SORT 39 (1) January-June 2015, 71-84

Discrete alpha-skew-Laplace distribution

S. Shams Harandi and M. H. Alamatsaz

Abstract

Classical discrete distributions rarely support modelling data on the set of whole integers. In this paper, we shall introduce a flexible discrete distribution on this set, which can, in addition, cover bimodal as well as unimodal data sets. The proposed distribution can also be fitted to positive and negative skewed data. The distribution is indeed a discrete counterpart of the continuous alpha-skew-Laplace distribution recently introduced in the literature. The proposed distribution can also be viewed as a weighted version of the discrete Laplace distribution. Several distributional properties of this class such as cumulative distribution function, moment generating function, moments, modality, infinite divisibility and its truncation are studied. A simulation study is also performed. Finally, a real data set is used to show applicability of the new model comparing to several rival models, such as the discrete normal and Skellam distributions.

MSC: 60E, 62E

Keywords: Discrete Laplace distribution, discretization, maximum likelihood estimation, uni-bimodality, weighted distribution.

1. Introduction

The traditional discrete distributions (geometric, Poisson, etc.) have limited applicability in modelling certain real situations such as data on the set of integers $\mathbb{Z} = \{0, \mp 1, \mp 2, \ldots\}$ or bimodal data sets. Thus, several researchers have attempted to develop new classes of discrete distributions to cover such situations. Recall that any continuous distribution on $\mathbb{R}$ with probability density function (pdf) f admits a discrete counterpart supported on the set of integers $\mathbb{Z} = \{0, \mp 1, \mp 2, \ldots\}$ whose probability mass function (pmf) is defined as

$$P(X = x) = \frac{f(x)}{\sum_{y=-\infty}^{\infty} f(y)}, \quad x \in \mathbb{Z}\,. \tag{1}$$

Corresponding author: alamatho@sci.ui.ac.ir (and mh_-alamatsaz@yahoo.com)
Department of Statistics, University of Isfahan, Isfahan, Iran. salimeh.shams2013@yahoo.com
Received: December 2013
Accepted: November 2014

For instance, Roy (2003) introduced a discrete version of normal distribution to cover discrete data on the whole set of integers $\mathbb{Z} = \{0, \mp 1, \mp 2, \dots\}$ and, similarly, Inusah and Kozubowski (2006) considered a discrete analogue of Laplace (DL) distribution. Kozubowski and Inusah (2006) proposed a discrete version of the skew Laplace (skewDL) distribution as a generalization of discrete Laplace distribution which is useful for unimodal data sets. Also, Barbiero (2014) and Jayakumar and Jacob (2012) introduced other discrete distributions based on skew Laplace and wrapped skew Laplace distributions on the integers, respectively.

The aim of this paper is to propose a more flexible distribution on $\mathbb{Z}$ which can cover unimodal as well as bimodal data. The new discrete distribution can also fit both positively and negatively skewed data. In fact, using (1), we provide a discrete version of the alpha-skew-Laplace distribution which was recently introduced by Shams Harandi and Alamatsaz (2013). The probability density function of the alpha-skew-Laplace distribution is

$$f(x;\alpha,\mu,\sigma) = \frac{1}{4\sigma(1+\alpha^2)}[1+(1-\frac{\alpha}{\sigma}(x-\mu))^2]e^{\frac{-|x-\mu|}{\sigma}}, \qquad x \in \mathbb{R}, \tag{2}$$

where $\alpha \in \mathbb{R}$ is the skewness parameter and $\mu \in \mathbb{R}$ and $\sigma > 0$ are its location and scale parameters, respectively. The discrete version of (2) which is considered here can be fitted to unimodal as well as bimodal data sets having positive as well as negative skewness.

The rest of the article is organized as follows. Section 2 introduces the discrete alpha-skew-Laplace ($DASL(p,\gamma)$) distribution and discusses some of its important features and properties. In Section 3, we shall provide some distributional properties such as moment generating function and moments. Maximum likelihood estimations of parameters involved will be discussed in section 4. Section 5 describes a simulation study. In Section 6, we shall consider some interesting modification of *DASL* distribution. In Section 7, we attempt to fit the proposed model and its special cases to a real data set and compare it with several rival models such as the discrete normal, DL, skewDL and Skellam distributions.

2. The family of discrete alpha-skew-Laplace distributions

In this section, we present the pmf of our new class of discrete distributions on $\mathbb{Z}$ by discretizing alpha-skew-Laplace distribution (2). We let $\mu = 0$ and use relation (1) to obtain

$$p(x;p,\alpha) = C(p,\alpha)[1+(1+\alpha x \log p)^2]p^{|x|}, \qquad x \in \mathbb{Z}, \tag{3}$$

where $C(p,\alpha) = \frac{1}{2}\frac{1-p}{1+p}\,[1+\alpha^2(\log p)^2\frac{p}{(1-p)^2}]^{-1}$, $0 < p = e^{-\frac{1}{\sigma}} < 1$ and $\alpha \in \mathbb{R}$.

Since $\sigma = (-\log p)^{-1}$, to simplify, we let $\gamma = \frac{\alpha}{\sigma}$. Then, we have

$$p(x;p,\gamma) = C(p,\gamma)[1+(1-\gamma x)^2]p^{|x|}, \qquad x \in \mathbb{Z}, \quad p \in (0,1), \gamma \in \mathbb{R}, \tag{4}$$

where $C(p,\gamma) = \frac{1}{2}\frac{1-p}{1+p}[1+\gamma^2\frac{p}{(1-p)^2}]^{-1}$. We denote this distribution by $X \sim DASL(p,\gamma)$.

