SORT 38 (2) July-December 2014, 161-180

Estimators for the parameter mean of Morgenstern type bivariate generalized exponential distribution using ranked set sampling

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Abstract

In situations where the sampling units in a study can be more easily ranked based on the measurement of an auxiliary variable, ranked set sampling provides unbiased estimators for the mean of a population that are more efficient than unbiased estimators based on simple random sampling. In this paper, we consider the Morgenstern type bivariate generalized exponential distribution and obtain several unbiased estimators for the mean parameter of its marginal distribution, based on different ranked set sampling schemes. The efficiency of all considered estimators are evaluated and several numerical illustrations are given.

MSC: 62D05; 62F07;62G30.

Keywords: Concomitants of order statistics, Morgenstern type bivariate generalized exponential distribution, ranked set sampling.

1. Introduction

Ranked set sampling (RSS) was first suggested by McIntyre (1952) for estimating mean pasture and forage yields. RSS is applicable whenever ranking of a set of sampling units can be done easily by a judgement method with respect to the variable of interest. Later, Takahasi and Wakimoto (1968) provided the statistical foundation and necessary mathematical properties of the method. They indicated that in situations where the sampling units in a study can be more easily ranked based on the measurement of an auxiliary

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Received: August 2013

Accepted: May 2014

variable, RSS provides unbiased estimators for the mean of a population, and these estimators are more efficient than unbiased estimators based on simple random sampling (SRS).

The RSS technique is composed of two stages in a sample selection procedure: At the first stage, n simple random samples of size n are drawn from a population and each sample is called a set. Then, each of the units are ranked from the smallest to the largest according to a variable of interest, say Y, in each set based on a low-level measurement such as a concomitant variable or previous experience. At the second stage, the first unit from the first set, the second unit from the second set and going on like this till the nth unit from the nth set are taken and measured according to the variable Y. The obtained sample is called a RSS. It can be noted that the units of this sample are independent order statistics but not identically distributed. The reader can refer to the book of Chen et al. (2004) for details of RSS and its applications.

Other schemes and modifications of RSS were investigated in the literature: A modified RSS procedure is introduced by Stokes (1980) and only the largest or the smallest judgment ranked unit is chosen for quantification in each set. In estimating the population mean, Samawi et al. (1996) suggested the extreme ranked set sampling (ERSS), Muttlak (1997) suggested the median RSS, Jemain and Al-Omari (2006) suggested double quartile ranked set samples, and Al-Odat and Al-Saleh (2001) suggested moving extreme ranked set sampling (MERSS). Yu and Tam (2002) considered the problem of estimating the mean of a population based on RSS with censored data. Al-Saleh and Al-Kadiri (2000) considered double RSS (DRSS), and Al-Saleh and Al-Omari (2002) generalized the DRSS to the multistage ranked set sampling (MSRSS) method. For the mean normal or exponential, Sinha et al. (1996) used median ranked set sampling (MRSS) to modify the RSS estimators Muttlak (2003) introduced percentile ranked set sampling (PRSS). Al-Nasser (2007) proposed a generalized robust sampling method called L ranked set sampling (LRSS) and showed that the estimator for the mean based on the LRSS is unbiased if the underlying distribution is symmetric. A robust extreme ranked set sampling (RERSS) is proposed by Al-Nasser and Mustafa (2009) for estimating the population mean.

RSS and its modifications are applied for estimating a parameter in a bivariate population (X,Y), where Y is the variable of interest and X is a concomitant variable that is not of direct interest but is relatively easy to measure or to order by judgment: Stokes (1977) studied RSS with concomitant variables. Barnett and Moore (1997) derived the best linear unbiased estimator (BLUE) for the mean of Y, based on a ranked set sample obtained using an auxiliary variable X. Al-Saleh and Al-Ananbeh (2007) estimated the means of the bivariate normal distribution using moving extremes RSS. Chacko and Thomas (2008) and Al-Saleh and Diab (2009) considered estimation of a parameter of Morgenstern type bivariate exponential distribution and Downton's bivariate exponential distribution, respectively. Tahmasebi and Jafari (2012) assumed the Morgenstern type bivariate uniform distribution and obtained several estimators for a scale parameter.

The distribution function of a Morgenstern type bivariate generalized exponential distribution (MTBGED) is defined as

$$F_{X,Y}(x,y) = (1-e^{-\theta_1 x})^{\alpha_1}(1-e^{-\theta_2 y})^{\alpha_2}[1+\lambda(1-(1-e^{-\theta_1 x})^{\alpha_1})(1-(1-e^{-\theta_2 y})^{\alpha_2})], \quad (1)$$
$$x,y>0,\ -1\le\lambda\le 1,\ \alpha_1,\alpha_2,\theta_1,\theta_2>0,$$

with the corresponding probability density function (pdf)

$$\begin{aligned} f_{X,Y}(x,y) &= \alpha_1\alpha_2\theta_1\theta_2 e^{-\theta_1 x-\theta_2 y}(1-e^{-\theta_1 x})^{\alpha_1-1}(1-e^{-\theta_2 y})^{\alpha_2-1} \\ &\quad \times\left\{1+\lambda[2(1-e^{-\theta_1 x})^{\alpha_1}-1][2(1-e^{-\theta_2 y})^{\alpha_2}-1]\right\}. \end{aligned} \quad (2)$$

Note that when (X,Y) has MTBGED, the marginal distribution of X and Y is the generalized exponential distribution with the expected values

$$\mu_x = \frac{B(\alpha_1)}{\theta_1}, \qquad \mu_y = \frac{B(\alpha_2)}{\theta_2},$$

respectively, where $B(\alpha)=\psi(\alpha+1)-\psi(1)$ and $\psi(.)$ is the digamma function. Also, the correlation coefficient between X and Y is obtained as (see Tahmasebi and Jafari, 2013)

$$\rho = \frac{\lambda D(\alpha_1)D(\alpha_2)}{\sqrt{C(\alpha_1)C(\alpha_2)}} = \lambda g(\alpha_1)g(\alpha_2), \quad (3)$$

where $D(\alpha)=B(2\alpha)-B(\alpha)$, $C(\alpha)=\psi'(1)-\psi'(\alpha+1)$, $\psi'(.)$ is the derivative of the digamma function, and $g(\alpha)=\frac{D(\alpha)}{\sqrt{C(\alpha)}}$.

In this paper, we consider estimation of the parameter μ_y when α_2 is known, and propose several estimator based on the RSS idea. Also, we suggest some improved version of these estimators. In Section 2, we present unbiased estimators for the parameter μ_y in MTBGED based on the RSS, LRSS, ERSS, MERSS, and MSRSS methods. We evaluate the efficiency of all considered estimators in Section 3.

2. Unbiased estimators for μ_y based on different RSS schemes

Suppose that the random variable (X,Y) has a MTBGED as defined in (1). In this section, we find unbiased estimators for the parameter μ_y based on different sampling schemes. In each case, first the general pattern of sampling is presented, and then an unbiased estimator with its variance is given for the parameter μ_y. Also, the efficiency of the proposed estimators are obtained.

