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I STATE that the present study, entitled "On the local metric dimension of graphs", presented by Gabriel Antonio Barragán-Ramírez for the award of the degree of Doctor has been carried out under my supervision at the Department of Computer Engineering and Mathematics of this university, and that it fulfils all the requirements to be eligible for the International Doctorate Award.

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Abstract

The metric dimension of a general metric space was introduced in 1953 but attracted little attention until, about twenty years later, it was applied to the distances between vertices of a graph. Since then it has been frequently used in graph theory, chemistry, biology, robotics and many other disciplines. Due to the variety of situations from which the problem of distinguishing the vertices of a graph can arise, several variants of the original concept of metric dimension have been appearing in specialized literature. In this thesis we study one of these variants, namely, the local metric dimension. Specifically, we focus on the problem of computing the local metric dimension of graphs. We first report on the state of the art on the local metric dimension and present some original results in which we characterize all graphs that reach some known bounds. Secondly, we obtain closed formulas and tight bounds on the local metric dimension of several families of graphs, including strong product graphs, corona product graphs, rooted product graphs and lexicographic product graphs. Finally, we introduce the study of simultaneous local metric dimension and we give some general results on this new research line.

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Chapter 1

Introduction

Graph theory is a relatively new and very prolific research area in mathematics. The causes of its popularity are manifold. We can not deny its recreational origins that seduced and challenged some brilliant mathematicians, among them, the two most prolific of all time, Leonard Euler and Paul Erdős, whose attention and solutions to what could be regarded as recreational problems opened the door to completely new study areas. However, nowadays a great part of graph theoretical results are published in applied science and engineering journals showing its relevancy to face industrial and other kinds of applied problems.

A good deal of the attractiveness of the theory lies in the deceptive simplicity of the general model, easy to comprehend and to apply to numerous situations. The diversity of problems that can be considered belonging to the theory provide occupation for a wide range of researchers. From the one that tries to prove his or her theorems from scratch and with a naive approach, to ones who apply tools consecrated in other branches of mathematics, such as Algebra or Analysis, thus creating new hybrid areas. From the dyed-in-the-wool purists to the most down-to-earth scientists, everybody can easily find something of interest in Graph theory.

Another reason for the popularity of Graph theory is the great amount of situations one can represent and study by means of a graph, essentially a symmetric relation. From relationships of friendship to the connectivity of computer nets passing through map colourings, industrial processes or board and strategy games, most of them let themselves be modelled by means of a graph. For instance, in computer networks, servers, hosts or hubs can be represented as vertices in a graph and edges can represent connections between them. Likewise, the Internet, social networks or transportation infrastructures are modelled by graphs, where the vertices represent web-pages, users and population centres, respectively; and the edges represent hyperlinks, personal relations, and roads, in that order.

When we use a graph as a model each vertex represents a defined object, but the storage and retrieval of the characteristics that define each of the vertices in order to distinguish one from the other would be costly and impractical. In 1953 Blumenthal [4] introduced the concept of *metric dimension* for general metric spaces. By the concept of metric dimension any metric space can be endowed with a coordinate system that relies only on the distance function of the space. Considering the metric structure of a graph, the concept of metric dimension was applied by Slater [57] who introduced the concept of *locating set* of a graph. Independently Harary and Melter [31] introduced the same concept with the name of *resolving sets* and calculated the metric dimension of a *tree* graph showing that it is possible to find a *metric basis* containing end-vertices only and giving an algorithm to calculate it.

We recall that the pair (M, d) is a *metric space* if M is a nonempty set whose elements are called *points* and d is a binary function in M with values in $\mathbb{R}^+ \cup \{0\}$ such that for every x, y, $z \in M$:

1. $d(x, y) = d(y, x)$
2. $d(x, y) = 0$ if and only if $x = y$
3. $d(x, y) \leq d(x, z) + d(z, y)$

Given a metric space (M, d), a set $B \subseteq M$ is a *metric generator* for M if, for each pair of points, $x, y \in M$ there exists a point $z \in B$ such that $d(z, x) \neq d(z, y)$. As an example of metric space we can think in the pair $(\mathbb{R}^2, d)$ where $\mathbb{R}^2$ is the set of pairs of real numbers and d is the Euclidean distance. We know that, for a point $z_1 \in \mathbb{R}^2$, and a positive number r, the set $Z_1 = \{x \in \mathbb{R}^2 : d(z_1, x) = r\}$ is a circumference. That means that a singleton set cannot be a metric generator for $\mathbb{R}^2$. If we choose a second point $z_2 \in \mathbb{R}^2$ and a positive number s such that $r - s < d(z_1, z_2) < r + s$ and consider the set $Z_2 = \{x \in \mathbb{R}^2 : d(z_2, x) = s\}$ we have that $|Z_1 \cap Z_2| = 2$ and, in consequence, no set of cardinality two can be a metric generator for $\mathbb{R}^2$. However, for every point $z_3 \in \mathbb{R}^2$, non-collinear with z_1 and z_2, we have that the set $W = \{z_1, z_2, z_3\}$ is a metric generator for $\mathbb{R}^2$. As the cardinality of W is minimum among the cardinalities of the sets which have the property of distinguishing any pair of points in $\mathbb{R}^2$, we say that the set W is a metric basis for $\mathbb{R}^2$ and that the metric dimension of $\mathbb{R}^2$ is equal to three (Fig. 1.1).

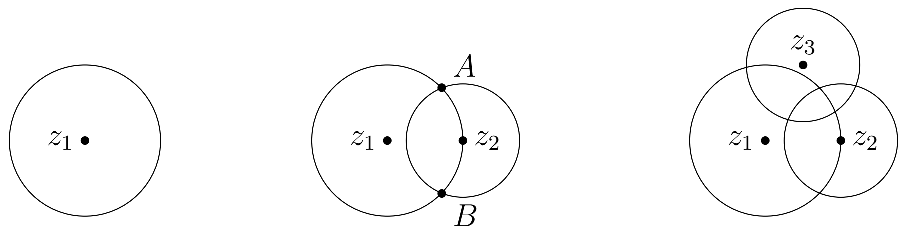

Figure 1.1: From the left, z_1 does not distinguish the vertices in the circumference; neither z_1 nor z_2 distinguishes A from B; every pair of vertices in $\mathbb{R}^2$ is distinguished by z_1, z_2 or z_3.

That means that every point in $\mathbb{R}^2$ is determined by its distances to any three non-collinear points. This system, is an alternative to the classical coordinate system and it relies only on the metric of the space. If the (ordered) set $U = \{z_1, z_2, z_3\}$ is a metric basis for a metric space we can consider for a point x in the space the vector $Code_U(x) = (d(z_1, x), d(z_2, x), d(z_3, x))$ as its coordinates in U since for any pair of points x, y, we have $x = y$ if and only if $Code_U(x) = Code_U(y)$.

Now, we consider an example in the domain of Graph Theory. In Figure 1.2 we have three copies of the *Petersen* graph. In each copy we have chosen a different set of vertices and calculated the coordinates of each vertex in the copy with respect to the correspondent set. The considered set in the first graph is the singleton of A. In the second graph the set is $\{A, B\}$. We can observe that in both cases there exist pairs of vertices non-distinguished by their distances to the vertices in the referenced set. In the third graph we observe that the set $\{A, B, C\}$ distinguishes any pair of vertices, hence the set $\{A, B, C\}$ is a metric generator for the Petersen graph. It can be proved that three is the minimum cardinality for a metric generator of the Petersen graph. Therefore, the metric dimension of the Petersen graph is equal to three.

A metric basis is used to give a coordinate system to a metric space. The metric dimension gives us an idea of how difficult it is to distinguish two different points considering only their distances to some other points that we

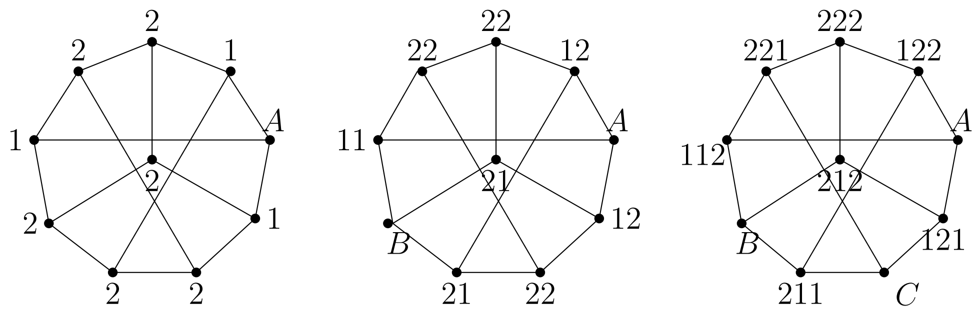

Figure 1.2: Three copies of the Petersen graph indicating the coordinates of each vertex with respect to different sets of vertices

can see as landmarks in the graph. This characterization of points reveals great applicability as it is an inbuilt parameter of the space. Slater [57] described the usefulness of these ideas when working with U.S. sonar and coast guard LORAN (long range aids to navigation). To be able to distinguish each vertex in a graph is useful when we are moving through it or trying to localize a vertex that requires special attention: an SOS point on road or in a subway station, an spoiled device in a computer network, a specific entry in a thesaurus. Khuller et al. [39] mention applications of the metric dimension in the premises of robot navigation in a graph-structured framework. Chartrand et. al.[11] inform that "the structure of a chemical compound is frequently viewed as a set of functional groups arrayed on a structure. From a graph-theoretic perspective, the structure is a labelled graph where the vertex and edge labels specify the atom and bond types, respectively". In the same article they say that the functional group responsible of the pharmacological properties of the compound is a subgraph of the graph representing the compound and that the relative position of this subgraph with respect to specific sets of atoms is relevant in drug discovery. This position can be specified by the distances vector and its study can be optimized by the use of metric generators [1]. Other applications of the metric dimension can be found in digital geometry [46] related with pattern recognition and image processing. Manuel et al. [45] have computed the metric dimension of honeycomb networks underlining their relevance due to their wide use in computer graphics, cellular phone base stations, image processing, and in chemistry as the representation of benzenoid hydrocarbons. The *network verification*

[1]Chartrand et al. call the metric generators *resolving sets*

problem consists of calculating the minimum number of queries that verify all edges and non-edges in a given graph, which is equivalent to determining the metric dimension of a graph [3]. Some variants of the *Mastermind game* let themselves be modelled through a Cartesian product of complete graphs (i.e. a *Hamming graph*) and the number of questions necessary to solve the game is bounded by the metric dimension of such a graph [14], [8]. Also for the *coin weighing problem*, the minimum number of weighings that are necessary to determine the number of coins of each weight from two fixed ones differs from the metric dimension [2] of a *hypercube graph* by at most one unit [36]. The metric dimension of a graph is also used to determine the graphs G such that there exists a wining strategy for a "cop" in the *game of cops and robbers* played on G, as proved in [9].

Due to the multiplicity of situations from which the problem of distinguishing the vertices of a graph can arise, several variants of the original concept of metric dimension have been appearing in specialized literature. Sometimes the same parameter is called in different ways, sometimes close names define quite different concepts. Some of the related notions with their specific features are listed below

- *Resolving dominating set* [6]: The metric generator is also a dominating set.
- *Independent resolving set*: The metric generator is also an independent set. Introduced in [12], we can find some application to the calculus of the total resolvability and weak total resolvability in [10].
- *Connected resolving set* [53]: The graph induced by the metric generator is connected.
- *Strong metric generator* [56], [41] : Two vertices not belonging to the generator are distinguished by some vertex in the metric generator which lies is a minimum-length path with both of them.
- *k-metric generator* [2, 21, 61]: Two vertices are distinguished by at least k vertices in the metric generator.
- *Locating-dominating set* [58]: The metric generator is a dominating set and any two vertices not belonging to the generator have different

[2] In [36], metric dimension of a metric space is called its rigidity.

(open) neighborhoods in the set.

- *Identifying code* [38]: The metric generator is a dominating set and any two vertices of the graph have different closed neighbourhoods in the set.

- *Resolving partition* [13]: Any two different vertices have different distance vectors to the sets of the partition.

- *Strong resolving partition* [59]: For every two vertices belonging to a set of the partition, there exists another set in the partition that strongly resolves the pair.

- *Total resolving set* [10]: Any two vertices in the graph are distinguished by a third, different of both, vertex in the set.

- *Weak total resolving set* [10] Any two vertices, one in the set and the other not, are resolved by a third, different of both, vertex in the set.

- *Adjacency resolving set* [35], [19]: Any two vertices that do not belong to the set, have different neighbourhoods in the set.

- *Simultaneous metric generator*[7]: the set is a metric generator for each member of a family of graphs with common vertex set.

- *Local metric generator* [47]: The set, not necessarily a metric generator, distinguishes any pair of adjacent vertices in the graph.

For the definitions of the concepts used above we refer to Chapter 1 of this work.

In 1979 Garey et. al. [27] proved that the problem of finding the metric dimension of a graph is NP-hard. Diaz et al. [16] proved that the calculus remain NP-hard even when we consider only bounded-degree planar graphs. Epstein et al. [18] proved that the calculus of the metric dimension in the following classes is also NP-hard: split graphs, bipartite graphs, co-bipartite graphs, line graphs of bipartite graphs. Finally Foucaud et al. [25] proved the NP-hardness of the problem for interval graphs. Positive results are that metric dimension is polynomial-time solvable on trees, and the existence of a log(n)-approximation algorithm for general graphs. Metric dimension can also be computed efficiently for co-graphs, k-edge-augmented trees, and wheels. Rodríguez-Velázquez and Fernau [23] proved that also the calculus of the adjacency dimension and the local metric dimension are NP-hard problems.

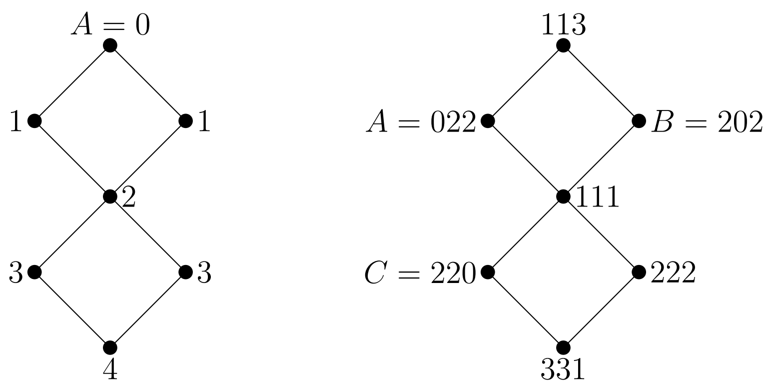

Figure 1.3: Two copies of the graph G showing the distance vectors of each vertex to a determined set. In the left side the set $\{A\}$ is a local metric basis. In the right side the set $\{A, B, C\}$ is a metric basis.

The scope of this work falls into the study of the local metric dimension of graphs. That is, we are concerned in distinguishing the pairs of adjacent vertices in a graph. This study was introduced by Okamoto et. al [47]. In their paper they established some general bounds that we review in Chapter 1 and also gave general constructions showing the relative independence of the study of the local metric dimension from the study of the metric dimension. We can see in Figure 1.3 an example which shows a graph with different local metric dimension and metric dimension. For the sake of clarity we draw two copies of the same graph G. The local metric dimension of the graph G is equal to one, as it is for every bipartite graph, whereas the metric dimension of G is equal to two. In the figure we show the vectors of distances of each vertex to a set of vertices. On the left to a singleton set that is a local metric basis and on the right to a set of two vertices that form a metric basis for G. It is easy to generalize this example in order to show that the difference between the metric dimension and the local metric dimension of a graph can be as big as we want. It suffices to take enough copies of an even-length cycle each one with a distinguished vertex that we proceed to identify. The metric dimension is at least the number of cycles whereas the local metric dimension remains equal to one.

The structure of this thesis is the following: After the introduction, in Chapter 2 we introduce most of the terminology and notation we shall need, we report on the state of the art on local metric dimension and present some original results. Chapter 3 is devoted to the strong product of graphs. We introduce the study of the graphs obtained by point attaching from elementary subgraphs in Chapter 4 and deepen the study of corona product graphs in Chapter 5. In Chapter 6 we deal with the generalized lexicographic product, and in Chapter 7 we study of the simultaneous local metric dimension. In Chapter Conclusions, we present some concluding remarks, summarize the contributions of this thesis, and give a list of future works. Finally, we present the bibliography.

Chapter 2

Basic concepts, notation and general results

2.1 Basics

For the basics concepts in graph theory and notation we globally follow the book of Diestel [17], and the classic book of Harary [30]. We would like to point out that for a graph G we always mean a finite, non-oriented, simple graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. If, for some $u, v \in V(G)$, $\{u, v\} \in E(G)$, we say that the vertices u and v are adjacent and we simplify the notation saying $uv \in E(G)$, otherwise we say that the vertices u and v are not adjacent. The complement of G is the graph G^c with $V(G^c) = V(G)$ and $uv \in E(G^c)$ if and only if $uv \notin E(G)$.

For a vertex u, the *open neighborhood* of u in G is $N_G(u) = \{v \in V(G) : uv \in E(G)\}$. The *degree of* u is $\delta_G(u) = |N_G(u)|$. Special values are $\delta(G) = \min\{\delta(u) : u \in V(G)\}$ and $\Delta(G) = \max\{\delta(u) : u \in V(G)\}$. The *closed neighborhood* of a vertex u is $N_G[u] = N_G(u) \cup \{u\}$.

An *independent set* in G is a set $X \subseteq V(G)$ such that $u, v \in X$ implies $uv \notin E(G)$. The *independence number* of G is $\alpha(G) = \max\{|X| : X$ is an independent set for $G\}$. As a dual for this notion we define a *clique* in G as a set $X \subseteq V(G)$ such that $u, v \in X$ implies $uv \in E(G)$. The parameter $\omega(G) = \max\{|X| : X$ is a clique for $G\}$ is the *clique number* of G. A set $X \subseteq V(G)$ is a dominating set for G if for each $u \in V(G)$, we have $N_G[u] \cap X \neq \emptyset$. The *dominating number* of G is $\gamma(G) = \max\{|X| : X$ is a dominating set for $G\}$.

Two vertices u, v are *true twins* if $N_G[u] = N_G[v]$. They are *false twins* if $N_G(u) = N_G(v)$ and *twins* if they are any of the previous. These three relations are equivalence relations. It is not difficult to see that the equivalence classes of the true-twin relations are cliques and those of the false-twin relations are independent sets. It follows that a class that contains both true twins and false twins has to be a singleton.

For a connected graph G, the distance between two vertices $u, v \in V(G)$ is denoted by $d_G(u, v)$ and the *diameter* of G is $D(G) = \max\{d_G(u, v) : u, v \in V(G)\}$. Given a vertex $u \in V(G)$ its *eccentricity* is $\varepsilon(u) = \max\{d_G(u, v) : v \in V(G)\}$. The *radius* of G is $r(G) = \min\{\varepsilon(u) : u \in V(G)\}$ and the center of G is $C(G) = \{u \in V(G) : \varepsilon(u) = r(G)\}$. The *girth* $\mathsf{g}(G)$ of the graph G is the length of its shortest cycle, if there is any, and ∞ in the case of acyclic graphs. Also we say that the diameter of a non-connected graph is ∞. Two vertices $u, v \in V(G)$ are *diametral* vertices if $d_G(u, v) = D(G)$.

We say that a vertex $u \in V(G)$ *distinguishes* two vertices $x, y \in V(G)$ if $d_G(u, x) \neq d_G(u, y)$. A *metric generator* for G is a set $B \subseteq V(G)$ with the property that, for each pair of vertices $x, y \in V(G)$, there exists a vertex $u \in B$ that distinguishes x, y. If for some metric generator $A \subseteq V(G)$, we have that $|A| = \min\{|B| : B \text{ is a metric generator for } G\}$, we say that A is a *metric basis* for G and, in this case, $\dim(G) = |A|$, is the *metric dimension* of G.

In this work we are focused on the *local metric dimension* of a graph that is defined as follows: A set $L \subseteq V(G)$ is said to be a *local metric generator* for G if for each pair of vertices $x, y \in V(G)$ *such that* $uv \in E(G)$, there exists a vertex $u \in L$ that distinguishes u and v. If for some local metric generator $M \subseteq V(G)$, we have that $|M| = \min\{|L| : L \text{ is a local metric generator for } G\}$, then we say that M is a *local metric basis* for G and $\dim_l(G) = |M|$, is the *local metric dimension* of G.

The concept of adjacency generator[1] was introduced by Jannesari and Omoomi [35] as a tool to study the metric dimension of lexicographic product graphs. An *adjacency generator* for G is a set $B \subset V(G)$ such that for each $x, y \in V(G) - B$ there exists $b \in B$ such that b is adjacent to exactly one of x and y. An adjacency generator whose cardinality is the minimum amomg the cardinalities of all the adjacency generators of G is called an *adjacency basis* of G, and its cardinality is the *adjacency dimension* of G, denoted by

[1] Adjacency generators were called adjacency resolving sets in [35].

adim(G) [35]. The concepts of *local adjacency generator*, *local adjacency basis* and *local adjacency dimension* are defined by analogy, and the local adjacency dimension of a graph G is denoted by $\mathrm{adim}_l(G)$. Fernau and Rodríguez-Velázquez in [23, 24] introduced the study of local adjacency generators and showed that the (local) metric dimension of the corona product of a graph of order n and some non-trivial graph H equals n times the (local) adjacency dimension of H. As a consequence of this strong relation they showed that the problem of computing the local metric dimension and the (local) adjacency dimension is an NP-hard problem.

As pointed out in [23, 24], any adjacency generator of a graph $G = (V, E)$ is also a metric generator in a suitably chosen metric space. Given a positive integer t, we define the distance function $d_{G,t} : V \times V \to \mathbb{N} \cup \{0\}$, as

$$d_{G,t}(x, y) = \min\{d_G(x, y), t\}. \tag{2.1}$$

From this definition is clear that any metric generator for $(V, d_{G,t})$ is a metric generator for $(V, d_{G,t+1})$ and, as a consequence, the metric dimension of $(V, d_{G,t+1})$ is less than or equal to the metric dimension of $(V, d_{G,t})$. In particular, the metric dimension of $(V, d_{G,1})$ equals $|V|-1$, the metric dimension of $(V, d_{G,2})$ equals $\mathrm{adim}(G)$ and, as $d_{G,D(G)} = d_G$, the metric dimension of $(V, d_{G,D(G)})$ equals $\dim(G)$.

Notice that B is an adjacency generator for G if and only if B is an adjacency generator for its complement G^c. This is justified by the fact that given an adjacency generator B for G, it holds that for every $x, y \in V - B$ there exists $b \in B$ such that b is adjacent to exactly one of x and y, and this property holds in G^c. Thus, $\mathrm{adim}(G) = \mathrm{adim}(G^c)$.

We say that $A \subseteq V(G)$ is a *local adjacency generator* for a graph G if, for every pair of vertices $u, v \in V(G)-A$ such that $uv \in E(G)$, there exists $w \in A$ such that $|N_G(w) \cap \{u, v\}| = 1$. If the set B is a local adjacency generator for G with the property that for every other local adjacency generator A, $|B| \leq |A|$, then we said that B is a *local adjacency basis* for G and its cardinality is the *local adjacency dimension* of G, denoted by $\mathrm{adim}_l(G)$.

Remark 2.1. *As every local adjacency generator of G is also a local metric generator for G,*

$$\dim_l(G) \leq \mathrm{adim}_l(G).$$

It is enough to consider the graph P_6 to see that the inequality in Remark

2.1 can be strict:

$$1 = \dim_l(P_6) < 2 = \mathrm{adim}_l(P_6).$$

The study of the local adjacency dimension of a graph arises in a natural way in the study of the local metric dimension in some graph operations in which the maximum distance between two vertices of a distinguished subgraph becomes equal to two (as in the corona product or the lexicographic product of graphs that we will define further in this work, see the chapters related to the lexicographic product and corona graphs).

From the definitions of the different variants of generators, we can observe: an adjacency generator is a metric generator; a metric generator is a local metric generator; a local adjacency generator is a local metric generator; an adjacency generator is a local adjacency generator. These facts show that the following inequalities hold:

(i) $\dim(G) \le \mathrm{adim}(G)$

(ii) $\dim_l(G) \le \dim(G) \le \dim_l(G) + \mathrm{adim}_l(G^c)$

(iii) $\dim_l(G) \le \mathrm{adim}_l(G)$

(iv) $\mathrm{adim}_l(G) \le \mathrm{adim}(G)$

Remark 2.2. *Let G be a graph. If there exists a local metric basis $B \subseteq V(G)$ such that $\varepsilon(b) < 3$ for every $b \in B$, then $\dim_l(G) = \mathrm{adim}_l(G)$.*

Proof. For a graph G suppose that there exists a local metric basis $B \subseteq V(G)$ such that $b \in B$ implies $\varepsilon(b) < 3$. Let $u, v \in V(G)$ such that $uv \in E(G)$. There exist $b \in B$ such that $d_G(b,u) \neq d_G(b,v)$, as $\varepsilon(b) < 3$, $\max\{d_G(b,u), d_G(b,v)\} \le 2$, then $d_{G,2}(b,u) = d_G(b,u) \neq d_G(b,v) = d_{G,2}(b,v)$ and we are done. □

Theorem 2.3. [23] *Let G be a non-empty graph of order t. The following assertions hold.*

(i) $\mathrm{adim}_l(G) = 1$ *if and only if G is a bipartite graph having only one non-trivial connected component G^* and $r(G^*) \le 2$.*

(ii) $\mathrm{adim}_l(G) = t - 1$ *if and only if $G \cong K_t$.*

As a comparative between metric dimension and local metric dimension we present Table 2.1 where: K_n is the *complete graph* on n vertices, N_n is the *empty graph* on n vertices, P_n is the *path graph* on n vertices, S_n is the *star graph* on n vertices, K_{n_1,n_2} is the *complete bipartite graph* on n_1+n_2 vertices, B_{n_1,n_2} is an arbitrary connected *bipartite graph* on n_1+n_2 vertices, C_{2n}, C_{2n+1} are the *cycle graph* on $2n$ and $2n+1$ vertices respectivelly, W_n is the *wheel graph* on n vertices.

G	$\dim(G)$	$\dim_l(G)$	$\mathrm{adim}(G)$	$\mathrm{adim}_l(G)$
K_n	$n-1$	$n-1$	$n-1$	$n-1$
N_n	$n-1$	0	$n-1$	0
P_n	1	1	$\lfloor \frac{2n+2}{5} \rfloor$	$\lfloor \frac{n}{4} \rfloor$ $(n \equiv 1(4))$ and $\lceil \frac{n}{4} \rceil$ i.o.c.
S_n	$n-2$	1	$n-2$	1
K_{n_1,n_2}	n_1+n_2-2	1	n_1+n_2-2	1
B_{n_1,n_2}	NP-hard	1	?	?
C_{2n}	2	1	$\lfloor \frac{2n+2}{5} \rfloor$	$\lceil \frac{n}{4} \rceil$
C_{2n+1}	2	2	$\lfloor \frac{2n+2}{5} \rfloor$	$\lceil \frac{n}{4} \rceil$
W_n	$n \notin \{4,7\}$, $\lfloor \frac{2n}{5} \rfloor$	$n > 5$, $\lceil \frac{n-1}{4} \rceil$	$\dim(W_n)$	$\dim_l(W_n)$

Table 2.1: A comparative between $\dim(G), \dim_l(G), \mathrm{adim}(G)$, and $\mathrm{adim}_l(G)$

2.2 General results

In this section we will present some of the first results about local metric dimension of graphs. First of all we would like to remark that Rodríguez-Velázquez and Hening [24] proved that in the general case the calculus of the local metric dimension of an arbitrary graph is NP-hard. We can, however, present some general results. Most of these results were obtained by Okamoto et al.[47] in their seminal article on the subject. They deal with the general case, most of their results are bounds that they proved to be tight. As a novelty, in subsequent sections, we give the characterization of the graphs in which some of these bounds are attained. We start with an useful result.

Theorem 2.4. [47] *Let G be a connected graph of order n. The following statements hold.*

- $\dim_l(G) = 1$ *if and only if G is bipartite.*
- $\dim_l(G) = n - 1$ *if and only if $G \cong K_n$.*
- $\dim_l(G) = n - 2$ *if and only if $\omega(G) = n - 1$.*

The following Remarks relate the local metric dimension of the graph with some special subgraphs.

Remark 2.5. [47] *For a graph G of order n and independence number α,* $\dim_l(G) \leq n - \alpha$*.*

Remark 2.6. [47] *If G is a nontrivial connected graph of order n and diameter $D(G)$, then* $\dim_l(G) \leq n - D(G)$*.*

The relation of the local metric dimension of the graph and its clique number is not trivial and it is studied in the following Theorems:

Theorem 2.7. [47] *If G is a nontrivial connected graph of order n and clique number ω, then* $\dim_l(G) \geq \lceil \log_2(\omega) \rceil$*. Furthermore, for each integer ω such that $\omega \geq 2$, there exists a connected graph G_ω with clique number ω such that* $\dim_l(G_\omega) = \lceil \log_2(\omega) \rceil$*.*

Theorem 2.8. [47] *If G is a nontrivial connected graph of order n with $\omega = \omega(G)$, then* $\dim_l(G) < n - 2^{n-\omega}$*. Furthermore, for each pair n, ω of integers with $2^{n-\omega} \leq \omega \leq n$, there exists a connected graph G of order n whose clique number is ω such that* $\dim_l(G) = n - 2^{n-\omega}$*.*

From Theorem 2.8 we have the following Corollary.

Corollary 2.9. [47] *If G is a nontrivial connected graph of order n and* $\dim_l(G) = n - k$*, then $\omega(G) \leq n - \lceil \log_2(k) \rceil$.*

We merge an observation and a theorem from Okamoto et. al [47] in the next theorem that involves the true twin classes of G.

Theorem 2.10. [47] *Let G be a nontrivial connected graph of order n having l true twin equivalence classes. If p of these l true twin equivalence classes consist of a single vertex, then* $n - l \leq \dim_l(G) \leq n - l + p$*.*

In fact Okamoto et al. [47] give a necessary and sufficient condition for $\dim_l(G) = n - l$. Prior to enunciate the result we need a pair of definitions.

For two subsets X, Y of $V(G)$ them define $d_G(X,Y) = \min\{d_G(u,v) : u \in X, v \in Y\}$. Let $\mathcal{U} = \{U_1, \ldots, U_l\}$ the set of true twin classes in G. For $\mathcal{V} \subseteq \mathcal{U}$ ordered as $\mathcal{V} = \{V_1, \ldots, V_r\}$ and $U_i \in \mathcal{U}$, define $code^*_{\mathcal{V}}(U)$ as the ordered $r - tuple$, $(a_1, \ldots, a_r)$ where

$$a_i = \begin{cases} d_G(V_i, U) & if \quad V_i \neq U \\ 1 & if \quad V_i = U \end{cases}$$

Theorem 2.11. [47] *Let G be non trivial connected graph with true twin equivalence classes $U_1, \ldots, U_l$, at least one of them non a singleton class, $|U_i| \geq |U_{i+1}|$ for $i = 1, \ldots, l-1$. Let $k = \max\{i : |U_i| > 1\}$. Consider the ordered set $\mathcal{S} = (U_1, \ldots, U_k)$, $\dim_l(G) = n - l$ if and only if $code^*_{\mathcal{S}}(U_i) \neq code^*_{\mathcal{S}}(U_j)$ for each U_i, U_j, such that $d_G(U_i, U_j) = 1$*

Finally, we recall that given two graphs G and H, its Cartesian product $G \square H$ is the graph defined as follows:

- $V(G \square H) = V(G) \times V(H)$
- $(u_1, v_1)(u_2, v_2) \in E(G \square H)$ if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(H)$.

For a deep study of graph products and their properties we refer to the book [29].

Okamoto et al. [47] calculate the local metric dimension of the Cartesian product of two graphs.

Theorem 2.12 ([47])**.** *For every connected graphs H and G,*

$$\dim_l(G \square H) = \max\{\dim_l(G), \dim_l(H)\}$$

Upper bounds using independent sets

In this section we present two novelties related with Remark 2.5. First we characterize the family of graphs G such that $\dim_l(G) = n - \alpha$ and second, we give another bound also related with the independence number. To begin with, we give the following remarks, definitions and results.

A family $\mathcal{A} = \{A_1, \ldots, A_k\}$ of subsets of a set A is a *clustered nested cover* of A, if $\mathcal{A}$ is a cover ($\cup A_i = A$) and for every $A_i, A_j \in \mathcal{A}$, $A_i \neq \emptyset \neq A_j$

and either $A_i \subseteq A_j$ or $A_j \subseteq A_i$ or $A_i \cap A_j = \emptyset$. For any non-empty set $A = \{a_1, \ldots, a_n\}$ we can give the following examples of clustered nested covers of the set A:

- The family whose only element is the set A, $\mathcal{A} = \{A\}$.
- The family elements of any partition of the set A. As examples:
  - The *total partition* family: $\mathcal{TP}(A) = \{\{a_1\}, \ldots, \{a_n\}\}$.
  - For any non-empty $B \subset A$, the family induced by B: $\mathcal{A}(B) = \{B, A - B\}$.
- The *nested* family: $\mathcal{N}(a_{i_1}, \ldots, a_{i_n}) = \{A_1 = \{a_{i_1}\}, \ldots, A_n = A\}$, where $A_{j+1} = A_j \cup \{a_{i_{j+1}}\}$, for $1 \leq j \leq n-1$.