Remark 1 Recall that for a distribution with pdf (pmf) f, we can construct a new distribution with pdf (pmf)

$$g(x;\Theta_1,\Theta_2) = \frac{w(x;\Theta_1,\Theta_2)}{E_{\Theta_1}[w(x;\Theta_1,\Theta_2)]}f(x;\Theta_1),$$

where Θ_1 and Θ_2 can be two vectors of parameters and w is called a weighted distribution of f. It is worth noting that pmf (4) can also be viewed as the weighted version of the discrete Laplace distribution of Inusah and Kozubowski (2006). To see this, it is sufficient to consider the weight function $w(x;\Theta_1,\Theta_2) = (1+(1-\gamma x)^2)$ with $\Theta_1 = p$, $\Theta_2 = \gamma$ and $f(x;p) = \frac{1-p}{1+p}p^{|x|}$.

Some special cases of this new class of discrete distributions are revealed below:

1. If $\alpha = 0$ in (3), or equivalently $\gamma = 0$ in (4), we obtain the discrete Laplace (DL) distribution.

2. If $\alpha \to \infty$ in (3), or equivalently $\gamma \to \infty$ in (4), we have

$$p(x;p,\gamma) \to \frac{(1-p)^3}{2p(1+p)}x^2p^{|x|}, \quad x \in \mathbb{Z}$$

   which is a symmetric and bimodal discrete distribution.

3. If $X \sim DASL(p,\gamma)$, then $-X \sim DASL(p,-\gamma)$.

4. By considering the continuous version of the alpha-skew-Laplace distribution of Shams and Alamatsaz (2013), we can conclude that $DASL(p,\gamma)$ is unimodal for $\log p < \gamma < -\log p$ and bimodal for $\gamma \geq -\log p$ or $\gamma \leq \log p$, respectively. Equivalently, if we consider the pmf in (3), then the distribution is unimodal for $-1 < \alpha < 1$ and bimodal for $\alpha \leq -1$ or $\alpha \geq 1$, respectively.

Figure 1 below illustrates several plots of $DASL(p,\gamma)$ distribution for selected values of the parameters p and γ which confirms our result on modality of the distribution. We note that for $\gamma < 0$, all plots are symmetric.

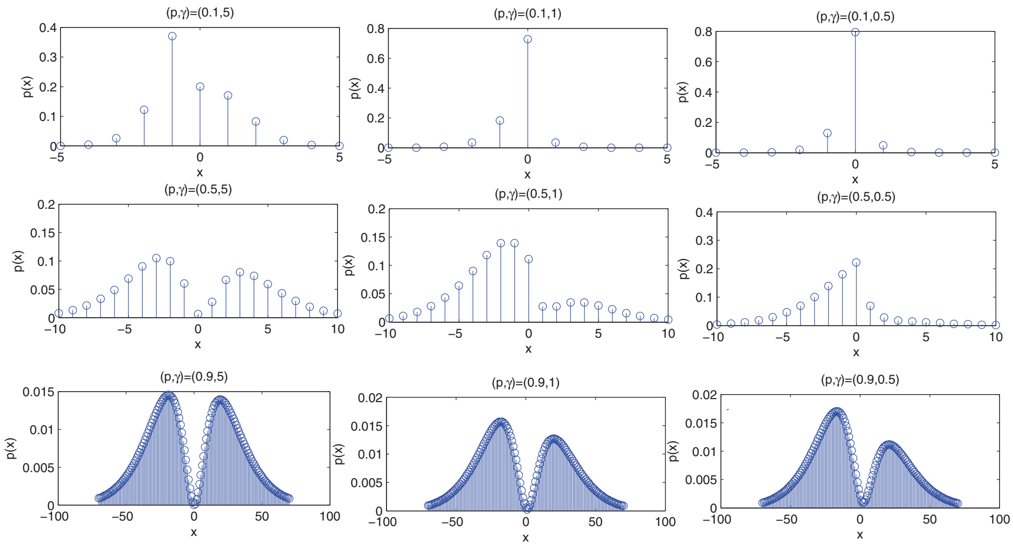

Figure 1: *Illustrations of pmf of DASL(p,γ) for different values of p and γ.*

The cumulative distribution function (cdf) of the random variable $X \sim DASL(p,\gamma)$ is given by

$$F(x;p,\gamma) = P(X \leq x)$$
$$= C(p,\gamma)\begin{cases} p^{-[x]}\left[\frac{2}{1-p} + \gamma^2 \frac{p^2([x]+1)^2 - p(2[x]^2+2[x]-1)+[x]^2}{(1-p)^3}\right. \\ \left. -2\gamma\frac{(1-p)[x]-p}{(1-p)^2}\right], & x<0 \\ \frac{2(1+p)}{1-p} + 2\gamma^2\frac{p(1+p)}{(1-p)^3} - p^{[x]+1}\left[\gamma^2\frac{(1-p)^2([x]+1)^2+2p(1-p)([x]+1)+p(1+p)}{(1-p)^3}\right. \\ \left. -2\gamma\frac{(1-p)([x]+1)+p}{(1-p)^2} + \frac{2}{1-p}\right], & x \geq 0. \end{cases}$$