2.1. RSS estimation

The procedure of RSS is described by Stokes (1977) for a bivariate random variable by the following steps:

Step 1. Randomly select n independent bivariate samples, each of size n.

Step 2. Rank the units within each sample with respect to variable X together with the Y variate associated.

Step 3. In the rth sample of size n, select the unit $(X_{(r)r}, Y_{[r]r})$, $r = 1, 2, \ldots, n$, where $X_{(r)r}$ is the measured observation on the variable X in the rth unit and $Y_{[r]r}$ is the corresponding measurement made on the study variable Y of the same unit.

Therefore, $Y_{[r]r}$, $r = 1, 2, 3, \ldots, n$, are the RSS observations made on the units of the RSS regarding the study variable Y which is correlated with the auxiliary variable X. Therefore, clearly $Y_{[r]r}$ is the concomitant of rth order statistic arising from the rth sample.

From Scaria and Nair (1999) the pdf of $Y_{[r]r}$ for $1 \leq r \leq n$ is given by

$$h_{[r]r}(y) = \alpha_2 \theta_2 e^{-\theta_2 y}(1 - e^{-\theta_2 y})^{\alpha_2 - 1}[1 + \delta_r(1 - 2(1 - e^{-\theta_2 y})^{\alpha_2})], \qquad 1 \leq r \leq n, \tag{4}$$

where $\delta_r = \frac{\lambda(n-2r+1)}{n+1}$ and its mean and variance of $Y_{[r]r}$ are obtained by Tahmasebi and Jafari (2013) as

$$E[Y_{[r]r}] = \frac{1}{\theta_2}[B(\alpha_2) - \delta_r D(\alpha_2)], \quad Var[Y_{[r]r}] = \frac{1}{\theta_2^2}[C(\alpha_2) + \delta_r(C(2\alpha_2) - C(\alpha_2))]. \tag{5}$$

Since $Y_{[r]r}$ and $Y_{[s]s}$ for $r \neq s$ are drawn from two independent samples, so we have

$$Cov(Y_{[r]r}, Y_{[s]s}) = 0, \quad r \neq s.$$

Theorem 1 *Based on the RSS procedure, an unbiased estimator for μ_y is given by*

$$\hat{\mu}_{RSS} = \frac{1}{n}\sum_{r=1}^{n} Y_{[r]r},$$

and its variance is

$$Var(\hat{\mu}_{RSS}) = \frac{C(\alpha_2)}{n\theta_2^2}. \tag{6}$$

Proof. Since $\sum_{r=1}^{n}\delta_r = \sum_{r=1}^{n}\frac{\lambda(n-2r+1)}{n+1} = 0$, using (5)

$$E\left(\hat{\mu}_{\mathrm{RSS}}\right) = \frac{1}{n}\sum_{r=1}^{n} E\left(Y_{[r]r}\right) = \frac{1}{n\theta_2}\sum_{r=1}^{n}\left(B(\alpha_2) - \delta_r D(\alpha_2)\right) = \frac{B(\alpha_2)}{\theta_2} = \mu_y,$$

and

$$Var\left(\hat{\mu}_{\mathrm{RSS}}\right) = \frac{1}{n}\sum_{r=1}^{n} Var\left(Y_{[r]r}\right) = \frac{1}{n^2\theta_2^2}\sum_{r=1}^{n}\left[C\left(\alpha_2\right) + \delta_r\left(C\left(2\alpha_2\right) - C\left(\alpha_2\right)\right)\right]$$

$$= \frac{1}{n^2\theta_2^2}\sum_{r=1}^{n}\left[C\left(\alpha_2\right) + \delta_r\left(C\left(2\alpha_2\right) - C\left(\alpha_2\right)\right)\right] = \frac{C(\alpha_2)}{n\theta_2^2}. \qquad \square$$

Now, we study the efficiency of $\hat{\mu}_{\mathrm{RSS}}$ relative to the BLUE of μ_y, $\tilde{\mu}$, based on $Y_{[r]r}$, $r = 1,2,3,\ldots,n$, for MTBGED, when λ is known. From David and Nagaraja (2003, p. 185) the BLUE of μ_y is derived as

$$\tilde{\mu} = \sum_{r=1}^{n} a_r Y_{[r]r},$$

where

$$a_r = \frac{H(\alpha_2,r)}{W(\alpha_2,r)}\Big(\sum_{j=1}^{n}\frac{[H(\alpha_2,j)]^2}{W(\alpha_2,j)}\Big)^{-1}, \;\; r = 1,2,3,\ldots,n,$$

$H(\alpha_2,r) = 1 - \frac{\delta_r D(\alpha_2)}{B(\alpha_2)}$ and $W(\alpha_2,r) = C(\alpha_2) + \delta_r[C(2\alpha_2) - C(\alpha_2)]$. The variance of $\tilde{\mu}$ is

$$Var[\tilde{\mu}] = \frac{v_2}{\theta_2^2},$$

where $v_2 = \left(\sum\limits_{r=1}^{n}\frac{[H(\alpha_2,r)]^2}{W(\alpha_2,r)}\right)^{-1}$, and therefore, the relative efficiency of $\hat{\mu}_{\mathrm{RSS}}$ to $\tilde{\mu}$ is given by

$$e_1 = e(\tilde{\mu} \mid \hat{\mu}_{\mathrm{RSS}}) = \frac{C(\alpha_2)}{n}\sum_{r=1}^{n}\frac{[H(\alpha_2,r)]^2}{W(\alpha_2,r)}.$$

In Section 3, we calculate the relative efficiency of $\hat{\mu}_{\mathrm{RSS}}$ to $\tilde{\mu}$, e_1, for some values of the parameters and sample size.

Remark 2 *We know that the correlation coefficient between X and Y in MTBGED is $\lambda g(\alpha_1)g(\alpha_2)$. So when α_1 and α_2 are known, by using the sample correlation coefficient q of the RSS observations $(X_{(r)r}, Y_{[r]r})$, $r = 1,2,3,\ldots,n$ an estimator for λ is given by*

$$\hat{\lambda} = \begin{cases} -1 & q < -g(\alpha_1)g(\alpha_2) \\ \frac{q}{g(\alpha_1)g(\alpha_2)} & -g(\alpha_1)g(\alpha_2) \leq q \leq g(\alpha_1)g(\alpha_2) \\ 1 & g(\alpha_1)g(\alpha_2) < q \end{cases}$$

Sometimes, k units of observations are censored in the RSS schemes. Let $Y_{[m_r]m_r}$, $r = 1,2,\ldots,n-k$, be the ranked set sample observations on the study variable Y, which results from censoring and ranking on the auxiliary variable X. We can represent the ranked set sample observations on the study variate Y as $p_1Y_{[1]1},\ p_2Y_{[2]2},\ldots,p_nY_{[n]n}$, where $p_r = 0$ if the rth unit is censored, and $p_r = 1$ otherwise. Consider k units are censored. Hence $\sum_{r=1}^{n} p_r = n-k$. if we write $m_r, r = 1,2,\ldots,n-k$, as the integers such that $1 \leq m_1 < m_2 < \cdots < m_{n-k} \leq n$ and $p_{m_r} = 1$, then

$$E\left(\frac{\sum_{r=1}^{n} p_r Y_{[r]r}}{n-k}\right) = \frac{1}{\theta_2}\left(B(\alpha_2) - \frac{D(\alpha_2)}{n-k}\sum_{r=1}^{n-k}\delta_{m_r}\right),$$

Therefore, the ranked set sample mean in the censored case is not an unbiased estimator for μ_y. However, we can construct an unbiased estimator based on this expected value.