Remark 2.13. *Let $\mathcal{A}$, $\mathcal{B}$ two covers of a set A such that $\mathcal{B} \subseteq \mathcal{A}$. If $\mathcal{A}$ is a clustered nested cover of A, then $\mathcal{B}$ is also a clustered nested cover of A.*

Let G be a graph, and $\mathcal{B} = \{B_i : B_i \subseteq V(G)\}$ a family of non-empty subsets of $V(G)$. Let $s = |\mathcal{B}|$ and consider a family of pairs $\mathcal{H} = \{(H_1, A_1) \ldots, (H_s, A_s)\}$, where H_i is a graph and $A_i \subseteq V(H_i)$ for $i = 1, \ldots, s$. We define the graph $G +_{\mathcal{B}} \mathcal{H}$ from the graph G and the graphs H_i by joining, by an edge, each element $v \in A_i$ to each vertex $u \in B_i$ for $1 \leq i \leq s$. Formally:

- $V(G +_{\mathcal{B}} \mathcal{H}) = V(G) \bigcup (\cup_{H_i \in \mathcal{H}} V(H_i))$
- $uv \in E(G +_{\mathcal{B}} \mathcal{H})$ if and only if either $uv \in E(G)$ or $u \in B_i, v \in A_i$

As examples of this construction we have:

- The *n-sun graph* (Figure 2.1) can be defined as
$$K_n +_{\{\{v_i, v_{i+1}\}: i=1,\ldots,n-1\} \cup \{\{v_n, v_1\}\}} \{(K_1, V(K_1)), \ldots, (K_1, V(K_1))\}.$$
- The *n-sunlet* graph as $C_n +_{\{\{v_i\}: i=1,\ldots,n\}} \{(K_1, V(K_1)), \ldots, (K_1, V(K_1))\}$. (Figure 2.1).
- Let G a graph of order n, the *Corona product* of the graph G and the family $\mathcal{H}$ is $G \odot \mathcal{H} = G +_{V(G)} \{(H_1, V(H_1)), \ldots, (H_n, V(H_n))\}$ (Figure 2.2).

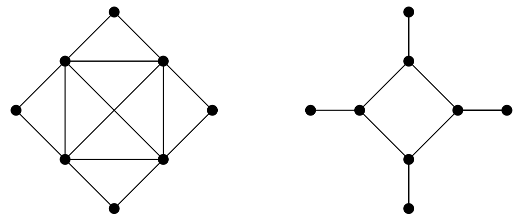

Figure 2.1: Left: 4-sun graph. Right: 4-sunlet graph

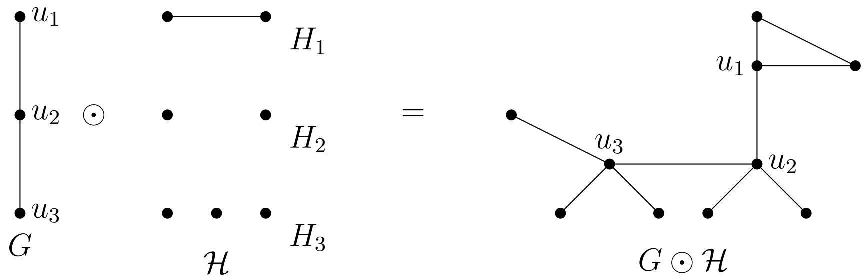

Figure 2.2: The corona product of G and the family $\mathcal{H} = \{H_1, H_2, H_3\}$

- The *join* of the graphs G and H is defined $G + H = G +_{\{\{V(G)\}\}} \{(H, V(H))\}$ (Figure 2.3).

Given a set A and family of subsets $\mathcal{A} = \{A_i \subseteq A\}$, we say that $\mathcal{A}$ *distinguishes* the elements of A if for each pair of different elements $a, b \in A$ there exists $A_i \in \mathcal{A}$ such that $|A_i \cap \{a, b\}| = 1$. Let $A = \{a_1, \ldots, a_n\}$ a set, examples of families of subsets that distinguish the elements of A are $\mathcal{TP}(A)$ and $\mathcal{N}(a_{i_1}, \ldots, a_{i_n})$.

Remark 2.14. *Let $\mathcal{A}$ be clustered nested cover of a set A. If $\mathcal{A}$ distinguishes the elements of A, then there exists $A_i \in \mathcal{A}$ such that $|A_i| = 1$.*

Proof. Suppose, for a contradiction, that $\mathcal{A}$ is a clustered nested cover of a set A that distinguishes the elements of A and, for every $A_i \in \mathcal{A}$, we have $|A_i| \geq 2$.Let $A_0 \in \mathcal{A}$ such that $|A_0| = \min\{|A_i| : A_i \in \mathcal{A}\}$. The elements in A_0 are not distinguished, which is a contradiction. $\square$

Lemma 2.15. *Let A be a set such that $|A| = n \geq 2$, and $\mathcal{A}$ a clustered nested cover of A. If $\mathcal{A}$ distinguishes the elements of the set A then $|\mathcal{A}| \geq n$.*

Proof. We proceed by induction over the cardinal of A. If $|A| = 2$, say $A = \{a, b\}$, then the clustered nested covers of A that distinguish the elements of A

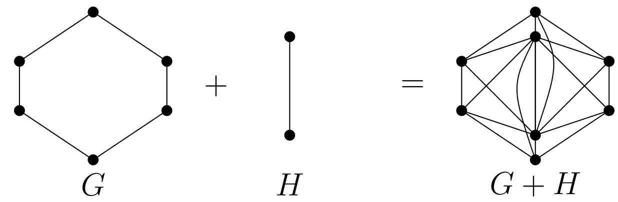

Figure 2.3: The join of G an H

are $\mathcal{A}_1 = \{\{a\},\{b\}\}$, $\mathcal{A}_2 = \{\{a\}, A\}$, $\mathcal{A}_3 = \{\{b\}, A\}$ and $\mathcal{A}_4 = \{\{a\},\{b\}, A\}$, and all of them have at least two elements. Let us suppose then that the result holds for any set of cardinality less than or equal $k \geq 2$. Let A be a set such that $|A| = k+1$ and $\mathcal{A}$ a clustered nested cover of A that distinguishes the elements of A. Consider $a \in A$ such that $\{a\} \in \mathcal{A}$, such an a exists by Remark 2.14. For each $A_i \in \mathcal{A} - \{\{a\}\}$ define $A_i^a = A - \{a\}$, and consider the family $\mathcal{A}^a = \{A_i^a : A_i \in \mathcal{A} - \{\{a\}\}\}$. The family $\mathcal{A}^a$ is a family of non-empty subsets of $A - \{a\}$ and for any $A_i, A_j \in \mathcal{A}$, if $A_i \subseteq A_j$, then $A_i^a \subseteq A_j^a$ and if $A_i \cap A_j = \emptyset$, then $A_i^a \cap A_j^a = \emptyset$. As $\cup A_i^a = A - \{a\}$, we can conclude that $\mathcal{A}^a$ is a clustered nested cover of $A - \{a\}$. Moreover, as $\mathcal{A}$ distinguishes the elements of the set A, for any $x, y \in A - \{a\} \subseteq A$ there exists $A_i \in \mathcal{A}$ such that $|A_i \cap \{x,y\}| = 1$. As $a \notin \{x,y\}$, $|(A_i - \{a\}) \cap \{x,y\}| = 1$, which implies that $\mathcal{A}^a$ distinguishes the elements of $A - \{a\}$. By the induction hypothesis, $|\mathcal{A}^a| \geq |A - \{a\}| \geq k$, thus $|\mathcal{A}| = |\mathcal{A}^a| + 1 \geq k+1$ and the result follows. □

Theorem 2.16. *Let G be a connected graph G of order n and independence number $\alpha(G)$. The following statements are equivalent:*

1. $\dim_l(G) = n - \alpha(G)$
2. $\operatorname{adim}_l(G) = n - \alpha(G)$
3. $G \cong K_r +_{\mathcal{B}} \mathcal{H}$, *where* $\mathcal{B} = \{B_i : B_i \subseteq V(K_r)\}$ *is a clustered nested cover of* $V(K_r)$, $V(K_r) \in \mathcal{B}$, $|\mathcal{B}| = s$ *and* $\mathcal{H} = \{(N_{n_1}, V(N_{n_1})), \ldots, ((N_{n_s}, V(N_{n_s}))\}$.

Proof. Let G be a connected graph G of order n. Suppose first that $\dim_l(G) = n - \alpha(G)$. Remark 2.1 implies that $n - \alpha(G) = \dim_l(G) \leq \operatorname{adim}_l(G) \leq n - \alpha(G)$ and then $\operatorname{adim}_l(G) = n - \alpha(G)$.

Suppose now that $\operatorname{adim}_l(G) = n - \alpha(G)$. Let $X \subseteq V(G)$ be an independent set such that $|X| = \alpha(G)$. By hypothesis, $Y = V(G) - X$ is a local adjacency basis for G. Consider the partition of X into false twin classes $X_1, \ldots, X_s$. For each X_i fix a vertex $x_i \in X_i$ and let $R = \{x_i : x_i \in X_i, 1 \leq i \leq s\}$. Consider the families $\mathcal{B} = \{B_i \subseteq Y : B_i = N_G(x_i), x_i \in R\}$ and $\mathcal{H} = \{(H_1, X_1)), \ldots, (H_s, X_s))\}$ where $V(H_i) = X_i$ and $H_i \cong N_{|X_i|}$, for $1 \leq i \leq s$.

Claim 1: *For each $B_i, B_j \in \mathcal{B}$ either $B_i \subseteq B_j$ or $B_j \subseteq B_i$ or $B_i \cap B_j = \emptyset$.* We proceed by contradiction. Suppose that there exist $B_i, B_j \in \mathcal{B}$ such that $\emptyset \notin \{B_i \cap B_j, B_i - B_j, B_j - B_i\}$. Let $u_1 \in B_i \cap B_j$, $u_2 \in B_i - B_j$ and $u_3 \in B_j - B_i$. We affirm that $(Y - \{u_1, u_2, u_3\}) \cup \{x_i, x_j\}$ is a local adjacency generator for G. For any $u, v \in V(G) - ((Y - \{u_1, u_2, u_3\}) \cup \{x_i, x_j\}) = (X - \{x_i, x_j\}) \cup \{u_1, u_2, u_3\}$, such that $uv \in E(G)$ we have $\{u, v\} \cap \{u_1, u_2, u_3\} \neq \emptyset$. If $|\{u, v\} \cap \{u_1, u_2, u_3\}| = 1$, say $u \in \{u_1, u_2, u_3\}$ and $v \in X - \{x_i, x_j\}$, then, for some $x \in \{x_i, x_j\}$, say x_i, $d_G(x_i, u) = 1 \neq 2 = d_G(x_i, v)$ and we are done. Otherwise $|\{u, v\} \cap \{u_1, u_2, u_3\}| = 2$ and, without loss of generality, $v \notin B_i$. Then $d_G(x_i, u) = 1 \neq 2 = d_G(x_i, v)$ and we are done. Hence $(Y - \{u_1, u_2, u_3\}) \cup \{x_i, x_j\}$ is a local adjacency basis for G and $|(Y - \{u_1, u_2, u_3\}) \cup \{x_i, x_j\}| = |Y| - 1$ which is a contradiction. Therefore for each $B_i, B_j \in \mathcal{B}$, $B_i \subseteq B_j$ or $B_j \subseteq B_i$ or $B_i \cap B_j = \emptyset$.

Claim 2. *Y is a clique in G.* Suppose, for a contradiction, that there exist $u_1, u_2 \in Y$ such that $u_1u_2 \notin E(G)$. As X is a maximal independent set for G, X is a dominating set for G. Let $v \in X$ such that $u_1v \in E(G)$. We affirm that $(Y - \{u_1, u_2\}) \cup \{v\}$ is a local adjacency generator for G. In order to see that let $x, y \in V(G) - ((Y - \{u_1, u_2\}) \cup \{v\}) = (X - \{v\}) \cup \{u_1, u_2\}$, such that $xy \in E(G)$. Without loss of generality, $x = u_1$, $y \in X - \{v\}$, hence $d_G(v, x) = 1 \neq 2 = d_G(v, y)$ and we are done. Thus $(Y - \{u_1, u_2\}) \cup \{v\}$ is a local adjacency generator for G and $|(Y - \{u_1, u_2\}) \cup \{v\}| = |Y| - 1$ which is a contradiction. Therefore Y is a clique in G.

Claim 3.*There exists $v \in X$ such that $N_G(v) = Y$.* If $|Y| = 1$, then $G \cong S_n$ (the *star graph* on n vertices) and result follows. Let us suppose then that $|Y| \geq 2$, and, for every $v \in X$, $N_G(v) \neq Y$. If $|Y| = 2$, then, by Claim 2, $Y \cong K_2$ and G is a tree of diameter equal to three. In this case, for any $u \in Y$, $\{u\}$ is a local metric basis of G. Since $\varepsilon(u) = 2$, Remark 2.2 implies that $\{u\}$ is a local adjacency basis for G and hence $\operatorname{adim}_l(G) = 1 < 2 = |Y| = n - \alpha(G)$ which is a contradiction. Thus $|Y| \geq 3$. Let $\widetilde{X} = \{v \in X :$ for every $v_i \in X$, $N_G(v_i) \subseteq N_G(v)$ or $N_G(v_i) \cap N_G(v) = \emptyset\}$ and fix $v_1 \in \widetilde{X}$ such that$|N_G(v_1)| = \max\{|N_G(v_i)| : v_i \in \widetilde{X}\}$. Let $u_1 \in$

$N_G(v_1)$ and $u_2 \in Y - N_G(v_1)$. We claim that $(Y - \{u_1, u_2\}) \cup \{v_1\}$ is a local adjacency basis for G. In order to prove that, consider a pair of adjacent vertices $x, y \in V(G) - ((Y - \{u_1, u_2\}) \cup \{v_1\}) = (X - \{v_1\}) \cup \{u_1, u_2\}$. We differentiate the following cases:

- Case 1: $x = u_1$. In this case $d_G(v_1, u_1) = 1 \neq 2 = d_G(v_1, u_2)$ and we are done.

- Case 2: $x = u_2$, $y \in X - \{v_1\}$. As $|Y| \geq 3$ either there exists $u_3 \in N_G(v_1) - \{u_1\}$, and Claim 2 implies that $d_G(u_3, u_2) = 1 \neq 2 = d_G(u_3, y)$, or $N_G(v_1) = \{u_1\}$ and, as $|N_G(v_1)|$ is maximum, $N_G(y) = \{u_2\}$. So, for $u_3 \notin N_G(y)$, $d_G(u_3, u_2) = 1 \neq 2 = d_G(u_3, y)$.

Hence $(Y - \{u_1, u_2\}) \cup \{v_1\}$ is a local adjacency generator for G and $|(Y - \{u_1, u_2\}) \cup \{v_1\}| = |Y| - 1$, which is a contradiction. Thus there exists $v \in X$ such that $N_G(v) = Y$. From the three claims above we have that Y is a clique, $\mathcal{B}$ is a clustered nested cover of Y, $Y \in \mathcal{B}$ and $G \cong K_{|Y|} +_{\mathcal{B}} \mathcal{H}$, where $\mathcal{H}$ has the form $\{(N_{n_1}, V(N_{n_1})), \ldots, ((N_{n_s}, V(N_{n_s}))\}$.

Now we suppose that $G \cong K_r +_{\mathcal{B}} \mathcal{N}$ as in the hypotheses. We have to prove that $\dim_l(G) = n - \alpha(G)$. By Lemma 2.5, $\dim_l(G) \leq n - \alpha(G)$ and the definition of G implies that $n - \alpha(G) = r$. Suppose, for a contradiction, that $\dim_l(G) < n - \alpha = r$, and let A be a local metric basis for G. Notice that, $V(K_r) - A \neq \emptyset$. Define $A_1 = A \cap V(K_r)$ and $A_2 = A - A_1$. As $V(K_r) \in \mathcal{B}$, there exist $H_i \in \mathcal{H}$ and $v_0 \in V(H_i)$ such that for every $u \in V(K_r)$, $uv_0 \in E(G)$. Therefore $A_2 \neq \emptyset$, because if $A \subseteq V(K_r)$ and $u_0 \in V(K_r) - A$, no vertex in A distinguishes u_0 and v_0. From the above considerations, $|A_1| \leq |V(K_r) - \{u_0\}| - |A_2| \leq r - 2$ and $|V(K_r) - A_1| \geq 2$. Consider the family $\mathcal{B}_{A_2} = \{N_G(v) - A_1 : v \in A_2\}$. Either $v_0 \in A_2$ or for each $x \in V(K_r) - A_1$, there exist $v \in A_2$ such that $xv \in E(G)$, in order to distinguish x and v_0. It is straightforward to see that $\mathcal{B}_{A_2}$ is a clustered nested cover of $V(K_r) - A_1$. For every $u_1, u_2 \in V(K_r) - A_1$ there exists $v \in A_2$ such that v distinguishes the pair, thus $|N_G(v) \cap \{u_1, u_2\}| = 1$. As $u_1, u_2 \notin A_1$, $|(N_G(v) - A_1) \cap \{u_1, u_2\}| = 1$. Thus the family $\mathcal{B}_{A_2}$ distinguishes the elements of $V(K_r) - A_1$. Lemma 2.15 implies that $|\mathcal{B}_{A_2}| \geq r - |A_1|$. By definition $|A_2| \geq |\mathcal{B}_{A_2}|$. Hence $|A| = |A_1| + |A_2| \geq r$ which is a contradiction and therefore $\dim_l(G) = n - \alpha(G)$. □

Corollary 2.17. *Let G be a connected graph of order n and independence number $\alpha(G)$. If* $\dim_l(G) = n - (\alpha(G) + 1)$, *then* $\operatorname{adim}_l(G) = n - (\alpha(G) + 1)$

Proof. Let G be a connected graph of order n and independence number $\alpha(G)$, such that $\dim_l(G) = n - (\alpha(G) + 1)$. If $\dim_l(G) \neq n - \alpha(G)$, then by Theorem 2.16, $\mathrm{adim}_l(G) \neq n - \alpha(G)$ then $n - (\alpha(G) + 1) = \dim_l(G) \leq \mathrm{adim}_l(G) \leq n - (\alpha(G) + 1)$ and the result follows. □

Remark 2.18. *The converse of Corollary 2.17 is not true.*

Proof. It is sufficient to consider $G \cong P_7$ to see that. In this case $|V(G)| = 7$, $\alpha(G) = 4$, $\mathrm{adim}_l(G) = 2 = |V(G)| - (\alpha(G) + 1) > 1 = \dim_l(G)$. □

As a second result in this section we present another upper bound for the local metric dimension of a graph in terms of its independent sets.

Let G be a connected graph with independence number $\alpha = \alpha(G)$ and let $S = \{s_1, \ldots, s_\alpha\} \subseteq V(G)$ be a maximal independent set. Let $Q(S) = \{N_G(u) \cap S : u \in V(G) - S\}$. For $A_i \in Q(S)$ let $U_{A_i} = \{x \in V(G) - S : N_G(x) \cap S = A_i\}$ and let α_i be the independence number of the graph induced in $G - S$ by U_{A_i}. For each $A_i \in Q(S)$ let $S_i \subseteq U_{A_i}$ an independent set of G such that $|S_i| = \alpha_i$.

Remark 2.19. *For each $A_i \in Q(S)$, $\alpha_i \leq |A_i|$*

Proof. Let us suppose, for a contradiction, that for some $A_i \in Q(S)$, $\alpha_i > |A_i|$ and let $S_i \subseteq U_{A_i}$ an independent set in the subgraph induced by U_{A_i} in G such that $|S_i| = \alpha_i$. The set $S' = (S - A_i) \cup S_i$ is an independent set in G and $|S'| > \alpha(G)$ which is a contradiction and result follows. □

Theorem 2.20. *Let G a graph of order n. If $\mathcal{S}$ is the family of maximal independent sets of G, then*

$$\mathrm{adim}_l(G) \leq n - \max_{S \in \mathcal{S}} \sum_{A_i \in Q(S)} \alpha_i$$

Proof. For each $S \in \mathcal{S}$ and $A_i \in Q(S)$ we choose a maximal independent set $S_i \subseteq U_{A_i}$. Let $B = \cup_{A_i \in Q(S)} S_i$. We claim that that $V(G) - B$ is a local adjacency generator for G. In order to see that, let $b_i, b_j \in B$ such that $b_i b_j \in E(G)$. As $b_i \in U_{A_i}$ and $b_j \in U_{A_j}$ and $A_i \neq A_j$, without loss of generality, there exists $s_1 \in A_i - A_j$. Hence, $s_1 b_i \in E(G)$ and $s_1 b_j \notin E(G)$, and we are done. □

We give an example of application of the bound. Let G be the graph in Figure 2.4(*co-domino graph*). $S_1 = \{v_1, v_4\}$ and $S_2 = \{v_2, v_5\}$ are two maximal independent sets. Let $A_1 = \{v_1\}$, $A_2 = \{v_4\}$, $A_3 = \{v_2\}$, $A_4 = \{v_5\}$, $A_5 = S_2$. Then $Q(S_1) = \{A_1, A_2\}$ and $Q(S_2) = \{A_3, A_4, A_5\}$. $U_{A_1} = \{v_2, v_6\}$, $U_{A_2} = \{v_3, v_5\}$, $U_{A_3} = \{v_1\}$, $U_{A_4} = \{v_4\}$, $U_{A_5} = \{v_3, v_6\}$. $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$ and $\alpha_5 = 2$. Therefore,

$$\sum_{A_i \in Q(S_1)} \alpha_i = \alpha_1 + \alpha_2 = 2,$$

$$\sum_{A_i \in Q(S_2)} \alpha_i = \alpha_3 + \alpha_4 + \alpha_5 = 4.$$

And in fact

$$2 = \operatorname{adim}_l(G) = n - \sum_{A_i \in Q(S_2)} \alpha_i$$

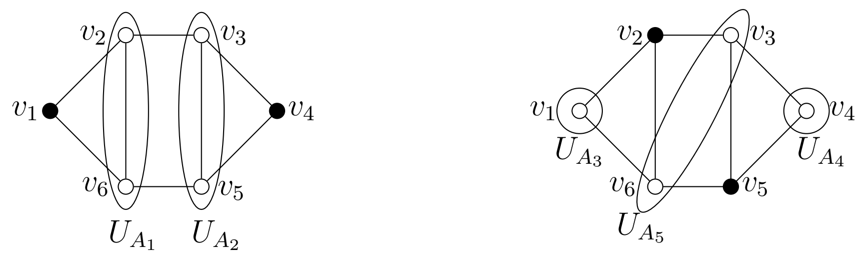

Figure 2.4: G with two $Q(S)$ structures. In black, maximal independent sets. On the left $S_1 = \{v_1, v_4\}$ and on the right $S_2 = \{v_2, v_5\}$.

Upper bounds using isometric subgraphs

In this section we work on Remark 2.6. First we remark that any minimal path of length equal to the diameter is an isometric subgraph (concept that we define further) of G and then Remark 2.6 is a particular case of our Remark 2.21 as so it is Lemma 2.22 from Jannesari et al. [34]. After the remarks we characterize the graphs G such that $\dim_l(G) = n - D(G)$.

For a graph G we say that H is an *isometric subgraph* of G if H is a subgraph of G and for every $u, v \in V(H)$, $d_H(u,v) = d_G(u,v)$. It turns out that an isometric graph is an induced subgraph but the converse is not always true, even if the isometric subgraph is connected. We can see an example in Figure 2.5.

Remark 2.21. *If H is an isometric subgraph of a nontrivial connected graph G, then* $\dim_l(G) \leq |V(G)| - |V(H)| + \dim_l(H)$.

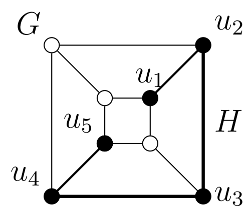

Figure 2.5: The set $\{u_1, u_2, u_3, u_4, u_5\}$ induces a connected subgraph H (thick lines) of G that is not isometric since $d_H(u_1, u_5) = 4 \neq 2 = d_G(u_1, u_5)$.

Proof. Let $B \subseteq V(H)$ be a local metric basis for H. We claim that $C = (V(G) - V(H)) \cup B$ is a local metric generator for G. Let $u, v \in V(G) - C = V(H) - B$ such that $uv \in E(G)$, then there exists $b \in B$ such that $d_G(b, u) = d_H(b, u) \neq d_H(b, v) = d_G(b, v)$ and we are done. □

Remark 2.21 is also valid for the metric dimension of a graph and Jannesari et al. [34] give the next bound.

Lemma 2.22. [34] *If G is a connected graph not a tree with order n and girth g,* $\dim(G) \leq n - \mathsf{g} + 2$.

Graphs which attain the bound in Lemma 2.22 are also characterized in [34]. By Remark 2.21, Lemma 2.22 is also valid for the local metric dimension of a graph.

Remark 2.23. *If $G \cong K_n$ or $G \cong C_{2r+1}$ then* $\dim_l(G) = |V(G)| - \mathsf{g}(G) + 2$ *and we conjecture that there are no more cases in which the equality holds.*

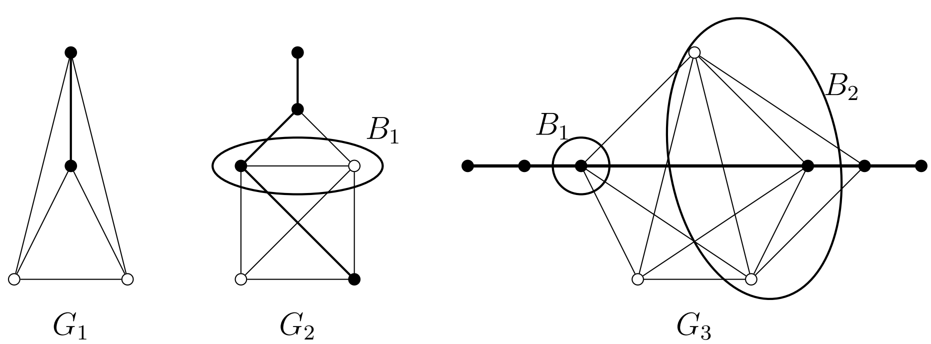

Figure 2.6: The three types of graph G such that $\dim_l(G) = n - D(G)$. In bold a diametral path of each.

Now we proceed to characterize the graphs G with $\dim_l(G) = n - D(G)$.

Theorem 2.24. *Let $B_1, B_2 \subseteq V(K_r)$ be such that, $B_1 \neq \emptyset$, $B_1 \cap B_2 = \emptyset$. For some $n_1, n_2 \geq 1$ let $H_1 \cong P_{n_1}$, $H_2 \cong P_{n_2}$ and let $u_1 \in V(H_1)$, $v_1 \in V(H_2)$ be pendant vertices of H_1 and H_2 respectively. Consider the following families of pairs $\mathcal{H}_1 = \{(H_1, \{u_1\})\}$, $\mathcal{H}_2 = \{(H_1, \{u_1\}), (H_2, \{v_1\})\}$. For a connected graph G, $\dim_l(G) = |V(G)| - D(G)$ if and only if one of the following conditions hold:*

1. $G \cong K_r$
2. $G \cong K_r +_{\{B_1\}} \mathcal{H}_1$
3. $B_2 \neq \emptyset$ *and* $G \cong K_r +_{\{B_1, B_2\}} \mathcal{H}_2$

Proof. For the first case, we remark that for a graph G or order n the following statements are equivalent [47]:

- $G \cong K_n$
- $\dim_l(G) = n - 1$

For the other cases we start by proving the sufficiency of the conditions. Suppose first that $G \cong K_r +_{\{B_1\}} \mathcal{H}_1$.

If $B_1 \neq V(K_r)$, then $D(G) = n_1 + 1$ and, for each $w_1 \in B_1$ and $w_2 \in V(G) - B_1$, the set $T = V(P_{n_1}) \cup \{w_1, w_2\}$ induces a path of length equal to the diameter. As T is an isometric subgraph of G, w_2 distinguishes any pair of adjacent vertices $x, y \in T - \{w_2\}$. Thus $C = (V(G) - T) \cup \{w_2\}$ is a local metric generator for G and $|C| = |V(G)| - (n_1 + 2) + 1 = |V(G)| - D(G)$. Now, let us suppose, for a contradiction, that there exists a local metric generator $B \subseteq V(G)$ for G such that $|B| \leq |V(G)| - (n_1 + 2) = r - 2$. If $B \subseteq V(K_r)$ then there exists $x, y \in V(K_r) - B$ such that no vertex in B is able to distinguish them. On the other hand, if $B \not\subseteq V(K_r)$ then $|B \cap V(K_r)| \leq r - 3$ and then either there exist $x, y \in (V(K_r) - B) \cap N(u_1)$ or there exist $x, y \in (V(K_r) - (B \cup N(u_1))$. In both cases no vertex in B is able to distinguish x and y, which is a contradiction. Therefore, $\dim_l(G) = n - D(G)$.

If $B_1 = V(K_r)$, then $D(G) = n_1$ and, for each $w \in B_1$, the path induced by the set $T_w = V(P_{n_1}) \cup \{w\}$ has length equal to the diameter. As any T_w is an isometric subgraph of G, $w \in V(K_r)$ distinguishes any pair of adjacent vertices $x, y \in V(G) - V(K_r) = V(P_{n_1})$. Hence $V(K_r)$ is a local metric generator for G. Let us suppose, for a contradiction, that there exists a local metric generator $B \subseteq V(G)$ for G such that $|B| \leq r-1$. If $B \subseteq V(K_r)$, then there exists $x \in V(K_r) - B$ and no vertex in B is able to distinguish the pair x, u_1. And if $B \not\subseteq V(K_r)$ then $|B \cap V(K_r)| \leq |V(K_r)| - 2$ and then there exist $x, y \in V(K_r) - B$. No vertex in B is able to distinguish x and y, which is a contradiction. Therefore $\dim_l(G) = |V(K_r)| = n - n_1 = n - D(G)$.

For the third condition, suppose that there exist $B_1, B_2 \subseteq V(K_r)$ such that $B_1 \neq \emptyset \neq B_2$, $B_1 \cap B_2 = \emptyset$ and $G \cong K_r +_{\{B_1,B_2\}} \mathcal{H}_2$. In this case $D(G) = n_1+n_2+1$. If $r = 2$ then $G \cong P_{n_1+n_2+2}$, hence $\dim_l(G) = 1 = |V(G)| - D(G)$ and we are done. Let us suppose $r \geq 3$. For fixed $w_1 \in B_1$, $w_2 \in B_2$, the set $T_{w_1w_2} = V(P_{n_1}) \cup \{w_1, w_2\} \cup V(P_{n_2})$ induces a shortest path of length equal to the diameter. As $T_{w_1w_2}$ is an isometric subgraph of G, w_1 distinguishes any pair of adjacent vertices $x, y \in T_{w_1w_2} - \{w_1\}$, thus $C = V(K_r) - \{w_2\}$ is a local metric generator for G and $|C| = r - 1 = n - D(G)$. Let us suppose, for a contradiction, that there exists $B \subseteq V(G)$ a local metric generator for G such that $|B| \leq r - 2$. If $B \subseteq V(K_r)$ then there exists $x, y \in V(G) - B$ such that no vertex in B is able to distinguish them, which is a contradiction. Thus, $B \not\subseteq V(K_r)$, which implies that $|B \cap V(K_r)| \leq r - 3$. We now differentiate the following cases:

Case 1: $V(P_{n_1}) \cap B = \emptyset$. In this case, $|(V(K_r) - \{w_2\}) - B| \geq 2$ and then either there exist $x, y \in (V(K_r) - B) \cap N(u_2)$ or there exist $x, y \in (V(K_r) - (B \cup N(u_2))$. In both cases no vertex in B is able to distinguish x and y, which is a contradiction.

Case 2: $V(P_{n_1}) \cap B \neq \emptyset \neq V(P_{n_2}) \cap B$. In this case $|B \cap V(K_r)| \leq r - 4$, thus either there exist $x, y \in (V(K_r) - B) \cap N(u_1)$ or there exist $x, y \in (V(K_r) - B) \cap N(u_2)$ or there exist $x, y \in (V(K_r) - (B \cup N(u_1) \cup N(u_2))$. In all these cases no vertex in B is able to distinguish x and y, which is a contradiction.

According to the two cases above, we conclude that $\dim_l(G) = |(V(G) - T) \cup \{w_1\}| = n - D(G)$.

Now we proceed to prove the necessity of the conditions. Let G be a connected graph of order n such that $\dim_l(G) = n - D(G)$. Let Y be a maximum clique in G. If $V(G) = Y$, then we are done. So let us suppose that $V(G) - Y \neq \emptyset$. Let $x_0, x_D \in V(G)$ be such that $d_G(x_0, x_D) = D(G)$ and let $T = \{x_0, \ldots, x_D\} \subseteq V(G)$ such that the subgraph induced by T is a shortest x_0x_D-path. We affirm that $V(G) = T \cup Y$. If $V(G) = T$ there is nothing to prove. So let suppose that $V(G) - T \neq \emptyset$. We proceed to prove the following three claims.

Claim 2.25. *For each* $z \in V(G)-T$*, there exists* $x_i \in T$ *such that* $\{x_i, x_{i+1}\} \subseteq N_G(z) \cap T \subseteq \{x_i, x_{i+1}, x_{i+2}\}$.

Let $x_{i_0} \in T$ be such that $d_G(z,T) = d_G(z, x_{i_0})$. If either $|N_G(z) \cap T| \leq 1$ or $N_G(z) \cap T = \{x_{i_0}, x_j\}$ and $x_j x_{i_0} \notin E(G)$, then x_{i_0} distinguishes any pair of adjacent vertices $u, v \in \{z\} \cup (T - \{x_{i_0}\})$. Hence, the set $B = V(G) - (\{z\} \cup (T - \{x_{i_0}\}))$ is a local metric generator for $V(G)$ and $|B| = n - (D(G)+1)$, which is a contradiction. Thus, there exists $x_i \in T$ such that $\{x_i, x_{i+1}\} \subseteq N_G(z)$. On the other hand, if there exist $x_i, x_j \in N_G(z) \cap T$ such that $j > i+2$, then $T' = x_0, \ldots, x_i, z, x_j, \ldots, x_D$ is a $x_0 x_D$-path in G and $l(T') < D(G)$, which is a contradiction. Hence, $N_G(z) \cap T \subseteq \{x_i, x_{i+1}, x_{i+2}\}$.

Claim 2.26. *For every* $z_1, z_2 \in V(G) - T$*,* $z_1 z_2 \in E(G)$ *and there exists* $x_i \in T$ *such that* $x_i, x_{i+1} \in N_G(z_1) \cap N_G(z_2)$.

