely $T = 1000$.

From Table 2, we observe that the MLE and AMLE compare very well with the Bayes estimator in terms of biases and MSEs. We also observe that the MSE, and biases of the MLE, and AMLE are very close. Comparing the two Bayes estimators based on two informative gamma priors clearly shows that the Bayes estimators based on prior 3 perform better than the Bayes estimators based on prior 2, in terms of both biases and MSEs. The Bayes estimators based on both priors perform better than the ones obtained using the noninformative prior 1.

We also computed the 95% C.I.'s for R based on the asymptotic distributions of the MLE and AMLE. We further compute Boot-p, and Boot-t C.I.'s, and the HPD credible intervals. In Table 3, we presented the average confidence credible lengths, and the corresponding coverage percentages. The nominal level for the C.I.'s or the credible intervals is 0.95 in each case. From Table 3, we observe that the bootstrap C.I.'s are wider than the other C.I.'s. We also observe that the HPD intervals provide the smallest average confidence credible lengths for different censoring schemes, and for different parameter values. The asymptotic C.I.'s MLE and AMLE are the second best CIs. It is also observed that Boot-p C.I.'s perform better than the Boot-t C.I.'s. From Table 3, it is evident that the the Boot-t credible intervals provide the most coverage probabilities in most cases considered.

Table 2: Biases and MSE of the MLE, AMLE, and Bayes estimates of R.

$(\alpha,\theta_1,\theta_2)$	C.S.		MLE	AMLE	BS		
					prior 1	prior 2	prior 3
(0.5 , 1 , 1)	$(r_1\ ,\ r_1)$	Bias	−0.051	−0.053	−0.064	−0.60	−0.055
		MSE	0.019	0.020	0.022	0.021	0.020
	$(r_1\ ,\ r_2)$	Bias	−0.046	−0.050	−0.060	−0.056	−0.051
		MSE	0.012	0.014	0.017	0.016	0.015
	$(r_1\ ,\ r_3)$	Bias	−0.032	−0.035	−0.047	−0.043	−0.039
		MSE	0.020	0.021	0.026	0.024	0.023
	$(r_2\ ,\ r_2)$	Bias	−0.015	−0.022	−0.024	−0.023	−0.021
		MSE	0.011	0.013	0.016	0.016	0.015
	$(r_2\ ,\ r_3)$	Bias	−0.013	−0.015	−0.021	−0.019	−0.018
		MSE	0.013	0.014	0.017	0.016	0.016
	$(r_3\ ,\ r_3)$	Bias	−0.031	−0.036	−0.047	−0.043	−0.038
		MSE	0.012	0.014	0.018	0.017	0.015
(1.5 , 1 , 1)	$(r_1\ ,\ r_1)$	Bias	−0.046	−0.048	−0.060	−0.057	−0.053
		MSE	0.021	0.021	0.025	0.024	0.023
	$(r_1\ ,\ r_2)$	Bias	−0.034	−0.039	−0.050	−0.047	−0.045
		MSE	0.017	0.018	0.023	0.022	0.021
	$(r_1\ ,\ r_3)$	Bias	−0.027	−0.028	−0.034	−0.031	−0.030
		MSE	0.013	0.014	0.018	0.016	0.015
	$(r_2\ ,\ r_2)$	Bias	−0.017	−0.023	−0.034	−0.031	−0.029
		MSE	0.010	0.011	0.017	0.016	0.014
	$(r_2\ ,\ r_3)$	Bias	−0.013	−0.017	−0.031	−0.025	−0.021
		MSE	0.008	0.009	0.015	0.012	0.011
	$(r_3\ ,\ r_3)$	Bias	−0.035	−0.039	−0.049	−0.047	−0.043
		MSE	0.021	0.023	0.024	0.024	0.022
(2.5 , 1 , 1)	$(r_1\ ,\ r_1)$	Bias	−0.029	−0.032	−0.036	−0.034	−0.032
		MSE	0.020	0.022	0.026	0.025	0.023
	$(r_1\ ,\ r_2)$	Bias	−0.022	−0.027	−0.039	−0.037	−0.034
		MSE	0.011	0.013	0.018	0.018	0.016
	$(r_1\ ,\ r_3)$	Bias	−0.017	−0.019	−0.025	−0.024	−0.022
		MSE	0.007	0.008	0.010	0.010	0.009
	$(r_2\ ,\ r_2)$	Bias	−0.016	−0.018	−0.024	−0.023	−0.022
		MSE	0.008	0.009	0.012	0.011	0.011
	$(r_2\ ,\ r_3)$	Bias	−0.014	−0.017	−0.025	−0.021	−0.020
		MSE	0.012	0.015	0.020	0.018	0.017
	$(r_3\ ,\ r_3)$	Bias	−0.033	−0.038	−0.050	−0.047	−0.045
		MSE	0.017	0.018	0.025	0.024	0.022

Table 3: Average confidence/credible length and coverage percentage for estimators of R.