3. Moments

The moment generating function of a random variable $X \sim DASL(p,\gamma)$ is given by

$$M_X(t) = E(e^{tX}) = C(p,\gamma)\left[\frac{2p}{e^t - p} + \frac{2}{1-pe^t} + 2\gamma\frac{pe^t}{(e^t-p)^2}\right.$$
$$\left. -2\gamma\frac{pe^t}{(pe^t-1)^2} + \gamma^2\frac{pe^t(p+e^t)}{(e^t-p)^3} + \gamma^2\frac{pe^t(pe^t+1)}{(1-pe^t)^3}\right], \quad |t| > \log p.$$

Replacing t in $M_X(t)$ by it, $i=\sqrt{-1}$, we can easily obtain the characteristic function of $DASL(p,\gamma)$.

To find the moments, using the combinatorial identity

$$\sum_{x=1}^{\infty} x^n p^x = \sum_{x=1}^{n} S(n,x)\frac{x!p^x}{(1-p)^{x+1}},$$

(see, e.g., formula (7.46), p. 337, of Graham et al., 1989), where

$$S(n,x)=\frac{1}{x!}\sum_{k=0}^{x-1}(-1)^k\binom{x}{k}(x-k)^n$$

is the Stirling number of the second kind, we obtain the n-th moment of $X\sim DASL(p,\gamma)$ for $n\geqslant 1$ as

$$\begin{aligned}\mu_n = E(X^n) = C(p,\gamma)\sum_{x=1}^{n}\frac{x!p^x}{(1-p)^{x+1}}&\Big[\Big(2S(n,x)+\gamma^2 S(n+2,x)\Big)\\ &\times\Big(1+(-1)^n\Big)-2\gamma S(n+1,x)\Big(1+(-1)^{n+1}\Big)\Big]\\ &+C(p,\gamma)\frac{p^{n+1}(n+1)!}{(1-p)^{n+2}}\Big\{[\gamma^2(1+(-1)^n)(S(n+2,n+1)\\ &+p\frac{n+2}{1-p}S(n+2,n+2))-2\gamma(1+(-1)^{n+1})S(n+1,n+1)]\Big\}.\end{aligned} \tag{5}$$

We can easily observe that for even n,

$$\begin{aligned}\mu_n = 2C(p,\gamma)\Big\{&\sum_{x=1}^{n}\frac{x!p^x}{(1-p)^{x+1}}\Big[2S(n,x)+\gamma^2 S(n+2,x)\Big]\\ &+\frac{p^{n+1}(n+1)!}{(1-p)^{n+2}}\Big[\gamma^2(S(n+2,n+1)+p\frac{n+2}{1-p}S(n+2,n+2))\Big]\Big\}.\end{aligned}$$

and for odd n,

$$\begin{aligned}\mu_n = -4\gamma C(p,\gamma)\Big\{&\sum_{x=1}^{n}\frac{x!p^x}{(1-p)^{x+1}}S(n+1,x)\\ &+\frac{p^{n+1}(n+1)!}{(1-p)^{n+2}}S(n+1,n+1)\Big\}\end{aligned}$$

In particular, we have

$$E(X) = -2\gamma \frac{p}{(1-p)^2}[1+\gamma^2 \frac{p}{(1-p)^2}]^{-1},$$

$$E(X^2) = p\Big[\frac{2}{(1-p)^2} + \gamma^2 \frac{(p^2+10p+1)}{(1-p)^4}\Big][1+\gamma^2 \frac{p}{(1-p)^2}]^{-1},$$

$$E(X^3) = -2\gamma p \frac{p^2+10p+1}{(1-p)^4}[1+\gamma^2 \frac{p}{(1-p)^2}]^{-1},$$

$$E(X^4) = \frac{p}{(1-p)^4}\Big[2(p^2+10p+1) + \frac{\gamma^2(p^4+56p^3+246p^2+56p+1)}{(1-p)^2}\Big]$$

$$[1+\gamma^2 \frac{p}{(1-p)^2}]^{-1}$$

and thus

$$Var(X) = \frac{p}{(1-p)^2}\Big[2+\gamma^2 \frac{p^2+10p+1}{(1-p)^2} - 4\gamma^2 \frac{p}{(1-p)^2+\gamma^2 p}\Big][1+\gamma^2 \frac{p}{(1-p)^2}]^{-1}.$$

Skewness and kurtosis of our distribution can be evaluated easily. But since their formulas are too long, they are omitted and we only show their behaviour by their graphs in Figure 2.