Theorem 3 *An unbiased estimator for μ_y based on the censored RSS is given by*

$$\hat{\mu}_{CRSS} = \frac{1}{w}\sum_{r=1}^{n-k} Y_{[m_r]m_r},$$

where $w = n-k+\left(1-\frac{B(2\alpha_2)}{B(\alpha_2)}\right)\sum_{r=1}^{n-k}\delta_{m_r}$, and its variance is

$$Var(\hat{\mu}_{CRSS}) = \frac{v_3}{\theta_2^2},$$

where $v_3 = \frac{1}{w^2}\sum_{r=1}^{n-k}[C(\alpha_2) + \delta_{m_r}(C(2\alpha_2) - C(\alpha_2))]$.

Proof

$$E(\hat{\mu}_{\mathrm{CRSS}}) = \frac{1}{w}\sum_{r=1}^{n-k} E(Y_{[m_r]m_r}) = \frac{\sum_{r=1}^{n-k}(B(\alpha_2) - \delta_{m_r}D(\alpha_2))}{(n-k-\frac{D(\alpha_2)}{B(\alpha_2)}\sum_{r=1}^{n-k}\delta_{m_r})\theta_2} = \frac{B(\alpha_2)}{\theta_2} = \mu_y,$$

and $Var(\hat{\mu}_{\mathrm{CRSS}})$ can be easily obtained from (5). □

2.2. LRSS estimation

Al-Nasser (2007) proposed a generalized robust sampling method called L ranked set sampling (LRSS) for estimating population mean. The procedure of LRSS with a concomitant variable is as follows:

Step 1. Randomly select n independent bivariate samples, each of size n.

Step 2. Rank the units within each sample with respect to variable X together with the Y variate associated.

Step 3. Select the LRSS coefficient, $k = [n\gamma]$, such that $0 \leq \gamma < .5$, where $[x]$ is the largest integer value less than or equal to x.

Step 4. For each of the first $k+1$ ranked samples of size n, select the unit $(X_{(k+1)r}, Y_{[k+1]r})$, $r = 1, 2, \ldots, k$.

Step 5. For each of the last $k+1$ ranked samples of size n, i.e., the $(n-k)$th to the nth ranked sample, select the unit $(X_{(n-k)r}, Y_{[n-k]r})$, $r = n-k+1, \ldots, n$.

Step 6. For $j = k+2, \ldots, n-k-1$, select the unit $(X_{(r)r}, Y_{[r]r})$, $r = k+1, \ldots, n-k$.

Note that this LRSS scheme leads to the RSS when $k = 0$, and to the traditional MRSS when $k = \left[\frac{n-1}{2}\right]$. Also, the PRSS could be considered as a special case of this scheme.

Theorem 4 *An unbiased estimator of μ_y in MTBGED based on LRSS scheme is given by*

$$\hat{\mu}_{LRSS} = \frac{1}{n}\left(\sum_{r=1}^{k} Y_{[k+1]r} + \sum_{r=k+1}^{n-k} Y_{[r]r} + \sum_{r=n-k+1}^{n} Y_{[n-k]r}\right),$$

with variance

$$Var(\hat{\mu}_{LRSS}) = Var(\hat{\mu}_{RSS}) = \frac{C(\alpha_2)}{n\theta_2^2}. \tag{7}$$

Proof. Since

$$\sum_{r=1}^{k} \delta_{k+1} = \frac{\lambda}{n+1}\sum_{r=1}^{k}(n-2(k+1)+1) = \frac{\lambda k}{n+1}(n-2k-1),$$
$$\sum_{r=1}^{k} \delta_{n-k} = \frac{\lambda}{n+1}\sum_{r=n-k+1}^{n}(n-2(n-k)+1) = \frac{\lambda k}{n+1}(-n+2k+1),$$
$$\sum_{r=k+1}^{n-k} \delta_{r} = \frac{\lambda}{n+1}\sum_{r=k+1}^{n-k}(n-2r+1) = 0,$$

we have

$$E\left(\hat{\mu}_{\text{LRSS}}\right) = \frac{1}{n}\left(\frac{kB\left(\alpha_2\right)}{\theta_2} - \frac{D\left(\alpha_2\right)}{\theta_2}\frac{\lambda k}{n+1}\left(n-2k-1\right) + \frac{kB\left(\alpha_2\right)}{\theta_2}\right.$$
$$\left. - \frac{D\left(\alpha_2\right)}{\theta_2}\frac{\lambda k}{n+1}\left(-n+2k+1\right) + \frac{(n-2k)B\left(\alpha_2\right)}{\theta_2}\right) = \frac{B\left(\alpha_2\right)}{\theta_2} = \mu_y,$$

and

$$Var\left(\hat{\mu}_{\text{LRSS}}\right) = \frac{1}{n^2}\left(\frac{kC\left(\alpha_2\right)}{\theta_2^2} - \frac{C(2\alpha_2) - C(\alpha_2)}{\theta_2}\frac{\lambda k}{n+1}\left(n-2k-1\right) + \frac{kC\left(\alpha_2\right)}{\theta_2^2}\right.$$
$$\left. - \frac{C(2\alpha_2) - C(\alpha_2)}{\theta_2^2}\frac{\lambda k}{n+1}\left(-n+2k+1\right) + \frac{(n-2k)C\left(\alpha_2\right)}{\theta_2^2}\right) = \frac{C\left(\alpha_2\right)}{n\theta_2^2}.$$

□

2.3. ERSS estimation

The extreme ranked set sampling (ERSS) method with concomitant variable, introduced by Samawi et al. (1996), can be described as follows:

Step 1. Select n random samples each of size n bivariate units from the population.

Step 2. If the sample size n is even, then select from $\frac{n}{2}$ samples the smallest ranked unit X together with the associated Y and from the other $\frac{n}{2}$ samples the largest ranked unit X together with the associated Y. These selected observations $(X_{(1)1}, Y_{[1]1})$, $(X_{(n)2}, Y_{[n]2}), (X_{(1)3},\ Y_{[1]3}), \ldots, (X_{(1)n-1}, Y_{[1]n-1}), (X_{(n)n}, Y_{[n]n})$ can be denoted by ERSS_1.