$(\alpha = 0.5, \quad \theta_1 = 1, \quad \theta_2 = 1)$							
C.S.	ML	AML	Boot-p	Boot-t	BS		
					prior 1	prior 2	prior 3
$(r_1\ ,\ r_1)$	0.376(0.925)	0.378(0.926)	0.389(0.937)	0.396(0.951)	0.351(0.950)	0.346(0.950)	0.337(0.947)
$(r_1\ ,\ r_2)$	0.373(0.922)	0.376(0.924)	0.386(0.942)	0.391(0.946)	0.349(0.945)	0.341(0.942)	0.335(0.942)
$(r_1\ ,\ r_3)$	0.361(0.924)	0.365(0.927)	0.372(0.947)	0.382(0.959)	0.342(0.945)	0.338(0.943)	0.334(0.942)
$(r_2\ ,\ r_2)$	0.354(0.947)	0.356(0.951)	0.368(0.948)	0.375(0.957)	0.337(0.952)	0.329(0.951)	0.325(0.948)
$(r_2\ ,\ r_3)$	0.347(0.932)	0.350(0.934)	0.361(0.939)	0.366(0.950)	0.331(0.949)	0.326(0.947)	0.321(0.944)
$(r_3\ ,\ r_3)$	0.377(0.930)	0.380(0.932)	0.389(0.932)	0.391(0.949)	0.345(0.936)	0.342(0.934)	0.338(0.934)
$(\alpha = 1.5, \quad \theta_1 = 1, \quad \theta_2 = 1)$							
C.S.	ML	AML	Boot-p	Boot-t	BS		
					prior 1	prior 2	prior 3
$(r_1\ ,\ r_1)$	0.333(0.941)	0.336(0.943)	0.345(0.947)	0.352(0.954)	0.321(0.948)	0.313(0.945)	0.304(0.947)
$(r_1\ ,\ r_2)$	0.317(0.943)	0.319(0.946)	0.327(0.946)	0.340(0.953)	0.311(0.951)	0.305(0.950)	0.301(0.948)
$(r_1\ ,\ r_3)$	0.308(0.946)	0.311(0.949)	0.320(0.951)	0.334(0.953)	0.301(0.957)	0.295(0.953)	0.285(0.951)
$(r_2\ ,\ r_2)$	0.299(0.940)	0.304(0.944)	0.312(0.944)	0.323(0.951)	0.288(0.954)	0.280(0.952)	0.277(0.950)
$(r_2\ ,\ r_3)$	0.290(0.939)	0.296(0.937)	0.306(0.946)	0.317(0.951)	0.282(0.953)	0.274(0.949)	0.272(0.948)
$(r_3\ ,\ r_3)$	0.322(0.940)	0.325(0.946)	0.332(0.949)	0.340(0.956)	0.313(0.946)	0.307(0.948)	0.303(0.942)
$(\alpha = 2.5, \quad \theta_1 = 1, \quad \theta_2 = 1)$							
C.S.	ML	AML	Boot-p	Boot-t	BS		
					prior 1	prior 2	prior 3
$(r_1\ ,\ r_1)$	0.255(0.935)	0.260(0.938)	0.272(0.949)	0.285(0.951)	0.247(0.947)	0.243(0.942)	0.236(0.946)
$(r_1\ ,\ r_2)$	0.243(0.941)	0.247(0.943)	0.254(0.944)	0.265(0.953)	0.238(0.947)	0.233(0.948)	0.226(0.946)
$(r_1\ ,\ r_3)$	0.221(0.946)	0.225(0.945)	0.237(0.947)	0.249(0.949)	0.215(0.945)	0.211(0.944)	0.205(0.948)
$(r_2\ ,\ r_2)$	0.208(0.947)	0.211(0.951)	0.218(0.948)	0.234(0.953)	0.204(0.951)	0.197(0.953)	0.189(0.950)
$(r_2\ ,\ r_3)$	0.190(0.941)	0.192(0.945)	0.203(0.946)	0.211(0.954)	0.183(0.943)	0.179(0.946)	0.175(0.948)
$(r_3\ ,\ r_3)$	0.244(0.940)	0.247(0.941)	0.255(0.942)	0.268(0.945)	0.241(0.943)	0.239(0.944)	0.232(0.946)

Table 4: Biases and MSE of the MLE and Bayes estimators of R and average confidence length and coverage percentage when α is known and $\theta_1 = \theta_2 = 1$.