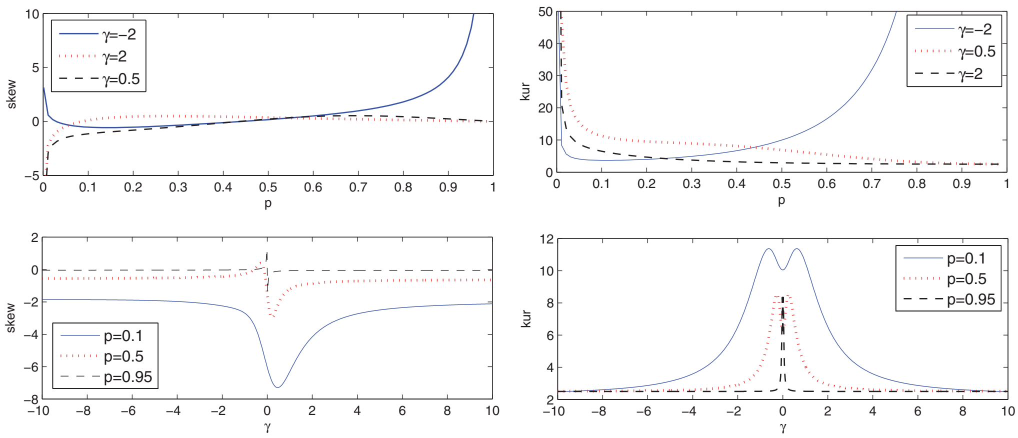

Figure 2: *Illustrations of the skewness and kurtosis as functions of p and γ.*

4. Maximum likelihood estimation

To apply maximum likelihood for estimating p and γ, assume that $x_1, x_2, \ldots, x_n$ are the observed values of a random sample of size n from a $DASL(p,\gamma)$ distribution. The log-likelihood function becomes

$$\ell(p,\gamma) = -n\log 2 + n\log(1-p) - n\log(1+p) - n\log(1+\gamma^2\frac{p}{(1-p)^2})$$
$$+\sum_{i=1}^{n}\log(1+(1-\gamma x_i)^2) + \sum_{i=1}^{n}|x_i|\log p.$$

Then, the likelihood equations for p and γ are given by

$$\frac{\partial\ell(p,\gamma)}{\partial p} = \frac{\sum_{i=1}^{n}|x_i|}{p} - \frac{2n}{1-p^2} - n\gamma^2\frac{1+p}{(1-p)^3+p\gamma^2(1-p)} = 0 \tag{6}$$

and

$$\frac{\partial\ell(p,\gamma)}{\partial\gamma} = -\frac{2n\gamma p}{(1-p)^2+p\gamma^2} - 2\sum_{i=1}^{n}x_i\frac{(1-\gamma x_i)}{1+(1-\gamma x_i)^2} = 0. \tag{7}$$

The solutions of likelihood equations (7) and (8) provide the maximum likelihood estimators (MLEs) of p and γ, which can be obtained by a numerical method such as the Newton-Raphson type procedure.

Since the MLEs of the unknown parameters (p,γ) can not be obtained in closed forms, it is not easy to derive the exact distributions of MLEs. One can show that the *DASL* family satisfies the regularity conditions which are fulfilled for parameters in the interior of the parameter space but not on the boundary (see, e.g., Ferguson, 1996, pp. 121). Hence, by using the simplest large sample approach, the MLE vector $(\hat{p},\hat{\gamma})$ is consistent and asymptotically normal, i.e.,

$$(\widehat{p}-p, \widehat{\gamma}-\gamma) \rightarrow N_2(\mathbf{0}, I^{-1}(\widehat{p},\widehat{\gamma})),$$

where I^{-1} is the variance covariance matrix of the unknown parameters (p,γ) and the covariance matrix I^{-1}, as the Fisher information matrix, can be obtained by

$$I^{-1}(p,\gamma) = \begin{bmatrix} -E\left(\frac{\partial^2\ell}{\partial p^2}\right) & -E\left(\frac{\partial^2\ell}{\partial p\partial\gamma}\right) \\ -E\left(\frac{\partial^2\ell}{\partial\gamma\partial p}\right) & -E\left(\frac{\partial^2\ell}{\partial\gamma^2}\right) \end{bmatrix}^{-1} = \begin{bmatrix} Var(\widehat{p}) & Cov(\widehat{p},\widehat{\gamma}) \\ Cov(\widehat{p},\widehat{\gamma}) & Var(\widehat{\gamma}) \end{bmatrix},$$

whose elements are evaluated by using the following expressions:

$$\frac{\partial^2 \ell}{\partial p^2} = -\frac{\sum_{i=1}^n |x_i|}{p^2} - 4n\frac{p}{(1-p^2)^2} - n\gamma^2 \frac{2(1-p)^2(2-p)+\gamma^2(p^2+2p-1)}{[(1-p)^3+p\gamma^2(1-p)]^2},$$

$$\frac{\partial^2 \ell}{\partial p \partial \gamma} = -2n\gamma \frac{1-p^2}{[(1-p)^2+\gamma^2 p]^2}$$

and

$$\frac{\partial^2 \ell}{\partial \gamma^2} = 2n\frac{p}{(1-p)^2+\gamma^2 p}[\frac{(1-p)^2-\gamma^2 p}{(1-p)^2+\gamma^2 p}] - 2\sum_{i=1}^n x_i^2 \frac{1-(1-\gamma x_i)^2}{[1+(1-\gamma x_i)^2]^2}.$$

To find expectations of the above expressions, we need to compute $E|X|$ and $E\{X^2 \frac{1-(1-\gamma X)^2}{[1+(1-\gamma X)^2]^2}\}$. The Fisher's information matrix can be computed using the approximation

$$I(\hat{p},\hat{\gamma}) = -\begin{bmatrix} \frac{\partial^2 \ell}{\partial p^2}|_{\hat{p},\hat{\gamma}} & \frac{\partial^2 \ell}{\partial p \partial \gamma}|_{\hat{p},\hat{\gamma}} \\ \frac{\partial^2 \ell}{\partial \gamma \partial p}|_{\hat{p},\hat{\gamma}} & \frac{\partial^2 \ell}{\partial \gamma^2}|_{\hat{p},\hat{\gamma}} \end{bmatrix},$$

as the observed Fisher's information matrix.