First, let us suppose, for a contradiction, that there exists $z_1, z_2 \in V(G) - T$ such that $|N_G(z_1) \cap N_G(z_2)| \leq 1$. Let $i_0 = \min\{i : x_i \in N_G(z_1) \cap T\}$, $i_1 = \max\{i : x_i \in N_G(z_1) \cap T\}$, $j_0 = \min\{i : x_i \in N_G(z_2) \cap T\}$, $j_1 = \max\{i : x_i \in N_G(z_2) \cap T\}$. Without loss of generality, Claim 2.25 implies that $i_1 \leq j_0$. We differentiate two cases:

- Case 1: $i_0+1 = j_1-1$. In this case we claim that the set $B = (V(G) - \{z_1, z_2\}) - (T - \{x_{i_0}, x_{j_1}\})$ is a local metric generator for G. In order to see that, let $u, v \in V(G) - B = \{z_1, z_2\} \cup (T - \{x_{i_0}, x_{j_1}\})$ such that $uv \in E(G)$. If $u, v \in T$, then x_{i_0} distinguishes u and v. If $u = z_1$, then $v = x_{i_0+1}$ and, as $d_G(x_{j_1}, z_1) = 2 \neq 1 = d_G(x_{j_1}, x_{j_1-1}) = d_G(x_{j_1}, x_{i_0+1})$, the vertex x_{j_1} distinguishes u and v. And if $u = z_2$, then $v = x_{j_1-1}$ and, as $d_G(x_{i_0}, z_2) = 2 \neq 1 = d_G(x_{i_0}, x_{i_0+1}) = d_G(x_{i_0}, x_{j_1-1})$, the vertex x_{i_0} distinguishes u and v. Finally, if $u = z_1$ and $v = z_2$, then the vertex x_{i_0} distinguishes u and v. Thus, B is a local metric generator for G and $|B| = n - (D(G)+1)$, which is a contradiction.

- Case 2: $i_0 + 1 < j_1 - 1$. In this case we claim that

  $$d_G(x_{j_1-1}, x_{i_0+2}) = d_G(x_{j_1-1}, z_1) - 1 = d_G(x_{j_1-1}, x_{i_0}) - 2$$

  and

  $$d_G(x_{i_0+1}, x_{j_1-2}) = d_G(x_{i_0+1}, z_2) - 1 = d_G(x_{i_0+1}, x_{j_1}) - 2.$$

First we remark that

$$d_G(x_{j_1-1}, x_{i_0}) = d_G(x_{j_1-1}, x_{i_0+2}) + d_G(x_{i_0+2}, x_{i_0}) \leq$$

$$d_G(x_{j_1-1}, x_{i_0+2}) + d_G(x_{i_0+2}, z_1) + d_G(z_1, x_{i_0}).$$

If

$$d_G(x_{j_1-1}, x_{i_0+2}) + d_G(x_{i_0+2}, x_{i_0}) <$$

$$d_G(x_{j_1-1}, x_{i_0+2}) + d_G(x_{i_0+2}, z_1) + d_G(z_1, x_{i_0}),$$

then

$$d_G(x_{i_0+2}, x_{i_0}) < d_G(x_{i_0+2}, z_1) + d_G(z_1, x_{i_0}) = 2$$

which is a contradiction with the fact that x_{i_0+2} and x_{i_0} are in a shortest path. The other equality is proved in a similar way and we are done.

Now we claim that the set $B = (V(G) - \{z_1, z_2\}) - (T - \{x_{i_0+1}, x_{j_1-1}\})$ is a local metric generator for G. In order to see that, let $u, v \in V(G) - B = \{z_1, z_2\} \cup (T - \{x_{i_0+1}, x_{j_1-1}\})$ such that $uv \in E(G)$. If $u, v \in T$, then x_{i_0+1} distinguishes u and v. If $u = z_1$, then $v \in \{x_{i_0}, x_{i_0+2}\}$. As $d_G(x_{j_1-1}, x_{i_0+2}) = d_G(x_{j_1-1}, z_1) - 1 = d_G(x_{j_1-1}, x_{i_0}) - 2$, the vertex x_{j_1} distinguishes u and v. And if $u = z_2$, then $v \in \{x_{j_1-2}, x_{j_1}\}$ and, as $d_G(x_{i_0+1}, x_{j_1-2}) = d_G(x_{i_0+1}, z_2) - 1 = d_G(x_{i_0+1}, x_{j_1}) - 2$ the vertex x_{i_0+1} distinguishes u and v. Thus, B is a local metric generator for G and $|B| = n - (D(G) + 1)$, which is a contradiction.

From to the cases above, we conclude that if $z_1, z_2 \in V(G) - T$, then $|N_G(z_1) \cap N_G(z_2)| \geq 1$.

Now suppose, for a contradiction, that for some $z_1, z_2 \in V(G) - T$, $z_1 z_2 \notin E(G)$. Let $x_i, x_{i+1} \in N_G(z_1) \cap N_G(z_2) \cap T$. We claim that $B = (V(G) - \{z_1, z_2\}) - (T - \{x_i, x_{i+1}\})$ is a local metric generator for G. In order to see that, let $u, v \in V(G) - B = \{z_1, z_2\} \cup (T - \{x_i, x_{i+1}\})$ such that $uv \in E(G)$. If $u, v \in T$, then x_i distinguishes u and v. If $u \in \{z_1, z_2\}$, then $v \in \{x_{i-1}, x_{i+2}\}$. Hence, if $v = x_{i-1}$, then x_{i+1} distinguishes u and v and, if $v = x_{i+2}$, then x_i distinguishes u and v. Thus B is a local metric generator for G and $|B| = n - (D(G) + 1)$, which is a contradiction. Therefore, for every $z_1, z_2 \in V(G) - T$, $z_1 z_2 \in E(G)$.

Claim 2.27. *There exists $x_i \in T$ such that, for every $z \in V(G) - T$, $x_i, x_{i+1} \in N_G(z)$*

Claim 2.26 implies that, if $|V(G) - T| \leq 2$, then there is nothing to prove. Let us suppose $|V(G) - T| \geq 3$. If for every $z, w \in V(G) - T$, $|N_G(z) \cap N_G(w) \cap T| = 3$, then for every $z, w \in V(G) - T$, $N_G(z) = N_G(w)$ and there is nothing to prove. Let us suppose, for a contradiction, that there exist $z_1, z_2, z_3 \in V(G) - T$ such that $|N_G(z_1) \cap N_G(z_2) \cap T| = 2$, say $x_i, x_{i+1} \in N_G(z_1) \cap N_G(z_2) \cap T$ and $x_i \notin N_G(z_3) \cap T$, being the case $x_{i+1} \notin N_G(z_3)$ symmetric. By Claims 2.25 and 2.26 there exist $x_j, x_{j+1} \in N_G(z_3) \cap N_G(z_1) \cap T$, so, as $x_i \notin N_G(z_3) \cap T$, $\{x_j, x_{j+1}\} = \{x_{i+1}, x_{i+2}\}$. Also there exist $x_k, x_{k+1} \in N_G(z_3) \cap N_G(z_2) \cap T$, so, as $x_i \notin N_G(z_3) \cap T$, we have $\{x_j, x_{j+1}\} = \{x_{i+1}, x_{i+2}\}$. Then $x_i, x_{i+1}, x_{i+2} \in N_G(z_1) \cap N_G(z_2) \cap T$ which is a contradiction and the result follows.

Let $x_i, x_{i+1} \in \bigcap_{z \in V(G)-T}(N_G(z) \cap T)$ and $Y = (V(G) - T) \cup \{x_i, x_{i+1}\}$. By the Claims above, Y is a clique in G and $V(G) = T \cup Y$.

Therefore, $\langle V(G) - Y \rangle_G \cong \langle T \rangle_G - \{x_i, x_{i+1}\}$, where x_i, x_{i+1} are two adjacent vertices in $\langle T \rangle_G$. We have three cases in function of the number of components of $V(G) - Y$

- Case 1: $V(G) - Y = \emptyset$. In this case $G \cong K_r$ and we are done.

- Case 2: $V(G) - Y$ has only one component $\langle V(G) - Y \rangle_G \cong P_{n_1}$. In this case there exists only one $y \in V(P_{n_1})$ such that for some $x \in Y$, $xy \in E(G)$ and such a y is a pendant vertex of P_{n_1}. For such a y,

$$G \cong K_r +_{\{N_G(y) \cap Y\}} \{(P_{n_1}, \{y\})\}$$

  and we are done.

- Case 3: $V(G) - Y$ has two components $L_1 \cong P_{n_1}$ and $L_2 \cong P_{n_2}$. In this case, for each $i \in \{1, 2\}$, there exists only one $y_i \in V(L_i)$ such that for some $x_i \in X$, $x_i y_i \in E(G)$ and such y_1, y_2 are pendant vertices of L_1 and L_2. Let $y_i \in V(L_i)$ such that $N_G(y_i) \cap Y \neq \emptyset$, for $i \in \{1, 2\}$. Since the subgraph of G induced by T is a shortest path, $B_1 \cap B_2 = \emptyset$. Hence

$$G \cong K_r +_{\{N_G(y_1) \cap Y, N_G(y_2) \cap Y\}} \{(L_1, \{y_1\}), (L_2, \{y_2\})\}$$

  and we are done.

□

The case $\dim_l(G) = n - 3$

In this subsection we will characterize the connected graphs G of order n such that $\dim_l(G) = n - 3$.

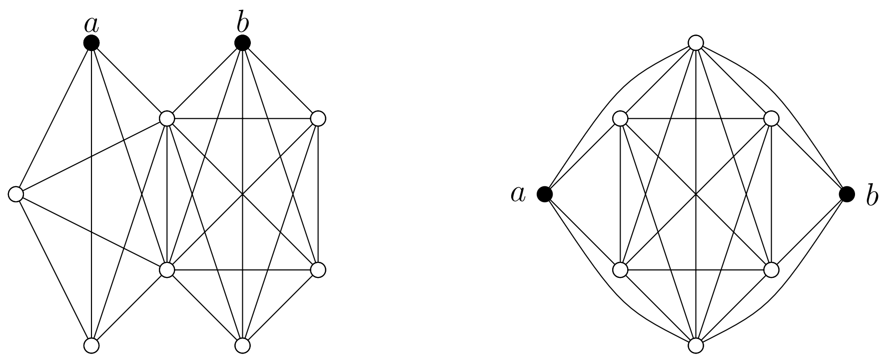

Figure 2.7: The two types of graphs G such that $\dim_l(G) = n - 3$ and $Q(S) = \{\{a\}, \{b\}, S\}$.

We start with some easy results.

Lemma 2.28. *If G is a connected graph of order n with $\dim_l(G) = n - 3$, then*

- $n \geq 4$
- $n = 4$ *if and only if G is bipartite.*
- *If $n = 5$, then G is not bipartite and $\omega(G) \leq 3$*

Proof. If $n \leq 3$, then $\dim_l(G) \leq 0$, which is impossible. If $n = 4$, then $\dim_l(G) = 1$ and, by Theorem 2.4, G is bipartite, and vice versa.

We now assume that $n = 5$. In this case, $\dim_l(G) = 2$, which implies that G is not bipartite. As G is connected $\omega(G) \geq 2$. If $\omega(G) \geq 4$ then, by Theorem 2.4, $\dim_l(G) \geq 3$, which is a contradiction. Thus, $\omega(G) \leq 3$. □

If G is a connected graph of order n with $\dim_l(G) = n-3$ and $\alpha(G) = 3$, then Theorem 2.16 characterizes G in the following way.

Remark 2.29. *Let G be a connected graph of order n and $\dim_l(G) = n-3$. If $\alpha(G) = 3$, then $\omega(G) = n-2$ and if A is a maximal clique in G and $\{u, v\} = V(G) - A$, then $N_G(u) \subseteq N_G(v)$ or $N_G(v) \subseteq N_G(u)$ or $N_G(u) \cap N_G(v) = \emptyset$.*

According to the remark above and Remark 2.5, from now on we consider graphs with $\alpha(G) = 2$. Moreover, since $D(G) \geq 4$ implies that $\alpha(G) \geq 3$, we restrict ourselves to the case $D(G) \leq 3$. Let $S = \{a, b\} \subseteq V(G)$ be a maximal independent set for G. We recall the notation used in Theorem 2.20 and see that, up to isomorphism, there are four possibilities for a set $Q(S)$

- Case 1: $Q(S) = \{\{a\}, \{b\}\}$
- Case 2: $Q(S) = \{S\}$
- Case 3: $Q(S) = \{\{a\}, S\}$
- Case 4: $Q(S) = \{\{a\}, \{b\}, S\}$

Each of these structures entails a type of connected graph G of order n, independence number equals 2 and local metric dimension equals $n - 3$. We characterize them in the following theorems.

Remark 2.30. *Let G be a connected graph of order $n \geq 6$ and independence number $\alpha = 2$. Let $S = \{a, b\} \subseteq V(G)$ be a maximum independent set for G. If $Q(S) = \{\{a\}, \{b\}\}$, then $D(G) = 3$*

Proof. It suffices to remark that $N_G(a) \cap N_G(b) = \emptyset$ implies $d_G(a, b) \geq 3$. □

If $D(G) = 3$, then Theorem 2.24 characterizes such graphs in the following way.

Theorem 2.31. *Let G be a connected graph of order $n \geq 6$ with $\dim_l(G) = n - 3$, $\alpha(G) = 2$ and $D(G) = 3$. Let B be a non-empty proper subset of $V(K_{n-2})$. Let $u_1 \in V(P_2)$ be a pendant vertex of P_2 and $u_2, u_3 \notin V(K_{n-2}) \cup V(P_2)$. If $\mathcal{H}_1 = \{(P_2, \{u_1\})\}$ and $\mathcal{H}_2 = \{(\langle u_2 \rangle, \{u_2\}), (\langle u_3 \rangle, \{u_3\})\}$, then either*

1. *$G \cong K_{n-2} +_{\{B\}} \mathcal{H}_1$ or*
2. *$G \cong K_{n-2} +_{\{B_1, V(K_{n-2}) - B_1\}} \mathcal{H}_2$.*

Theorem 2.32. *Let G be a connected graph of order n with $\alpha(G) = 2$. Let $S = \{a, b\} \subseteq V(G)$ be a maximum independent set for G such that $Q(S) = \{S\}$ and let $H = \langle V(G) - S \rangle_G$ and $|V(H)| = n - 2$. Then $\dim_l(G) = n - 3$ if and only if $\omega(H) = n - 3$.*

Proof. Let G, H, $S = \{a, b\}$ as in the hypotheses. We would like to recall that for every $v \in V(H)$, $va, vb \in E(G)$, as $Q(S) = \{S\}$.

First we suppose that $\omega(H) = n - 3$. In this case, we have $\omega(G) = n - 2$ and Remark 2.4 implies that $\dim_l(G) \leq n - 3$. Let us suppose, for a contradiction that there exists a local metric basis $C \subseteq V(G)$ such that $|C| \leq n-4$. Let $K \subseteq V(G)$ be a maximal clique in H and $\{u_0\} = V(H)-K$. If $C \subseteq K$, then there exists $v \in K-C$ and no vertex is C is able to distinguish v and a, which is a contradiction. Thus $C \cap \{a, b, u_0\} \neq \emptyset$ and $|K - C| \geq 2$. Let $v_1, v_2 \in K - C$. If $v_1, v_2 \in N_G(u_0)$ or $v_1, v_2 \notin N_G(u_0)$, then no vertex in C is able to distinguish v_1 and v_2, which is a contradiction. Thus, by pigeon hole principle the only possibility is $K - C = \{v_1, v_2\}$ and without loss of generality $v_1 \in N_G(u_0)$ and $v_2 \notin N_G(u_0)$. In this case, however, no vertex in $V(H)$ is either able to distinguish v_1 and a or able to distinguish v_1 and b. Hence $C \cap \{a, b\} \neq \emptyset$, say $a \in C$. Since, $K - C = \{v_1, v_2\}$ and $|K| = n - 3$, $|C \cap K| = n - 5$. Now, since $a \in C - K$, we have $|C| = n - 4$, which implies that $b, u_0 \notin C$. Thus, no vertex in C is able to distinguish v_1 and v_2, which is a contradiction. Therefore $\dim_l(G) = n - 3$.

Now, let us suppose that $\dim_l(G) = n-3$. Notice that $\alpha(H) \leq \alpha(G) \leq 2$. If $\alpha(H) = 1$ then $\omega(G) = n-1$ and then Remark 2.4 implies that $\dim_l(G) = n - 2$, which is a contradiction. Hence $\alpha(H) = 2$. Set $n' = n - 2 = |V(H)|$. Let us suppose, towards a contradiction, that $\operatorname{adim}_l(H) < n' - \alpha(H)$ and let $B \subseteq V(H)$ be a local adjacency basis for H, then $B \cup \{a\}$ is a local adjacency generator for G and $\operatorname{adim}_l(G) \leq \operatorname{adim}_l(H) + 1 < n' - \alpha(H) + 1 = n - 3$, which is a contradiction. Thus, $\operatorname{adim}_l(H) = n' - 2$. By Remark 2.4, either $H \cong K_{n_1} \cup K_{n_2}$ or $\omega(H) = n' - 1$. Let us suppose, for a contradiction that $H \cong K_{n_1} \cup K_{n_2}$, $2 \leq n_1 \leq n_2$, and let $u_1 \in V(K_{n_1})$, $v_1 \in V(K_{n_2})$. Any vertex $u_2 \in V(K_{n_1}) - \{u_1\}$ distinguishes v_1 and a and also distinguishes v_1 and b. Any vertex $v_2 \in V(K_{n_2}) - \{v_1\}$ distinguishes u_1 and a and also distinguishes u_1 and b. Hence $V(H) - \{a, b, v_1, v_2\}$ is a local metric generator for G. Thus, $\dim_l(G) \leq n - 4$, which is a contradiction. Therefore $\omega(H) = n' - 1 = n - 3$. □

Theorem 2.33. *Let G be a connected graph of order $n \geq 6$ such that $\alpha(G) = 2$ and diameter $D(G) = 2$. Let $S = \{a, b\} \subseteq V(G)$ be a maximal independent set for G such that $Q(S) = \{\{a\}, S\}$, and let $H = \langle V(G) - S \rangle_G$. Then $\dim_l(G) = n - 3$ if and only if there exists $u_0 \in V(H)$ such that $V(H) - \{u_0\}$ is a maximal clique in H and one of the following conditions holds*

1. $N_H(u_0) \subseteq N_H(b)$
2. $N_H(b) \subseteq N_H(u_0)$
3. $N_H(b) \cap N_H(u_0) = \emptyset$
4. $N_H(u_0) \cup N_H(b) = V(H)$

Proof. Let G, H, S, $Q(S)$ as in the hypotheses. In order to prove the sufficiency of the conditions we suppose that there exists $u_0 \in V(H)$ such that $V(H) - \{u_0\}$ is a maximal clique in H. Let $v_0 \in V(H) - \{u_0\}$ such that $u_0v_0 \notin E(G)$. As $d_G(a,x) = 1 \neq 2 = d_G(a,b)$ for $x \in \{u_0, v_0\}$, the set $B = V(G) - \{u_0, v_0, b\}$ is a local metric generator for G. Let us suppose, for a contradiction that there exists a local metric basis $C \subseteq V(G)$ such that $|C| < |B| = n-3$. Consider the sets $B_1 = N_H(u_0) - N_H(b)$, $B_2 = N_H(b) - N_H(u_0)$, $B_3 = N_H(b) \cap N_H(u_0)$, $B_4 = V(H) - (N_H(u_0) \cup N_H(b))$. Each of the four conditions in the hypotheses implies that one of these sets is empty. In fact, condition number i implies $B_i = \emptyset$, for $i \in \{1,2,3,4\}$. In the case that $B_i \neq \emptyset$ we have that B_i is a true twin class of $V(H)$, for $i \in \{1,2,3,4\}$. Thus, $|B_i - C| \leq 1$ for $i \in \{1,2,3,4\}$. As $\emptyset \in \{B_1, B_2, B_3, B_4\}$ and $|C| \leq n-4$, we have that the set $\{a, b, u_0\} - C$ is non-empty. We consider the following cases:

Case 1: $a \notin C$. In this case, for $v_1 \in B_1 - C$, no vertex in C is able to distinguish v_1 and a. Thus $B_1 \subseteq C$. Condition $|C| \leq n-4$ implies $\{b, u_0\} - C \neq \emptyset$. We consider the following subcases:

Case 1.1. $b \notin C$. In this case, for $v_3 \in B_3 - C$, no vertex in C is able to distinguish v_3 and a. Thus $B_3 \subseteq C$. Also, for $v_2 \in B_2 - C$ and $v_4 \in B_4 - C$ no vertex in C is able to distinguish v_2 and v_4 then $|(B_3 \cup B_4) - C| \leq 1$. Thus $|V(G) - C| = |V(H) - C| + |\{u_0, a, b\} - C| \leq 2$ which is a contradiction.

Case 1.2. $u_0 \notin C$. In this case, for $v_{14} \in (B_1 \cup B_4) - C$, no vertex in C is able to distinguish v_{14} and a. Thus, $(B_1 \cup B_4) \subseteq C$. Thus $|V(G) - C| = |V(H) - C| + |\{u_0, a, b\} - C| \leq 2$, which is a contradiction.

Case 2: $b \notin C$. In this case, for $v_1 \in B_1 - C$ and $v_3 \in B_3 - C$, no vertex in C is able to distinguish v_1 and v_3. Thus $|(B_1 \cup B_3) - C| \leq 1$. Also for $v_2 \in B_2 - C$ and $v_4 \in B_4 - C$, no vertex in C is able to distinguish v_2 and v_2. Thus $|(B_2 \cup B_4) - C| \leq 1$. Hence $|V(H) - C| \leq 2$ and then $\{a, u_0\} - C \neq \emptyset$. If $a \notin C$ we arrive to a contradiction as in Case 1.1. On the other hand, if $u_0 \notin C$ then, for $u, v \in V(H) - C$, no vertex in C is able to distinguish u and v. Thus $|V(H) - C| \leq 1$. Therefore $|V(G) - C| = |V(H) - C| + |\{u_0, a, b\} - C| \leq 3$, which is a contradiction.

Case 3: $u_0 \notin C$. In this case, for $v_2 \in B_2 - C$ and $v_3 \in B_3 - C$, no vertex in C is able to distinguish v_2 and v_3. Thus $|(B_2 \cup B_3) - C| \leq 1$. Also for $v_1 \in B_1 - C$ and $v_4 \in B_4 - C$, no vertex in C is able to distinguish v_1 and v_4. Thus $|(B_1 \cup B_4) - C| \leq 1$. Hence $|V(H) - C| \leq 2$ and then $\{a, b\} - C \neq \emptyset$. If $a \notin C$ we arrive to a contradiction as in Case 1.2. On the other hand, if $u_0 \notin C$, then we arrive to the contradiction in Case 2.

According to the three cases above we have $\{a, b, u_0\} \subseteq C$, which is a contradiction. Therefore, $\dim_l(G) = n - 3$.

In order to prove the necessity, we assume that $\dim_l(G) = n - 3$. Let $S = \{a, b\} \subseteq V(G)$ be a maximal independent set for G such that $Q(S) = \{\{a\}, S\}$.

If $\alpha(H) = 1$, then $V(H) \cup \{a\}$ is a clique in G and $\omega(G) = n - 1$ and Remark 2.4 implies, $\dim_l(G) = n - 2$ which is a contradiction. Thus, $\alpha(H) = 2$ and, if we write $n' = n - 2 = |V(H)|$, Theorem 2.3 implies that $\operatorname{adim}_l(H) \leq n' - 2$. Let us suppose, for a contradiction, that $\operatorname{adim}_l(H) < n' - 2$. If B is a local adjacency basis for H, then $B_1 = B \cup \{a\}$ is a local metric generator for G and $|B_1| = |B| + 1 < n - 3$, which is a contradiction. Thus, $\operatorname{adim}_l(H) = n' - 2$ and so Theorem 2.16 implies that $\dim_l(H) = n' - 2$, and by Remark 2.4, either there exist n_1, n_2 such that $2 \leq n_1 \leq n_2$ and $H \cong K_{n_1} \cup K_{n_2}$ or there exists $u_0 \in V(H)$ such that $V(H) - \{u_0\}$ is a maximal clique in H. Let us suppose, for a contradiction, that there exist n_1, n_2 such that $2 \leq \min\{n_1, n_2\}$ and $H \cong K_{n_1} \cup K_{n_2}$. Let $u_0 \in V(K_{n_1})$ and $v_0 \in V(K_{n_2})$. Any vertex $u_1 \in V(K_{n_1}) - \{u_0\}$ distinguishes u_0 and v_0 and also distinguishes v_0 and a. Also, any vertex $v_1 \in V(K_{n_2}) - \{v_0\}$ distinguishes u_0 and a, so that $B - \{u_0, v_0, a, b\}$ is a local metric basis for G. Thus, $\dim_l(G) \leq n - 4$, which is a contradiction. Hence there exists $u_0 \in V(H)$ such that $V(H) - \{u_0\}$ is a maximal clique in H. Let us consider the partition of $V(H)$ in the following sets: $B_1 = N_H(b) - N_H(u_0)$, $B_2 = N_H(u_0) - N_H(b)$, $B_3 = N_H(u_0) \cap N_H(b)$, $B_4 = V(H) - (N_H(u_0) \cup N_H(b))$. If all of them are non empty then, we claim that for any $v_i \in B_i$, $i \in \{1, 2, 3, 4\}$, the set $B - \{v_1, v_2, v_3, v_4\}$ is a local metric generator for G. In order to see that, just consider that b distinguishes x_1 and x_2, where $x_1 \in \{v_1, v_3\}$ and $x_2 \in \{v_2, v_4\}$ and the vertex u_0 distinguishes y_1 and y_2 where $y_1 \in \{v_2, v_3\}$ and $y_2 \in \{v_1, v_4\}$. Therefore $\dim_l(G) \leq n - 4$, which is a contradiction. Hence, for some $i \in \{1, 2, 3, 4\}$, $B_i = \emptyset$. $\square$

We prove the following Lemma prior to tackling the fourth case.

Lemma 2.34. *Let G be a connected graph and let $S \subseteq V(G)$ be a maximal independent set of G such that $\dim_l(G) = n - |Q(S)|$. Let $A_i, A_j \in Q(S)$ and $x, y \in U_{A_i}$. If $|U_{A_j}| \geq 2$, then $N_G[x] \cap U_{A_j} = N_G[y] \cap U_{A_j}$.*

Proof. Let G be a connected graph and $S \subseteq V(G)$ a maximal independent set such that $\dim_l(G) = n - |Q(S)|$. By Theorem 2.20, we have that, for each $A_i \in Q(S)$, the set U_{A_i} is a clique in G. Thus, for each $A_i \in Q(S)$ and $x, y \in U_{A_i}$, $N_G[x] \cap U_{A_i} = U_{A_i} = N_G[y] \cap U_{A_i}$. Consider now the case $A_i \neq A_j$ with $|U_{A_j}| \geq 2$. Suppose, for a contradiction that there exist $x, y \in U_{A_i}$ and $z \in U_{A_j}$ such that $xz \in E(G)$ and $yz \notin E(G)$. Let $w \in U_{A_j} - \{z\}$. For each $A_k \in Q(S) - \{A_i, A_j\}$ we choose $u_{k_0} \in U_{A_k}$. Consider the set $B = \{x, y, w\} \cup \{u_{k_0} : A_k \in Q(S) - \{A_i, A_j\}\}$, we claim that $V(G) - B$ is a local metric generator for G. In order to see that, let $u, v \in B$ such that $uv \in E(G)$. Then $u \in A_{k_1}$ and $v \in A_{k_2}$. If $k_1 \neq k_2$ then, without loss of generality there exist $a \in A_{k_1} - A_{k_2}$ and $d_G(a, u) = 1 \neq 2 = d_G(a, v)$. And if $k_1 = k_2$ then, without loss of generality, $u = x$ and $v = y$ and $d_G(z, u) = 1 \neq 2 = d_G(z, v)$. Thus, $V(G) - B$ is a local metric generator for G, so that $\dim_l(G) \leq n - |B| = n - (|Q(s)| + 1)$, which is a contradiction. □

Theorem 2.35. *Let G be a connected graph of order $n \geq 6$ with $\alpha(G) = 2$ and $D(G) = 2$. Let $S = \{a, b\} \subseteq V(G)$ be a maximal independent set for G such that $Q(S) = \{\{a\}, \{b\}, S\}$ and for each $A_i \in Q(S)$, $|U_{A_i}| \geq 2$. Then $\dim_l(G) = n - 3$ if and only if the following conditions hold*

1. *For each $A_i \in Q(S)$ and $x, y \in U_{A_i}$, $N_G[x] = N_G[y]$.*

2. *For each $c \in \{a, b\}$, there exist $x \in U_{\{c\}}$ and $y \in U_S$, $xy \in E(G)$.*

Proof. Let $S = \{a, b\} \subseteq V(G)$ be a maximal independent set for G such that $Q(S) = \{\{a\}, \{b\}, S\}$ and for each $A_i \in Q(S)$, $|U_{A_i}| \geq 2$. First, we prove the sufficiency of the conditions. By condition 1, for each $A_i \in Q(S)$, U_{A_i} is a clique in G. Theorem 2.20 implies that $\dim_l(G) \leq n - \sum \alpha_i = n - 3$. Let us suppose, for a contradiction, that there exists a local metric generator $C \subseteq V(G)$ such that $|C| \leq |V(G)| - 4$. Condition 1 implies that for each $A_i \in Q(S)$, U_{A_i} is a true twin class in G. Thus, for each $A_i \in Q(S)$, $|U_{A_i} - C| \leq 1$. Hence $S - C \neq \emptyset$, say $a \in S - C$ and consider the following cases.

- Case 1: There exists $u \in U_{\{a\}}$ and $v \in U_{\{b\}}$ such that $uv \notin E(G)$. In this case, Condition 1 implies that, for every $u \in U_{\{a\}}$ and $v \in U_{\{b\}}$, $uv \notin E(G)$. If $U_{\{a\}} - C \neq \emptyset$, then Condition 2 implies that for $u \in U_{\{a\}} - C$, no vertex in C is able to distinguish u and a, which is a contradiction. Thus, $U_{\{a\}} \subseteq C$ and, since $|C| \leq n-4$, we have that $b \notin C$. If $U_{\{b\}} - C \neq \emptyset$, then for $v \in U_{\{b\}} - C$, no vertex in C is able to distinguish v and b, which is a contradiction. Thus $U_{\{b\}} \subseteq C$ and $V(G) - C \subseteq \{a, b\} \cup U_S$, hence $|C| \leq n-3$, which is a contradiction.

- Case 2: There exists $u \in U_{\{a\}}$ and $v \in U_{\{b\}}$ such that $uv \in E(G)$. In this case, Condition 1 implies that, for every $u \in U_{\{a\}}$ and $v \in U_{\{b\}}$, $uv \in E(G)$. If $U_{\{b\}} - C \neq \emptyset$, then for $v \in U_{\{b\}} - C$, no vertex in C is able to distinguish v and b, which is a contradiction. Thus $U_{\{b\}} \subseteq C$ and, since $|C| \leq n-4$, we have that $b \notin C$. If $U_{\{a\}} - C \neq \emptyset$, then for $u \in U_{\{a\}} - C$, no vertex in C is able to distinguish u and a, which is a contradiction. Thus $U_{\{a\}} \subseteq C$ and $V(G) - C \subseteq \{a, b\} \cup U_S$, hence $|C| \leq n-3$, which is a contradiction.

According to the two cases above we conclude that $\dim_l(G) = n-3$, as required.

From now on, we assume that $\dim_l(G) = n-3$. By Theorem 2.20, for each $A_i \in Q(S)$, U_{A_i} is a clique. Lemma 2.34 implies that for each $A_i, A_j \in Q(S)$ and $x, y \in U_{A_i}$, $N_G[x] \cap U_{A_j} = N_G[y] \cap U_{A_j}$ and then for each $x, y \in U_{A_i}$, $N_G[x] = N_G[y]$. Let us suppose, for a contradiction that there exists $x \in U_a$, $y \in U_S$ such that $xy \notin E(G)$. Lemma 2.34 implies that for each $x \in U_a$, $y \in U_S$, $xy \notin E(G)$. Let $u \in U_{\{a\}}$, $v \in U_{\{b\}}$ and $w \in U_S$, and consider the following cases.

- Case 1: For each $x \in U_{\{a\}}$ and each $y \in U_{\{b\}}$, $xy \notin E(G)$. In this case, x and b are at distance three in G, which is a contradiction.

- Case 2: For each $x \in U_{\{a\}}$ and each $y \in U_{\{b\}}$, $xy \in E(G)$. In this case we claim that $B = V(G) - \{u, v, w, a\}$ is a local metric generator for G. In order to see this, let $x, y \in V(G) - B = \{u, v, w, a\}$ such that $xy \in E(G)$. Let $u_1 \in U_a - \{u\}$ and $w_1 \in U_S - \{w\}$. We have: $d_G(u_1, a) = 1 \neq 2 = d_G(u_1, w) \neq 1 = d_G(u_1, v)$, $d_G(w_1, a) = 1 \neq 2 = d_G(w_1, u)$, $d_G(b, v) = 1 \neq d_G(b, u)$. Then B is a local metric generator for G and $|B| = n-4$, which is a contradiction.

According to the two cases above we conclude that for each $x \in U_a$, $y \in U_S$, $xy \in E(G)$ and, by symmetry, for each $x \in U_b$, $y \in U_S$, $xy \in E(G)$. $\square$

Chapter 3

The local metric dimenson of strong product graphs

3.1 Introduction

The strong product of two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is the graph $G \boxtimes H = (V, E)$, such that $V = V(G) \times V(H)$ and two vertices $(a, b), (c, d) \in V$ are adjacent in $G \boxtimes H$ if and only if

$a = c$ and $bd \in E_2$, or

$b = d$ and $ac \in E_1$, or

$ac \in E_1$ and $bd \in E_2$.

We would like to point out that the Cartesian product $G \square H$ is a subgraph of $G \boxtimes H$ and also that $K_r \boxtimes K_s = K_{rs}$.

One of our tools will be a well-known result, which states the relationship between the vertex distances in $G \boxtimes H$ and the vertex distances in the factor graphs.