	C.S		MLE	BS	ABS	Exact con.	
$\alpha = 1$ (exponential case)	(r_1, r_1)	Bias	−0.0016	−0.0015	−0.0026	Mean	0.405
		MSE	0.0122	0.0112	0.0157	Cov.Prob.	0.949
	(r_1, r_2)	Bias	−0.0014	−0.0013	−0.0023	Mean	0.405
		MSE	0.0123	0.0113	0.0158	Cov.Prob.	0.949
	(r_1, r_3)	Bias	0.0007	0.0007	0.0009	Mean	0.405
		MSE	0.0111	0.0101	0.0137	Cov.Prob.	0.957
	(r_2, r_2)	Bias	0.0028	0.0027	0.0061	Mean	0.405
		MSE	0.0122	0.0112	0.0189	Cov.Prob.	0.944
	(r_2, r_3)	Bias	−0.0037	−0.0035	−0.0059	Mean	0.407
		MSE	0.0112	0.0103	0.0181	Cov.Prob.	0.958
	(r_3, r_3)	Bias	−0.0036	−0.0034	−0.0049	Mean	0.405
		MSE	0.0122	0.0112	0.0149	Cov.Prob.	0.950
$\alpha = 2$ (Rayleigh case)	(r_1, r_1)	Bias	0.0029	0.0027	0.0069	Mean	0.405
		MSE	0.0126	0.0116	0.0173	Cov.Prob.	0.947
	(r_1, r_2)	Bias	0.0016	0.0015	0.0028	Mean	0.406
		MSE	0.0113	0.0104	0.0141	Cov.Prob.	0.953
	(r_1, r_3)	Bias	−0.0003	−0.0003	−0.0017	Mean	0.404
		MSE	0.0110	0.0101	0.0128	Cov.Prob.	0.948
	(r_2, r_2)	Bias	−0.0024	−0.0023	−0.0038	Mean	0.406
		MSE	0.0114	0.0105	0.0133	Cov.Prob.	0.950
	(r_2, r_3)	Bias	0.0008	0.0007	0.0019	Mean	0.406
		MSE	0.0115	0.0106	0.0166	Cov.Prob.	0.948
	(r_3, r_3)	Bias	0.0047	0.0045	0.0086	Mean	0.405
		MSE	0.0124	0.0114	0.0167	Cov.Prob.	0.944
$\alpha = 2.5$	(r_1, r_1)	Bias	−0.0039	−0.0037	−0.0080	Mean	0.404
		MSE	0.0128	0.0117	0.0193	Cov.Prob.	0.945
	(r_1, r_2)	Bias	−0.0029	−0.0027	−0.0045	Mean	0.405
		MSE	0.0119	0.0110	0.0132	Cov.Prob.	0.948
	(r_1, r_3)	Bias	0.0002	0.0002	0.016	Mean	0.404
		MSE	0.0110	0.0101	0.0129	Cov.Prob.	0.946
	(r_2, r_2)	Bias	0.0059	0.0057	0.0092	Mean	0.407
		MSE	0.0118	0.0109	0.0153	Cov.Prob.	0.956
	(r_2, r_3)	Bias	0.0021	0.0020	0.0031	Mean	0.405
		MSE	0.0123	0.0112	0.0153	Cov.Prob.	0.944
	(r_3, r_3)	Bias	0.0024	0.0023	0.0055	Mean	0.405
		MSE	0.0122	0.0112	0.0152	Cov.Prob.	0.941

Now let us consider the case when the common shape parameter α is known. In this case, we obtain the MLE of R using (20). Since we do not have any prior information on R, we prefer to use the non-informative prior i.e $a_1 = b_1 = a_2 = b_2 = 0$ to compute Bayes estimates. Under the same prior distributions, we compute Bayes estimates and approximate Bayes estimates of R using (22) and (23), respectively. We report the average biases and MSEs based on 2000 replications. The results are reported in Table 4. From Table 4, comparing the MLE, Bayes and approximate Bayes estimators, we observe that Bayes estimators provides the smallest biases and MSE's. The MLE's are the best second estimators. Comparing different censoring schemes, we observe that the scheme (r_1, r_3) provides the smallest biases and MSEs.

5.2. Example (real data set)

Here we present a data analysis of the strength data reported by Badar and Priest (1982). This data, represent the strength measured in GPA for single carbon fibers, and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 20mm (Data Set 1) and 10mm (Data Set 2). These data have been used previously by Raqab and Kundu (2005), Kundu and Gupta (2006) and Kundu and Raqab (2009). The data are presented in Tables 5 and 6.

Table 5: Data Set 1 (gauge lengths of 20 mm).