The normal approximation can then be used to construct confidence intervals for p and q to test hypothesis of the kind $H_0 : p = p_0$ and $H_0 : \gamma = \gamma_0$, respectively, as

$$(\hat{p} - z_{\alpha/2} I(\hat{p}), \hat{p} + z_{\alpha/2} I(\hat{p}))$$

and

$$(\hat{\gamma} - z_{\alpha/2} I(\hat{\gamma}), \hat{\gamma} + z_{\alpha/2} I(\hat{\gamma})).$$

where $I(\hat{p})$ and $I(\hat{\gamma})$ refer to the roots of diagonal elements of the inverse Fisher's information matrix.

5. Simulation

Here, we assess the performance of the maximum-likelihood estimate given by Equations (7) and (8) with respect to the sample size n. The simulation study assessment is based on the inversion method with 1000 iterations.

Table 1: *MLEs of p and γ in DASL(p,γ) for different values of n.*

γ	-2							
p	0.25				0.6			
n	$\hat{p}$	$\hat{\gamma}$	$\hat{Var}(\hat{p})$	$\hat{Var}(\hat{\gamma})$	$\hat{p}$	$\hat{\gamma}$	$\hat{Var}(\hat{p})$	$\hat{Var}(\hat{\gamma})$
100	0.2521	−2.0304	0.0009	0.1699	0.5994	−2.3095	0.0004	1.3711
200	0.2499	−2.0199	0.0004	0.0796	0.5997	−2.0784	0.0002	0.2564
400	0.2501	−2.0081	0.0002	0.0324	0.5999	−2.0431	0.0001	0.1072
γ	-0.5							
n	$\hat{p}$	$\hat{\gamma}$	$\hat{Var}(\hat{p})$	$\hat{Var}(\hat{\gamma})$	$\hat{p}$	$\hat{\gamma}$	$\hat{Var}(\hat{p})$	$\hat{Var}(\hat{\gamma})$
100	0.2411	−0.5797	0.0023	0.1270	0.5974	−0.5193	0.0023	0.0413
200	0.2451	−0.5363	0.0011	0.0467	0.6002	−0.5046	0.0005	0.0069
400	0.2481	−0.5126	0.0004	0.0121	0.5999	−0.5031	0.0002	0.0034
γ	0.5							
n	$\hat{p}$	$\hat{\gamma}$	$\hat{Var}(\hat{p})$	$\hat{Var}(\hat{\gamma})$	$\hat{p}$	$\hat{\gamma}$	$\hat{Var}(\hat{p})$	$\hat{Var}(\hat{\gamma})$
100	0.2434	0.5697	0.0020	0.0935	0.5994	0.5160	0.0010	0.0165
200	0.2461	0.5378	0.0011	0.0458	0.5997	0.5083	0.0004	0.0069
400	0.2475	0.5106	0.0005	0.0125	0.5998	0.5048	0.0002	0.0032
γ	1.5							
n	$\hat{p}$	$\hat{\gamma}$	$\hat{Var}(\hat{p})$	$\hat{Var}(\hat{\gamma})$	$\hat{p}$	$\hat{\gamma}$	$\hat{Var}(\hat{p})$	$\hat{Var}(\hat{\gamma})$
100	0.2523	1.5352	0.0014	0.1430	0.6002	1.6370	0.0004	0.3117
200	0.2512	1.5146	0.0006	0.0599	0.6001	1.5402	0.0002	0.0845
400	0.2507	1.5079	0.0003	0.0293	0.6001	1.5231	0.0001	0.0356

These results are presented in Table 1 accompanied by their estimated variances ($\hat{Var}$), for different values of n. Table 1 shows how the MLEs and estimated variances of parameters vary with respect to n. The difference between real and estimated values of the parameters are not too large and, thus, the method works well.

6. Some special cases

In this section, we consider the distribution of the random variable $X \sim DASL(p,\gamma)$ truncated at zero. This distribution is an important case, because it is a weighted version of geometric distribution and may be useful in fitting count or time data sets.

Let $Y = X|X \geq 0$, then the pmf of Y is given by

$$p_Y(y; p,\gamma) = C^*(p,\gamma)(1+(1-\gamma y)^2)p^y,\ y = 0,1,2,\ldots,$$

where ${C^*}^{-1}(p,\gamma) = \frac{2}{1-p} - 2\gamma\frac{p}{(1-p)^2} + \gamma^2\frac{p(1+p)}{(1-p)^3}$, $0 < p < 1$ and $\gamma \in \mathbb{R}$. This distribution is called weighted geometric distribution and is denoted by $Y \sim WGD(p,\gamma)$.