Remark 3.1. [29] *Let G and H be two connected graphs. Then*

$$d_{G\boxtimes H}((a, b), (c, d)) = \max\{d_G(a, c), d_H(b, d)\}.$$

For the remainder of the chapter, definitions will be introduced whenever a concept is needed.

3.2 General bounds

We begin by giving general bounds for the local metric dimension of strong product graphs.

Theorem 3.2. *Let G and H be two connected graphs of order $n_1 \geq 2$ and $n_2 \geq 2$, respectively. Then*

$$3 \leq \dim_l(G \boxtimes H) \leq n_1 \cdot \dim_l(H) + n_2 \cdot \dim_l(G) - \dim_l(G) \cdot \dim_l(H).$$

Proof. Let $V(G)$ and $V(H)$ be the set of vertices of G and H, respectively. We claim that $S = (V(G) \times S_2) \cup (S_1 \times V(H))$ is a local metric generator for $G \boxtimes H$, where S_1 and S_2 are local metric basis for G and H, respectively.

Let $(u_i, v_j), (u_k, v_l) \in V(G) \times V(H) - S$ be two adjacent vertices of $G \boxtimes H$. If $i = k$, then v_j and v_l are adjacent in H and there exists $b \in S_2$ such that $d_{G\boxtimes H}((u_i, b), (u_i, v_j)) = d_H(b, v_j) \neq d_H(b, v_l) = d_{G\boxtimes H}((u_i, b), (u_k, v_l))$. So, (u_i, v_j) and (u_k, v_l) are distinguished by $(u_i, b) \in (V(G) \times S_2) \subset S$. Analogously, if $j = l$, then u_i and u_k are adjacent in G and there exists $a \in S_1$ such that $d_G(a, u_i) \neq d_G(a, u_k)$ and, as above, (u_i, v_j) and (u_k, v_l) are distinguished by $(a, v_j) \in (S_1 \times V(H)) \subset S$. Finally, if $u_i u_k \in E_1$ and $v_j v_l \in E_2$, then for any $a \in S_1$ such that $d_G(a, u_i) \neq d_G(a, u_k)$ we have

$$d_{G\boxtimes H}((u_i, v_j), (a, v_j)) = d_G(u_i, a) \neq$$

$$d_G(u_k, a) = \max\{d_G(u_k, a), 1\} = d_{G\boxtimes H}((a, v_j), (u_k, v_l)).$$

Thus, (u_i, v_j) and (u_k, v_l) are distinguished by $(a, v_j) \in S_1 \times V(H) \subset S$. Then we conclude that S is a local metric generator for $G \boxtimes H$ and, as a consequence, $\dim_l(G \boxtimes H) \leq |S| = n_1 \cdot \dim_l(H) + n_2 \cdot \dim_l(G) - \dim_l(G) \cdot \dim_l(H)$.

To prove the lower bound, let B be a local metric basis of $G \boxtimes H$. Given $(u_1, v_1) \in B$, chose $u^* \in N_G(u_1)$, $v^* \in N_H(v_1)$ and define

$$W = \{(u^*, v_1), (u_1, v^*), (u^*, v^*)\}.$$

Since (u_1, v_1) is not able to distinguish any pair of adjacent vertices in W, there exists $(u_2, v_2) \in B - \{(u_1, v_1)\}$. Let

$$q = \min_{(a,b)\in W}\{d_{G\boxtimes H}((u_2, v_2), (a, b))\}.$$

Now, as $d_{G\boxtimes H}((a,b),(u_2,v_2)) \in \{q, q+1\}$ for every $(a,b) \in W$, by Dirichlet's box principle, there are two vertices $(x_1,y_1),(x_2,y_2) \in W$ such that

$$d_{G\boxtimes H}((u_2,v_2),(x_1,y_1)) = d_{G\boxtimes H}((u_2,v_2),(x_2,y_2)).$$

Hence, $B - \{(u_1,v_1),(u_2,v_2)\} \neq \emptyset$, and the result follows. □

Since $K_{n_1} \boxtimes K_{n_2} \cong K_{n_1 \cdot n_2}$ and for any complete graph K_n, $\dim_l(K_n) = n-1$, we deduce

$$\dim_l(K_{n_1} \boxtimes K_{n_2}) = n_1 \cdot \dim_l(K_{n_2}) + n_2 \cdot \dim_l(K_{n_1}) - \dim_l(K_{n_1}) \cdot \dim_l(K_{n_2}).$$

Therefore, the upper bound is tight. Examples of non-complete graphs, where the upper bound is attained, can be derived from Theorem 3.7.

In order to show that the lower bound is tight, consider two paths P_t and $P_{t'}$, where $t' \leq t \leq 2t'-1$, $V(P_t) = \{u_1, u_2, \ldots, u_t\}$ and $u_i u_{i+1} \in E(P_t)$, for every $i \in \{1, \ldots, t-1\}$. Also, take $v_1, v_{t'} \in V(P_{t'})$ such that $d_{P_{t'}}(v_1, v_{t'}) = t'-1$. It is not difficult to check that $\{(u_1,v_1),(u_{t'},v_{t'}),(u_t,v_1)\}$ is a local metric generator for $P_t \boxtimes P_{t'}$, so that Theorem 3.2 leads to $\dim_l(P_t \boxtimes P_{t'}) = 3$.

3.3 The case of adjacency k-resolved graphs

Now we will give some results involving the diameter or the radius of G. Given two vertices x and y in a connected graph $G = (V, E)$, the interval $I[x,y]$ between x and y is defined as the collection of all vertices which lie on some shortest xy path. Given a nonnegative integer k, we say that G *is adjacency k-resolved* if for every two adjacent vertices $x, y \in V$, there exists $w \in V$ such that

$d_G(y,w) \geq k$ and $x \in I[y,w]$, or

$d_G(x,w) \geq k$ and $y \in I[x,w]$.

For instance, the path and the cycle graphs of order n ($n \geq 2$) are adjacency $\lceil \frac{n}{2} \rceil$-resolved, the two-dimensional grid graphs $P_r \Box P_t$ are adjacency $\left(\lceil \frac{r}{2} \rceil + \lceil \frac{t}{2} \rceil\right)$-resolved, and the hypercube graphs Q_k are adjacency k-resolved.

Theorem 3.3. *Let H be an adjacency k-resolved graph of order n_2 and let G be a non-trivial graph of diameter $D(G) < k$. Then* $\dim_l(G \boxtimes H) \leq n_2 \cdot \dim_l(G)$.

Proof. Let $V(G) = \{u_1, u_2, ..., u_{n_1}\}$ and $V(H) = \{v_1, \dots, v_{n_2}\}$ be the set of vertices of G and H, respectively. Let S_1 be a local metric generator for G. We will show that $S = S_1 \times V(H)$ is a local metric generator for $G \boxtimes H$. Let $(u_i, v_j), (u_r, v_l)$ be two adjacent vertices of $G \boxtimes H$. We differentiate the following two cases.

Case 1. $j = l$. Since $u_i u_r \in E(G)$ and S_1 is a local metric generator for G, there exists $u \in S_1$ such that $d_G(u_i, u) \neq d_G(u_r, u)$. Hence,

$$d_{G\boxtimes H}((u_i, v_j), (u, v_j)) = d_G(u_i, u) \neq d_G(u_r, u) = d_{G\boxtimes H}((u_r, v_j), (u, v_j)).$$

Case 2. $v_j v_l \in E(H)$. Since H is adjacency k-resolved, there exists $v \in V(H)$ such that $(d_H(v, v_l) \geq k$ and $v_j \in I[v, v_l])$ or $(d_H(v, v_j) \geq k$ and $v_l \in I[v, v_j])$. Say $d_H(v, v_l) \geq k$ and $v_j \in I[v, v_l]$. In such a case, as $D(G) < k$, for every $u \in S_1$ we have

$$\begin{aligned} d_{G\boxtimes H}((u_i, v_j), (u, v)) &= \max\{d_G(u_i, u), d_H(v_j, v)\} \\ &< d_H(v, v_l) \\ &= \max\{d_G(u, u_r), d_H(v, v_l)\} \\ &= d_{G\boxtimes H}((u_r, v_l), (u, v)). \end{aligned}$$

Therefore, S is a local metric generator for $G \boxtimes H$. □

Lemma 3.4. *Let H be a connected bipartite graph of order greater than or equal to three. Then H is adjacency k-resolved for any $k \in \{2, \dots, r(H)\}$.*

Proof. Let $x, y, w \in V(H)$ such that $xy \in E(H)$ and $d_H(x, w) = k$, for some $k \in \{2, .., r(H)\}$. Since H does not have cycles of odd length, $d_H(w, y) \neq k$. Thus, either $d_H(w, y) = d_H(w, x) + d_H(x, y) = k+1$ or $d_H(w, x) = d_H(w, y) + d_H(y, x) = k$. Therefore, the result follows. □

Now we derive a consequences of combining Theorem 3.3 and Lemma 3.4.

Theorem 3.5. *Let G and H be two connected non-trivial graphs. If H is bipartite and $D(G) < r(H)$, then $\dim_l(G \boxtimes H) \leq |V(H)| \dim_l(G)$.*

As we will show in Theorem 3.11, the above inequality is tight.

3.4 The role of true twin equivalence classes

With the definition of true twin vertices in mind we state the following results.

Lemma 3.6. *Let G and H be two non-trivial connected graphs of order n_1 and n_2, having t_1 and t_2 true twin equivalent classes, respectively. Then the vertex set of $G \boxtimes H$ is partitioned into t_1t_2 true twin equivalent classes.*

Proof. First of all, we would point out that for any $a \in V(G)$ and $b \in V(H)$ it holds

$$N_{G\boxtimes H}[(a,b)] = \{(x,y): \ x \in N_G[a], y \in N_H[b]\} = N_G[a] \times N_H[b].$$

Now, since the result immediately holds for complete graphs, we assume that $G \not\cong K_{n_1}$ or $H \not\cong K_{n_2}$. Let $U_1, U_2, ..., U_{t_1}$ and $U'_1, U'_2, ..., U'_{t_2}$ be the true twin equivalence classes of G and H, respectively. Since each U_i (and U'_j) induces a clique and its vertices have identical closed neighborhoods, for every $a, c \in U_i$ and $b, d \in U'_j$,

$$N_{G\boxtimes H}[(a,b)] = N_G[a] \times N_H[b] = N_G[c] \times N_H[d] = N_{G\boxtimes H}[(c,d)].$$

Hence, $V(G) \times V(H)$ is partitioned as $V(G) \times V(H) = \bigcup_{j=1}^{t_2} \left(\bigcup_{i=1}^{t_1} U_i \times U'_j \right)$, where $U_i \times U'_j$ induces a clique in $G \boxtimes H$ and its vertices have identical closed neighborhoods. Moreover, for any $(a,b) \in U_i \times U'_j$ and $(c,d) \in U_k \times U'_l$, where $i \neq k$ or $j \neq l$, we have

$$N_{G\boxtimes H}[(a,b)] = N_G[a] \times N_H[b] \neq N_G[c] \times N_H[d] = N_{G\boxtimes H}[(c,d)].$$

Therefore, the true twin equivalence classes of $G \boxtimes H$ are of the form $U_i \times U'_j$, where $i \in \{1, .., t_1\}$ and $j \in \{1, .., t_2\}$. □

We would point out that the above result was indirectly obtained in [52], proof of Theorem 2.3.

Theorem 2.10 and Lemma 3.6 directly lead to the next result.

Theorem 3.7. *Let G and H be two non-trivial connected graphs of order n_1 and n_2, having t_1 and t_2 true twin equivalence classes, respectively. Then*

$$\dim_l(G \boxtimes H) \geq n_1n_2 - t_1t_2.$$

By Theorems 2.4, 3.2 and 3.7 we deduce the following result.

Theorem 3.8. *Let G and H be two non-trivial connected graphs of order n_1 and n_2, having t_1 and t_2 true twin equivalence classes, respectively. Then the following assertions hold:*

(i) *If $\dim_l(G) = n_1 - t_1$ and $\dim_l(H) = n_2 - t_2$, then $\dim_l(G \boxtimes H) = n_1 n_2 - t_1 t_2$.*

(ii) *If $\dim_l(G) = n_1 - t_1$ and H is bipartite, then $n_2(n_1 - t_1) \leq \dim_l(G \boxtimes H) \leq n_2(n_1 - t_1) + t_1$.*

Since any complete graph K_n has only one true twin equivalence class, Theorem 3.8 leads to the next result.

Corollary 3.9. *Let H be a connected graph of order $n' \geq 2$ having t true twin equivalent classes. Then for any integer $n \geq 2$,*

$$\dim_l(K_n \boxtimes H) = nn' - t.$$

In particular, if H does not have true twin vertices, then

$$\dim_l(K_n \boxtimes H) = n'(n-1).$$

Note that if H is an adjacency k-resolved graph, for $k \geq 2$, then H does not have true twin vertices. Therefore, Theorems 3.7 and 3.3 lead to the following result.

Theorem 3.10. *Let H be an adjacency k-resolved graph of order n_2 and let G be a non-trivial connected graph of order n_1, having t_1 true twin equivalence classes and diameter $D(G) < k$. If $\dim_l(G) = n_1 - t_1$, then $\dim_l(G \boxtimes H) = n_2(n_1 - t_1)$.*

Our next result can be deduced from Corollary 3.4 and Theorem 3.10 or from Theorems 3.7 and 3.5.

Theorem 3.11. *Let H be connected bipartite graph of order n_2 and let G be a non-trivial connected graph of order n_1, having t_1 true twin equivalence classes. If $\dim_l(G) = n_1 - t_1$ and $D(G) < r(H)$, then $\dim_l(G \boxtimes H) = n_2(n_1 - t_1)$.*

3.5 The particular case of $P_t \boxtimes G$

In this section we assume that t is an integer greater than or equal to two and $V(P_t) = \{u_1, u_2, \ldots, u_t\}$, where $u_i u_{i+1} \in E(P_t)$, for every $i \in \{1, \ldots, t-1\}$. In the proof of the next lemma we will use the notation $\mathcal{B}_r(x)$ for the closed ball of center $x \in V(G)$ and radius $r \geq 0$, *i.e.*,

$$\mathcal{B}_r(x). = \{y \in V(G) :\ d_G(x, y) \leq r\}.$$

Lemma 3.12. *Let G be a connected graph and let $t \geq 1$ be an integer. Let $u_{i_1}, u_{i_2}, \ldots, u_{i_b}$ be the first components of the elements in a local metric basis of $P_t \boxtimes G$, where $i_1 \leq i_2 \leq \cdots \leq i_b$. Then the following assertions hold.*

(i) $i_2 \leq D(G) + 1$ *and* $i_{b-1} \geq t - D(G)$.

(ii) *For any* $l \in \{1, \ldots, b-2\}$, $i_{l+2} \leq 2D(G) + i_l$.

(iii) $i_3 \leq 2D(G) + 1$.

Proof. Let B be a local metric basis of $P_t \boxtimes G$ and let $u_{i_1}, u_{i_2}, \ldots, u_{i_b}$ be the first components of the elements in B, where $i_1 \leq i_2 \leq \cdots \leq i_b$. First of all, notice that $|B| = b$ and, by Theorem 3.2, $b \geq 3$.

We first proceed to prove (i). Suppose, for the contrary, that $i_2 > D(G)+1$. Let $y, z \in V(G)$ such that $(u_{i_1}, y) \in B$ and $z \in N_G(y)$. If $i_1 \neq 1$, then no vertex in B is able to distinguish (u_1, y) and (u_1, z). Now, if $i_1 = 1$, then no vertex in B is able to distinguish (u_2, y) and (u_2, z). So, in both cases we get a contradiction. The proof of $i_{b-1} \geq t - D(G)$ is deduced by symmetry. Hence, (i) follows.

To prove (ii) we proceed by contradiction. Suppose that $i_{l+2} > 2D(G)+i_l$ for some $l \in \{1, \ldots, b-2\}$. In such a case we have that $i_{l+1} > D(G) + i_l$ or $i_{l+2} > D(G) + i_{l+1}$. We suppose that $i_{l+1} > D(G) + i_l$, being the second case analogous. We now take $y, z \in V(G)$ such that $(u_{i_{l+1}}, y) \in B$ and $z \in N_G(y)$. Notice that $(u_{i_l+D(G)}, y)$ and $(u_{i_l+D(G)}, z)$ are adjacent.We differentiate the following cases for $(u_{i_k}, w) \in B$. If $k \leq l$, then $i_l + D(G) - i_k \geq D(G)$ and so

$$d_{P_t \boxtimes G}((u_{i_k}, w), (u_{i_l+D(G)}, y)) = i_l + D(G) - i_k = d_{P_t \boxtimes G}((u_{i_k}, w), (u_{i_l+D(G)}, z)).$$

If $k = l+1$ and $i_{l+1} \neq i_{l+2}$, then $w = y$ and since $i_{l+1} > D(G) + i_l$, we have

$$d_{P_t \boxtimes G}((u_{i_k}, w), (u_{i_l+D(G)}, y)) = i_k - i_l - D(G) = d_{P_t \boxtimes G}((u_{i_k}, w), (u_{i_l+D(G)}, z)).$$

If $k = l+1$ and $i_{l+1} = i_{l+2}$, then from the assumption $i_{l+2} > 2D(G) + i_l$ we have that $i_k - i_l - D(G) > D(G)$ and so

$$d_{P_t \boxtimes G}((u_{i_k}, w), (u_{i_l+D(G)}, y)) = i_k - i_l - D(G) = d_{P_t \boxtimes G}((u_{i_k}, w), (u_{i_l+D(G)}, z)).$$

If $k \geq l+2$, then the assumption $i_{l+2} > 2D(G) + i_l$ leads to $i_k - i_l - D(G) > D(G)$ and so

$$d_{P_t \boxtimes G}((u_{i_k}, w), (u_{i_l+D(G)}, y)) = i_k - i_l - D(G) = d_{P_t \boxtimes G}((u_{i_k}, w), (u_{i_l+D(G)}, z)).$$

Hence, no vertex in B is able to distinguish $(u_{i_l+D(G)}, y)$ from $(u_{i_l+D(G)}, z)$, which is a contradiction. Therefore, the proof of (ii) is complete.

Finally, we proceed to prove (iii). If $i_1 = 1$, then by (ii) we obtain $i_3 \leq 2D(G) + 1$. Hence, we assume that $i_1 > 1$. For contradiction purposes, suppose that $i_3 > 2D(G) + 1$. We differentiate two cases for $(u_{i_1}, v_1), (u_{i_2}, v_2) \in B$.

Case 1: $i_1 + i_2 - 2 > d_G(v_1, v_2)$. In this case $|\mathcal{B}_{i_1-1}(v_1) \cap \mathcal{B}_{i_2-1}(v_2)| \geq 2$ and so we take $\alpha, \beta \in \mathcal{B}_{i_1-1}(v_1) \cap \mathcal{B}_{i_2-1}(v_2)$ such that $\alpha\beta \in E(G)$. For the pair of adjacent vertices $(u_1, \alpha), (u_1, \beta)$ we have

$$d_{P_t \boxtimes G}((u_{i_1}, v_1), (u_1, \alpha)) = i_1 - 1 = d_{P_t \boxtimes G}((u_{i_1}, v_1), (u_1, \beta))$$

and

$$d_{P_t \boxtimes G}((u_{i_2}, v_2), (u_1, \alpha)) = i_2 - 1 = d_{P_t \boxtimes G}((u_{i_2}, v_2), (u_1, \beta)).$$

So, neither (u_{i_1}, v_1) nor (u_{i_2}, v_2) distinguishes (u_1, α) from (u_1, β). Furthermore, for $i_r \geq i_3 > 2D(G) + 1$ and $(u_{i_r}, v_r) \in B$ we have

$$d_{P_t \boxtimes G}((u_{i_r}, v_r), (u_1, \alpha)) = i_r - 1 = d_{P_t \boxtimes G}((u_{i_r}, v_r), (u_1, \beta)).$$

Therefore, no vertex $(u_{i_r}, v_r) \in B$ distinguishes (u_1, α) from (u_1, β), which is a contradiction.

Case 2: $i_1 + i_2 - 2 \leq d(v_1, v_2)$. In this case we have

$$(D(G) + 2 - i_1) + (D(G) + 2 - i_2) = 2D(G) + 2 - (i_1 + i_2 - 2) \geq$$

$$2D(G) + 2 - d(v_1, v_2) \geq D(G) + 2.$$

Hence, there exist $\alpha, \beta \in \mathcal{B}_{D(G)+2-i_1}(v_1) \cap \mathcal{B}_{D(G)+2-i_2}(v_2)$ such that $\alpha\beta \in E(G)$. For the pair of adjacent vertices $(u_{D(G)+2}, \alpha), (u_{D(G)+2}, \beta)$ we have

$$d_{P_t \boxtimes G}((u_{i_1}, v_1), (u_{D(G)+2}, \alpha)) = D(G) + 2 - i_1 = d_{P_t \boxtimes G}((u_{i_1}, v_1), (u_{D(G)+2}, \beta))$$

and

$$d_{P_t \boxtimes G}((u_{i_2}, v_2), (u_{D(G)+2}, \alpha)) = D(G) + 2 - i_2 = d_{P_t \boxtimes G}((u_{i_2}, v_2), (u_{D(G)+2}, \beta))$$

So, neither (u_{i_1}, v_1) nor (u_{i_2}, v_2) distinguishes $(u_{D(G)+2}, \alpha)$ from $(u_{D(G)+2}, \beta)$. For $i_r \geq i_3 > 2D(G) + 1$ and $(u_{i_r}, v_r) \in B$ we have

$$d_{P_t \boxtimes G}((u_{i_r}, v_r), (u_{D(G)+2}, \alpha)) = i_r - (D(G)+2) = d_{P_t \boxtimes G}((u_{i_r}, v_r), (u_{D(G)+2}, \beta)).$$

Thus, no vertex $(u_{i_r}, v_r) \in B$ distinguishes $(u_{D(G)+2}, \alpha)$ from $(u_{D(G)+2}, \beta)$, which is a contradiction. □

Theorem 3.13. *For any connected G and any integer $t \geq 2D(G) + 1$,*

$$\dim_l(P_t \boxtimes G) \geq \left\lceil \frac{t-1}{D(G)} \right\rceil + 1.$$

Proof. Let B be a local metric basis of $P_t \boxtimes G$ and let $u_{i_1}, u_{i_2}, \ldots, u_{i_b}$ be the first components of the elements in B, where $i_1 \leq i_2 \leq \cdots \leq i_b$. We differentiate two cases.

Case 1. b odd. In this case $b - 1$ is even and by Lemma 3.12 (i) and (ii) we have

$$i_2 \leq D(G) + 1,\ i_4 \leq 3D(G) + 1,\ \ldots,\ i_{b-1} \leq (b-2)D(G) + 1.$$

Case 2. b even. In this case $b - 1$ is odd and by Lemma 3.12 (iii) and (ii) we have

$$i_3 \leq 2D(G) + 1,\ i_5 \leq 4D(G) + 1,\ \ldots,\ i_{b-1} \leq (b-2)D(G) + 1.$$

According to the two cases above and Lemma 3.12 (i) we have

$$t - D(G) \leq i_{b-1} \leq (b-2)D(G) + 1.$$

Therefore, $b \geq \frac{t-1}{D(G)} + 1$. □

From now on we say that a set $W \subset V(G \boxtimes H)$ *resolves* the set $X \subseteq V(G \boxtimes H)$ if every pair of adjacent vertices in X is distinguished by some element in W.

Lemma 3.14. *Let G and H be two connected nontrivial graphs such that H is bipartite. Let $u_1, u_2, u_3 \in V(G)$ and $v_1, v_2 \in V(H)$ such that $u_2 \in I_G[u_1, u_3]$, $d_G(u_1, u_2) \leq d_H(v_1, v_2) = D(H)$ and $d_G(u_2, u_3) \geq D(H)$. Then, for any shortest path P from u_1 to u_2, the set $B = \{(u_1, v_1), (u_2, v_2), (u_3, v_1)\}$ resolves $V(P) \times V(H)$.*

Proof. Let P be a shortest path form u_1 to u_2 and let $(u_i, v_j), (u_k, v_l) \in V(G \boxtimes H)$ be two adjacent vertices such that $u_i, u_k \in V(P)$. Without lost of generality, we assume that $d_G(u_i, u_1) \leq d_G(u_k, u_1)$. Notice that from this assumption we have that $d_G(u_i, u_3) \geq d_G(u_k, u_3)$. We differentiate the following two cases:

Case 1: $u_i u_k \in E(G)$. As $d_G(u_2, u_3) \geq D(H)$ and $u_i, u_k \in V(P)$, we have $D(H) \leq d_G(u_3, u_k) < d_G(u_3, u_i)$ and so $d_{G\boxtimes H}((u_3, v_1), (u_i, v_j)) = d_G(u_3, u_i) > d_G(u_3, u_k) = d_{G\boxtimes H}((u_3, v_1), (u_k, v_l))$.

Case 2: $i = k$. In this case $v_j v_l \in E(H)$ and, as H is a bipartite graph, $d_H(v_1, v_j) \neq d_H(v_1, v_l)$ and $d_H(v_2, v_j) \neq d_H(v_2, v_l)$. We assume, without lost of generality, that $d_H(v_1, v_j) < d_H(v_1, v_l)$. Notice that

$$d_H(v_1, v_j)+d_H(v_j, v_2) \geq d_H(v_1, v_2) = D(H) \geq d_G(u_1, u_2) = d_G(u_1, u_i)+d_G(u_i, u_2).$$

Hence, $d_H(v_1, v_j) \geq d_G(u_1, u_i)$ or $d_H(v_j, v_2) > d_G(u_2, u_i)$. If $d_H(v_1, v_j) \geq d_G(u_1, u_i)$, then

$$d_{G\boxtimes H}((u_1, v_1), (u_i, v_j)) = d_H(v_1, v_j) < d_H(v_1, v_l) = d_{G\boxtimes H}((u_1, v_1), (u_k, v_l)).$$

Now, if $d_H(v_j, v_2) > d_G(u_2, u_i)$, then $d_H(v_l, v_2) \geq d_G(u_2, u_i) = d_G(u_2, u_k)$ and so

$$d_{G\boxtimes H}((u_2, v_2), (u_i, v_j)) = d_H(v_2, v_j) \neq d_H(v_2, v_l) = d_{G\boxtimes H}((u_2, v_2), (u_k, v_l)).$$

According to the cases above, the result follows. □

Theorem 3.15. *For any connected bipartite graph G and any integer $t \geq 2D(G) + 1$,*

$$\dim_l(P_t \boxtimes G) = \left\lceil \frac{t-1}{D(G)} \right\rceil + 1.$$

Proof. Let G and P_t be as in the hypotheses. From $\alpha = \left\lfloor \frac{t-1}{D(G)} \right\rfloor$ and two diametral vertices $a, b \in V(G)$ we define a set B_α as follows.

If $\alpha = \frac{t-1}{D(G)}$, then

$$B_\alpha = \{(u_1, a), (u_{D(G)+1}, b), (u_{2D(G)+1}, a), (u_{3D(G)+1}, b), \ldots, (u_{\alpha D(G)+1}, b)\}$$

for α is odd and

$$B_\alpha = \{(u_1, a), (u_{D(G)+1}, b), (u_{2D(G)+1}, a), (u_{3D(G)+1}, b), \ldots, (u_{\alpha D(G)+1}, a)\}$$

for α even.

If $\alpha < \frac{t-1}{D(G)}$, then

$$B_\alpha = \{(u_1, a), (u_{D(G)+1}, b), (u_{2D(G)+1}, a), (u_{3D(G)+1}, b), \dots, (u_{\alpha D(G)+1}, b), (u_t, a)\}$$

for α odd and

$$B_\alpha = \{(u_1, a), (u_{D(G)+1}, b), (u_{2D(G)+1}, a), (u_{3D(G)+1}, b), \dots, (u_{\alpha D(G)+1}, a), (u_t, b)\}$$

for α even.

We would point out that, in any case, $|B_\alpha| = \left\lceil \frac{t-1}{D(G)} \right\rceil + 1$.

We will show that B_α is a local metric generator for $P_t \boxtimes G$. In order to see that, let (u_i, v_j) and (u_k, v_l) be two adjacent vertices belonging to $V(P_t \boxtimes G) - B_\alpha$. We consider, without lost of generality, that $i \leq k$ and we differentiate the following three cases for k.

- $1 \leq k \leq D(G) + 1$. Let $T_1 = \{u_1, \dots, u_{D(G)+1}\} \times V(G)$. In this case $(u_i, v_j), (u_k, v_l) \in T_1$ and, by Lemma 3.14 we have that

  $$\{(u_1, a), (u_{D(G)+1}, b), (u_{2D(G)+1}, a)\} \subset B_\alpha$$

  resolves T_1.

- $pD(G) + 2 \leq k \leq (p+1)D(G) + 1$, for some integer $p \in \{1, \dots, \alpha - 1\}$. Let $T_p = \{u_{pD(G)+1}, \dots, u_{(p+1)D(G)+1}\} \times V(G)$. In this case $(u_i, v_j), (u_k, v_l) \in T_p$ and we can take $x, y \in \{a, b\}$ so that

  $$X_p = \{(u_{(p-1)D(G)+1}, x), (u_{pD(G)+1}, y), (u_{(p+1)D(G)+1}, x)\} \subset B_\alpha$$

  thus, by Lemma 3.14 we can conclude that X_p resolves T_p.

- $\alpha D(G) + 2 \leq k \leq t$. Let $T_t = \{u_{\alpha D(G)+1}, \dots, u_t\} \times V(G)$. As above, $(u_i, v_j), (u_k, v_l) \in T_t$ and we can take $x, y \in \{a, b\}$ so that the set $X_t = \{(u_{(\alpha-1)D(G)+1}, x), (u_{\alpha D(G)+1}, y), (u_t, x)\}$ is a subset of B_α. Thus, by Lemma 3.14 we can conclude that X_t resolves T_t.

According to the three cases above we have $\dim_l(P_t \boxtimes G) \leq \left\lceil \frac{t-1}{D(G)} \right\rceil + 1$. Therefore, by Theorem 3.13 we conclude the proof. □

The authors of [52] conjectured that for any integers t and t' such that $2 \leq t' < t$, the metric dimension of $P_t \boxtimes P_{t'}$ equals $\left\lceil \frac{t+t'-2}{t'-1} \right\rceil$. We are now able to prove the conjecture.

Theorem 3.16. *For any integers t and t' such that $2 \leq t' < t$,*

$$\dim(P_t \boxtimes P_{t'}) = \left\lceil \frac{t+t'-2}{t'-1} \right\rceil.$$

Proof. As pointed out in Section 3.2, for $t' \leq t \leq 2t'-1$, $\dim_l(P_t \boxtimes P_{t'}) = 3$. Now, since $\dim_l(P_t \boxtimes P_{t'}) \leq \dim(P_t \boxtimes P_{t'})$, if $t \geq 2t'-1$, then by Theorem 3.15 we obtain the lower bound $\dim(P_t \boxtimes P_{t'}) \geq \left\lceil \frac{t+t'-2}{t'-1} \right\rceil$. The upper bound was obtained in [52]. Therefore, the result follows. □

3.6 The particular case of $C_t \boxtimes G$

In this section we assume that t is an integer greater than or equal to three and $V(C_t) = \{u_1, u_2, \dots, u_t\}$, where $u_1u_t \in E(C_t)$ and $u_iu_{i+1} \in E(C_t)$, for every $i \in \{1, \dots, t-1\}$.

Lemma 3.17. *Let G be a connected graph and let $t \geq 3$ be an integer. Let $u_{i_1}, u_{i_2}, \dots, u_{i_b}$ be the first components of the elements in a local metric basis of $C_t \boxtimes G$, where $i_1 \leq i_2 \leq \dots \leq i_b$. Then for any $l \in \{1, \dots, b\}$, $d_{C_t}(u_{i_{l+2}}, u_{i_l}) \leq 2D(G)$, where the subscripts of i are taken modulo b.*

Proof. Let B be a local metric basis of $C_t \boxtimes G$ and let $u_{i_1}, u_{i_2}, \dots, u_{i_b}$ be the first components of the elements in B, where $i_1 = 1 \leq i_2 \leq \dots \leq i_b$. First of all, notice that $|B| = b$ and, by Theorem 3.2, $b \geq 3$.

We proceed by contradiction. Suppose that $d_{C_t}(u_{i_{l+2}}, u_{i_l}) > 2D(G)$ for some $l \in \{1, \dots, b\}$. In such a case we have that $d_{C_t}(u_{i_{l+1}}, u_{i_l}) > D(G)$ or $d_{C_t}(u_{i_{l+2}}, u_{i_{l+1}}) > D(G)$. We suppose that $d_{C_t}(u_{i_{l+1}}, u_{i_l}) > D(G)$, being the second case analogous. We now take $y, z \in V(G)$ such that $(u_{i_{l+1}}, y) \in B$ and $z \in N_G(y)$. Notice that $(u_{i_l+D(G)}, y)$ and $(u_{i_l+D(G)}, z)$ are adjacent. We differentiate the following cases for $(u_{i_k}, w) \in B$. If $k \neq l+1$, then $d_{C_t}(u_{i_l+D(G)}, u_{i_k}) \geq D(G)$ and so

$$d_{C_t \boxtimes G}((u_{i_k}, w), (u_{i_l+D(G)}, y)) = d_{C_t}(u_{i_l+D(G)}, u_{i_k}) = d_{C_t \boxtimes G}((u_{i_k}, w), (u_{i_l+D(G)}, z)).$$

If $k = l+1$ and $i_{l+1} \neq i_{l+2}$ then $w = y$ and since $d_{C_t}(u_{i_{l+1}}, u_{i_l}) > D(G)$, we have

$$d_{C_t \boxtimes G}((u_{i_k}, w), (u_{i_l+D(G)}, y)) = d_{C_t}(u_{i_k}, u_{i_l+D(G)}) = d_{C_t \boxtimes G}((u_{i_k}, w), (u_{i_l+D(G)}, z)).$$

If $k = l+1$ and $i_{l+1} = i_{l+2}$ then from the assumption $d_{C_t}(u_{i_{l+2}}, u_{i_l}) > 2D(G)$ we have that $d_{C_t}(u_{i_k}, u_{i_l+D(G)}) > D(G)$ and so

$$d_{C_t \boxtimes G}((u_{i_k}, w), (u_{i_l+D(G)}, y)) = d_{C_t}(u_{i_k}, u_{i_l+D(G)}) = d_{C_t \boxtimes G}((u_{i_k}, w), (u_{i_l+D(G)}, z)).$$

Hence, no vertex in B is able to distinguish the adjacent vertices $(u_{i_l+D(G)}, y)$ and $(u_{i_l+D(G)}, z)$, which is a contradiction. Therefore, the proof is complete. □

Theorem 3.18. *For any connected graph G and any integer $t \geq 3$,*

$$\dim_l(C_t \boxtimes G) \geq \left\lceil \frac{t}{D(G)} \right\rceil.$$

Proof. If $3D(G) \geq t \geq 3$, then $\left\lceil \frac{t}{D(G)} \right\rceil \leq 3$ and, by Theorem 3.2, the result follows. From now on we take $t > 3D(G)$. Let $u_{i_1}, u_{i_2}, \ldots, u_{i_b}$ be the first components of the elements in a local metric basis B of $C_t \boxtimes G$, where $i_1 = 1 \leq i_2 \leq \cdots \leq i_b$. First of all, notice that $t+1-i_{b-1} = d_{C_t}(u_{i_1}, u_{i_{b-1}})$ and so Lemma 3.17 leads to $i_{b-1} \geq t+1-2D(G)$. We now differentiate two cases.