Step 3. If n is odd then select from $\frac{n-1}{2}$ samples the smallest ranked unit X together with the associated Y and from the other $\frac{n-1}{2}$ samples the largest ranked unit X together with the associated Y and from one sample the median of the sample for actual measurement. In this case the selected observations $(X_{(1)1}, Y_{[1]1}), (X_{(n)2}, Y_{[n]2})$, $(X_{(1)3}, Y_{[1]3}), \ldots, (X_{(n)n-1}, Y_{[n]n-1}), (\frac{X_{(1)n}+X_{(n)n}}{2}, \frac{Y_{[1]n}+Y_{[n]n}}{2})$ can be denoted ERSS_2 and $(X_{(1)1}, Y_{[1]1}), (X_{(n)2}, Y_{[n]2}), (X_{(1)3}, Y_{[1]3}), \ldots, (X_{(n)n-1}, Y_{[n]n-1}), (X_{(\frac{n+1}{2})n}, Y_{[\frac{n+1}{2}]n})$ can be denoted by ERSS_3.

Theorem 5 *(i) if n is even, then an unbiased estimator for μ_y using $ERSS_1$ is defined as*

$$\hat{\mu}_{ERSS_1} = \frac{1}{n}\sum_{r=1}^{n/2}\left(Y_{[1]2r-1} + Y_{[n]2r}\right),$$

with variance

$$Var(\hat{\mu}_{ERSS_1}) = Var(\hat{\mu}_{RSS}) = \frac{C(\alpha_2)}{n\theta_2^2}.$$

(ii) If n is odd then unbiased estimators for μ_y using $ERSS_2$ and $ERSS_3$ are obtained as

$$\hat{\mu}_{ERSS_2} = \frac{1}{n}\sum_{r=1}^{(n-1)/2}(Y_{[1]2r-1} + Y_{[n]2r}) + \frac{Y_{[1]n} + Y_{[n]n}}{2n},$$
$$\hat{\mu}_{ERSS_3} = \frac{1}{n}\sum_{r=1}^{(n-1)/2}(Y_{[1]2r-1} + Y_{[n]2r}) + \frac{Y_{[\frac{n+1}{2}]n}}{n},$$

with variance

$$Var(\hat{\mu}_{ERSS_2}) = \frac{v_4}{\theta_2^2}, \tag{8}$$
$$Var(\hat{\mu}_{ERSS_3}) = Var(\hat{\mu}_{ERSS_1}) = \frac{C(\alpha_2)}{n\theta_2^2}, \tag{9}$$

respectively, where $v_4 = \frac{1}{2n^2}\{(2n-1)C(\alpha_2) + \frac{4\lambda^2 D^2(\alpha_2)}{(n+1)^2(n+2)}\}$.

Proof. (i) Since

$$\sum_{r=1}^{n/2}\delta_1 = \frac{\lambda n(n-1)}{2(n+1)}, \quad \sum_{r=1}^{n/2}\delta_n = \frac{\lambda n(-n+1)}{2(n+1)},$$

we have

$$E(\hat{\mu}_{\mathrm{ERSS}_1}) = \frac{1}{n}(\frac{nB(\alpha_2)}{2\theta_2} - \frac{D(\alpha_2)}{\theta_2}\frac{\lambda n(n-1)}{2(n+1)} + \frac{nB(\alpha_2)}{2\theta_2} - \frac{D(\alpha_2)}{\theta_2}\frac{\lambda n(-n+1)}{2(n+1)}) = \frac{B(\alpha_2)}{\theta_2},$$

$$Var(\hat{\mu}_{\mathrm{ERSS}_1}) = \frac{1}{n^2}(\frac{nC(\alpha_2)}{2\theta_2^2} + \frac{C(2\alpha_2) - C(\alpha_2)}{\theta_2^2}\frac{\lambda n(n-1)}{2(n+1)} + \frac{nC(\alpha_2)}{2\theta_2^2}$$
$$+ \frac{C(2\alpha_2) - C(\alpha_2)}{\theta_2^2}\frac{\lambda n(-n+1)}{2(n+1)}) = \frac{C(\alpha_2)}{n\theta_2^2}.$$

(ii) In the estimator $\hat{\mu}_{\mathrm{ERSS}_2}$, it is easy to see that $Y_{[1]1}, Y_{[n]2}, Y_{[1]3}, \ldots, Y_{[n]n-1}$ are independent of $Y_{[1]n}$ and $Y_{[n]n}$, but the random variables $Y_{[1]n}$ and $Y_{[n]n}$ are dependent. From Scaria and Nair (1999) the joint density function of $(Y_{[1]n}, Y_{[n]n})$ is given by

$$h_{[1,n]n}(z,w) = (\alpha_2\theta_2)^2 e^{-\theta_2(z+w)}[(1-e^{-\theta_2 z})(1-e^{-\theta_2 w})]^{\alpha_2-1}\{1+\frac{2\lambda(n-1)}{n+1}[(1-e^{-\theta_2 w})^{\alpha_2} - (1-e^{-\theta_2 z})^{\alpha_2}] + \delta_{1,n}[1-2(1-e^{-\theta_2 w})^{\alpha_2}][1-2(1-e^{-\theta_2 z})^{\alpha_2}]\},$$

where $\delta_{1,n} = \frac{\lambda^2(-n^2+n+2)}{(n+1)(n+2)}$. Therefore,

$$\begin{aligned} Cov(Y_{[1]n}, Y_{[n]n}) &= E[Y_{[1]n}Y_{[n]n}] - E[Y_{[1]n}]E[Y_{[n]n}] = \frac{D^2(\alpha_2)}{\theta_2^2}[\delta_{1,n} - \delta_1\delta_n] \\ &= \frac{\lambda^2 D^2(\alpha_2)}{\theta_2^2}[\frac{-n^2+n+2}{(n+1)(n+2)} + (\frac{n-1}{n+1})^2] = \frac{4\lambda^2 D^2(\alpha_2)}{(n+1)^2(n+2)\theta_2^2}. \end{aligned}$$