This can be used as a discrete lifetime distribution which contains geometric distribution by setting $\gamma = 0$. The survival and failure rate functions of this random variable are given by

$$R_Y(y;p,\gamma) = P(Y > y) = C^*(p,\gamma)\frac{p^{y+1}}{1-p}[2+2\gamma\frac{yp-y-1}{1-p}$$
$$+\gamma^2\frac{y^2p^2+(-2y^2-2y+1)p+(y+1)^2}{(1-p)^2}],\ y = 0,1,\ldots$$

and

$$H_Y(y;p,\gamma) = (1-p)\frac{1+(1-\gamma y)^2}{2p+2p\gamma\frac{yp-y-1}{1-p}+\gamma^2 p\frac{y^2p^2+(-2y^2-2y+1)p+(y+1)^2}{(1-p)^2}},\ y = 0,1,\ldots,$$

respectively. The behaviour of the failure rate function of $X \sim WGD(p,\gamma)$ is described in Figure 3. As we can see the failure rate function of WGD distribution can be increasing or U-shaped. Further, we note that if $\gamma = 0$, WGD distribution will reduce to the geometric distribution with constant failure rate function which depends only on p.

Another important structural property of a distribution, both in theory and application, is its infinite divisibility. We refer, for example, to the monograph of Steutel and Van Harn (2004) for a good and complete introduction of the subject. Since most of the well-known distributions possess this property, one has to be concerned with the infinite divisibility or non-infinite divisibility property of any distribution newly introduced. Here, we note that $WGD(p,\gamma)$ distribution is not infinitely divisible. To see this, we first recall the following interesting result from the above-mentioned monograph (page 56).

Lemma 1 *If p_k, $k \in \mathbb{Z}_+$ is infinitely divisible, then we have $p_k \leq 1/e$, for all $k \in \mathbb{N}$.*

Now, we can show that $p_Y(y;p,\gamma) > 1/e$ for some values of $y \in \mathbb{N}$, p and γ. For instance, take $y = 1$, $p = 0.1$ and $\gamma = 10$. Then, we see that $p_Y(1;0.1,10) \simeq 0.5525 > 1/e \simeq 0.3679$. Thus, a $WGD(p,\gamma)$ distribution is not infinitely divisible in general. In the case $\gamma = 0$, however, we have the geometric distribution, with probability of success p, which is obviously infinitely divisible.

It is also worth noting that, we can describe the distribution of the random variable $Z = |X|$ as a new distribution on the set of non-negative integers as follows:

$$p^*(z;p,\gamma) = P(|X| = z) = 2C(p,\gamma)\begin{cases} 1, & z = 0 \\ \{2+\gamma^2z^2\}p^z, & z = 1,2,\ldots. \end{cases}$$

where p and γ are given as before. This distribution is called a generalized geometric distribution and denoted by $Z \sim GGD(p,\gamma)$. It is worth mentioning that if $Z \sim GGD(p,\gamma)$,

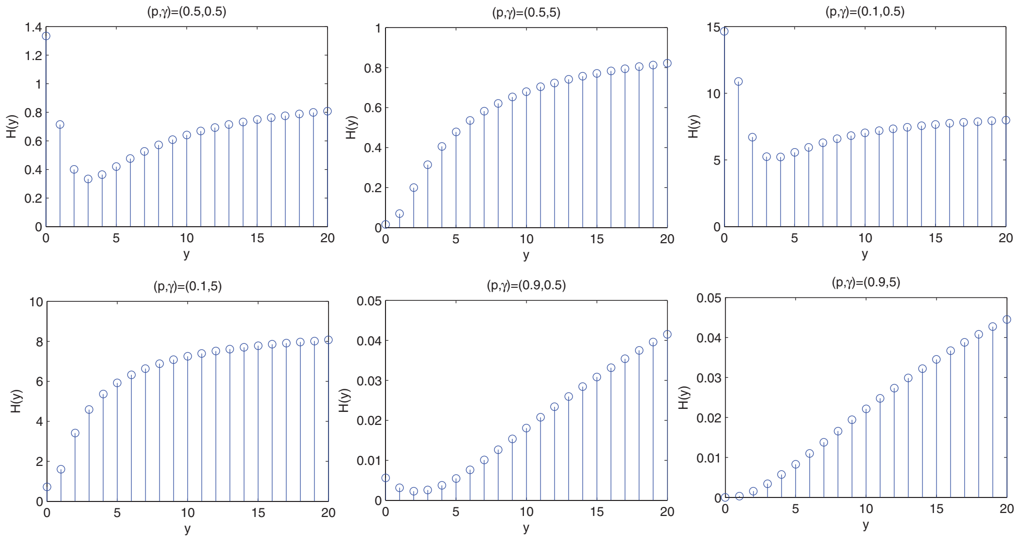

Figure 3: *Illustrations of the failure rate function of $X \sim WGD(p,\gamma)$ for some selected values of p and γ.*

then we have

1. If $\gamma = 0$, $Z \sim \frac{1-p}{1+p}\begin{cases} 1, & z=0 \\ 2p^z, & z=1,2,\dots \end{cases}$, whose truncation at zero is a geometric distribution with parameter p and support on $\{1,2,\dots\}$.

2. $GGD(p,\gamma) \cong GGD(p,-\gamma)$.

3. If $\gamma \longrightarrow \mp\infty$, then $p^*(z;p,\gamma) \longrightarrow \frac{(1-p)^3}{p(1+p)} z^2 p^z, \quad z=1,2,\dots.$

7. Application and comparison

In this section, we attempt to examine application and advantage of $DASL(p,\gamma)$ and $WGD(p,\gamma)$ distributions comparing to several rival models using some real data sets.