Case 1. b even. In this case $b-1$ is odd and by Lemma 3.17 we have

$$i_3 \leq 2D(G)+1,\ i_5 \leq 4D(G)+1,\ \ldots,\ i_{b-1} \leq (b-2)D(G)+1.$$

Hence, $t+1-2D(G) \leq i_{b-1} \leq (b-2)D(G)+1$, so that $b \geq \frac{t}{D(G)}$.

Case 2. b odd. By Lemma 3.17 we have

$$i_3 \leq D(G)+1,\ i_4 \leq 3D(G)+1,\ \ldots,\ i_b \leq (b-1)D(G)+1.$$

Now, since $t+i_2-i_b = d_{C_t}(u_{i_2}, u_b) \leq 2D(G)$, we have

$$i_2 \leq 2D(G) - t + i_b \leq (b+1)D(G) - t + 1.$$

Hence,

$$i_2 \leq (b+1)D(G)-t+1,\ i_4 \leq (b+3)D(G)-t+1,\ \ldots,\ i_{b-1} \leq (2b-2)D(G)-t+1.$$

Thus, $t+1-2D(G) \leq i_{b-1} \leq (2b-2)D(G)-t+1$, so that $b \geq \frac{t}{D(G)}$. □

Theorem 3.19. *For any connected bipartite graph G and any integer $t \geq 4D(G)$,*

$$\dim_l(C_t \boxtimes G) \leq \left\lceil \frac{t}{D(G)} \right\rceil + 1.$$

Furthermore, if $\left\lceil \frac{t}{D(G)} \right\rceil$ is even, then

$$\dim_l(C_t \boxtimes G) = \left\lceil \frac{t}{D(G)} \right\rceil.$$

Proof. Let G and C_t be as in the hypotheses. From $\alpha = \left\lceil \frac{t}{D(G)} \right\rceil$ and two diametral vertices $a, b \in V(G)$ we define a set B_α as follows. If α is even, then

$$B_\alpha = \{(u_1, a), (u_{D(G)+1}, b), (u_{2D(G)+1}, a), (u_{3D(G)+1}, b), \ldots, (u_{(\alpha-1)D(G)+1}, b)\}$$

and, if α is odd, then

$$B_\alpha = \{(u_1, a), (u_{D(G)+1}, b), (u_{2D(G)+1}, a), (u_{3D(G)+1}, b), \ldots,$$

$$(u_{(\alpha-1)D(G)+1}, a), (u_{(\alpha-1)D(G)+1}, b)\}.$$

Notice that $|B_\alpha| = \alpha$, for α even, and $|B_\alpha| = \alpha + 1$, for α odd. We will show that B_α is a local metric generator for $C_t \boxtimes G$. In order to see that, let (u_i, v_j), (u_k, v_l) be a pair of adjacent vertices belonging to $V(C_t \boxtimes G) - B_\alpha$. We consider, without lost of generality, that $i \le k$ and we differentiate the following three cases for k.

- $2 \le k \le D(G) + 1$. Let $T_1 = \{u_1, \ldots, u_{D(G)+1}\} \times V(G)$. In this case $(u_i, v_j), (u_k, v_l) \in T_1$ and, by Lemma 3.14, the set

  $$\{(u_1, a), (u_{D(G)+1}, b), (u_{2D(G)+1}, a)\}$$

  resolves T_1.

- $pD(G)+2 \le k \le (p+1)D(G)+1$, for some integer $p \in \{1, ..., \alpha-2\}$. Let $T_p = \{u_{pD(G)+1}, \ldots, u_{(p+1)D(G)+1}\} \times V(G)$. In this case $(u_i, v_j), (u_k, v_l) \in T_p$ and we can take $x, y \in \{a, b\}$ such that

  $$X_p = \{(u_{(p-1)D(G)+1}, x), (u_{pD(G)+1}, y), (u_{(p+1)D(G)+1}, x)\}$$

  is a subset of B_α. Thus, by Lemma 3.14 we can conclude that X_p resolves T_p.

- $(\alpha - 1)D(G) + 2 \le k \le t + 1$. Let $T_t = \{u_{(\alpha-1)D(G)+1}, \ldots, u_{t+1}\} \times V(G)$. In this case, $(u_i, v_j), (u_k, v_l) \in T_t$ and we take the set $X_t = \{(u_{(\alpha-1)D(G)+1}, b), (u_1, a), (u_{D(G)+1}, b)\} \subset B_\alpha$. By Lemma 3.14 we can conclude that X_t resolves T_t.

According to the three cases above B_α is a local metric generator for $C_t \boxtimes G$ and so $\dim_l(C_t \boxtimes G) \le |B_\alpha|$. Therefore, by Theorem 3.18 we conclude the proof. □

Chapter 4

The local metric dimension of graphs from the local metric dimension of their primary subgraphs

4.1 Introduction

In this chapter we show that the computation of the local metric dimension of a graph with cut vertices is reduced to the computation of the local metric dimension of the so-called primary subgraphs. The main results are applied to specific constructions including bouquets of graphs, rooted product graphs, corona product graphs, block graphs and chain of graphs.

Let $G[\mathcal{H}]$ be a connected graph constructed from a family of pairwise disjoint (non-trivial) connected graphs $\mathcal{H} = \{G_1, \ldots, G_k\}$ as follows. Select a vertex of G_1, a vertex of G_2, and identify these two vertices. Then continue in this manner inductively. More precisely, suppose that we have already used $G_1, \ldots, G_i$ in the construction, where $2 \leq i \leq k-1$. Then select a vertex in the already constructed graph (which may in particular be one of the already selected vertices) and a vertex of G_{i+1}; we identify these two vertices. Note that any graph $G[\mathcal{H}]$ constructed in this way has a tree-like structure, the $G_i's$ being its building stones (see Figure 4.1).

We will briefly say that $G[\mathcal{H}]$ is obtained by *point-attaching* from $\mathcal{H} = \{G_1, \ldots, G_k\}$ and that $G_i's$ are the *primary subgraphs* of $G[\mathcal{H}]$. We will also

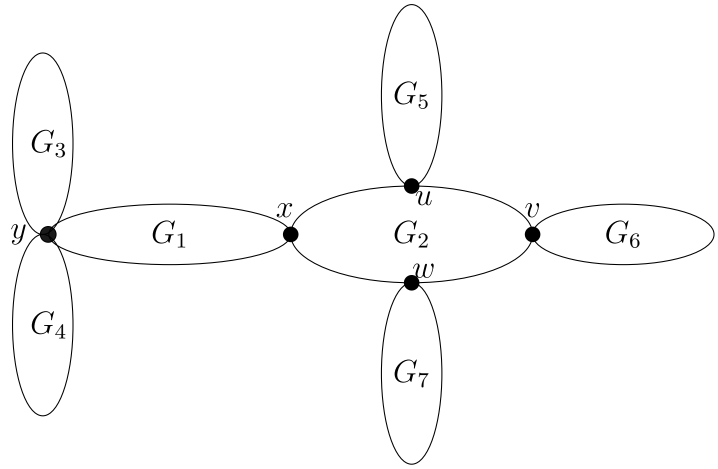

Figure 4.1: A graph $G[\mathcal{H}]$ obtained by point-attaching from $\mathcal{H} = \{G_1, G_2, ..., G_7\}$

say that the vertices of $G[\mathcal{H}]$ obtained by identifying two vertices of different primary subgraphs are the *attachment vertices* of $G[\mathcal{H}]$. Our definition and terminology is equivalent to the one previously introduced in [15] where the authors obtained an expression that reduces the computation of the Hosoya polynomials of a graph with cut vertices to the Hosoya polynomial of the so-called primary subgraphs. The reader is referred to [44] for a study on the metric dimension of graphs from primary subgraphs.

To begin with the study of the local metric dimension of $G[\mathcal{H}]$ we need some additional terminology. Given an attachment vertex x of $G[\mathcal{H}]$ and a primary subgraph G_j such that $x \in V(G_j)$, we define the subgraph $G_j(x^+)$ of $G[\mathcal{H}]$ as follows. We remove from $G[\mathcal{H}]$ all the edges connecting x with vertices in G_j, then $G_j(x^+)$ is the connected component which has x as a vertex. For instance, Figure 4.2 shows the subgraph $G_1(x^+)$ of the graph $G[\mathcal{H}]$ shown in Figure 4.1.

Let $J_{\mathcal{H}} \subseteq [k]$ be the set of subscripts such that $j \in J_{\mathcal{H}}$ whenever G_j is a non-bipartite primary subgraph of $G[\mathcal{H}]$. Note that $J_{\mathcal{H}} = \emptyset$ if and only if $G[\mathcal{H}]$ is bipartite, *i.e.*, $J_{\mathcal{H}} = \emptyset$ if and only if $\dim_l(G[\mathcal{H}]) = 1$. From now on we assume that $J_{\mathcal{H}} \neq \emptyset$.

Now, let C_j be the set composed by attachment vertices of $G[\mathcal{H}]$ belonging to $V(G_j)$ such that $x \in C_j$ whenever $G_j(x^+)$ is not bipartite. For instance, if G_2, G_3 and G_7 are the non-bipartite primary subgraphs of the graph shown in Figure 4.1, then $C_2 = \{x, w\}$.

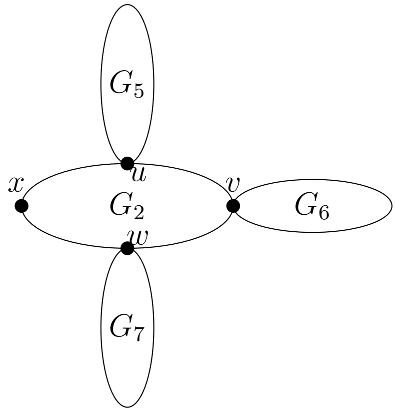

Figure 4.2: The subgraph $G_1(x^+)$ of the graph $G[\mathcal{H}]$ shown in Figure 4.1.

For any $j \in J_\mathcal{H}$ we define

$$\alpha_j = \max_{B \in \mathcal{B}(G_j)} \{|C_j \cap B|\},$$

where $\mathcal{B}(G_j)$ is the set of local metric bases of G_j, *i.e.*, α_j is the maximum cardinality of a set $\{x_1, \ldots, x_{\alpha_j}\} \subseteq V(G_j)$ composed by attachment vertices of $G[\mathcal{H}]$ belonging simultaneously to a local metric basis of G_j such that for every $l \in \{1, \ldots, \alpha_j\}$ the subgraph $G_j(x_l^+)$ is not bipartite.

4.2 Main results

Theorem 4.1. *For any non-bipartite graph $G[\mathcal{H}]$ obtained by point-attaching from a family of connected graphs $\mathcal{H} = \{G_1, \ldots, G_k\}$,*

$$\dim_l(G[\mathcal{H}]) \leq \sum_{j \in J_\mathcal{H}} (\dim_l(G_j) - \alpha_j).$$

Proof. For any $j \in J_\mathcal{H}$ we take $B_j \in \mathcal{B}(G_j)$ and $M_j \subseteq B_j \cap C_j$ such that $|M_j| = \alpha_j$. We claim that $B = \bigcup_{j \in J_\mathcal{H}} (B_j - M_j)$ is a local metric generator for $G[\mathcal{H}]$.

First of all, note that by the structure of $G[\mathcal{H}]$ we have that for any $v \in M_j$ there exists a non-bipartite primary subgraph G_r, which is a subgraph of $G_j(v^+)$, such that $B_r - M_r \neq \emptyset$. To see this we take a non-bipartite primary subgraph G_{j_1}, which is a subgraph of $G_j(v^+)$, next, if $B_{j_1} = M_{j_1}$, then we take $v_1 \in V(G_{j_1})$ and, as above, we take a non-bipartite primary subgraph G_{j_2}, which is a subgraph of $G_j(v_1^+)$, and if $B_{j_2} = M_{j_2}$ then we repeat this process until obtain a non-bipartite primary subgraph G_{j_t}, which is a subgraph of $G_j(v_{t-1}^+)$ such that $|B_{j_t}| > |M_{j_t}|$ (at worst, we will arrive to a subgraph $G_j(v_{t-1}^+)$ containing only one non-bipartite primary subgraph). With this fact in mind, we differentiate the following cases for two adjacent vertices $x, y \in V(G_i)$.

Case 1. $i \in J_{\mathcal{H}}$. If the pair x, y is distinguished by some $u \in B_i - M_i$, then we are done. Now, if the pair x, y is distinguished by $v \in M_i$, then we take G_r as a non-bipartite primary subgraph of $G_i(v^+)$ such that $B_r - M_r \neq \emptyset$. Since the pair x, y is distinguished by any vertex of $G_i(v^+)$, it is also distinguished by any $u \in B_r - M_r$.

Case 2. $i \in [k] - J_{\mathcal{H}}$. In this case, we take $j \in J_{\mathcal{H}}$ such that $B_j - M_j \neq \emptyset$ and, since G_i is bipartite, the pair x, y is distinguished by any $u \in B_j - M_j$.

Hence, B is a local metric generator for $G[\mathcal{H}]$ and, as a consequence,

$$\dim_l(G[\mathcal{H}]) \leq |B| = \sum_{j \in J_{\mathcal{H}}} (|B_j| - |M_j|) = \sum_{j \in J_{\mathcal{H}}} (\dim_l(G_j) - \alpha_j).$$

Therefore, the result follows. □

Theorem 4.2. *Let $G[\mathcal{H}]$ be a non-bipartite graph obtained by point-attaching from a family of connected graphs $\mathcal{H} = \{G_1, ..., G_k\}$. If for each $j \in [k]$ it holds that any minimal local metric generator for G_j is minimum, then*

$$\dim_l(G[\mathcal{H}]) = \sum_{j \in J_{\mathcal{H}}} (\dim_l(G_j) - \alpha_j).$$

Proof. Since $G[\mathcal{H}]$ is a non-bipartite graph, any vertex belonging to a local metric basis of $G[\mathcal{H}]$ distinguishes every pair of adjacent vertices included in a bipartite primary subgraph of $G[\mathcal{H}]$. Hence, we take a local metric basis A of $G[\mathcal{H}]$ which does not contain vertices belonging to the bipartite primary subgraphs of $G[\mathcal{H}]$. *i.e.*, for any $i \in [k] - J_{\mathcal{H}}$ it holds $A \cap V(G_i) = \emptyset$. Now, for each $j \in J_{\mathcal{H}}$ we define $A_j = A \cap V(G_j)$.

We claim that $C_j \cup A_j$ is a local metric generator for G_j. Suppose that there exist two adjacent vertices $x, y \in V(G_j)$ which are not distinguished by the elements of A_j. In such a case, there exists $x_r \in A_r$, $r \in J_{\mathcal{H}} - \{j\}$, which distinguishes x, y, and so there must exists $v \in C_j$ such that G_r is a subgraph of $G_j(v^+)$ and, as a result, v distinguishes the pair x, y. Hence, $C_j \cup A_j$ is a local metric generator for G_j.

Moreover, if $j \in J_{\mathcal{H}}$, then for any attachment vertex $w \in C_j$ it holds that $|A \cap V(G_j(w^+))| > 0$, as $G_j(w^+)$ is not bipartite. Hence, given two adjacent vertices $x, y \in V(G_j)$, which are distinguished by w, there exists $w' \in A_r \cap V(G_j(w^+)$, $r \in J_{\mathcal{H}} - \{j\}$, which distinguishes x, y, and so the minimality of A leads to $C_j \cap A_j = \emptyset$.

Now, if any minimal local metric generator for G_j is minimum, then there exists a set $C'_j \subseteq C_j$ such that $C'_j \cup A_j$ is a local metric basis for G_j. Thus, $|C'_j| + |A_j| = |C'_j \cup A_j| = \dim_l(G_j)$. Therefore,

$$\dim_l(G[\mathcal{H}]) = |A| = \sum_{j \in J_{\mathcal{H}}} |A_j| = \sum_{j \in J_{\mathcal{H}}} (\dim_l(G_j) - |C'_j|) \geq \sum_{j \in J_{\mathcal{H}}} (\dim_l(G_j) - \alpha_j).$$

We conclude the proof by Theorem 4.1. □

For any $j \in J_{\mathcal{H}}$ we define $\Gamma(G_j)$ as the family of local metric generators for G_j, and

$$\rho_j = \min_{S \subseteq V(G_j)} \{|S| : \ S \cup C_j \in \Gamma(G_j)\}.$$

Also, any set for which the above minimum is attained will be denoted by R_j. Notice that such a set is not necessarily unique.

With the above notation in mind we can state our next result.

Theorem 4.3. *For any non-bipartite graph $G[\mathcal{H}]$ obtained by point-attaching from a family of connected graphs $\mathcal{H} = \{G_1, ..., G_k\}$,*

$$\dim_l(G[\mathcal{H}]) = \sum_{j \in J_{\mathcal{H}}} \rho_j.$$

Proof. We will show that $X = \bigcup_{j \in J_{\mathcal{H}}} R_j$ is a local metric generator for $G[\mathcal{H}]$.

First of all, note that by the structure of $G[\mathcal{H}]$ we have that for any $v \in C_j$, $j \in J_{\mathcal{H}}$, there exists a non-bipartite primary subgraph G_i, which is a subgraph of $G_j(v^+)$, such that $R_i \neq \emptyset$. To see this we take a non-bipartite primary subgraph G_{j_1}, which is a subgraph of $G_j(v^+)$, next, if $R_{j_1} = \emptyset$, then we take $v_1 \in V(G_{j_1}) - \{v\}$ and, as above, we take a non-bipartite primary subgraph G_{j_2}, which is a subgraph of $G_j(v_1^+)$, and if $R_{j_2} = \emptyset$ then we repeat this process until obtain a non-bipartite primary subgraph G_{j_t}, which is a subgraph of $G_j(v_{t-1}^+)$ such that $R_{j_t} \neq \emptyset$ (at worst, we will arrive to a subgraph $G_j(v_{t-1}^+)$ containing only one non-bipartite primary subgraph). Hence, $X \neq \emptyset$ and, as a result, if G_i is bipartite, then any pair of adjacent vertices $x, y \in V(G_i)$ is distinguished by any vertex belonging to X.

Now, if x, y are adjacent in a non-bipartite primary subgraph G_j, then there exists $v \in R_j \cup C_j$ which distinguishes them. In the case that $v \in C_j$, we know that there exists a primary subgraph of $G_j(v^+)$, such that $R_i \neq \emptyset$ and any $w \in R_i$ also distinguishes x, y. As a result, X is a local metric generator for $G[\mathcal{H}]$. Therefore,

$$\dim_l(G[\mathcal{H}]) \leq |X| = \sum_{j \in J_{\mathcal{H}}} \rho_j.$$

It remains to show that $\dim_l(G[\mathcal{H}]) \geq |X| = \sum_{j \in J_{\mathcal{H}}} \rho_j$. Since $G[\mathcal{H}]$ is a non-bipartite graph, any vertex belonging to a local metric basis of $G[\mathcal{H}]$ distinguishes every pair of adjacent vertices included in a bipartite primary subgraph of $G[\mathcal{H}]$. Hence, we take a local metric basis A of $G[\mathcal{H}]$ which does not contain vertices belonging to the bipartite primary subgraphs of $G[\mathcal{H}]$ *i.e.*, for any $i \in [k] - J_{\mathcal{H}}$ it holds $A \cap V(G_i) = \emptyset$. For each $j \in J_{\mathcal{H}}$ we define $A_j = A \cap V(G_j)$. Note that $A_j \cup C_j$ is a local metric generator for G_j and, by the minimality of A, we have $A_j \cap C_j = \emptyset$. Hence, $|A_j| \geq |R_j| = \rho_j$. Therefore,

$$\dim_l(G[\mathcal{H}]) = |A| = \sum_{j \in J_{\mathcal{H}}} |A_j| \geq \sum_{j \in J_{\mathcal{H}}} \rho_j.$$

□

If G_j is the only non-bipartite primary subgraph of $G[\mathcal{H}]$, then $|J_{\mathcal{H}}| = 1$ and $\rho_j = \dim_l(G_j)$. Then we obtain the following particular case of Theorem 4.3.

Corollary 4.4. *Let $G[\mathcal{H}]$ be a graph obtained by point-attaching from the family of connected graphs $\mathcal{H} = \{G_1, ..., G_k\}$. If G_j is the only non-bipartite primary subgraph of $G[\mathcal{H}]$, then*

$$\dim_l(G[\mathcal{H}]) = \dim_l(G_j).$$

It is well-known that that a unicyclic graph G is bipartite if and only if its cycle has even length. For the case of non-bipartite unicyclic graphs we can apply Corollary 4.4 to deduce that for any non-bipartite unicyclic graph G it holds that $\dim_l(G) = 2$.

There are other cases in which ρ_j and α_j are very easy to obtain. For instance, if $C_j = \{v\}$, then $\rho_j = \dim_l(G_j) - \alpha_j$, where $\alpha_j = 1$ if v belongs to a local metric basis for G_i and $\alpha_j = 0$ in otherwise. Also, if $C_j = V(G_j)$, then $\rho_j = 0$ and $\alpha_j = \dim_l(G_j)$.

The remain sections of this chapter are devoted to derive some consequences of Theorem 4.3. We also give several families of graphs where the equality of Theorem 4.1 is achieved.

4.3 Rooted product graphs

Rooted product graphs can be constructed as follows. Let G be a graph of order n and let $\mathcal{H}$ be a sequence of n graphs $H_1, H_2, \ldots, H_n$. In each of these graphs a particular vertex v_i is selected. This vertex will be called the root of the graph H_i. The *rooted product graph* $G \circ \mathcal{H}$, is the graph obtained by identifying the root of the graph H_i with the i-th vertex of G, as defined by Godsil and Mckay [28]. Clearly, any rooted product graph is obtained by point-attaching from $G, H_1, H_2, ..., H_n$. Therefore, as a consequence of Theorem 4.3 we obtain a formula for the local metric dimension of any rooted product graph. To begin with, we consider the case where every H_i is a bipartite graph.

Corollary 4.5. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a sequence of n connected bipartite graphs $H_1, H_2, \ldots, H_n$. Then for any rooted product graph $G \circ \mathcal{H}$,*

$$\dim_l(G \circ \mathcal{H}) = \dim_l(G).$$

If every H_i is non-bipartite, the result can be expressed as follows.

Corollary 4.6. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a sequence of n connected non-bipartite graphs $H_1, H_2, \ldots, H_n$. Then for any rooted product graph $G \circ \mathcal{H}$,*

$$\dim_l(G \circ \mathcal{H}) = \sum_{j=1}^{n} (\dim_l(H_j) - \alpha_j).$$

Note that in this case $\alpha_j = 1$ if the root of H_j belongs to a local metric basis of H_j and $\alpha_j = 0$ in otherwise.

Now we will restrict ourselves to a particular case of rooted product graphs where the sequence $H_1, H_2, \ldots, H_n$ consists of n isomorphic graphs of order n', and will be using in each of them the same root vertex v. The resulting rooted product graph is denoted by the expression $G \circ_v H$. In this case Corollary 4.6 is simplified as follows.

Remark 4.7. *Let H be a connected non-bipartite graph and let v be a vertex of H.*

(i) *If v does not belong to any metric basis for H, then for any connected graph G of order n,*

$$\dim_l(G \circ_v H) = n \cdot \dim_l(H)$$

(ii) *If v belongs to a metric basis for H, then for any connected graph G of order $n \geq 2$,*

$$\dim_l(G \circ_v H) = n \cdot (\dim_l(H) - 1)$$

Lemma 4.8. *If H is a connected graph of order n' with clique number $\omega(H) = n' - 1$, and G is a connected graph of order $n \geq 2$, then for any $v \in V(H)$,*

$$\dim_l(G \circ_v H) = n(n' - 3).$$

Proof. Since H has clique number $\omega(H) = n' - 1$, by Theorem 2.4 we have $\dim_l(H) = n' - 2$. To conclude the proof by Remark 4.7 we need to prove that any vertex of H belongs to a local metric basis. With this aim, we consider three vertices $v_i, v_j, v_k \in V(H)$ and a maximum clique Q of H such that $v_i \notin V(Q)$, $v_j \in N_H(v_i)$ and $v_k \notin N_H(v_i)$ (Here $N_H(x)$ denotes the set of neighbours that x has in H). Then we have the following:

- Since v_i distinguishes the pair of adjacent vertices v_j, v_k, the set $B_i = V(H) - \{v_j, v_k\}$ is a local metric basis of H.
- Since $v_iv_k \notin E(H)$, the set, $B_j = V(H) - \{v_i, v_k\}$ is a local metric basis of H.
- Since v_k distinguishes the pair of adjacent vertices v_i, v_j, the set $B_k = V(H) - \{v_i, v_j\}$ is a local metric basis of H.

Therefore, any vertex of H belongs to a local metric basis. □

The equality $\dim_l(G \circ_v H) = n(n' - 3)$ is not exclusive for connected graphs of order n' with clique number $\omega(H) = n' - 1$. Consider for instance the graph $H = \langle v \rangle + (K_r \cup K_s)$, $r \geq 2$ and $s \geq 2$, *i.e.*, H is the graph $K_r \cup K_s$ together with all the edges joining an isolated vertex v to every vertex of $K_r \cup K_s$. In this case the order of H is $n' = r + s + 1$, while its local metric dimension is $\dim_l(H) = n' - 3$. Note however, that the vertex v can not be in any local metric basis. Hence, in this particular case for any connected graph G of order $n \geq 2$, the local metric dimension of the rooted product graph $G \circ_v H$ is calculated from Remark 4.7, giving

$$\dim_l (G \circ_v H) = n \cdot \dim_l(H) = n(n' - 3).$$

Proposition 4.9. *Let G be a connected graph of order $n \geq 2$. Let H be a connected non-bipartite graph of order n' and let $v \in V(H)$. Then the following assertions hold.*

(i) $n \leq \dim_l(G \circ_v H) \leq n(n' - 2)$.

(ii) $\dim_l(G \circ_v H) = n$ *if and only if* $\dim_l(H) = 2$ *and the root vertex v belongs to any local metric basis of H.*

(iii) $\dim_l(G \circ_v H) = n(n' - 2)$ *if and only if $H \cong K_{n'}$.*

(iv) *If $H \ncong K_{n'}$, then* $\dim_l(G \circ_v H) \leq n(n' - 3)$.

Proof. Remark 4.7 directly leads to the lower bound. Note that $\dim_l(H) \geq 2$, as H is not bipartite. Now, if v belongs to a local metric basis of H and $\dim_l(H) = 2$, then Remark 4.7 (ii) leads to $\dim_l(G \circ_v H) = n$. Otherwise, if v does not belong to any local metric basis of H, then Remark 4.7 leads to $\dim_l(G \circ_v H) \geq 2n$. This proves (ii).

Now, if $H \cong K_{n'}$, then $\dim_l(H) = n' - 1$ and, since v belongs to a local metric basis of H, Remark 4.7 (ii) leads to $\dim_l(G \circ_v H) = n(n' - 2)$. On the other hand, if H is a connected non-complete graph of order n', then we have $\dim_l(H) \leq n' - 2$. So, Remark 4.7 leads to the upper bound.

Note that if $\dim_l(H) = n' - 2$, then Theorem 2.4 and Lemma 4.8 lead to $\dim_l(G \circ_v H) \leq n(n' - 3)$. Thus, (iii) and (iv) follows. □

4.4 Unicyclic graphs

A graph H is said to be *unicyclic* if it is connected and contains exactly one cycle. It is easy to see that unyciclic graphs are obtained by point attaching of one cycle and some trees. If H is an unicyclic graph then H is bipartite if and only if its cycle has even length. For the case of non-bipartite unicyclic graphs we can apply Corollary 4.4 to deduce that for any non-bipartite unicyclic graph H it holds that $\dim_l(H) = 2$.

4.5 Block graphs

We say that $B \subseteq V(G)$ is a *block* of G is B induces a maximal two connected subgraph of G. A *block graph* is a graph whose blocks are cliques. Since any block graph is obtained by point-attaching from $H_1 = K_{t_1}, H_2 = K_{t_2}, \ldots, H_k = K_{t_k}$, as a consequence of Theorem 4.3 we obtain a formula for the local metric dimension of any block graph.

Corollary 4.10. *Let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$, such that for each $H_i \in \mathcal{H}$, there exists t_i such that $H_i \cong K_{t_i}$. If at least two of the t_i's are greater than two, for any block graph $G[\mathcal{H}]$,*

$$\dim_l(G[\mathcal{H}]) = \sum_{j=1}^{k} \max\{t_j - 1 - \delta_j, 0\}.$$

Where δ_j is the number of attachment vertices of H_j.

4.6 Cactus graphs

A *cactus graph* is a graph obtained by point-attaching in which $\mathcal{H} = \{C_{i_1}, C_{i_2}, \ldots, C_{i_n}\}$ where C_{i_j} are cycle graphs. If all the primary graphs of the family are even cycles the resulting cactus graph is bipartite and therefore its local metric dimension equals one. If there are exactly one odd cycle in the family, the dimension of the resulting cactus graphs is two. In order to calculate the local metric dimension of a cactus graph when the number of odd cycles in the family is greater or equal to two we prune the graph $G[\mathcal{H}]$ in the following sense: Let $\mathcal{H}$ be a family of connected graphs, not all of them bipartite, $G[\mathcal{H}']$ is a *pruned* $G[\mathcal{H}]$ if

- $\mathcal{H}' \subseteq \mathcal{H}$.
- $G[\mathcal{H}']$ is a graph obtained by point-attaching of the family $\mathcal{H}'$
- $G[\mathcal{H}']$ is a connected induced subgraph of $G[\mathcal{H}]$.
- If a graph $H_i \in \mathcal{H}'$, H_i has only one attachment vertex, the H_i is a non-bipartite graph.

It is easy to see that $\dim_l(G[\mathcal{H}]) = \dim_l(G[\mathcal{H}'])$.

Corollary 4.11. *If $G[\mathcal{H}]$, is a cactus graph in which at least two members of $\mathcal{H}$ are odd cycles then,*

$$\dim_l(G[\mathcal{H}]) = l'$$

where l' is the number of elementary graphs of $G[\mathcal{H}']$ that have only one attachment vertex.

4.7 Bouquet of graphs

Let $\mathcal{H} = \{G_1, \dots, G_k\}$ be a finite sequence of pairwise disjoint connected graphs and let $x_i \in V(G_i)$. By definition, the *bouquet* $\mathcal{H}_x$ of the graphs in $\mathcal{H}$ with respect to the vertices $\{x_i\}_{i=1}^k$ is obtained by identifying the vertices $x_1, \dots, x_k$ with a new vertex x. Clearly, the bouquet $\mathcal{H}_x$ is a graph obtained by point-attaching from $G_1, \dots, G_k$. Therefore, as a consequence of Theorem 4.3 we obtain the following result.

Corollary 4.12. *Let $\mathcal{H} = \{G_1, \dots, G_k\}$ be a finite sequence of pairwise disjoint connected graphs and let $x_i \in V(G_i)$ such that $J_{\mathcal{H}} \neq \emptyset$. If $\mathcal{H}_x$ is the bouquet obtained from $\mathcal{H}$ by identifying the vertices $x_1, \dots, x_k$ with a new vertex x, then*

$$\dim_l(\mathcal{H}_x) = \sum_{j \in J_{\mathcal{H}}} (\dim_l(G_j) - \delta_j).$$

Note that in this case $\delta_i = 1$ if x_i belongs to a local metric basis of G_i and $\delta_i = 0$ in otherwise.

4.8 Chain of graphs

Let $\mathcal{H} = \{G_1, \dots, G_k\}$ be a finite sequence of pairwise disjoint connected non-trivial graphs and let $x_i, y_i \in V(G_i)$. By definition, the *chain* $\mathcal{C}(\mathcal{H})$ of the graphs in $\mathcal{H}$ with respect to the set of vertices $\{y_1, x_k\} \cup \left(\cup_{i=2}^{k-1} \{x_i, y_i\}\right)$ is the connected graph obtained by identifying the vertex y_i with the vertex x_{i+1} for $i \in [k-1]$. Clearly, the chain $\mathcal{C}(\mathcal{H})$ is a graph obtained by point-attaching from $G_1, \dots, G_k$.

For every $j \in J_{\mathcal{H}}$ we say that x_j is *replaceable* in $\mathcal{C}(\mathcal{H})$ if and only if there exists a local metric basis B_j of G_j such that $x_j \in B_j$ and there exists $k < j$ such that G_k is a non-bipartite primary graph. Analogously, we say that y_j is *replaceable* in $\mathcal{C}(\mathcal{H})$ if and only if there exists a local metric basis B'_j of

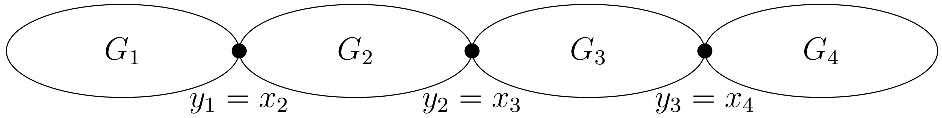

Figure 4.3: A chain $\mathcal{C}(\mathcal{H})$ obtained by point-attaching from $\mathcal{H} = \{G_1, G_2, G_3, G_4\}$.