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958
1.966	1.997	2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179
2.224	2.240	2.253	2.270	2.272	2.274	2.301	2.301	2.359	2.382
2.382	2.426	2.434	2.435	2.478	2.490	2.511	2.514	2.535	2.554
2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684	2.697	2.726
2.770	2.773	2.800	2.809	2.818	2.821	2.848	2.880	2.954	3.012
3.067	3.084	3.090	3.096	3.128	3.233	3.433	3.585	3.585	

Table 6: Data Set 2 (gauge lengths of 10 mm).

1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397	2.445
2.454	2.474	2.518	2.522	2.525	2.532	2.575	2.614	2.616	2.618
2.624	2.659	2.675	2.738	2.740	2.856	2.917	2.928	2.937	2.937
2.977	2.996	3.030	3.125	3.139	3.145	3.220	3.223	3.235	3.243
3.264	3.272	3.294	3.332	3.346	3.377	3.408	3.435	3.493	3.501
3.537	3.554	3.562	3.628	3.852	3.871	3.886	3.971	4.024	4.027
4.225	4.395	5.020							

Kundu and Gupta (2006) analyzed these data sets using two-parameter Weibull distribution after subtracting 0.75 from both these data sets. After subtracting 0.75 from all the points of these data sets, Kundu and Gupta (2006) observed that the Weibull distributions with equal shape parameters fit to both these data sets. We have generated two different progressively censored samples using two different sampling schemes from above data sets in Tables 5 and 6. The generated data and corresponding censored

Table 7: Data and the corresponding censored schemes.

i, j	1	2	3	4	5	6	7	8	9	10
x_i	1.312	1.479	1.552	1.803	1.944	1.858	1.966	2.027	2.055	2.098
r_i	1	0	1	2	0	0	3	0	1	50
y_j	1.901	2.132	2.257	2.361	2.396	2.445	2.373	2.525	2.532	2.575
r'_j	0	2	1	0	1	1	2	0	0	44

schemes have been presented in Table 7. The ML, AML and Bayes estimations of R become 0.176, 0.179 and 0.328; and the corresponding 95% C.I.'s become (0.069, 0.283), (0.076, 0.284) and (0.097, 0.527) respectively. We also obtain the 95% Boot-p and Boot-t confidence intervals as (0.064, 0.310) and (0.063, 0.342) respectively.

6. Some concluding remarks

Based on progressively censored samples, this paper considers estimation of $R = P(Y < X)$ by different methods when X and Y are two independent Weibull distributions with different scale parameters, but having the same shape parameter. It is observed that the MLE of R can be obtained using an iterative procedure. The proposed AMLE of R can be obtained explicitly. It is observe that the MSE, and biases of the MLE, and AMLE are very close. The Bayes estimate of R, and the corresponding credible interval can be obtained using the Gibbs sampling technique. It is also observe that the MLE and AMLE compare very well with the Bayes estimator in terms of biases and MSEs. Note that the results for exponential and Rayleigh distributions can be obtained as special cases with different scale parameters.

Appendix

Proof of Theorem 2

On using Theorem 1 and applying delta method, we can describe the asymptotic distribution of $\widehat{R} = g(\widehat{\alpha}, \widehat{\theta}_1, \widehat{\theta}_2)$, where $g(\alpha, \theta_1, \theta_2) = \theta_1/(\theta_1 + \theta_2)$ as the following:

$$\sqrt{m}(\widehat{R} - R) \xrightarrow{D} N(0, B),$$

where $B = \mathbf{b}'\mathbf{A}^{-1}\mathbf{b}$, with

$$\mathbf{b} = \begin{pmatrix} \frac{\partial g}{\partial \alpha} \\ \frac{\partial g}{\partial \theta_1} \\ \frac{\partial g}{\partial \theta_2} \end{pmatrix} = \frac{1}{(\theta_1+\theta_2)^2} \begin{pmatrix} 0 \\ \theta_2 \\ -\theta_1 \end{pmatrix},$$

$$\mathbf{A}^{-1} = \frac{1}{u} \begin{pmatrix} a_{22}a_{33} & -a_{12}a_{33} & -a_{22}a_{13} \\ -a_{21}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{21}a_{13} \\ -a_{22}a_{31} & a_{21}a_{13} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}.$$

and $u = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22}$. Therefore

$$B = \mathbf{b}'\mathbf{A}^{-1}\mathbf{b} = \frac{1}{u(\theta_1+\theta_2)^4}\left[(a_{11}a_{22} - a_{12}^2)\theta_1^2 - 2a_{12}a_{13}\theta_1\theta_2 + (a_{11}a_{33} - a_{13}^2)\theta_2^2\right].$$

The proof is thus obtained.

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