Example. The following data set is obtained[1] based on a recent local research carried out on the extent of success of Iranian universities in transferring technology to industry and their effective factors. Out of 500 questionaries distributed, 111 were returned. The data below show the difference between the desired and the existing state values on each sample; a positive number shows the extent of positive improvement and a negative number shows the extent of negative improvement.

1. The data is part of an unpublished research by A. Rafiei, an MSc student.

−1, 1, 3, 3, 1, 1, −20, −14, −15, −14, −5, −6, 0, 7, 4, 1, 9, 13, 5, 4, 1, 1, 1, 14, 5, −2, 4, 5, 3, 3, 12, 3, 4, 4, 5, −1, 5, 3, 4, 3, 4, 6, 6, 7, 0, 0, 0, 13, 0, 10, 15, 2, 2, 5, 11, 2, 2, −16, 2, 8, −7, −7, 2, −2, 9, 6, 11, −5, −5, 13, 13, 1, 14, 0, 8, −5, −2, 10, 2, 10, 8, −3, 10, 12, 14, 12, 11, 11, 8, 0, −2, 4, 6, 0, 0, 7, 8, 1, 9, −1, 9, 6, 11, 0, 7, 7, 10, 4, 7, 9, 28.

Thus, it is logical to compare our distribution with some similar distributions such as *SkewDL* and *ADSLaplace* distributions. *SkewDL* distribution was introduced by Kozubowski and Inusah (2006) and has the following pmf

$$p(x;p,q) = \frac{(1-p)(1-q)}{1-pq} \begin{cases} q^{-x}, & x = \ldots,-2,-1 \\ p^{x}, & x = 0,1,\ldots. \end{cases}$$

ADSLaplace distribution of Barbiero (2014) has pmf

$$p(x;p,q) = \frac{1}{\log(pq)} \begin{cases} \log(p)[q^{-(x+1)}(1-q)], & x = \ldots,-2,-1 \\ \log(q)[p^{x}(1-p)], & x = 0,1,\ldots. \end{cases}$$

Furthermore, we shall also consider the Skellam (Skellam, 1946) with pmf

$$p(x;\mu_1,\mu_2) = e^{-\mu_1-\mu_2}(\mu_1/\mu_2)I_x(2\sqrt{\mu_1\mu_2}), \quad x \in \mathbb{Z}\,.$$

where $I_x(2\sqrt{\mu_1\mu_2})$ is modified Bessel function of the first kind, and discrete normal (Roy, 2003) distribution with pmf:

$$p(x;\mu,\sigma) = \Phi(x+1,\mu,\sigma) - \Phi(x,\mu,\sigma), \quad x \in \mathbb{Z}\,.$$

where $\Phi(.,\mu,\sigma)$ is the cdf of normal distribution with mean μ and variance σ^2, respectively.

The results of comparison are illustrated in Table 2. We have also obtained maximum likelihood estimates and their estimation of standard errors for the parameters involved. We note that under regularity conditions, the standard error of the parameter estimators can be asymptotically computed by root square of the diagonal elements of the inverted Fisher's matrix. The Kolomogrov-Smirnov (K-S) statistic and Akaike information criterion as AIC=$-2\log L + 2k$, where k, the number of parameters in the model, n, the sample size, and L, the maximized value of the likelihood function for the estimated model, are used to compare the estimated models.

Since $DASL(p,\gamma)$ distribution is an extension of DL distribution, in our iterative algorithm of Newton-Raphson, we have used $\gamma = 0$ and the MLE of parameters of DL distribution as initial values to find the MLEs of the parameters. As one can see from Table 2, our model is preferable comparing to other models. Also, Figure 4 shows distribution plots of the data and the models in question.

Table 2: Comparing criterions for the rival distributions.

Model	*Parameter estimaties*	k	$K-S$	$\log L$	AIC
DL	$\widehat{p}=0.8496$, $\text{S.E}(\widehat{p})=0.0108$	1	0.3605	-389.054	780.107
ADSLaplace	$\widehat{p}=0.8787$, $\text{S.E}(\widehat{p})=0.0085$ $\widehat{q}=0.7530$, $\text{S.E}(\widehat{q})=0.0394$	2	0.2162	-379.864	763.728
SkewDL	$\widehat{p}=0.8732$, $\text{S.E}(\widehat{p})=0.0092$ $\widehat{q}=0.7605$, $\text{S.E}(\widehat{q})=0.0368$	2	0.1973	-377.098	758.196
dnormal	$\widehat{\mu}=4.2048$, $\text{S.E}(\widehat{\mu})=0.2271$ $\widehat{\sigma}=6.9769$, $\text{S.E}(\widehat{\sigma})=0.1607$	2	0.1423	-373.189	750.378
DASL	$\widehat{p}=0.7258$, $\text{S.E}(\widehat{p})=0.0225$ $\widehat{\gamma}=-0.3120$, $\text{S.E}(\widehat{\gamma})=0.0785$	2	0.1193	-366.131	736.262
Skellam	$\widehat{\mu_1}=26.0389$, $\text{S.E}(\widehat{\mu_1})=0.3599$ $\widehat{\mu_2}=22.3355$, $\text{S.E}(\widehat{\mu_2})=0.3087$	2	0.1421	-373.102	750.204