G_j such that $y_j \in B'_j$ and there exists $k > j$ such that G_k is a non-bipartite primary subgraph. We say that x_j and y_j are *simultaneously replaceable* in $\mathcal{C}(\mathcal{H})$ if both are replaceable in $\mathcal{C}(\mathcal{H})$ and there exists a local metric basis of G_j containing both x_j and y_j.

The formula for $\dim_l(\mathcal{C}(\mathcal{H}))$ is directly obtained from Theorem 4.3. In this case we have the following possibilities for the value of ρ_j.

- If $1 \in J_{\mathcal{H}}$ and y_1 is replaceable in $\mathcal{C}(\mathcal{H})$, then $\rho_1 = \dim_l(G_1) - 1$.
- If $1 \in J_{\mathcal{H}}$ and y_1 is not replaceable in $\mathcal{C}(\mathcal{H})$, then $\rho_1 = \dim_l(G_1)$.
- If $k \in J_{\mathcal{H}}$ and x_k is replaceable in $\mathcal{C}(\mathcal{H})$, then $\rho_k = \dim_l(G_1) - 1$.
- If $k \in J_{\mathcal{H}}$ and x_k is not replaceable in $\mathcal{C}(\mathcal{H})$, then $\rho_k = \dim_l(G_1)$.

For $j \in J_{\mathcal{H}} \cap \{2, ..., k-1\}$ we have the following possibilities.

- If neither x_j nor y_j is replaceable in $\mathcal{C}(\mathcal{H})$, then either $\rho_j = \dim_l(G_j)$ or $\rho_j = \dim_l(G_j) - 1$.
- If x_j and y_j are simultaneously replaceable in $\mathcal{C}(\mathcal{H})$, then $\rho_j = \dim_l(G_j) - 2$.
- If x_j and y_j are not simultaneously replaceable in $\mathcal{C}(\mathcal{H})$ and x_j (or y_j) is replaceable in $\mathcal{C}(\mathcal{H})$, then $\rho_j = \dim_l(G_j) - 1$.

Chapter 5

The local metric dimension of corona product graphs

5.1 Introduction

Let G be a graphs of order n and let $\mathcal{H} = \{H_1, H_2, ..., H_n\}$ be a family of graphs. Recall that the corona product $G \odot \mathcal{H}$ is defined as the graph obtained from G and $\mathcal{H}$ by taking one copy of G and joining by an edge each vertex from H_i with the i-th vertex of G, [26]. The join $G+H$ is defined as the graph obtained from disjoint graphs G and H by taking one copy of G and one copy of H and joining by an edge each vertex of G with each vertex of H. Notice that the particular case of corona graph $K_1 \odot H$ is isomorphic to the join graph $K_1 + H$. We can obtain any corona graph $G \odot \mathcal{H}$ by point-attaching from $G, K_1 + H_1, K_1 + H_2, ..., K_1 + H_n$. Note that if H_i is a non-trivial graph, then the primary subgraph $K_1 + H_i$ is not bipartite. In fact, we can see the corona graph as a particular case of rooted product graph.

If there exists a graph H such that $H_i \cong H$, for any $H_i \in \mathcal{H}$, then we denote $G \odot \mathcal{H}$ by $G \odot H$, for simplicity. The corona product $G \odot H$ was defined by Frucht and Harary in [26]. Recalling our notation for rooted product graphs

$$G \odot H \cong G \circ_w (w + H)$$

Figure 5.1 shows two examples of corona product graphs where the factors are non-trivial.

The metric dimension and related parameters have been studied for the case of corona graphs. For instance, the metric dimension was studied in

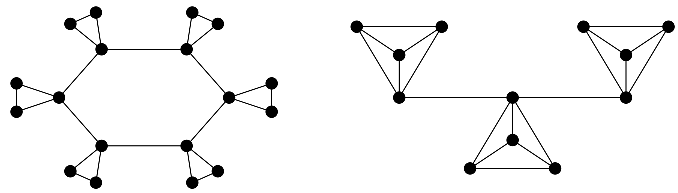

Figure 5.1: From the left, we show the corona graphs $C_6 \odot K_2$ and $P_3 \odot K_3$.

[33] and [60], the strong metric dimension was studied in [42], the partition dimension was studied in [51] and the simultaneous metric dimension was studied in [50]. In this chapter we study the local metric dimension. The chapter is organized as follows: In Section 5.2 we give closed formulae for $\dim_l(G \odot H)$ in terms of $\dim_l(G)$ and $\dim_l(K_1 \odot H)$. Then, we establish lower and upper bounds for $\dim_l(G \odot H)$ by using the orders of G and H, and in Section 5.3 we characterize all graphs when the bounds are attained. Finally, in Section 5.4 we investigate the value of $\dim_l(G \odot H)$ when H is a bipartite graph of radius three, and in particular, we compute $\dim_l(G \odot T)$ when T is a tree.

5.2 General results

From Theorem 4.3 we deduce the following result.

Corollary 5.1. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a sequence of n non-empty graphs $H_1, H_2, \dots, H_n$. Then for any corona product graph $G \circ \mathcal{H}$,*

$$\dim_l(G \odot \mathcal{H}) = \sum_{j=1}^{n} (\dim_l(K_1 + H_j) - \alpha_j).$$

Note that in this case $\alpha_j = 1$ if the vertex of K_1 belongs to a local metric basis of $K_1 + H_j$ and $\alpha_j = 0$ in otherwise.

From now on we consider the case of corona product graphs where the sequence $H_1, \dots, H_n$ consists of n isomorphic graphs of order n'. To begin with, we consider some straightforward cases. If H is an empty graph, then $K_1 \odot H$ is a star graph and $\dim_l(K_1 \odot H) = 1$. Moreover, if H is a complete graph of order n, then $K_1 \odot H$ is a complete graph of order $n + 1$ and $\dim_l(K_1 \odot H) = n$.

Theorem 5.2. *Let G be a connected non-trivial graph. For any empty graph H,*

$$\dim_l(G \odot H) = \dim_l(G).$$

Proof. Let B be a local metric basis for G. Since in $G \odot H$ every pair of adjacent vertices of G is distinguished by some vertex of B and every vertex of B distinguishes every pair of adjacent vertices composed by one vertex of G and one vertex of H, we conclude that B is a local metric generator for $G \odot H$.

Now, suppose that A is a local metric basis for $G \odot H$ such that $|A| < |B|$. Since H is an empty graph, if there exists $x \in A \cap V_i$, for some i, then the pairs of vertices of $G \odot H$ which are distinguished by x can be distinguished also by v_i. So, we consider the set A' obtained from A by replacing by v_i each vertex $x \in A \cap V_i$, where $i \in \{1, ..., n\}$. Thus, A' is a local metric generator for G and $|A'| \leq |A| < |B| = \dim_l(G)$, which is a contradiction. Therefore, B is a local metric basis for $G \odot H$. □

We present now the main result on the local metric dimension of corona graphs $G \odot H$ for the case where H is a non-empty graph. We would pont out that this result can be derived from Theorem 4.3 (or Corollary 5.1). Even so, we include the proof because we will use these ideas in Chapter 7.

Theorem 5.3. *Let H be a non-empty graph. The following assertions hold.*

(i) *If the vertex of K_1 does not belong to any local metric basis for $K_1 + H$, then for any connected graph G of order n,*

$$\dim_l(G \odot H) = n \cdot \dim_l(K_1 + H).$$

(ii) *If the vertex of K_1 belongs to a local metric basis for $K_1 + H$, then for any connected graph G of order $n \geq 2$,*

$$\dim_l(G \odot H) = n(\dim_l(K_1 + H) - 1).$$

Proof. If $n = 1$, then $G \odot H \cong K_1 + H$ and we are done. We consider $n \geq 2$. Let S_i be a local metric basis for $\langle v_i \rangle + H_i$ and let $S_i' = S_i - \{v_i\}$. Note that $S_i' \neq \emptyset$ because H_i is a non-empty graph and v_i does not distinguish any pair of adjacent vertices belonging to V_i. In order to show that $X = \cup_{i=1}^{n} S_i'$ is a local metric generator for $G \odot H$ we differentiate the following cases for two adjacent vertices x, y.

Case 1. $x, y \in V_i$. Since v_i does not distinguish x, y, there exists $u \in S_i'$ such that $d_{G\odot H}(x,u) = d_{\langle v_i\rangle + H_i}(x,u) \neq d_{\langle v_i\rangle + H_i}(y,u) = d_{G\odot H}(y,u)$.

Case 2. $x \in V_i$ and $y = v_i$. For $u \in S_j'$, $j \neq i$, we have $d_{G\odot H}(x,u) = 1 + d_{G\odot H}(y,u) > d_{G\odot H}(y,u)$.

Case 3. $x = v_i$ and $y = v_j$. For $u \in S_j'$, we have $d_{G\odot H}(x,u) = 2 = d_{G\odot H}(x,y) + 1 > 1 = d_{G\odot H}(y,u)$.

Hence, X is a local metric generator for $G \odot H$.

Now we shall prove (i). If the vertex of K_1 does not belong to any local metric basis for $K_1 + H$, then $v_i \notin S_i$ for every $i \in \{1, ..., n\}$ and, as a consequence,

$$\dim_l(G \odot H) \leq |X| = \sum_{i=1}^{n} |S_i'| = \sum_{i=1}^{n} \dim_l(\langle v_i\rangle + H_i) = n \cdot \dim_l(K_1 + H).$$

Now we need to prove that $\dim_l(G\odot H) \geq n\cdot\dim_l(K_1+H)$. In order to do this, let W be a local metric basis for $G \odot H$ and let $W_i = V_i \cap W$. Consider two adjacent vertices $x, y \in V_i - W_i$. Since no vertex $a \in W - W_i$ distinguishes the pair x, y, there exists $u \in W_i$ such that $d_{\langle v_i\rangle + H_i}(x,u) = d_{G\odot H}(x,u) \neq d_{G\odot H}(y,u) = d_{\langle v_i\rangle + H_i}(y,u)$. So we conclude that $W_i \cup \{v_i\}$ is a local metric generator for $\langle v_i\rangle + H_i$. Now, since v_i does not belong to any local metric basis for $\langle v_i\rangle + H_i$, we have that $|W_i| + 1 = |W_i \cup \{v_i\}| > \dim_l(\langle v_i\rangle + H_i)$ and, as a consequence, $|W_i| \geq \dim_l(\langle v_i\rangle + H_i)$. Therefore,

$$\dim_l(G \odot H) = |W| \geq \sum_{i=1}^{n} |W_i| \geq \sum_{i=1}^{n} \dim_l(\langle v_i\rangle + H_i) = n \cdot \dim_l(K_1 + H),$$

and the proof of (i) is complete.

Finally, we shall prove (ii). If the vertex of K_1 belongs to a local metric basis for K_1+H, then we assume that $v_i \in S_i$ for every $i \in \{1, ..., n\}$. Suppose that there exists B such that B is a local metric basis for $G\odot H$ and $|B| < |X|$. In such a case, there exists $i \in \{1, ..., n\}$ such that the set $B_i = B \cap V_i$ satisfies $|B_i| < |S_i'|$. Now, since no vertex of $B - B_i$ distinguishes the pairs of adjacent vertices belonging to V_i, the set $B_i \cup \{v_i\}$ must be a local metric generator for $\langle v_i\rangle + H_i$. So, $\dim_l(\langle v_i\rangle + H_i) \leq |B_i| + 1 < |S_i'| + 1 = |S_i| = \dim_l(\langle v_i\rangle + H_i)$, which is a contradiction. Hence, X is a local metric basis for $G \odot H$ and, as a consequence,

$$\dim_l(G \odot H) = |X| = \sum_{i=1}^{n} |S'_i| = \sum_{i=1}^{n} (\dim_l(\langle v_i \rangle + H_i) - 1) = n(\dim_l(K_1 + H) - 1).$$

The proof of (ii) is now complete. □

As a direct consequence of Theorem 5.3 we obtain the following results.

Corollary 5.4. *The following assertions hold for any connected graph G of order $n \geq 2$.*

(i) *For any integer $t \geq 2$, $\dim_l(G \odot K_t) = n(t-1)$.*

(ii) *For any positive integers r and s, $\dim_l(G \odot K_{r,s}) = n$.*

(iii) *Let $t \geq 4$ be an integer. If $t \equiv 1(4)$, then $\dim_l(G \odot P_t) = n \left\lfloor \frac{t}{4} \right\rfloor$ and if $t \not\equiv 1(4)$, then $\dim_l(G \odot P_t) = n \left\lceil \frac{t}{4} \right\rceil$.*

(iv) *For any integer $t \geq 4$, $\dim_l(G \odot C_t) = n \left\lceil \frac{t}{4} \right\rceil$.*

Proof. (i) If $H \cong K_t$, then $K_1 + K_t \cong K_{t+1}$ and the vertex of K_1 can belong to a local metric basis for $K_1 + K_t$. Thus,

$$\dim_l (G \odot K_t) = n \cdot (\dim_l (K_{t+1}) - 1) = n \cdot (t-1).$$

(ii) If $H = (U_1 \cup U_2, E) \cong K_{r,s}$ then for every $a \in U_1$ (or $a \in U_2$) the set $\{a, v\}$ is a local metric basis for $\langle v \rangle + H$. Therefore,

$$\dim_l (G \odot K_{r,s}) = n \cdot (\dim_l (K_1 + K_{r,s}) - 1) = n.$$

(iii) Notice that a set B is a local metric basis for $K_1 + P_t$ if and only if for every pair of adjacent vertices $x, y \in V(P_t)$, vertex x is adjacent to an element of B or vertex y is adjacent to an element of B. Thus, for any subgraph H' of P_t isomorphic to P_4, we have $B \cap V(H') \neq \emptyset$. With this observation in mind, we consider the following two cases.

Case 1. $4 \leq t \leq 5$. In this case we have that $\dim_l(\langle v \rangle + P_t) = 2$ and v belongs to any local metric basis. Thus, $\dim_l (G \odot P_t) = n = n \left\lfloor \dfrac{t}{4} \right\rfloor$.

Case 2. $t \geq 6$. For $t = 4k + r$, where $0 \leq r \leq 3$, we obtain

$$\dim_l(K_1 + P_t) = \begin{cases} k, \text{ if } r = 0 \text{ or } r = 1 \\ \\ k+1, \text{ if } r = 2 \text{ or } r = 3 \end{cases} \tag{5.1}$$

Therefore, since in this case vertex v does not belong to any local metric basis for $\langle v \rangle + P_t$, we obtain

$$\dim_l(G \odot P_t) = n \cdot \dim_l(K_1 + P_t) = \begin{cases} n \cdot \left\lfloor \frac{t}{4} \right\rfloor, \text{ if } t \equiv 1(4) \\ \\ n \cdot \left\lceil \frac{t}{4} \right\rceil, \text{ if } t \not\equiv 1(4). \end{cases}$$

(iv) If $4 \leq t \leq 5$, then $\dim_l(\langle v \rangle + C_t) = 2$. Since v belongs to any local metric basis for $\langle v \rangle + C_4$ and v does not belong to any local metric basis for $\langle v \rangle + C_5$, we have $\dim_l(G \odot C_4) = n$ and $\dim_l(G \odot C_5) = 2n = n \left\lceil \frac{5}{4} \right\rceil$.

Now we consider the case where $t \geq 6$. As in the proof of (iii), for any local metric basis B of $\langle v \rangle + C_t$ and any subgraph H' of C_t, isomorphic to P_4, we have $B \cap V(H') \neq \emptyset$. Hence, for $t = 4k + r$, where $0 \leq r \leq 3$, we deduce

$$\dim_l(K_1 + C_t) = \begin{cases} k, \text{ if } r = 0 \\ \\ k+1, \text{ otherwise.} \end{cases} \tag{5.2}$$

Then, since for $t \geq 6$ vertex v does not belong to any local metric basis for $\langle v \rangle + C_t$, $\dim_l(G \odot C_t) = n \cdot \dim_l(K_1 + C_t) = n \cdot \left\lceil \frac{t}{4} \right\rceil$.

□

Since any metric generator is a local metric generator, the local metric dimension of a graph G is at most equal to the metric dimension of G, *i.e.*, $\dim_l(G) \leq \dim(G)$. For instance, for the complete graph of order $n \geq 2$, $\dim_l(K_n) = \dim(K_n) = n - 1$, and for any bipartite graph G, different from a path, $\dim_l(G) = 1 < \dim(G)$. As an illustrative example where the local metric dimension can be significantly smaller than the metric dimension, we can take the complete bipartite graph $K_{r,s}$ of order $r + s \geq 4$, where $\dim_l(K_{r,s}) = 1 < r + s - 2 = \dim(K_{r,s})$. Similar examples can be derived for corona graphs. For instance, it was shown in [60] that for any connected graph G of order $n \geq 2$ and any integers $r \geq 2$ and $s \geq 1$ ($r + s \geq 4$), $\dim(G \odot K_r) = n(r-1)$ and $\dim(G \odot K_{r,s}) = n(r+s-2)$. Thus, according to Corollary 5.4 (i) and (ii), $\dim_l(G \odot K_r) = n(r-1) = \dim(G \odot K_r)$ and $\dim_l(G \odot K_{r,s}) = n < n(r+s-2) = \dim(G \odot K_{r,s})$.

Corollary 5.5. *For any connected graph H and any connected graph G of order $n \geq 2$,*

$$\dim_l(G \odot H) \geq n \cdot \dim_l(H).$$

Proof. Let B be a local metric basis for K_1+H. Since the vertex v of K_1 does not distinguish any pair of adjacent vertices $x, y \in V(H)$, $B - \{v\}$ is a local metric generator for H. Thus, if $v \in B$, then $\dim_l(K_1 + H) - 1 \geq \dim_l(H)$ and, if $v \notin B$, then $\dim_l(K_1 + H) \geq \dim_l(H)$. Therefore, Theorem 5.3 leads to $\dim_l(G \odot H) \geq n \cdot \dim_l(H)$. □

Now we will give some results involving the diameter or the radius of H.

Corollary 5.6. *For any graph H of diameter two and any connected graph G of order $n \geq 2$,*

$$\dim_l(G \odot H) = n \cdot \dim_l(H).$$

Proof. Since H has diameter two, for every $x, y \in V(H)$ it follows $d_H(x,y) = d_{K_1+H}(x,y)$. So, if the vertex of K_1 does not belong to any local metric basis for $K_1 + H$, then every local metric basis for H is a local metric basis for $K_1 + H$ and vice versa. Hence, in such a case, Theorem 5.3 (i) leads to $\dim_l(G \odot H) = n \cdot \dim_l(H)$.

Now we suppose that there exists a local metric basis B of $K_1 + H$ such that the vertex v of K_1 belongs to B. Since v does not distinguish any pair of vertices of H, $B' = B - \{v\}$ is a local metric generator for H. Moreover, if there exists $A \subset V(H)$ such that $|A| < |B'|$ and A is a local metric basis for H, then $A \cup \{v\}$ is a local metric generator for $K_1 + H$, which is a contradiction because $|A| + 1 < |B'| + 1 = |B| = \dim_l(K_1 + H)$. Therefore, B' is a local metric basis for H and, as a result, $\dim_l(K_1+H) = 1+\dim_l(H)$. So, by Theorem 5.3 (ii) we obtain $\dim_l(G \odot H) = n \cdot \dim_l(H)$. □

Lemma 5.7. *Let H be a graph of radius $r(H)$. If $r(H) \geq 4$ then the vertex of K_1 does not belong to any local metric basis for $K_1 + H$.*

Proof. Let B be a local metric basis for K_1+H. We suppose that the vertex v of K_1 belongs to B. Note that $v \in B$ if and only if there exists $u \in V(H) - B$ such that $B \subset N_{K_1+H}(u)$.

Now, if $r(H) \geq 4$, then we take $u' \in V(H)$ such that $d_H(u, u') = 4$ and a shortest path $uu_1u_2u_3u'$. In such a case for every $b \in B - \{v\}$ we will have that $d_{K_1+H}(b, u_3) = d_{K_1+H}(b, u') = 2$, which is a contradiction. Hence, v does not belong to any local metric basis for $K_1 + H$. □

The converse of Lemma 5.7 is not true. In Figure 5.2 we show a graph H of radius three where the vertex of K_1 does not belong to any local metric basis for $K_1 + H$.

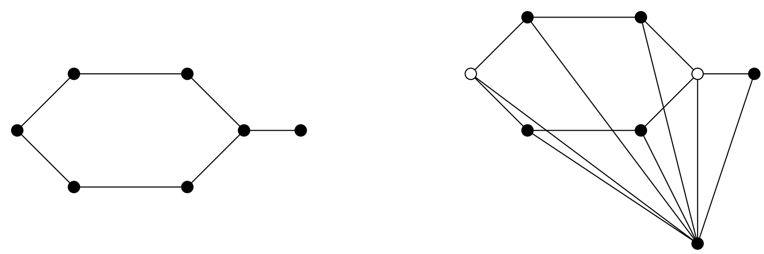

Figure 5.2: A graph H and the join graph $K_1 + H$. White vertices form a local metric basis for $K_1 + H$.

The following result is a direct consequence of Theorem 5.3 (i) and Lemma 5.7.

Theorem 5.8. *For any connected graph G of order n and any graph H of radius $r(H) \geq 4$,*

$$\dim_l(G \odot H) = n \cdot \dim_l(K_1 + H).$$

Another consequence of Theorem 5.3 is the following result.

Corollary 5.9. *For any non-empty graph H of order $n' \geq 2$ and any connected graph G of order $n \geq 2$,*

$$n \leq \dim_l(G \odot H) \leq n(n' - 1).$$

The aim of the next section is the study of the limit cases of Corollary 5.9.

5.3 Extremal values

Theorem 5.10. *Let H be a graph of order n' and let G be a connected graph of order $n \geq 2$. Then $\dim_l(G \odot H) = n(n' - 1)$ if and only if $H \cong K_{n'}$.*

Proof. By Theorem 5.3 we conclude that $\dim_l(G \odot H) = n(n' - 1)$ if and only if exactly one of the following cases hold:
Case a: the vertex v of K_1 does not belong to any local metric basis for $K_1 + H$ and $\dim_l(K_1 + H) = n' - 1$.
Case b: the vertex v of K_1 belongs to a local metric basis for $K_1 + H$ and $\dim_l(K_1 + H) = n'$.

We first consider Case a. By Theorem 2.4 $\dim_l(K_1 + H) = n' - 1$ if and only if $\omega(H) = n' - 1$. Let $V(H) = \{u_1, u_2, ...u_{n'}\}$. If $\langle V(H) - \{u_1\}\rangle$ is a clique and $u_iu_1 \in E(H)$, then $\{v\} \cup V(H) - \{u_1, u_i\}$ is a local metric basis for $K_1 + H$, which is a contradiction. Hence u_1 is an isolated vertex of H. So, $H \cong K_1 \cup K_{n'-1}$, which is a contradiction, as $\{v, u_3, \ldots, u_{n'}\}$ is a local metric basis of $\langle v\rangle + H$.

Finally, by Theorem 2.4 we deduce that Case b holds if and only if $H \cong K_{n'}$. □

The *center* of a connected graph G is the set of vertices of G with eccentricity equal to the radius of G.

Theorem 5.11. *Let H be a non-empty graph and let G be a connected graph of order $n \geq 2$. Then $\dim_l(G \odot H) = n$ if and only if H is a bipartite graph having only one non-trivial connected component H^* and $r(H^*) \leq 2$.*

Proof. Since $\langle v\rangle + H$ is not bipartite, by Theorem 2.4 we deduce $\dim_l(\langle v\rangle + H) \geq 2$. So, if $\dim_l(G \odot H) = n$, then by Theorem 5.3 we have that $\dim_l(\langle v\rangle + H) = 2$ and v belongs to a local metric basis for $\langle v\rangle + H$, say $B = \{u, v\}$. So, $B \cap V(H) = \{u\}$ must be a local metric generator for H and, by Theorem 2.4, we conclude that H is a bipartite graph having only one non-trivial connected component. Moreover, if the non-trivial component of H has radius $r > 2$, then there exists $u_3 \in V(H)$ such that $d_H(u, u_3) = 3$ and, as a consequence, for any shortest path $uu_1u_2u_3$ we have $d_{\langle v\rangle+H}(u, u_2) = d_{\langle v\rangle+H}((u, u_3)$, *i.e.*, the pair of adjacent vertices u_2, u_3 is not distinguished by the elements of B, which is a contradiction. Therefore, $r \leq 2$.

Conversely, let H be a bipartite graph where having only one non-trivial component H^*. Let $r(H^*) \leq 2$, let a be a vertex belonging to the center of H^* and let v be the vertex of K_1. Since H is a triangle free graph, a distinguishes every pair of adjacent vertices $x, y \in V(H^*)$. So, $\{v, a\}$ is a local metric generator for $K_1 + H$, which is a local metric basis because $\dim_l(K_1 + H) \geq 2$. We conclude the proof by Theorem 5.3 (ii). □

5.4 The value of $\dim_l(G \odot H)$ when H is a bipartite graph of radius three

Theorems 5.8 and 5.11 suggest to consider the case where H is a bipartite graph of radius three. To do that, we need the following additional notation. For any $a \in V(H)$, we denote

$$N_H^{(i)}(a) = \{w \in V(H) : \ d_H(w, a) = i\}.$$

We also define $N_H^{(i)}[a] = N_H^{(i)}(a) \cup \{a\}$. Note that $N_H^{(1)}(a) = N_H(a)$ and $N_H^{(1)}[a] = N_H[a]$. Given two sets $A, B \subset V(H)$ we say that A dominates B if every vertex in $B - A$ is adjacent to some vertex belonging to A. From now on we will use the notation $A \succ B$ to indicate that A dominates B. For every $x \in C(H)$, let $\eta(x) = \min\left\{|A| : \ A \subseteq N_H(x) \text{ and } A \succ N_H^{(2)}(x)\right\}$ and let

$$\delta'(H) = \min_{x \in C(H)} \{\eta(x)\}.$$

Lemma 5.12. *For any bipartite graph H of radius three,*

$$\dim_l(K_1 + H) \leq \delta'(H) + 1.$$

Moreover, $\dim_l(K_1 + H) = \delta'(H) + 1$ if and only if the vertex of K_1 belongs to a local metric basis for $K_1 + H$.

Proof. Let u be a vertex belonging to the center of H and $A \subseteq N_H(u)$ such that $A \succ N_H^{(2)}(u)$ and $|A| = \delta'(H)$. Let us show that $B = A \cup \{v\}$ is a local metric generator for $\langle v \rangle + H$. We first note that since H is bipartite, for two adjacent vertices $x, y \notin B$ it follows $d_H(u, x) \neq d_H(u, y)$. Hence, without loss of generality, we may consider the following three cases for two adjacent vertices $x, y \notin B$.

Case 1: $x = u$ and y is adjacent to u. In this case for every $z \in A$ it follows $d_{K_1+H}(x, z) = 1$ and $d_{K_1+H}(y, z) = 2$.

Case 2: $d_H(u, x) = 1$ and $d_H(u, y) = 2$. In this case $y \in N_H^{(2)}(u)$ and there exists $x' \in A$ which is adjacent to y and, since H is a bipartite graph, x' is not adjacent to x. So, $d_{K_1+H}(x,x') = 2$ and $d_{K_1+H}(y,x') = 1$.

Case 3: $d_H(u,x) = 2$ and $d_H(u,y) = 3$. In this case $x \in N_H^{(2)}(u)$ and there exists $x' \in A$ such that $ux'xy$ is a shortest path in H. So, $d_{K_1+H}(x,x') = 1$ and $d_{K_1+H}(y,x') = 2$.

Thus, B is a local metric generator for $K_1 + H$ and, as a consequence, $\dim_l(K_1 + H) \leq \delta'(H) + 1$.

Moreover, if $\dim_l(K_1 + H) = \delta'(H) + 1$, then B is a local metric basis for $K_1 + H$ which contains the vertex of K_1.

Conversely, let S be a local metric basis for $K_1 + H$ which contains the vertex v of K_1. In this case there exists $w \in V(H)$ such that $N_H(w) \supset S - \{v\}$. If $w \notin C(H)$, then there exists $w' \in V(H)$ such that $d_H(w,w') \geq 4$ and for every shortest path $ww_1w_2w_3w'$ from w to w' the pair of vertices w_3, w' is not resolved in $K_1 + H$ by any $s \in S$, which is a contradiction. Hence, $w \in C(H)$ and $S - \{v\} \succ N_H^{(2)}(w)$. The minimality of the cardinality of S leads to $|S - \{v\}| = \delta'(H)$. Therefore, $\delta'(H) + 1 = |S| = \dim_l(K_1 + H)$. $\square$

As a direct consequence of Theorem 5.3 and Lemma 5.12 we obtain the following result.

Theorem 5.13. *Let H be a bipartite graph of radius three and let G be a connected graph of order $n \geq 2$. Then*

$$\dim_l(G \odot H) \leq n \cdot \delta'(H).$$

The maximum value of $\dim_l(G \odot H)$.

In this section we show that the above bound is attained for a subfamily of bipartite graphs of diameter three that does not contain a square (a subgraph isomorphic to $K_{2,2}$). In such a case, the girth of H must be six and $H = (U_1 \cup U_2, E)$ satisfies the following property:

♦ For any $i \in \{1,2\}$ and any two distinct vertices $a, b \in U_i$, $|N_H(a) \cap N_H(b)| = 1$.

Therefore, H is the incidence graph of a finite projective plane. So, we have two possibilities (see, for instance, [5]):

(P1) $H = (U_1 \cup U_2, E)$ is the incidence graph of a degenerate projective plane. In this case $|U_1| = |U_2| = t$, $t \geq 3$, and H is a pseudo sphere graph S_t (also called near pencil) defined as follows: we consider $t-1$ path graphs of order 4 and we identify one extreme of each one of the $t-1$ path graphs in one pole a and all the other extreme vertices of the paths in a pole b. In particular, S_3 is the cycle graph C_6.

(P2) $H = (U_1 \cup U_2, E)$ is the incidence graph of a non-degenerate projective plane of order q. In this case H is a regular graph of degree $\delta_H = q+1$ and $|U_1| = |U_2| = q^2+q+1$. Note that $|U_1| = |U_2| = \delta_H^2 - \delta_H + 1$.

In the case (P1) the set $B = \{a, b\}$ composed by both poles of the pseudo sphere is a dominating set of S_t. Thus, B is a local metric basis for $\langle v \rangle + S_r$ and $N_{S_t}(a) \cap N_{S_t}(b) = \emptyset$. Also, there are no local metric generators composed by two vertices at distance two, so the vertex v does not belong to any local metric basis for $\langle v \rangle + S_t$ and, by Theorem 5.3 (i), we obtain that for any connected graph G of order $n \geq 2$, $\dim_l(G \odot S_t) = 2n$.

The rest of this section covers the study of case (P2), *i.e.*, the case where H is the incidence graph of a non-degenerate projective plane.

Lemma 5.14. *For any bipartite graph $H \ncong S_t$ of diameter three and girth six,*

$$\delta'(H) = \delta_H.$$

Proof. Let $x \in U_i$, $i \in \{1, 2\}$. Since for any $y, z \in N_H(x)$ we have $N_H(y) \cap N_H(z) = \{x\}$, we deduce that for any $A \subseteq N_H(x)$,

$$\left|N_H^{(2)}(x)\right| = |U_i - \{x\}| \geq \left|\bigcup_{y \in A} (N_H(y) - \{x\})\right| = \sum_{y \in A} (|N_H(y)| - 1) = (\delta_H - 1)\,|A|.$$

Therefore, since $|U_i| = \delta_H^2 - \delta_H + 1$, we have that $A \succ N_H^{(2)}(x)$ if and only if $A = N_H(x)$. □

Lemma 5.15. *Let $H = (U_1 \cup U_2, E) \ncong S_t$ be a bipartite graph of diameter three and girth six. For any local metric basis B of $K_1 + H$, either $B \cap U_1 = \emptyset$ or $B \cap U_2 = \emptyset$.*

Proof. We proceed by contradiction. Suppose that $B_1 = B \cap U_1 \neq \emptyset$ and $B_2 = B \cap U_2 \neq \emptyset$. We differentiate two cases.

Case 1: $B_1 \cup N_H(B_2) \neq U_1$ or $B_2 \cup N_H(B_1) \neq U_2$. We take, without loss of generality, $x \in U_1$ such that $x \notin B_1 \cup N_H(B_2)$. Since B is a local metric basis for $K_1 + H$ and $N_H(x) \cap B_2 = \emptyset$, the set $N_H(x)$ must be dominated by B_1. Moreover, since H is a square free graph, for any $b \in B_1$ there exists only one vertex $y_b \in N_H(x) \cap N_H(b)$. Thus, $\delta_H = |N_H(x)| \leq |B_1|$. On the other hand, by Lemmas 5.12 and 5.14 we have $|B \cap (U_1 \cup U_2)| \leq \delta_H$. Hence, the assumption $B_2 = B \cap U_2 \neq \emptyset$ leads to $|B_1| \leq \delta_H - 1$, which is a contradiction with the fact that $|B_1| \geq \delta_H$.

Case 2: $B_1 \cup N_H(B_2) = U_1$ and $B_2 \cup N_H(B_1) = U_2$. If $|B_1| = |B_2| = 1$, then $\delta_H^2 - \delta_H + 1 = |U_1| = |B_1 \cup N_H(B_2)| \leq 1 + \delta_H$, which is a contradiction for $\delta_H > 2$. Thus, without loss of generality, we assume that $|B_2| \geq 2$. Let $a, b \in B_2$ and let $c \in U_1$ such that $\{c\} = N_H(a) \cap N_H(b)$. We define $B_1' = B_1 \cup \{c\}$, $B_2' = B_2 - \{a, b\}$ and $B' = B_1' \cup B_2'$. Note that $|B'| < |B|$. We take two adjacent vertices x, y such that $x \in U_1 - B_1'$ and $y \in U_2 - B_2'$. Now, if $y \in \{a, b\}$, then $c \in B'$ distinguishes the pair x, y and if $y \notin \{a, b\}$, then there exists $y' \in B_1 \subseteq B'$ such that y' is adjacent to y. Thus, B' is a local metric basis for $K_1 + H$, which is a contradiction.