Also, $Y_{[1]1}, Y_{[n]2}, Y_{[1]3}, \ldots, Y_{[n]n-1}$ and $Y_{[\frac{n+1}{2}]n}$ are all independent in $\hat{\mu}_{\mathrm{ERSS}_3}$. Since

$$\sum_{r=1}^{(n-1)/2} \delta_1 = \frac{\lambda(n-1)^2}{2(n+1)}, \qquad \sum_{r=1}^{(n-1)/2} \delta_n = \frac{-\lambda(n-1)^2}{2(n+1)}, \qquad \delta_{(n+1)/2} = 0,$$

we have

$$\begin{aligned} E(\hat{\mu}_{\mathrm{ERSS}_2}) &= \frac{1}{n}\Bigg(\frac{(n-1)B(\alpha_2)}{2\theta_2} - \frac{D(\alpha_2)}{\theta_2}\frac{\lambda(n-1)^2}{2(n+1)} + \frac{(n-1)B(\alpha_2)}{2\theta_2} + \frac{D(\alpha_2)}{\theta_2}\frac{\lambda(n-1)^2}{2(n+1)} \\ &\quad + \frac{B(\alpha_2)}{2\theta_2} - \frac{D(\alpha_2)}{2\theta_2}\frac{\lambda(n-1)}{(n+1)} + \frac{B(\alpha_2)}{2\theta_2} + \frac{D(\alpha_2)}{2\theta_2}\frac{\lambda(n-1)}{(n+1)}\Bigg) = \frac{B(\alpha_2)}{\theta_2}, \\ E(\hat{\mu}_{\mathrm{ERSS}_3}) &= \frac{1}{n}\Bigg(\frac{(n-1)B(\alpha_2)}{2\theta_2} - \frac{D(\alpha_2)}{\theta_2}\frac{\lambda(n-1)^2}{2(n+1)} + \frac{(n-1)B(\alpha_2)}{2\theta_2} \\ &\quad + \frac{D(\alpha_2)}{\theta_2}\frac{\lambda n(n-1)^2}{2(n+1)} + \frac{B(\alpha_2)}{\theta_2}\Bigg) = \frac{B(\alpha_2)}{\theta_2}, \\ Var(\hat{\mu}_{\mathrm{ERSS}_2}) &= \frac{1}{n^2}\Bigg(\frac{(n-1)C(\alpha_2)}{2\theta_2^2} + \frac{C(2\alpha_2)-C(\alpha_2)}{\theta_2^2}\frac{\lambda(n-1)^2}{2(n+1)} + \frac{(n-1)C(\alpha_2)}{2\theta_2^2} \\ &\quad - \frac{C(2\alpha_2)-C(\alpha_2)}{\theta_2^2}\frac{\lambda(n-1)^2}{2(n+1)} + \frac{C(\alpha_2)}{4\theta_2^2} + \frac{C(2\alpha_2)-C(\alpha_2)}{4\theta_2^2}\frac{\lambda(n-1)}{2(n+1)} \\ &\quad + \frac{C(\alpha_2)}{4\theta_2^2} - \frac{C(2\alpha_2)-C(\alpha_2)}{4\theta_2^2}\frac{\lambda(n-1)}{2(n+1)} + \frac{1}{2}Cov(Y_{[1]n}, Y_{[n]n})\Bigg) \\ &= \frac{1}{2\theta_2^2 n^2}\left\{(2n-1)C(\alpha_2) + \frac{4\lambda^2 D^2(\alpha_2)}{(n+1)^2(n+2)}\right\}, \end{aligned}$$

$$Var(\hat{\mu}_{\text{ERSS}_3}) = \frac{1}{n^2}\left(\frac{(n-1)C(\alpha_2)}{2\theta_2^2} + \frac{C(2\alpha_2)-C(\alpha_2)}{\theta_2^2}\frac{\lambda(n-1)^2}{2(n+1)} + \frac{(n-1)C(\alpha_2)}{2\theta_2^2} - \frac{C(2\alpha_2)-C(\alpha_2)}{\theta_2^2}\frac{\lambda(n-1)^2}{2(n+1)} + \frac{C(\alpha_2)}{\theta_2^2}\right) = \frac{C(\alpha_2)}{n\theta_2^2}. \qquad \square$$

By using (6) and (8) the efficiency of $\hat{\mu}_{\text{RSS}}$ relative to the estimator $\hat{\mu}_{\text{ERSS}_2}$ is given by

$$e_2 = e(\hat{\mu}_{\text{ERSS}_2} \mid \hat{\mu}_{\text{RSS}}) = \frac{2nC(\alpha_2)}{(2n-1)C(\alpha_2) + \frac{4\lambda^2 D^2(\alpha_2)}{(n+1)^2(n+2)}}.$$

Note that e_2 decreases with respect to $|\lambda|$ for fixed n. Also, $\lim_{n\to\infty} e_2 = 1$. In Section 3, we calculate the relative efficiency of $\hat{\mu}_{\text{ERSS}_2}$ to $\hat{\mu}_{\text{RSS}}$, e_2, for some values of the parameters and sample size.

2.4. MERSS estimation

Al-Odat and Al-Saleh (2001) suggested the MERSS, and Al-Saleh and Al-Ananbeh (2007) used the concept of MERSS with concomitant variable for the estimation of the means of the bivariate normal distribution. The procedure of MERSS with concomitant variable in MTBGED is as follows:

Step 1. Select n units each of size n from the population using SRS. Identify by judgment the minimum of each set with respect to the variable X together with the associated Y.

Step 2. Repeat step 1, but for the maximum.

Note that the $2n$ pairs of set $\{(X_{(1)r}, Y_{[1]r}), (X_{(n)r}, Y_{[n]r}); r = 1,2,\ldots,n\}$ that are obtained using the above procedure, are independent but not identically distributed.

Theorem 6 *An unbiased estimator for μ_y based on MERSS is given by*

$$\hat{\mu}_{MERSS} = \frac{1}{2n}\sum_{r=1}^{n}(Y_{[1]r} + Y_{[n]r}),$$

and its variance is

$$Var(\hat{\mu}_{MERSS}) = \frac{C(\alpha_2)}{2n\theta_2^2} = \frac{1}{2}Var(\hat{\mu}_{RSS}).$$

Proof. The proof is similar to proof of Theorem 5, part (i). □

2.5. MSRSS estimation

Al-Saleh and Al-Kadiri (2000) have considered DRSS to increase the efficiency of the RSS estimator without increasing the set size n. Al-Saleh and Al-Omari (2002) generalized DRSS to MSRSS. The MSRSS scheme can be described as follows:

Step 1. Randomly selected n^{l+1} sample units from the population, where l is the number of stages, and n is the set size.

Step 2. Allocate the n^{l+1} selected units randomly into n^{l-1} sets, each of size n^2.

Step 3. For each set in Step 2, apply the procedure of ranked set sampling method with respect to variable X to obtain a (judgment) ranked set, of size n; this step yields n^{l-1} (judgment) ranked sets, of size n each.

Step 4. Without doing any actual quantification on these ranked sets, repeat Step 3 on the n^{l-1} ranked sets to obtain n^{l-2} second stage (judgment) ranked sets, of size n each.

Step 5. This process is continued, without any actual quantification, until we end up with the lth stage (judgement) ranked set of size n.

Step 6. Finally, the n identified in step 5 are now quantified for the variable X together with the associated Y. Show the value measured for (X,Y) on the units selected at the rth stage of the MSRSS by $(X^{(l)}_{(r)r}, Y^{(l)}_{[r]r})$, $r = 1,...n$.