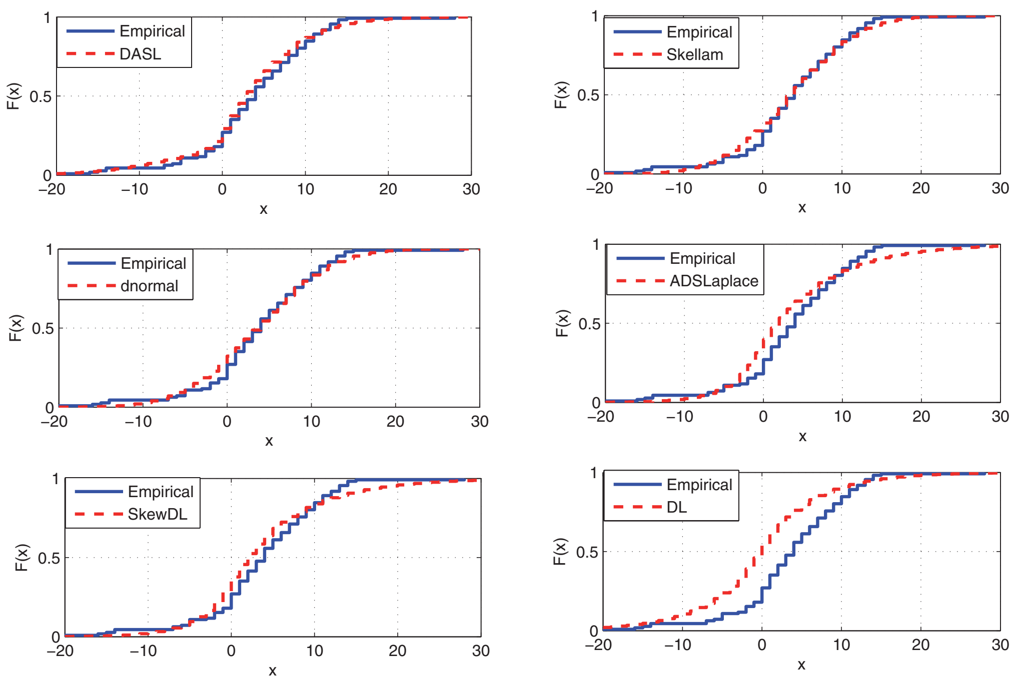

Figure 4: *Plots of empirical distribution functions for the data set and the fitted distributions.*

In addition, we can use the likelihood ratio (LR) test statistic to confirm our claim. To do this, we consider the following test of hypotheses

$$H_0 : \gamma = 0(DL(p)) \qquad v.s \qquad H_1 : \gamma \neq 0(DASL(p,\gamma)).$$

Observed value of the likelihood ratio (LR) test statistic is 43.845 while its tabulated value equals $\chi_1^2 = 3.84$. Thus the null hypothesis is rejected.

On the other hand, Figure 4 shows our different fitted distribution functions and the empirical distribution of the data set. From these plots, we can see that our distribution function better fits the data set.

Acknowledgments

The authors would like to sincerely thank two anonymous reviewers for their careful reading of the paper and their valuable points and comments which led to considerable improvements. The first author is also grateful to the Graduate Office of the University of Isfahan for their support.

References

Barbiero, A. (2014). An alternative discrete skew Laplace distribution. *Statistical Methodology*, 16, 47–67.

Ferguson, T. S. (1996). *A Course in Large Sample Theory*. Chapman and Hall, London.

Graham, R. L., Knuth, D. E. and Patashnik, O. (1989). *Concrete Mathematics*. Addison-Wesley Publishing Company, Reading, MA.

Gradshteyn, I. S. and Ryzhik, I. M. (2007). *Table of Integrals, Series, and Products*, 7th ed. Academic Press, San Diego, CA.

Inusah, S. and Kozubowski, T. J. (2006). A discrete analogue of the Laplace distribution. *Journal of Statistical Planning and Inference*, 136, 1090–1102.

Jayakumar, K. and Jacob, S. (2012). Wrapped skew Laplace distribution on integers: A new probability model for circular data. *Open Journal of Statistics*, 2, 106–114.

Kozubowski, T. J. and Inusah, S. (2006). A skew Laplace distribution on integers. *Annals of the Institute of Statistical Mathematics*, 58, 555–571.

Lawless, J. F. (2003). *Statistical Models and Methods for Lifetime Data*, 2nd ed. John Wiley and Sons, New York.

Molina, J. A. D and Arteaga, A. R. (2007). On the infinite divisibility of some skewed symmetric distributions, *Statistics and Probability Letters*, 77, 644–648.

Roy, D. (2003). The discrete normal distribution. *Communications in Statistics: Theory and Methods*, 32, 1871–1883.

Shams Harandi, S. and Alamatsaz, M. H. (2013). Alpha-Skew-Laplace distribution. *Statistics and Probability Letters*, 83, 774–782.

Skellam, J. G. (1946). The frequency distribution of the difference between two Poisson variates belonging to different populations. *Journal of the Royal Statistical Society, Series A*, 109, 296.

Steutel, F. W and Van Harn, K. (2004). *Infinite Divisibility of Probability Distributions on the Real Line*. Marcel Dekker, Inc, New York.

Weisberg, S. (2005). *Applied Linear Regression*, 3rd edition. John Wiley and Sons, New York.