Since both cases lead to a contradiction, the proof is complete. □

Lemma 5.16. *Let $H \ncong S_t$ be a bipartite graph of diameter three and girth six. Then the vertex of K_1 belongs to any local metric basis for $K_1 + H$.*

Proof. Let B be a local metric basis for $\langle v \rangle + H$. We proceed by contradiction. Suppose that $v \notin B$. By Lemmas 5.12 and 5.14 we have $|B| \leq \delta_H$. By Lemma 5.15 we can assume that $B \subset B_1$. Now, if $|B| \leq \delta_H - 1$, then

$$|N_H(B)| = \left| \bigcup_{b \in B} N_H(b) \right| \leq \sum_{b \in B} |N_H(b)| = (\delta_H - 1)\delta_H < |U_2|,$$

which is a contradiction because if there exist two adjacent vertices x, y such that $x \in U_1 - B$ and $y \in U_2 - N_H(B)$, then the pair x, y is not distinguished by the elements of B. Hence, we conclude $|B| = \delta_H$.

Now, if there exists $a \in U_2$ such that $N_H(a) = B$, then the pair of adjacent vertices a, v is not distinguished by the elements of B, which is a contradiction. Thus, let $b, b' \in B$, $a \in N_H(b) \cap N_H(b')$ and $x_a \in N_H(a) - B$. Since B is a local metric basis and H is a square free graph, for every $y, z \in N_H(x_a)$, there exist two vertices $b_y \in (B - \{b, b'\}) \cap N_H(y)$ and $b_z \in (B -$

$\{b, b'\}) \cap N_H(z)$ such that $b_y \neq b_z$. Hence,

$$\delta_H - 1 = |N_H(x_a) - a| \leq |B - \{b, b'\}| = \delta_H - 2,$$

which is a contradiction. Therefore, v must belong to B. □

Theorem 5.17. *Let $H \not\cong S_t$ be a bipartite graph of diameter three and girth six. Then for any connected graph G of order $n \geq 2$,*

$$\dim_l(G \odot H) = n \cdot \delta_H.$$

Proof. By Lemma 5.16 we know that the vertex of K_1 belongs to every local metric basis for $K_1 + H$, by Lemmas 5.12 and 5.14 we have $\dim_l(K_1 + H) = \delta_H + 1$ and by Theorem 5.3 (ii) we conclude $\dim_l(G \odot H) = n \cdot \delta_H$. □

Let $\pi = (P, L)$ be a finite non-degenerate projective plane of order q, where P is the set of points and L is the set of lines. Given two sets $P' \subset P$ and $L' \subset L$, we say that $P' \cup L'$ satisfies the property $\mathcal{G}$, if for any point p_0 and any line l_0 such that $p_0 \in l_0$ we have

- there exists $p \in P'$ such that $p \in l_0$, or
- there exists $l \in L'$ such that $p_o \in l$.

We define $\Upsilon(\pi) = \min\{|P' \cup L'|$ such that $P' \cup L'$ satisfies the property $\mathcal{G}\}$.

We have that if H is the incidence graph of π, then a set $P' \cup L'$ satisfies the property $\mathcal{G}$ if and only if $P' \cup L' \cup \{v\}$ is a local metric generator for $\langle v \rangle + H$. Therefore, according to Lemmas 5.12, 5.14 and 5.16 we conclude

$$\Upsilon(\pi) = \delta_H = q.$$

Note that if $P' \cup L'$ satisfies the property $\mathcal{G}$ and its cardinality is the minimum among all the sets satisfying this property, then either $P' = \emptyset$ and L' is the set of lines incident to one point or $L' = \emptyset$ and P' is the set composed by all the points laying on one line.

The minimum value of $\dim_l(G \odot H)$.

As a direct consequence of Theorems 5.3 and 5.11 we derive the following result.

Remark 5.18. *For any connected graph H of radius $r(H) \geq 3$ and any connected graph G of order $n \geq 2$,*

$$\dim_l(G \odot H) \geq 2n.$$

In this section we study the limit case of the above bound for the case where H is bipartite.

Lemma 5.19. *If H is a graph of radius three and $\dim_l(K_1 + H) = 2$, then the vertex of K_1 does not belong to any local metric basis for $K_1 + H$.*

Proof. Let $\{a, b\}$ be a local metric basis for $\langle v \rangle + H$. Since $r(H) = 3$, no vertex of H distinguishes every pair of adjacent vertices of H. Thus, $a \neq v$ and $b \neq v$. □

Theorem 5.20. *Let $H = (U_1, U_2, E)$ be a bipartite graph of radius three and let G be a connected graph of order n. Then $\dim_l(G \odot H) = 2n$ if and only if $\dim_l(K_1 + H) = 2$ or for some $i \in \{1, 2\}$, there exist $a, b \in U_i$ such that $N_H(a) \cup N_H(b) = U_j$, where $j \in \{1, 2\} - \{i\}$.*

Proof. By Theorem 5.3 we know that $\dim_l(G \odot H) = 2n$ if and only if either $\dim_l(\langle v \rangle + H) = 2$ and v does not belong to any local metric basis for $\langle v \rangle + H$ or $\dim_l(\langle v \rangle + H) = 3$ and there exists a local metric basis B of $\langle v \rangle + H$ such that $v \in B$.

If $\dim_l(\langle v \rangle + H) = 2$, then we are done (note that by Lemma 5.19 we have that v does not belong to any local metric basis for $\langle v \rangle + H$).

Let $B = \{a, b, v\}$ be a local metric basis of $\langle v \rangle + H$. Since $v \in B$, we have $N_H(a) \cap N_H(b) \neq \emptyset$. So, a and b must belong to the same color class, set $a, b \in U_1$. Hence, if there exists $y \in U_2 - (N_H(a) \cup N_H(b))$, then for every $x \in N_H(y)$, the pair x, y is not distinguished in $\langle v \rangle + H$ by the elements of B, which is a contradiction and, as a consequence, $N_H(a) \cup N_H(b) = U_2$.

Conversely, if there exists $a, b \in U_i$ such that $N_H(a) \cup N_H(b) = U_j$, where $j \in \{1, 2\} - \{i\}$, then for every $y \in U_j$ and $x \in N_H(y)$, the pair x, y is distinguished by a or by b. So, $\{a, b, v\}$ is a local metric generator for $\langle v \rangle + H$ and, as a consequence, $\dim_l(\langle v \rangle + H) \leq 3$. Therefore, either $\dim_l(\langle v \rangle + H) = 2$ or $\{a, b, v\}$ is a local metric basis of $\langle v \rangle + H$. □

Consider the following decision problem. The input is an arbitrary bipartite graph $H = (U_1 \cup U_2, E)$ of radius three. The problem consists in deciding whether H satisfies $\dim_l(K_1 + H) = 2$, or not. According to the next remark we deduce that the time complexity of this decision problem is at most $O(|U_1|^2|U_2|^2)$. Although this remark is straightforward, we include the proof for completeness.

Remark 5.21. *Let $H = (U_1, U_2, E)$ be a bipartite graph of radius three. Consider the following statements:*

(i) *For some $i \in \{1, 2\}$, there exist $a, b \in U_i$ such that $\{N_H(a), N_H(b)\}$ is a partition of U_j, where $j \in \{1, 2\} - \{i\}$.*

(ii) *There exist two vertices $a \in U_1$ and $b \in U_2$ such that for every edge $xy \in E$, where $x \in U_1$ and $y \in U_2$, it follows $y \in N_H(a)$ or $x \in N_H(b)$.*

Then $\dim_l(K_1 + H) = 2$ *if and only if* (i) *or* (ii) *holds.*

Proof. We first note that since $K_1 + H$ is not bipartite, Theorem 2.4 leads to $\dim_l(K_1 + H) \geq 2$.

(Sufficiency) If (i) holds, then $\{a, b\} \succ U_j$ and $N_H(a) \cap N_H(b) = \emptyset$. Hence, $\{a, b\}$ is a local metric basis of $K_1 + H$ and, as a consequence, $\dim_l(K_1 + H) = 2$.

Now, if (ii) holds, it is is straightforward that $\{a, b\}$ is a local metric basis of $K_1 + H$ and, as a consequence, $\dim_l(K_1 + H) = 2$.

(Necessity) Let $\{a, b\}$ be a local metric basis for of $\langle v \rangle + H$. By Lemma 5.19 we know that $v \notin \{a, b\}$. Then we have two possibilities.

Case 1. a and b belong to the same color class of H, say $a, b \in U_1$. Since for every $x \in V(H)$ the pair x, v must be distinguished by a or by b, we conclude that $N_H(a) \cap N_H(b) = \emptyset$. Also, since every pair of adjacent vertices $x \in U_1$ and $y \in U_2$ must be distinguished by a or by b, we conclude that $y \sim a$ or $y \sim b$ and, as a result, $\{a, b\} \succ U_2$. Hence, we conclude that $\{N_H(a), N_H(b)\}$ is a partition of U_2.

Case 2: a and b belong to different color classes of H, say $a \in U_1$ and $b \in U_2$. Since $\{a, b\}$ is a local metric basis for $\langle v \rangle + H$, for every edge $xy \in E$, where $x \in U_1$ and $y \in U_2$, it follows $y \in N_H(a)$ or $x \in N_H(b)$. □

Note that if $H = (U_1 \cup U_2, E)$ is a bipartite graph of diameter $D(H) = 3$, then for any $i \in \{1, 2\}$ and $x, y \in U_i$ we have $N_H(x) \cap N_H(y) \neq \emptyset$. Hence, we deduce the following consequence of Remark 5.21.

Corollary 5.22. *Let H be a bipartite graph where $D(H) = r(H) = 3$. If $B = \{a, b\}$ is a local metric basis for $K_1 + H$, the a and b belong to different color classes.*

Other direct consequence of Remark 5.21 is the following.

Corollary 5.23. *Let $H = (U_1, U_2, E)$ be a bipartite graph of radius three. If for some $i \in \{1, 2\}$, there exist $a \in U_i$ such that $\delta_H(a) = |U_j| - 1$, where $j \in \{1, 2\} - \{i\}$, then $\dim_l(K_1 + H) = 2$.*

Closed formulae for $\dim_l(G \odot H)$ when H is a tree of radius three.

In order to study the particular case when H is a tree of radius three, we introduce the following additional notation. Let T be a tree of radius three. For the particular case when $C(T) = \{u\}$ we consider the forest $F_u = \cup_{w \in N_T(u)} T_w$ composed of all the rooted trees $T_w = (V_w, E_w)$, of root $w \in N_T(u)$, obtained by removing the central vertex u from T. The height of T_w is $h_w = \max_{x \in V(T_w)}\{d(w, x)\}$. We denote by $\varsigma(T)$ the number of trees in F_u with h_w equal to two, *i.e.*, $\varsigma(T) = |S(T)|$, where

$$S(T) = \{w \in N_T(u) : \; h_w = 2\}.$$

Note that if $h_w \neq 1$, for every $w \in N_T(u)$, then $\varsigma(T) = \delta'(T)$. So, as the following result shows, the bound $\dim_l(G \odot T) \leq n \cdot \delta'(T)$ is tight.

Theorem 5.24. *Let T be a tree of radius three and center $C(T)$. The following assertions hold for any connected graph G of order $n \geq 2$.*

(i) *If $|C(T)| = 2$, then $\dim_l(G \odot T) = 2n$*

(ii) *If $C(T) = \{u\}$, then*

$$\dim_l(G \odot T) = \begin{cases} n \cdot (\varsigma(T) + 1), & \text{if there exists } w \in N_T(u) \text{ such that } h_w = 1, \\ n \cdot \varsigma(T), & \text{otherwhise.} \end{cases}$$

Proof. It is well-known that the center of a tree consists of either a single vertex or two adjacent vertices.

We first consider the case where $C(T)$ consists of two adjacent vertices, say $C(T) = \{u', u''\}$. Note that in this case, if we remove the edge $\{u', u''\}$

from T, we obtain two rooted trees $T' = (V', E')$ and $T'' = (V'', E'')$, with roots u' and u'', respectively, where the distance from the root to the leaves is at most two. Hence, in $K_1 + T$ every pair of adjacent vertices $x, y \in V'$ is distinguished by u' and every pair of adjacent vertices $x, y \in V''$ is distinguished by u''. Also, for every $x \in V' - \{u'\}$ the pair v, x is distinguished by u'' and for every $x \in V'' - \{u''\}$, the pair v, x is distinguished by u', where v is the vertex of K_1. So, $C(T)$ is a local metric generator for $K_1 + T$. Hence, $\dim_l(K_1 + T) \leq 2$ and, since $K_1 + T$ is not bipartite, by Theorem 2.4 we conclude that $\dim_l(K_1 + T) = 2$. Now, in this case, if the vertex of K_1 belongs to a local metric basis for $K_1 + T$, then there exists $z \in V(T)$ such that z distinguishes any pair of adjacent vertices $x, y \in V(T)$, and as a consequence $r(T) \leq 2$, which is a contradiction. Thus, we conclude that the vertex of K_1 does not belong to any local metric basis for $K_1 + T$. Therefore, as a consequence of Theorem 5.3 (i) we obtain $\dim_l(G \odot T) = 2n$.

Now let us consider the case where the center of T consists of a single vertex, say $C(T) = \{u\}$. Let B be a local metric basis for $K_1 + T$. We first note that for every rooted tree $T_w = (V_w, E_w)$ of height two we have $|B \cap V_w| = 1$, due to the fact that in $K_1 + T$ the vertex $w \in N_T(u)$ distinguishes every pair of adjacent vertices $x, y \in V_w$ and no vertex of $V(K_1 + T) - V_w$ distinguishes a pair of adjacent vertices where one vertex is a leaf. Hence, $\dim_l(K_1 + T) \geq \varsigma(T)$. Now we differentiate the following cases.

Case 1. There exists $w \in N_T(u)$ such that $h_w = 1$. In this case, the subgraph of T induced by the set $X = \cup_{h_w \leq 1} V_w \cup \{u\}$ is a tree of root u and height two. Hence, as above we conclude that $|B \cap X| = 1$. So, $\dim_l(K_1 + T) \geq \varsigma(T) + 1$. In order to show that the set $A = \{u\} \cup S(T)$ is a local metric basis for $K_1 + T$ we only need to observe that $N_T(w) \cap N_T(u) = \emptyset$ and, as a consequence, for every $x \in V(T)$ the pair x, v is distinguished by some $z \in A$. Thus, $\dim_l(K_1 \odot T) = \varsigma(T) + 1$.

Moreover, since for every metric basis A of $K_1 + T$ we have $|A \cap X| = 1$ and for every rooted tree $T_w = (V_w, E_w)$ of height two, $|A \cap V_w| = 1$, we conclude that the vertex of K_1 does not belong to any local metric basis for $K_1 + T$. Therefore, as a consequence of Theorem 5.3 (i) we obtain $\dim_l(G \odot T) = n(\varsigma(T) + 1)$.

Case 2. For every $w \in N_T(u)$, $h_w \neq 1$. In this case we define

$$\varphi(T_w) = |\{z \in N_{T_w}(w) : \ \delta_T(z) \geq 2\}|.$$

Suppose there exists $w_i \in N_T(u)$ such that $\varphi(T_{w_i}) = 1$. With this assumption we define

$$A' = \{z\} \cup S(T) - \{w_i\},$$

where $z \in V_{w_i}$ and $\delta_T(z) \geq 2$. Note that every pair of adjacent vertices $x, y \in \{u\} \cup V_{w_i}$ is distinguished by z. So, by analogy to Case 1 we show that A' is a local metric basis for $K_1 + T$ and the vertex of K_1 does not belong to any local metric basis for $K_1 + T$. Therefore, as a consequence of Theorem 5.3 (i) we obtain $\dim_l(G \odot T) = n \cdot \varsigma(T)$.

On the other hand, if for every $w \in S(T)$ it follows $\varphi(T_w) \geq 2$, then w is the only vertex of V_w which distinguishes every pair of adjacent vertices $x, y \in V_w$. Thus, in such a case $S(T)$ is a subset of any local metric basis for $K_1 + T$ and, as a consequence, the only two local metric basis for $K_1 + T$ are $\{u\} \cup S(T)$ and $\{v\} \cup S(T)$. Therefore, as a consequence of Theorem 5.3 (ii) we obtain $\dim_l(G \odot T) = n \cdot \varsigma(T)$. □

Chapter 6

The local metric dimension of lexicographic product graphs

6.1 Introduction

In this chapter we study the problem of finding the local metric dimension of the lexicographic product of graphs in terms of parameters of the graphs involved in the product. Let G be a graph of order n, and let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ be an ordered family composed by n graphs. The *lexicographic product* of G and $\mathcal{H}$ is the graph $G \circ \mathcal{H}$, such that $V(G \circ \mathcal{H}) = \bigcup_{u_i \in V(G)} (\{u_i\} \times V(H_i))$ and $(u_i, v_r)(u_j, v_s) \in E(G \circ \mathcal{H})$ if and only if $u_i u_j \in E(G)$ or $i = j$ and $v_r v_s \in E(H_i)$. Figure 6.1 shows the lexicographic product of P_3 and the family composed by $\{P_4, K_2, P_3\}$, and the lexicographic product of P_4 and the family $\{H_1, H_2, H_3, H_4\}$, where $H_1 \cong H_4 \cong K_1$ and $H_2 \cong H_3 \cong K_2$. In general, we can construct the graph $G \circ \mathcal{H}$ by taking one copy of each $H_i \in \mathcal{H}$ and joining by an edge every vertex of H_i with every vertex of H_j for every $u_i u_j \in E(G)$. Note that $G \circ \mathcal{H}$ is connected if and only if G is connected.

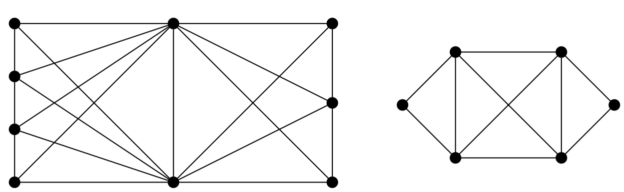

Figure 6.1: The lexicographic product graphs $P_3 \circ \{P_4, K_2, P_3\}$ and $P_4 \circ \{H_1, H_2, H_3, H_4\}$, where $H_1 \cong H_4 \cong K_1$ and $H_2 \cong H_3 \cong K_2$.

As a particular case, we will focus on the standard concept of lexicographic product graph, where $H_i \cong H$ for every $i \in \{1, \dots, n\}$, which is denoted as $G \circ H$ for simplicity . Another particular case of lexicographic product graphs is the join graph. The *join graph* $G + H$ is defined as the graph obtained from disjoint graphs G and H by taking one copy of G and one copy of H and joining by an edge each vertex of G with each vertex of H [30, 62]. Note that $G + H \cong K_2 \circ \{G, H\}$. The join operation is commutative and associative. Now, for the sake of completeness, Figure 6.2 illustrates two examples of join graphs.

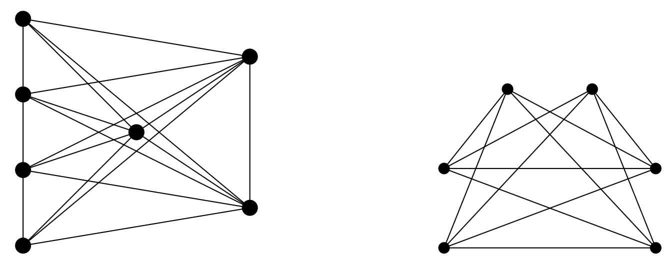

Figure 6.2: Two join graphs: $P_4 + C_3 \cong K_2 \circ \{P_4, C_3\}$ and $N_2 + N_2 + N_2 \cong K_3 \circ N_2$.

Moreover, complete k-partite graphs,

$$K_{p_1,\dots,p_k} \cong K_n \circ \{N_{p_1}, \dots, N_{p_k}\} \cong N_{p_1} + \cdots + N_{p_k},$$

are typical examples of join graphs. The particular case illustrated in Figure 6.2 (right hand side), is no other than the complete 3-partite graph $K_{2,2,2}$.

The relation between distances in a lexicographic product graph and those in its factors is presented in the following remark, for which it is necessary to recall (2.1).

Remark 6.1. *If G is a connected graph and (u_i, b) and (u_j, d) are vertices of $G \circ \mathcal{H}$, then*

$$d_{G \circ \mathcal{H}}((u_i, b), (u_j, d)) = \begin{cases} d_G(u_i, u_j), & \text{if } i \neq j, \\ \\ d_{H_i,2}(b, d), & \text{if } i = j. \end{cases}$$

We would point out that the remark above was stated in [29, 32] for the case where $H_i \cong H$ for all $H_i \in \mathcal{H}$.

The lexicographic product has been studied from different points of view in the literature. One of the most common researches focuses on finding relationships between the value of some invariant in the product and that of its factors. In this sense, we can find in the literature a large number of investigations on diverse topics. For instance, the metric dimension and related parameters have been studied in [20, 22, 35, 43, 48, 54].

6.2 Main results

From now on we denote by Θ the set of graphs H satisfying that for every local adjacency basis B, there exists $v \in V(H)$ such that $B \subseteq N_H(v)$. Notice that the only local adjacency basis of an empty graph N_r is the empty set, and so $N_r \in \Theta$. Moreover, $K_1 \cup K_2 \in \Theta$. In fact, a non-connected graph $H \in \Theta$ if and only if $H \cong N_r$ or $H \cong N_r \cup G$, where G is a connected graph in Θ. We denote by Φ the family of empty graphs. Notice that $\Phi \subset \Theta$. On the other hand, it is readily seen that no graph of radius greater than or equal to four belongs to Θ. As we will see in Proposition 6.15, if $H \in \Theta$ is a connected graph different from a tree, then $\mathtt{g}(H) \leq 6$.

In order to state our main result (Theorem 6.2) we need to introduce some additional notation. Let $\mathcal{U} = \{U_1, U_2, \ldots, U_k\}$ be the set of non-singleton true twin equivalence classes of a graph G. For the remainder of this paper we will assume that G is connected and has order $n \geq 2$, and $\mathcal{H} = \{H_1, \ldots, H_n\}$. We now define the following sets and parameters:

- $T(G) = \bigcup_{j=1}^{k} U_j$.
- $V_E = \{u_i \in V(G) - T(G) :\ H_i \in \Phi\}$.
- $I = \{u_i \in V(G) :\ H_i \in \Theta\}$.
- For any $I_j = I \cap U_j \neq \emptyset$, we can choose some $u \in I_j$ and set $I'_j = I_j - \{u\}$. We define the set $X_E = I - \bigcup_{I'_j \neq \emptyset} I'_j$.
- We say that two vertices $u_i, u_j \in X_E$ satisfy the relation $\mathcal{R}$ if and only if $u_i u_j \in E(G)$ and $d_G(u, u_i) = d_G(u, u_j)$ for all $u \in V(G) - (V_E \cup \{u_i, u_j\})$.
- We define $\mathcal{A}$ as the family of sets $A \subseteq X_E$ such that for every pair of vertices $u_i, u_j \in X_E$ satisfying $\mathcal{R}$ there exists a vertex in A that distinguishes them.
- $\varrho(G, \mathcal{H}) = \min\limits_{A \in \mathcal{A}} \{|A|\}$.

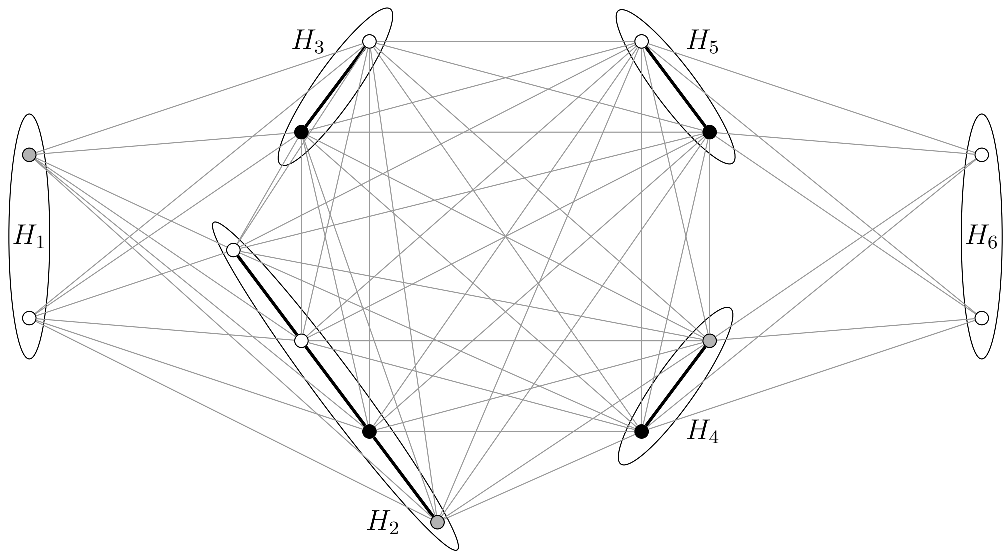

Figure 6.3: The graph $G \circ \mathcal{H}$, where G is the right-hand graph shown in Figure 6.1 and $\mathcal{H}$ is the family composed by the graphs $H_1 \cong H_6 \cong N_2$, $H_2 \cong P_4$, $H_3 \cong H_4 \cong H_5 \cong K_2$. The set of black- and grey-coloured vertices is a local metric basis of $G \circ \mathcal{H}$.

With the aim of clarifying what this notation means, we proceed to show an example where we explain the role of these parameters when constructing a local metric generator W for a lexicographic product graph. Let G be the right-hand graph shown in Figure 6.1 and let $\mathcal{H}$ be the family composed by the graphs $H_1 \cong H_6 \cong N_2$, $H_2 \cong P_4$, $H_3 \cong H_4 \cong H_5 \cong K_2$. Figure 6.3 shows the graph $G \circ \mathcal{H}$. Consider any $H_i \notin \Phi$. Note that the restriction of any local metric basis of $G \circ \mathcal{H}$ to the vertices of $\langle \{u_i\} \times V(H_i) \rangle \cong H_i$ must be a local adjacency generator for $\langle \{u_i\} \times V(H_i) \rangle$, as two adjacent vertices of $\langle \{u_i\} \times V(H_i) \rangle$ are not distinguished by any vertex outside $u_i \times V(H_i)$, so we can assume that the black-coloured vertices belong to W. Moreover, $U_1 = \{u_2, u_3\}$ and $U_2 = \{u_4, u_5\}$ are the non-singleton true twin equivalence classes of G. Since $u_4, u_5 \in I \cap U_2$, we have that no pair of non-black-coloured vertices in $(u_4 \times V(H_4)) \cup (u_5 \times V(H_5))$ is distinguished by any black-coloured vertex, so we add to W the grey-coloured vertex corresponding to the copy of H_4 and, by analogy, we add to W the grey-coloured vertex corresponding to the copy of H_2. Besides, note that the white-coloured vertices of the copies of H_3 and H_5 are only distinguished by themselves and by vertices from the copies of H_1 and H_6, so we need to add one more vertex to W, e.g. the grey-coloured vertex in the copy of H_1. Note that, according to our previous definitions, we have $V_E = \{u_1, u_6\}$ and we take $I'_1 = \{u_2\}$ and $I'_2 = \{u_4\}$. Thus, $X_E = \{u_1, u_3, u_5, u_6\}$. Therefore, since $u_1 \in X_E$ distinguishes the pair u_3, u_5, the sole pair of vertices from X_E satisfying $\mathcal{R}$, we take $A = \{u_1\}$ and conclude that $\varrho(G, \mathcal{H}) = 1$. Notice that,

$$\sum_{i=1}^{6} \operatorname{adim}_l(H_i) = 4, \sum_{I\cap U_j \neq \emptyset} (|I \cap U_j| - 1) = 2 \text{ and}$$

$$\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^{6} \operatorname{adim}_l(H_i) + \sum_{I\cap U_j \neq \emptyset} (|I \cap U_j| - 1) + \varrho(G, \mathcal{H}) = 7.$$

Theorem 6.2. *Let G be a connected graph of order $n \geq 2$, let $\{U_1, U_2, \ldots, U_k\}$ be the set of non-singleton true twin equivalence classes of G and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a family of graphs. Then*

$$\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^{n} \operatorname{adim}_l(H_i) + \sum_{I\cap U_j \neq \emptyset} (|I \cap U_j| - 1) + \varrho(G, \mathcal{H}).$$

Proof. We will first construct a local metric generator for $G \circ \mathcal{H}$. To this end, we need to introduce some notation. Let $V(G) = \{u_1, \ldots, u_n\}$ and let S_i be a local adjacency basis of H_i, where $i \in \{1, \ldots, n\}$. For any $I_j = I \cap U_j \neq \emptyset$, we choose $u \in I_j$ and set $I'_j = I_j - \{u\}$. Now, for every $u_i \in I'_j \neq \emptyset$, let $v_i \in V(H_i)$ such that $S_i \subseteq N_{H_i}(v_i)$. Finally, we consider a set $A \subseteq X_E$ achieving the minimum in the definition of $\varrho(G, \mathcal{H})$ and, for each $u_i \in A$, we choose one vertex $y_i \in V(H_i) - S_i$ such that $S_i \subseteq N_{H_i}(y_i)$. We claim that the set

$$S = \left(\bigcup_{S_i \neq \emptyset} (\{u_i\} \times S_i) \right) \cup \left(\bigcup_{I'_j \neq \emptyset} \{(u_i, v_i) :\ u_i \in I'_j\} \right) \cup \left(\bigcup_{u_i \in A} \{(u_i, y_i)\} \right)$$

is a local metric generator for $G \circ \mathcal{H}$. We differentiate the following four cases for two adjacent vertices (u_i, v), $(u_j, w) \in V(G \circ \mathcal{H}) - S$.

Case 1. $i = j$. In this case $vw \in E(H_i)$. Since S_i is a local adjacency basis of H_i, there exists $x \in S_i$ such that $d_{H_i,2}(x, v) \neq d_{H_i,2}(x, w)$ and so for $(u_i, x) \in \{u_i\} \times S_i \subset S$ we have $d_{G\circ\mathcal{H}}((u_i, x), (u_i, v)) = d_{H_i,2}(x, v) \neq d_{H_i,2}(x, w) = d_{G\circ\mathcal{H}}((u_i, x), (u_i, w))$.

Case 2. $i \neq j$, $u_i, u_j \in U_l$ and $u_i \notin I_l$. For any $y \in S_i - N_{H_i}(v)$ we have that $(u_i, y) \in \{u_i\} \times S_i \subseteq S$ and $d_{G\circ\mathcal{H}}((u_i, y), (u_i, v)) = 2 \neq 1 = d_{G\circ\mathcal{H}}((u_i, y), (u_j, w))$.

Case 3. $i \neq j$, $u_i, u_j \in U_l$ and $u_i, u_j \in I_l$. If $v = v_i$ and $w = v_j$, then $(u_i, v_i) \in S$ or $(u_j, v_j) \in S$. If $v \neq v_i$ or $w \neq v_j$ (say $v \neq v_i$) then either $S_i \subseteq N_{H_i}(v)$, in which case $d_{G\circ\mathcal{H}}((u_i, v_i), (u_i, v)) = 2 \neq 1 = d_{G\circ\mathcal{H}}((u_i, v_i), (u_j, w))$, or there exists $y \in S_i - N_{H_i}(v)$ such that $(u_i, y) \in \{u_i\} \times S_i \subseteq S$ and $d_{G\circ\mathcal{H}}((u_i, y), (u_i, v)) = 2 \neq 1 = d_{G\circ\mathcal{H}}((u_i, y), (u_j, w))$.

Case 4. $i \neq j$ and $N_G[u_i] \neq N_G[u_j]$. Notice that, in this case, $u_i \sim u_j$. If $u_i \notin I$, then $S_i \neq \emptyset$ and there exists $y \in S_i - N_{H_i}(v)$ such that $(u_i, y) \in \{u_i\} \times S_i \subseteq S$ and $d_{G\circ\mathcal{H}}((u_i, y), (u_i, v)) = 2 \neq 1 = d_{G\circ\mathcal{H}}((u_i, y), (u_j, w))$. Now, assume that $u_i, u_j \in I$. If $u_i \in I'_l$ or $u_j \in I'_l$ for some l (say $u_i \in I'_l$), then $d_{G\circ\mathcal{H}}((u_i, v_i), (u_i, v)) = 2 \neq 1 = d_{G\circ\mathcal{H}}((u_i, v_i), (u_j, w))$ or there exists $y \in S_i$ such that $d_{G\circ\mathcal{H}}((u_i, y), (u_i, v)) = 2 \neq 1 = d_{G\circ\mathcal{H}}((u_i, y), (u_j, w))$. Finally, if $u_i, u_j \notin \bigcup I'_l$, then by the construction of S there exists $u_l \in A \cup (V(G) - X_E)$ such that $d_G(u_l, u_i) \neq d_G(u_l, u_j)$. Since $u_l \in \{x : (x, y) \in S\}$, there exists $y \in V(H_l)$ such that $d_{G\circ\mathcal{H}}((u_l, y), (u_i, v)) \neq d_{G\circ\mathcal{H}}((u_l, y), (u_j, w))$.