For $\lambda > 0$, let $Y^{(l)}_{[n]r}, r = 1,2,\ldots,n$, be the value measured on the units selected at the rth stage of the unbalanced MSRSS (Similar to suggestion by Chacko and Thomas, 2008). It is easily to see that each $Y^{(l)}_{[n]r}$ is the concomitant of the largest order statistic of n^r independently and identically distributed bivariate random variables with MTBGED, and therefore, the pdf of $Y^{(l)}_{[n]r}$ is given by

$$h^{(l)}_{[n]r}(y) = \alpha_2\theta_2 e^{-\theta_2 y}(1-e^{-\theta_2 y})^{\alpha_2-1}[1+\frac{\lambda(n^l-1)}{n^l+1}(2(1-e^{-\theta_2 y})^{\alpha_2}-1)].$$

Thus the mean and variance of $Y^{(l)}_{[n]r}$ for $r = 1,2,\ldots,n$, are given as

$$E[Y^{(l)}_{[n]r}] = \mu_y \xi_{n^l}, \quad Var[Y^{(l)}_{[n]r}] = \frac{\gamma_{n^l}}{\theta_2^2}, \tag{10}$$

respectively, where $\xi_{n^l} = 1+\lambda\frac{(n^l-1)D(\alpha_2)}{(n^l+1)B(\alpha_2)}$ and $\gamma_{n^l} = C(\alpha_2)+\lambda\frac{(n^l-1)}{n^l+1}(C(\alpha_2)-C(2\alpha_2))$.

Theorem 7 *If α_2 and λ are known then the BLUE of μ_y is*

$$\hat{\mu}_{MSRSS} = \frac{1}{n\xi_{n^l}} \sum_{r=1}^{n} Y_{[n]r}^{(l)}, \tag{11}$$

with variance

$$Var(\hat{\mu}_{MSRSS}) = \frac{\gamma_{n^l}}{n\xi_{n^l}^2 \theta_2^2}. \tag{12}$$

Proof. It can easily be proved using (10). □

If we take $l = 1$ in (11) and (12), then we get the BLUE of μ_y based the usual single stage unbalanced RSS (URSS) as

$$\hat{\mu}_{\text{URSS}} = \frac{1}{n\xi_n} \sum_{r=1}^{n} Y_{[n]r},$$

where its variance is given as

$$Var(\hat{\mu}_{\text{URSS}}) = \frac{\gamma_n}{n\xi_n^2 \theta_2^2}. \tag{13}$$

If we let $l \to \infty$ in the MSRSS method described above, then $Y_{[n]r}^{(\infty)}$, $r = 1, 2, \ldots, n$ are unbalanced steady-state ranked set samples (USSRSS) of size n with the following pdf (Al-Saleh, 2004):

$$h_{[n]r}^{(\infty)}(y) = \alpha_2 \theta_2 e^{-\theta_2 y}(1 - e^{-\theta_2 y})^{\alpha_2 - 1}[1 + \lambda(2(1 - e^{-\theta_2 y})^{\alpha_2} - 1)].$$

The mean and variance of $Y_{[n]r}^{(\infty)}$ are obtained as

$$E[Y_{[n]r}^{(\infty)}] = \mu_y Z(\alpha_2, \lambda), \quad Var[Y_{[n]r}^{(\infty)}] = \frac{I(\alpha_2, \lambda)}{\theta_2^2}, \tag{14}$$

where $Z(\alpha_2, \lambda) = 1 + \lambda \frac{D(\alpha_2)}{B(\alpha_2)}$ and $I(\alpha_2, \lambda) = C(\alpha_2) + \lambda(C(\alpha_2) - C(2\alpha_2))$.

Theorem 8 *The BLUE of μ_y based on USSRSS is given by*

$$\hat{\mu}_{USSRSS} = \frac{1}{nZ(\alpha_2, \lambda)} \sum_{r=1}^{n} Y_{[n]r}^{(\infty)},$$

with variance

$$Var(\hat{\mu}_{USSRSS}) = \frac{I(\alpha_2,\lambda)}{n(Z(\alpha_2,\lambda))^2\theta_2^2}. \tag{15}$$

Proof. It can easily be proved using (14). □

From (6), (13), and (15), we get efficiency of unbiased estimators $\hat{\mu}_{\text{USSRSS}}$ and $\hat{\mu}_{\text{URSS}}$ relative to $\hat{\mu}_{\text{RSS}}$ as

$$e_3 = e(\hat{\mu}_{\text{URSS}} \mid \hat{\mu}_{\text{RSS}}) = \frac{C(\alpha_2)\xi_n^2}{\gamma_n},$$

$$e_4 = e(\hat{\mu}_{\text{USSRSS}} \mid \hat{\mu}_{\text{RSS}}) = \frac{C(\alpha_2)(Z(\alpha_2,\lambda))^2}{I(\alpha_2,\lambda)}.$$

Note that e_4 does not depend on the value of n. In Section 3, we calculate the relative efficiencies of estimators for μ_y based on the MSRSS scheme to $\hat{\mu}_{\text{RSS}}$ for some values of the parameters and sample size.

3. Efficiency of estimators

In this section, we compare the efficiency of the proposed estimators in Section 2 for μ_y based on different RSS schemes; usual RSS, ERSS, and MSRSS. These evaluations are based numerical computation, and we did not consider LRSS and MERSS schemes. Here, we consider $n = 2(2)10(5)25$, $\alpha_2 = 0.8, 1.0, 2.0, 5$, and $\lambda = \pm.25, \pm.5, \pm.75, \pm1$.

In Table 1, we calculate the relative efficiency e_1 of $\hat{\mu}_{\text{RSS}}$ to $\tilde{\mu}$, and we can conclude that i) $\tilde{\mu}$ is more efficient than $\hat{\mu}_{\text{RSS}}$, ii) the efficiency increases with respect to $|\lambda|$ for fixed n and α, iii) the efficiency increases with respect to n for fixed λ and α, and iv) the efficiency decreases with respect to α for fixed λ and n.

In Table 1, we calculate the relative efficiency e_2 of $\hat{\mu}_{\text{ERSS}_2}$ to $\hat{\mu}_{\text{RSS}}$, and we can conclude that i) $\hat{\mu}_{\text{ERSS}_2}$ is more efficient than $\hat{\mu}_{\text{RSS}}$, ii) the efficiency decreases with respect to $|\lambda|$ and α for fixed n, iii) the efficiency decreases with respect to n for fixed λ and α, iv) the efficiency closes to one for very large n, and v) the efficiency decreases with respect to α for fixed λ and n. Also, $\hat{\mu}_{\text{ERSS}_2}$ is more efficient than $\tilde{\mu}$.

In Tables 2 and 3, for different values for l, we calculate the relative efficiency of $\hat{\mu}_{\text{MSRSS}}$ to $\hat{\mu}_{\text{RSS}}$,

$$e_5 = e(\hat{\mu}_{\text{MRRSS}} \mid \hat{\mu}_{\text{RSS}}) = \frac{C(\alpha_2)\xi_{n^l}^2}{\gamma_{n^l}}.$$

Table 1: *Comparing the efficiency of estimations.*

		α_2							
		0.8		1.0		2.0		5.0	
n	λ	e_1	e_2	e_1	e_2	e_1	e_2	e_1	e_2
2	0.25	1.0049	1.3326	1.0039	1.3326	1.0019	1.3325	1.0008	1.3325
2	0.50	1.0195	1.3304	1.0157	1.3303	1.0077	1.3300	1.0032	1.3298
2	0.75	1.0440	1.3267	1.0353	1.3264	1.0174	1.3258	1.0073	1.3255
2	1.00	1.0786	1.3216	1.0629	1.3211	1.0310	1.3200	1.0130	1.3194
4	0.25	1.0088	1.1428	1.0070	1.1428	1.0035	1.1428	1.0015	1.1428
4	0.50	1.0353	1.1426	1.0283	1.1426	1.0139	1.1426	1.0058	1.1425
4	0.75	1.0801	1.1423	1.0640	1.1422	1.0314	1.1422	1.0131	1.1422
4	1.00	1.1443	1.1418	1.1149	1.1418	1.0561	1.1417	1.0234	1.1416
6	0.25	1.0104	1.0909	1.0084	1.0909	1.0041	1.0909	1.0017	1.0909
6	0.50	1.0421	1.0908	1.0337	1.0908	1.0166	1.0908	1.0069	1.0908
6	0.75	1.0958	1.0908	1.0764	1.0908	1.0375	1.0908	1.0156	1.0907
6	1.00	1.1731	1.0907	1.1375	1.0907	1.0669	1.0906	1.0278	1.0906
8	0.25	1.0114	1.0667	1.0091	1.0667	1.0045	1.0667	1.0019	1.0667
8	0.50	1.0459	1.0666	1.0367	1.0666	1.0181	1.0666	1.0076	1.0666
8	0.75	1.1045	1.0666	1.0834	1.0666	1.0408	1.0666	1.0170	1.0666
8	1.00	1.1893	1.0666	1.1501	1.0666	1.0729	1.0666	1.0303	1.0666
10	0.25	1.0120	1.0526	1.0096	1.0526	1.0048	1.0526	1.0020	1.0526
10	0.50	1.0483	1.0526	1.0386	1.0526	1.0190	1.0526	1.0080	1.0526
10	0.75	1.1101	1.0526	1.0878	1.0526	1.0430	1.0526	1.0179	1.0526
10	1.00	1.1996	1.0526	1.1582	1.0526	1.0767	1.0526	1.0319	1.0526
15	0.25	1.0128	1.0345	1.0103	1.0345	1.0051	1.0345	1.0021	1.0345
15	0.50	1.0517	1.0345	1.0414	1.0345	1.0204	1.0345	1.0085	1.0345
15	0.75	1.1180	1.0345	1.0940	1.0345	1.0460	1.0345	1.0192	1.0345
15	1.00	1.2142	1.0345	1.1696	1.0345	1.0821	1.0345	1.0341	1.0345
20	0.25	1.0132	1.0256	1.0106	1.0256	1.0053	1.0256	1.0022	1.0256
20	0.50	1.0535	1.0256	1.0428	1.0256	1.0211	1.0256	1.0088	1.0256
20	0.75	1.1221	1.0256	1.0973	1.0256	1.0475	1.0256	1.0198	1.0256
20	1.00	1.2219	1.0256	1.1756	1.0256	1.0849	1.0256	1.0353	1.0256
25	0.25	1.0135	1.0204	1.0108	1.0204	1.0054	1.0204	1.0022	1.0204
25	0.50	1.0546	1.0204	1.0436	1.0204	1.0215	1.0204	1.0090	1.0204
25	0.75	1.1247	1.0204	1.0993	1.0204	1.0485	1.0204	1.0202	1.0204
25	1.00	1.2267	1.0204	1.1793	1.0204	1.0866	1.0204	1.0360	1.0204
30	0.25	1.0137	1.0169	1.0110	1.0169	1.0054	1.0169	1.0023	1.0169
30	0.50	1.0553	1.0169	1.0442	1.0169	1.0218	1.0169	1.0091	1.0169
30	0.75	1.1264	1.0169	1.1007	1.0169	1.0492	1.0169	1.0205	1.0169
30	1.00	1.2299	1.0169	1.1818	1.0169	1.0878	1.0169	1.0365	1.0169

Table 2: *Comparing the efficiency of estimations.*

		$\alpha=0.8$					$\alpha=1.0$				
		l					l				
n	λ	1	2	5	13	∞	1	2	5	13	∞
2	0.25	1.120	1.223	1.365	1.392	1.392	1.108	1.201	1.327	1.350	1.350
2	0.50	1.250	1.482	1.827	1.894	1.894	1.225	1.430	1.728	1.785	1.786
2	0.75	1.392	1.784	2.410	2.539	2.540	1.350	1.691	2.220	2.326	2.327
2	1.00	1.546	2.133	3.151	3.372	3.373	1.485	1.988	2.823	2.999	3.000
4	0.25	1.223	1.340	1.391	1.392	1.392	1.201	1.305	1.349	1.350	1.350
4	0.50	1.482	1.765	1.892	1.894	1.894	1.430	1.675	1.784	1.786	1.786
4	0.75	1.784	2.293	2.536	2.540	2.540	1.691	2.122	2.323	2.327	2.327
4	1.00	2.133	2.954	3.366	3.373	3.373	1.988	2.665	2.994	3.000	3.000
6	0.25	1.270	1.368	1.392	1.392	1.392	1.242	1.329	1.350	1.350	1.350
6	0.50	1.592	1.834	1.894	1.894	1.894	1.525	1.734	1.785	1.786	1.786
6	0.75	1.977	2.424	2.539	2.540	2.540	1.856	2.231	2.326	2.327	2.327
6	1.00	2.437	3.174	3.372	3.373	3.373	2.242	2.842	2.999	3.000	3.000
8	0.25	1.296	1.378	1.392	1.392	1.392	1.265	1.338	1.350	1.350	1.350
8	0.50	1.655	1.860	1.894	1.894	1.894	1.580	1.756	1.786	1.786	1.786
8	0.75	2.092	2.473	2.540	2.540	2.540	1.953	2.272	2.327	2.327	2.327
8	1.00	2.622	3.259	3.373	3.373	3.373	2.395	2.909	3.000	3.000	3.000
10	0.25	1.313	1.383	1.392	1.392	1.392	1.280	1.342	1.350	1.350	1.350
10	0.50	1.697	1.872	1.894	1.894	1.894	1.616	1.767	1.786	1.786	1.786
10	0.75	2.168	2.497	2.540	2.540	2.540	2.017	2.291	2.327	2.327	2.327
10	1.00	2.746	3.299	3.373	3.373	3.373	2.496	2.941	3.000	3.000	3.000
15	0.25	1.337	1.388	1.392	1.392	1.392	1.302	1.347	1.350	1.350	1.350
15	0.50	1.757	1.884	1.894	1.894	1.894	1.668	1.777	1.786	1.786	1.786
15	0.75	2.279	2.521	2.540	2.540	2.540	2.110	2.311	2.327	2.327	2.327
15	1.00	2.930	3.340	3.373	3.373	3.373	2.645	2.974	3.000	3.000	3.000
20	0.25	1.350	1.389	1.392	1.392	1.392	1.313	1.348	1.350	1.350	1.350
20	0.50	1.789	1.889	1.894	1.894	1.894	1.695	1.781	1.786	1.786	1.786
20	0.75	2.339	2.529	2.540	2.540	2.540	2.160	2.318	2.327	2.327	2.327
20	1.00	3.030	3.354	3.373	3.373	3.373	2.726	2.985	3.000	3.000	3.000
25	0.25	1.358	1.390	1.392	1.392	1.392	1.320	1.349	1.350	1.350	1.350
25	0.50	1.809	1.891	1.894	1.894	1.894	1.712	1.783	1.786	1.786	1.786
25	0.75	2.376	2.533	2.540	2.540	2.540	2.191	2.321	2.327	2.327	2.327
25	1.00	3.093	3.361	3.373	3.373	3.373	2.777	2.990	3.000	3.000	3.000
30	0.25	1.363	1.391	1.392	1.392	1.392	1.325	1.349	1.350	1.350	1.350
30	0.50	1.822	1.892	1.894	1.894	1.894	1.724	1.784	1.786	1.786	1.786
30	0.75	2.402	2.535	2.540	2.540	2.540	2.213	2.323	2.327	2.327	2.327
30	1.00	3.137	3.365	3.373	3.373	3.373	2.812	2.993	3.000	3.000	3.000

Table 3: *Comparing the efficiency of estimations.*

		$\alpha = 2.0$					$\alpha = 5.0$				
		l					l				
n	λ	1	2	5	13	∞	1	2	5	13	∞
2	0.25	1.078	1.144	1.231	1.247	1.247	1.053	1.096	1.153	1.163	1.163
2	0.50	1.161	1.301	1.496	1.533	1.533	1.107	1.198	1.320	1.342	1.342
2	0.75	1.247	1.473	1.799	1.862	1.862	1.163	1.305	1.500	1.537	1.537
2	1.00	1.338	1.659	2.144	2.240	2.240	1.221	1.418	1.696	1.748	1.748
4	0.25	1.144	1.216	1.247	1.247	1.247	1.096	1.143	1.163	1.163	1.163
4	0.50	1.301	1.462	1.532	1.533	1.533	1.198	1.299	1.342	1.342	1.342
4	0.75	1.473	1.741	1.860	1.862	1.862	1.305	1.466	1.536	1.537	1.537
4	1.00	1.659	2.056	2.237	2.240	2.240	1.418	1.647	1.747	1.748	1.748
6	0.25	1.173	1.233	1.247	1.247	1.247	1.115	1.154	1.163	1.163	1.163
6	0.50	1.365	1.500	1.533	1.533	1.533	1.238	1.322	1.342	1.342	1.342
6	0.75	1.577	1.806	1.862	1.862	1.862	1.369	1.504	1.537	1.537	1.537
6	1.00	1.813	2.154	2.240	2.240	2.240	1.508	1.701	1.748	1.748	1.748
8	0.25	1.189	1.239	1.247	1.247	1.247	1.126	1.158	1.163	1.163	1.163
8	0.50	1.401	1.514	1.533	1.533	1.533	1.261	1.331	1.342	1.342	1.342
8	0.75	1.638	1.830	1.862	1.862	1.862	1.405	1.518	1.537	1.537	1.537
8	1.00	1.902	2.191	2.240	2.240	2.240	1.560	1.721	1.748	1.748	1.748
10	0.25	1.199	1.242	1.247	1.247	1.247	1.133	1.160	1.163	1.163	1.163
10	0.50	1.424	1.521	1.533	1.533	1.533	1.275	1.335	1.342	1.342	1.342
10	0.75	1.677	1.842	1.862	1.862	1.862	1.429	1.525	1.537	1.537	1.537
10	1.00	1.960	2.208	2.240	2.240	2.240	1.593	1.731	1.748	1.748	1.748
15	0.25	1.214	1.245	1.247	1.247	1.247	1.142	1.162	1.163	1.163	1.163
15	0.50	1.458	1.528	1.533	1.533	1.533	1.296	1.339	1.342	1.342	1.342
15	0.75	1.734	1.853	1.862	1.862	1.862	1.462	1.532	1.537	1.537	1.537
15	1.00	2.045	2.226	2.240	2.240	2.240	1.641	1.741	1.748	1.748	1.748
20	0.25	1.222	1.246	1.247	1.247	1.247	1.147	1.163	1.163	1.163	1.163
20	0.50	1.476	1.530	1.533	1.533	1.533	1.307	1.340	1.342	1.342	1.342
20	0.75	1.764	1.857	1.862	1.862	1.862	1.480	1.534	1.537	1.537	1.537
20	1.00	2.090	2.232	2.240	2.240	2.240	1.666	1.744	1.748	1.748	1.748
25	0.25	1.227	1.246	1.247	1.247	1.247	1.150	1.163	1.163	1.163	1.163
25	0.50	1.486	1.531	1.533	1.533	1.533	1.314	1.341	1.342	1.342	1.342
25	0.75	1.782	1.859	1.862	1.862	1.862	1.491	1.535	1.537	1.537	1.537
25	1.00	2.118	2.235	2.240	2.240	2.240	1.682	1.746	1.748	1.748	1.748
30	0.25	1.230	1.247	1.247	1.247	1.247	1.152	1.163	1.163	1.163	1.163
30	0.50	1.494	1.532	1.533	1.533	1.533	1.318	1.341	1.342	1.342	1.342
30	0.75	1.795	1.860	1.862	1.862	1.862	1.498	1.536	1.537	1.537	1.537
30	1.00	2.138	2.237	2.240	2.240	2.240	1.692	1.746	1.748	1.748	1.748

Note that e_5 is the relative efficiency of $\hat{\mu}_{\text{USSRSS}}$ to $\hat{\mu}_{\text{RSS}}$, i.e. e_4, when $l = \infty$. We can conclude that i) $\hat{\mu}_{\text{MSRSS}}$ is more efficient than $\hat{\mu}_{\text{RSS}}$, ii) the efficiency increases with respect to $\lambda > 0$ for fixed n and α, iii) the efficiency increases with respect to n for fixed λ and α, and iv) the efficiency decreases with respect to α for fixed λ and n. Also, the efficiency increases when the number of stages, l, increases, and $\hat{\mu}_{\text{USSRSS}}$ is more efficient than $\hat{\mu}_{\text{MSRSS}}$ for all l.

Acknowledgements

The authors would like to thank an anonymous reviewer of this journal for many constructive suggestions and comments for improving this manuscript.

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