In conclusion, S is a local metric generator for $G \circ \mathcal{H}$ and, as a result,

$$\dim_l(G \circ \mathcal{H}) \leq |S| = \sum_{i=1}^{n} \operatorname{adim}_l(H_i) + \sum_{I_j \neq \emptyset} (|I_j| - 1) + \varrho(G, \mathcal{H}).$$

It remains to show that

$$\dim_l(G \circ \mathcal{H}) \geq \sum_{i=1}^{n} \operatorname{adim}_l(H_i) + \sum_{I_j \neq \emptyset} (|I_j| - 1) + \varrho(G, \mathcal{H}).$$

To this end, we take a local metric basis W of $G \circ \mathcal{H}$ and for every $u_i \in V(G)$ we define the set $W_i = \{y : (u_i, y) \in W\}$. As for any $u_i \in V(G)$ and two adjacent vertices $v, w \in V(H_i)$, no vertex outside $\{u_i\} \times W_i$ distinguishes (u_i, v) and (u_i, w), we can conclude that W_i is a local adjacency generator for H_i. Hence,

$$|W_i| \geq \operatorname{adim}_l(H_i), \text{ for all } i \in \{1, \ldots, n\}. \tag{6.1}$$

Now suppose, for the purpose of contradiction, that there exist $u_i, u_j \in I \cap U_l$ such that $|W_i| = \operatorname{adim}_l(H_i)$ and $|W_j| = \operatorname{adim}_l(H_i)$. In such a case, there exist $v_i \in V(H_i) - W_i$ and $v_j \in V(H_j) - W_j$ such that $W_i \subseteq N_{H_i}(v_i)$

and $W_j \subseteq N_{H_j}(v_j)$, which is a contradiction. Hence, if $|I \cap U_l| \geq 2$, then $|\{u_i \in I \cap U_l : \ |W_i| \geq \text{adim}_l(H_i) + 1\}| \geq |I \cap U_l| - 1$ and, as a consequence,

$$\sum_{u_i \in I \cap T(G)} |W_i| \geq \sum_{u_i \in I \cap T(G)} \text{adim}_l(H_i) + \sum_{I \cap U_j \neq \emptyset} (|I \cap U_j| - 1). \tag{6.2}$$

On the other hand, assume that $\varrho(G, \mathcal{H}) \neq \emptyset$. We claim that

$$\sum_{u_j \in X_E} |W_j| \geq \sum_{u_j \in X_E} \text{adim}_l(H_j) + \varrho(G, \mathcal{H}). \tag{6.3}$$

To see this, we will prove that for any pair of vertices u_i, u_j satisfying $\mathcal{R}$ there exists $u_r \in X_E$ such that $|W_r| \geq \text{adim}_l(H_r)+1$. If $|W_i| = \text{adim}_l(H_i)+1$ or $|W_j| = \text{adim}_l(H_j) + 1$, then we are done. Suppose that $|W_i| = \text{adim}_l(H_i)$ and $|W_j| = \text{adim}_l(H_j)$. Since W_i and W_j are local adjacency bases of H_i and H_j, respectively, there exist $v \in V(H_i)$ and $w \in V(H_j)$ such that $\{u_i\} \times W_i \subseteq N_{\langle \{u_i\} \times V(H_i) \rangle}(u_i, v)$ and $\{u_j\} \times W_j \subseteq N_{\langle \{u_j\} \times V(H_j) \rangle}(u_j, w)$. Thus, there exists $(u_r, y) \in \{u_r\} \times W_r$, $r \neq i, j$, which distinguishes the pair $(u_i, v), (u_j, w)$, and so $d_G(u_r, u_i) \neq d_G(u_r, u_j)$. Hence, since u_i, u_j satisfy $\mathcal{R}$, we can claim that $u_r \in V_E \subseteq X_E$ and so $|W_r| > 0 = \text{adim}_l(H_r)$. In consequence, (6.3) holds. Therefore, (6.1), (6.2) and (6.3) lead to

$$\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^{n} |W_i| \geq \sum_{i=1}^{n} \text{adim}_l(H_i) + \sum_{I \cap U_j \neq \emptyset} (|I \cap U_j| - 1) + \varrho(G, \mathcal{H}),$$

as required. □

From now on we proceed to obtain some particular cases of this main result. To begin with, we consider the case $\varrho(G, \mathcal{H}) = 0$.

Corollary 6.3. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a family of graphs. If for any pair of adjacent vertices $u_i, u_j \in V(G)$, not belonging to the same true twin equivalence class, $H_i \notin \Theta$ or $H_j \notin \Theta$, or there exists $u_l \in V(G) - \{u_i, u_j\}$ such that $H_l \notin \Phi$ and $d_G(u_l, u_i) \neq d_G(u_l, u_j)$, then*

$$\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^{n} \text{adim}_l(H_i) + \sum_{I \cap U_j \neq \emptyset} (|I \cap U_j| - 1).$$

In particular, if $\mathcal{H} \cap \Phi = \emptyset$, then $\varrho(G, \mathcal{H}) = 0$, and so we can state the following result, which is a particular case of Corollary 6.3.

Remark 6.4. *For any connected graph G of order $n \geq 2$ and any family $\mathcal{H} = \{H_1, \ldots, H_n\}$ composed by non-empty graphs,*

$$\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^{n} \operatorname{adim}_l(H_i) + \sum_{I \cap U_j \neq \emptyset} (|I \cap U_j| - 1).$$

If $G \cong K_n$, then $\sum_{I \cap U_j \neq \emptyset}(|I \cap U_j| - 1) = \max\{0, |I| - 1\}$, $|X_E| \in \{0, 1\}$, which implies that $\varrho(G, \mathcal{H}) = 0$, and so Theorem 6.2 leads to the following.

Corollary 6.5. *For any integer $n \geq 2$ and any family $\mathcal{H} = \{H_1, \ldots, H_n\}$ of graphs,*

$$\dim_l(K_n \circ \mathcal{H}) = \sum_{i=1}^{n} \operatorname{adim}_l(H_i) + \max\{0, |I| - 1\}.$$

Furthermore, the following assertions hold for a graph H.

- *If $H \in \Theta$, then $\dim_l(K_n \circ H) = n \cdot \operatorname{adim}_l(H) + n - 1$.*
- *If $H \notin \Theta$, then $\dim_l(K_n \circ H) = n \cdot \operatorname{adim}_l(H)$.*

Notice that, in the general case, $\sum_{I \cap U_j \neq \emptyset}(|I \cap U_j| - 1) = 0$ if and only if each true twin equivalence class of G contains at most one vertex u_i such that $H_i \in \Theta$. Thus, we can state the following corollary.

Corollary 6.6. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a family of graphs. Then $\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^{n} \operatorname{adim}_l(H_i)$ if and only if for every two adjacent vertices $u_i, u_j \in I$, not belonging to the same true twin equivalence class, there exists $u \in V(G) - (V_E \cup \{u_i, u_j\})$ such that $d_G(u, u_i) \neq d_G(u, u_j)$ and each true twin equivalence class of G contains at most one vertex u_i such that $H_i \in \Theta$.*

A particular case of the result above is stated in the next remark.

Remark 6.7. *Let G be a connected bipartite graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a family of graphs. If $\mathcal{H} \nsubseteq \Theta$, then*

$$\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^{n} \operatorname{adim}_l(H_i).$$

Corollary 6.8. *Let G be a connected bipartite graph of order n, let H be a non-empty graph, and let $\mathcal{H}$ be a family composed by n graphs. If $\mathcal{H} - \Phi = \{H\}$, then*

$$\dim_l(G \circ \mathcal{H}) = \begin{cases} \operatorname{adim}_l(H) + 1, & \text{if } H \in \Theta; \\ \operatorname{adim}_l(H), & \text{otherwise.} \end{cases}$$

Proof. If $G \cong K_2$, then $\varrho(G,\mathcal{H}) = 0$, $\sum_{I\cap U_j\neq\emptyset}(|I\cap U_j|-1) = 1$ whenever $H\in\Theta$, and $\sum_{I\cap U_j\neq\emptyset}(|I\cap U_j|-1) = 0$ whenever $H\notin\Theta$. On the other hand, if $G\not\cong K_2$, then $\sum_{I\cap U_j\neq\emptyset}(|I\cap U_j|-1) = 0$, $\varrho(G,\mathcal{H}) = 1$ whenever $H\in\Theta$, and $\varrho(G,\mathcal{H}) = 0$ whenever $H\notin\Theta$. Since in any case $\sum_{i=1}^{n}\mathrm{adim}_l(H_i) = \mathrm{adim}_l(H)$, the result follows from Theorem 6.2. □

Our next result concerns the case of a family $\mathcal{H}$ composed by empty graphs.

Remark 6.9. *For any connected graph G of order $n\geq 2$ and any family $\mathcal{H}$ composed by n graphs,*

$$\dim_l(G\circ\mathcal{H}) \geq \dim_l(G).$$

In particular, if $\mathcal{H}\subset\Phi$, then

$$\dim_l(G\circ\mathcal{H}) = \dim_l(G).$$

Proof. Let W be a local metric basis of $G\circ\mathcal{H}$ and let $W_G = \{u:\ (u,v)\in W\}$ be the projection of W onto G. If there exist two adjacent vertices $u_i,u_j\in V(G)-W_G$ not distinguished by any vertex in W_G, then no pair of vertices $(u_i,v)\in\{u_i\}\times V(H_i)$, $(u_j,w)\in\{u_j\}\times V(H_j)$ is distinguished by elements of W, which is a contradiction. Thus, W_G is a local metric generator for G, so $\dim_l(G\circ\mathcal{H}) = |W| \geq |W_G| \geq \dim_l(G)$.

Now, we assume that $\mathcal{H}\subset\Phi$ and proceed to show that $\dim_l(G\circ\mathcal{H})\leq\dim_l(G)$. Let A be a local metric basis of G. For each $H_l\in\mathcal{H}$ we select one vertex y_l and we define the set $A' = \{(u_l,y_l) : u_l\in A\}$. Let (u_i,v) and (u_j,w) be two adjacent vertices of $G\circ\mathcal{H}$. Since $u_i\sim u_j$, there exists $u_l\in A$ such that $d_G(u_i,u_l)\neq d_G(u_j,u_l)$. Now, if $l\neq i,j$, then we have $d_{G\circ\mathcal{H}}((u_l,y_l),(u_i,v)) = d_G(u_i,u_l) \neq d_G(u_j,u_l) = d_{G\circ\mathcal{H}}((u_l,y_l),(u_j,w))$. If $l=i$, then $d_{G\circ\mathcal{H}}((u_l,y_l),(u_i,v)) = 2 \neq 1 = d_{G\circ\mathcal{H}}((u_l,y_l),(u_j,w))$. Since the case $l=j$ is analogous to the previous one, we can conclude that A' is a local metric generator for $G\circ\mathcal{H}$ and, as a consequence, $\dim_l(G\circ\mathcal{H})\leq\dim_l(G)$. Therefore, the proof is complete. □

In general, the converse of Corollary 6.9 does not hold. For instance, we take G as the graph shown in Figure 6.4, $H_1\cong H_5\cong K_2$ and $H_2,H_3,H_4\in\Phi$.

In this case, we have that, for instance, $\{u_1, u_5\}$ is a local metric basis of G, whereas for any $y \in V(H_1)$ and $y' \in V(H_5)$, the set $\{(u_1, y), (u_5, y')\}$ is a local metric basis of $G \circ \mathcal{H}$, so $\dim_l(G \circ \mathcal{H}) = \dim_l(G) = 2$.

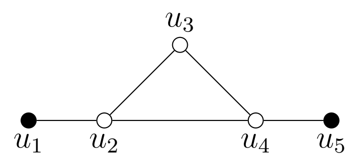

Figure 6.4: The set $\{u_1, u_5\}$ is a local metric basis of this graph.

As a direct consequence of Theorems 2.3 and 6.2 we deduce the following two results.

Theorem 6.10. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a family composed by non-empty graphs. Then $\dim_l(G \circ \mathcal{H}) = n$ if and only if each true twin equivalence class of G contains at most one vertex u_i such that $H_i \in \Theta$ and each $H_i \in \mathcal{H}$ is a bipartite graph having only one non-trivial connected component H_i^* and $r(H_i^*) \leq 2$.*

Theorem 6.11. *Let G be a connected true twins free graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a family composed by non-empty graphs of order n_i. Then $\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^{n} n_i - n$ if and only if $H_i \cong K_{n_i}$, for all $H_i \in \mathcal{H}$.*

6.3 The local adjacency dimension of H versus the local metric dimension of $K_1 + H$

From now on we denote by Θ' the set of graphs H satisfying that there exists a local metric basis of $K_1 + H$ which contains the vertex of K_1.

Proposition 6.12. *Let H be a graph. Then $H \in \Theta'$ if and only if $H \in \Theta$.*

Proof. Let $H \in \Theta'$, and B a local metric basis of $\langle u \rangle + H$ such that $u \in B$. Since u does not distinguish any pair of vertices of H, $B - \{u\}$ is a local adjacency generator for H, and so $\dim_l(\langle u \rangle + H) - 1 \geq \operatorname{adim}_l(H)$. Now, if there exists a local adjacency basis A of H such that $A \not\subseteq N_H(v)$ for all $v \in V(H)$, then A is a local metric basis of $\langle u \rangle + H$ and so $\dim_l(\langle u \rangle + H) = \operatorname{adim}_l(H)$, which is a contradiction. Therefore, $H \in \Theta$.

Now, let $H \in \Theta$. Suppose that there exists a local metric basis W of $\langle u\rangle + H$ such that $u \notin W$. In such a case, for every vertex $x \in V(H)$ there exists $y \in W$ such that $y \notin N_H(x)$, which implies that W is not a local adjacency basis of H, as $H \in \Theta$. Thus, since W is a local adjacency generator for H, we conclude that $\dim_l(\langle u\rangle + H) = |W| \geq \operatorname{adim}_l(H) + 1$. Therefore, for any local adjacency basis A of H, $A \cup \{u\}$ is a local adjacency basis of $\langle u\rangle + H$. □

Theorem 6.13. [23] *Let H be a non-empty graph. The following assertions hold.*

(i) *If $H \notin \Theta'$, then* $\operatorname{adim}_l(H) = \dim_l(K_1 + H)$.

(ii) *If $H \in \Theta'$, then* $\operatorname{adim}_l(H) = \dim_l(K_1 + H) - 1$.

(iii) *If H has radius $r(H) \geq 4$, then* $\operatorname{adim}_l(H) = \dim_l(K_1 + H)$.

As the following result shows, we can express all our previous results in terms of the local adjacency dimension of the graphs $K_1 + H_i$, where $H_i \in \mathcal{H}$, i.e., Theorem 6.14 is analogous to Theorem 6.2.

Theorem 6.14. *Let G be a connected graph of order $n \geq 2$, and $\mathcal{H} = \{H_1, \ldots, H_n\}$ a family of graphs. Then*

$$\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^{n} \dim_l(K_1 + H_i) - \tau + \varrho(G, \mathcal{H}),$$

where τ is the number of non-singleton true twin equivalence classes of G having at least one vertex u_i such that $H_i \in \Theta'$.

Proof. Notice that, by Proposition 6.12, the parameter $\varrho(G, \mathcal{H})$ can be redefined in terms of Θ'. The result immediately follows from Proposition 6.12 and Theorems 6.2 and 6.13. □

Lemma 6.15. *Let H be a connected graph different from a tree. If $H \in \Theta$, then* $\mathsf{g}(H) \leq 6$.

Proof. Let A be local adjacency basis of H. Since $H \in \Theta$, we consider v as the vertex of H such that $A \subseteq N_H(v)$. Let $N_i(v) = \{u \in V(H) : d_H(v, u) = i\}$. Since $A \subseteq N_1(v)$, we have that $N_3(v)$ is an independent set and $N_i(v) = \emptyset$, for all $i \geq 4$. Therefore, $\mathsf{g}(H) \leq 6$. □

By Proposition 6.12, Theorem 6.14 and Lemma 6.15 we can derive the following consequence of Theorem 6.14 (or equivalently, Theorem 6.2).

Corollary 6.16. *Let G be a connected graph of order $n \geq 2$, and $\mathcal{H} = \{H_1, \ldots, H_n\}$ a family composed by connected graphs. If each $H_i \in \mathcal{H}$ has radius $r(H_i) \geq 4$, or H_i is not a tree and it has girth $\mathtt{g}(H_i) \geq 7$, then*

$$\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^{n} \dim_l(K_1 + H_i) = \sum_{i=1}^{n} \operatorname{adim}_l(H_i).$$

Proposition 6.17. [23] *For any integer $n \geq 4$, $\operatorname{adim}_l(C_n) = \left\lceil \frac{n}{4} \right\rceil$.*

From Corollary 6.16 and Proposition 6.17 we deduce the following result.

Proposition 6.18. *Let G be a connected graph of order $t \geq 2$, and $\mathcal{H} = \{C_{n_1}, \ldots, C_{n_t}\}$ a family composed by cycles of order at least 7. Then*

$$\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^{t} \left\lceil \frac{n_i}{4} \right\rceil.$$

6.4 On the local adjacency dimension of lexicographic product graphs

By a simple transformation of Theorem 6.2 we obtain an analogous result on the local adjacency dimension of lexicographic product graphs, which we will state without proof. To this end, we consider again some of our previous notation. As above, let $\{U_1, U_2, \ldots, U_k\}$ be the set of non-singleton true twin equivalence classes of a connected graph G of order $n \geq 2$, and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a family of graphs. Recall that

$$V_E = \{u_i \in V(G) - T(G) : \, H_i \in \Phi\},$$

$$I = \{u_i \in V(G) : \, H_i \in \Theta\}$$

and, for any $I_j = I \cap U_j \neq \emptyset$, we can choose some $u \in I_j$ and set $I'_j = I_j - \{u\}$. Moreover, recall that $X_E = I - \bigcup_{I'_j \neq \emptyset} I'_j$. Now, we say that two vertices $u_i, u_j \in X_E$ satisfy the relation $\mathcal{R}'$ if and only if $u_i u_j \in E(G)$ and $d_{G,2}(u, u_i) = d_{G,2}(u, u_j)$ for all $u \in V(G) - (V_E \cup \{u_i, u_j\})$. We define $\mathcal{A}'$ as the family of sets $A \subseteq X_E$ such that for every pair of vertices $u_i, u_j \in X_E$ satisfying $\mathcal{R}'$ there exists a vertex in A which is adjacent to exactly one of them. Finally, we define $\varrho'(G, \mathcal{H}) = \min_{A \in \mathcal{A}'} \{|A|\}$.

Theorem 6.19. *Let G be a connected graph of order $n \geq 2$, let $\{U_1, U_2, \ldots, U_k\}$ be the set of non-singleton true twin equivalence classes of G and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a family of graphs. Then*

$$\operatorname{adim}_l(G \circ \mathcal{H}) = \sum_{i=1}^{n} \operatorname{adim}_l(H_i) + \sum_{I \cap U_j \neq \emptyset} (|I \cap U_j| - 1) + \varrho'(G, \mathcal{H}).$$

Let $G \cong P_4$ where $V(P_4) = \{u_1, u_2, u_3, u_4\}$ and $u_i u_{i+1} \in E(G)$, for $i \in \{1, 2, 3\}$. If $H_1 \cong H_2 \cong H_4 \cong P_3$ and $H_3 \cong N_3$, then $\dim_l(G \circ \mathcal{H}) = 3 < 4 = \operatorname{adim}_l(G \circ \mathcal{H})$. Notice that $\varrho(G, \mathcal{H}) = 0$ and $\varrho'(G, \mathcal{H}) = 1$. However, if $H_2 \cong H_3 \cong P_3$ and $H_1 \cong H_4 \cong N_3$, then $\varrho(G, \mathcal{H}) = \varrho'(G, \mathcal{H}) = 1$ and $\dim_l(G \circ \mathcal{H}) = 3 = \operatorname{adim}_l(G \circ \mathcal{H})$.

We already know that for any graph G of diameter less than or equal to two, $\dim_l(G) = \operatorname{adim}_l(G)$. However, the previous example shows that the above mentioned equality is not restrictive to graphs of diameter at most two, as $D(G \circ \mathcal{H}) = D(P_4) = 3$.

Notice that $\varrho'(G, \mathcal{H}) \geq \varrho(G, \mathcal{H})$, which is a direct consequence of Theorems 6.2 and 6.19, as well as the fact that $\operatorname{adim}_l(G) \geq \dim_l(G)$ for any graph G. The next result corresponds to the case $\varrho(G, \mathcal{H}) = \varrho'(G, \mathcal{H})$.

Theorem 6.20. *Let G be a connected graph of order $n \geq 2$, and $\mathcal{H} = \{H_1, \ldots, H_n\}$ a family of graphs. Then $\dim_l(G \circ \mathcal{H}) = \operatorname{adim}_l(G \circ \mathcal{H})$ if and only if $\varrho(G, \mathcal{H}) = \varrho'(G, \mathcal{H})$.*

We now characterize the case $\varrho(G, \mathcal{H}) = \varrho'(G, \mathcal{H}) = 0$. The symmetric difference of two sets U and W will be denoted by $U \triangledown W$.

Theorem 6.21. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a family of graphs. Then the following assertions are equivalent.*

(i) $\dim_l(G \circ \mathcal{H}) = \operatorname{adim}_l(G \circ \mathcal{H}) = \sum_{i=1}^{n} \operatorname{adim}_l(H_i) + \sum_{I \cap U_j \neq \emptyset} (|I \cap U_j| - 1)$.

(ii) *For any pair of adjacent vertices $u_i, u_j \in V(G)$, not belonging to the same true twin equivalence class, $H_i \notin \Theta$ or $H_j \notin \Theta$, or there exists $u_l \in N_G(u_i) \triangledown N_G(u_j) - \{u_i, u_j\}$ where H_l is not empty.*

Proof. By Theorems 6.2, 6.19 and 6.20, we only need to show that $\varrho'(G, \mathcal{H}) = 0$ if and only if (ii) holds.

($(i) \Rightarrow (ii)$) If $\varrho'(G,\mathcal{H}) = 0$, then for every two adjacent vertices $u_i, u_j \in I$, not belonging to the same true twin equivalence class, there exists $u_l \in V(G) - (V_E \cup \{u_i, u_j\})$ such that $d_{G,2}(u_l, u_i) \neq d_{G,2}(u_l, u_j)$, which implies that $u_l \in N_G(u_i) \triangledown N_G(u_j)$ and H_l is not empty. Now, if $u_i, u_j \notin I$, then $H_i \notin \Theta$ or $H_j \notin \Theta$.

($(ii) \Rightarrow (i)$) If for any pair of adjacent vertices $u_i, u_j \in V(G)$, not belonging to the same true twin equivalence class, $H_i \notin \Theta$ or $H_j \notin \Theta$, or there exists $u_l \in N_G(u_i) \triangledown N_G(u_j)$ where H_l is not empty, then no pair of adjacent vertices satisfy $\mathcal{R}'$ and $V(G) - X_E$ is a local adjacency generator for G, which implies that $\varrho'(G,\mathcal{H}) = 0$. □

Chapter 7

The simultaneous local metric dimension of graphs

7.1 Introduction

The simultaneous metric dimension was introduced in the framework of the navigation problem proposed in [39], where navigation was studied in a graph-structured framework in which the navigating agent (which was assumed to be a point robot) moves from node to node of a "graph space". The robot can locate itself by the presence of distinctively labeled "landmark" nodes in the graph space. On a graph, there is neither the concept of direction nor that of visibility. Instead, it was assumed in [39] that a robot navigating on a graph can sense the distances to a set of landmarks. Evidently, if the robot knows its distances to a sufficiently large set of landmarks, its position on the graph is uniquely determined. This suggests the following problem: given a graph G, what are the fewest number of landmarks needed, and where should they be located, so that the distances to the landmarks uniquely determine the robot's position on G? Indeed, the problem consists in determining the metric dimension and a metric basis of G. Now, consider the following extension of this problem, introduced by Ramírez-Cruz, Oellermann and Rodríguez-Velázquez in [49]. Suppose that the topology of the navigation network may change within a range of possible graphs, say $G_1, G_2, \ldots, G_k$. This scenario may reflect several situations, for instance the simultaneous use of technologically differentiated redundant sets of landmarks, the use of a dynamic network whose links change over time, etc. In this case, the above mentioned problem becomes to determine the minimum cardinality of a set S which must be simultaneously a metric generator for each graph G_i, $i \in \{1, \ldots, k\}$. So, if S is a solution for this problem, then each robot can be uniquely determined by the distance to the elements of S, regardless of the graph G_i that models the network at each moment. Such sets we called *simultaneous metric generators* in [49], where, by analogy, a *simultaneous metric basis* was defined as a minimum cardinality simultaneous metric generator and this cardinality was called the *simultaneous metric dimension* of the graph family $\mathcal{G}$, denoted by $\mathrm{Sd}(\mathcal{G})$.

As pointed out by [47], a number of applications arise where only neighbouring vertices need to be distinguished. Such applications were the basis for the introduction of the local metric dimension. Here, we consider the necessity of distinguishing neighbouring vertices in a multiple topology scenario, so we deal with the problem of finding the minimum cardinality of a set S which must be simultaneously a local metric generator for each graph G_i, $i \in \{1, \ldots, k\}$.

Given a family $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ of connected graphs $G_i = (V, E_i)$ on a common vertex set V, we define a *simultaneous local metric generator* for $\mathcal{G}$ as a set $S \subset V$ such that S is simultaneously a local metric generator for each G_i. We say that a minimum simultaneous local metric generator for $\mathcal{G}$ is a *simultaneous local metric basis* of $\mathcal{G}$, and its cardinality the *simultaneous local metric dimension* of $\mathcal{G}$, denoted by $\mathrm{Sd}_l(\mathcal{G})$ or explicitly by $\mathrm{Sd}_l(G_1, G_2, \ldots, G_k)$. An example is shown in Figure 7.1, where $\{u_3, u_4\}$ is a simultaneous local metric basis of $\{G_1, G_2, G_3\}$.

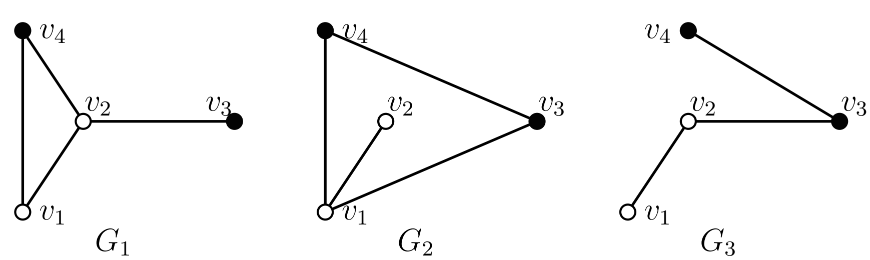

Figure 7.1: The set $\{v_3, v_4\}$ is a simultaneous local metric basis of $\{G_1, G_2, G_3\}$. Thus, $\mathrm{Sd}_l(G_1, G_2, G_3) = 2$.

It will be useful to define the *Simultaneous local adjacency dimension* of a family $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ of connected graphs $G_i = (V, E_i)$ on a common vertex set V, as the cardinal of minimum set $S \subseteq V$ such that S

is simultaneously a local adjacency generator for each G_i. We denote this parameter as $\mathrm{Sad}_l\,\mathcal{G}$.

As usual a set $A \subseteq V(G)$ is a *vertex cover* for G if for every $uv \in E(G)$, $u \in A$ or $v \in A$. The vertex cover number of G, denoted by $\beta(G)$ is the minimum cardinal of a vertex cover of G.

The chapter is organized as follows. In Section 7.2 we obtain some general results on the simultaneous local metric dimension of graph families. Section 7.3 is devoted to the case of graph families obtained by small changes on a graph, while in Sections 7.4 and 7.5 we study the particular cases of families of corona graphs and families of lexicographic product graphs, respectively. Finally, in Section 7.6 we show that the problem of computing the simultaneous local metric dimension of graph families is NP-Hard, even when restricted to families of tadpole graphs.

7.2 Basic results

Remark 7.1. *For any family $\mathcal{G} = \{G_1, \ldots, G_t\}$ of connected graphs on a common vertex set V and let $G' = (V, \cup E(G_i))$. The following results hold:*

(i) $\mathrm{Sd}_l(\mathcal{G}) \geq \max\limits_{i\in\{1,\ldots,k\}} \{\dim_l(G_i)\}$.

(ii) $\mathrm{Sd}_l(\mathcal{G}) \leq \mathrm{Sd}(\mathcal{G})$.

(iii) $\mathrm{Sd}_l(\mathcal{G}) \leq \min\left\{\beta(G'), \sum_{i=1}^{k} \dim_l(G_i)\right\}$.

Proof. (i) is deduced directly from the definition of simultaneous local metric dimension. Let B be a simultaneous metric basis of $\mathcal{G}$ and let $u, v \in V - B$, be two vertices not in B such that $u \sim_{G_i} v$ in some G_i. Since in G_i there exists $x \in B$ such that $d_{G_i}(u, x) \neq d_{G_i}(v, x)$, B is a simultaneous local metric generator for $\mathcal{G}$, so (ii) holds. Finally, (iii) is obtained from the following facts: (a) the union of local metric generators for all graphs in $\mathcal{G}$ is a simultaneous local metric generator for $\mathcal{G}$, which implies that $\mathrm{Sd}_l(\mathcal{G}) \leq \sum_{i=1}^{k} \dim_l(G_i)$; (b) any vertex cover of G' is a local metric generator of G_i, for every $G_i \in \mathcal{G}$, which implies that $\mathrm{Sd}_l(\mathcal{G}) \leq \beta(G')$. □

The inequalities above are tight. For example, the graph family $\mathcal{G}$ shown in Figure 7.1 satisfies $\mathrm{Sd}_l(\mathcal{G}) = \mathrm{Sd}(\mathcal{G})$, whereas $\mathrm{Sd}_l(\mathcal{G}) = 2 = \dim_l(G_1) =$

$\dim_l(G_2) = \max_{i\in\{1,2,3\}}\{\dim_l(G_i)\}$. Moreover, the family $\mathcal{G}$ shown in Figure 7.2 satisfies $\mathrm{Sd}_l(\mathcal{G}) = 3 = |V| - 1 < \sum_{i=1}^{6} \dim_l(G_i) = 12$, whereas the family $\mathcal{G} = \{G_1, G_2\}$ shown in Figure 7.3 satisfies $\mathrm{Sd}_l(\mathcal{G}) = 4 = \dim_l(G_1) + \dim_l(G_2) < |V| - 1 = 7$.

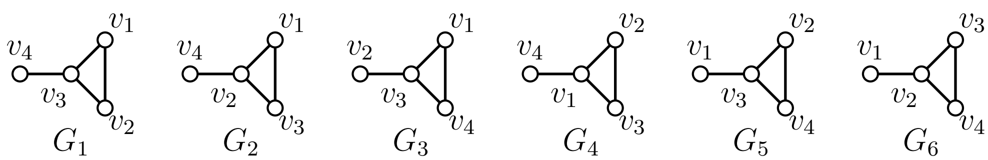

Figure 7.2: The family $\mathcal{G} = \{G_1, \ldots, G_6\}$ satisfies $\mathrm{Sd}_l(\mathcal{G}) = |V| - 1 = 3$.

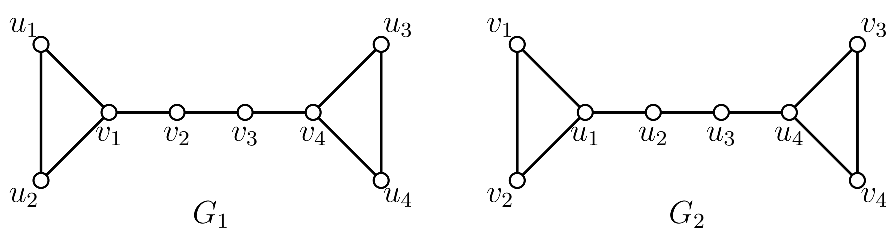

Figure 7.3: The family $\mathcal{G} = \{G_1, G_2\}$ satisfies $\mathrm{Sd}_l(\mathcal{G}) = \dim_l(G_1) + \dim_l(G_2) = 4$.

We now analyse the extreme cases of the bounds given in Remark 7.1.

Corollary 7.2. *Let $\mathcal{G}$ be a family of connected graphs on a common vertex set. If $K_n \in \mathcal{G}$, then*

$$\mathrm{Sd}_l(\mathcal{G}) = n - 1.$$

As shown in Figure 7.2, the converse of Corollay 7.2 does not hold. In general, the cases for which the upper bound $\mathrm{Sd}_l(\mathcal{G}) \leq |V| - 1$ is reached are summarised in the next result.

Theorem 7.3. *Let $\mathcal{G}$ be a family of connected graphs on a common vertex set V. Then $\mathrm{Sd}_l(\mathcal{G}) = |V| - 1$ if and only if for every $u, v \in V$, there exists a graph $G_{uv} \in \mathcal{G}$ such that u and v are true twins in G_{uv}.*

Proof. We first note that for any connected graph $G = (V, E)$ and any vertex $v \in V$, it holds that $V - \{v\}$ is a local metric generator for G. So, if

$\mathrm{Sd}_l\,\mathcal{G} = |V| - 1$, then for any $v \in V$, the set $V - \{v\}$ is a simultaneous local metric basis of $\mathcal{G}$ and, as a consequence, for every $u \in V - \{v\}$ there exists a graph $G_{uv} \in \mathcal{G}$ such that the set $V - \{u, v\}$ is not a local metric generator for G_{uv}, *i.e.*, u and v are adjacent in G_{uv} and $d_{G_{u,v}}(u, x) = d_{G_{u,v}}(v, x)$ for every $x \in V - \{u, v\}$. So, u and v are true twins in $G_{u,v}$.

Conversely, if for every $u, v \in V$ there exists a graph $G_{uv} \in \mathcal{G}$ such that u and v are true twins in G_{uv}, then for any simultaneous local metric basis B of $\mathcal{G}$ it holds that $u \in B$ or $v \in B$. Hence, all but one element of V must belong to B. Therefore $|B| \geq |V| - 1$, which implies that $\mathrm{Sd}_l\,\mathcal{G} = |V| - 1$. □

Notice that Corollary 7.2 is obtained directly from the previous result. Now, the two following results concern the limit cases of item (i) of Remark 7.1.

Theorem 7.4. *If $\mathcal{G}$ is a family of connected bipartite graphs on a common vertex set V, then*

$$\mathrm{Sd}_l(\mathcal{G}) = 1.$$

Proof. The result follows directly from the fact that for any $v \in V$, the set $\{v\}$ is a local metric basis of every $G_i \in \mathcal{G}$. □

Paths, trees and even-order cycles are bipartite. The following result covers the case of families composed by odd-order cycles.

Theorem 7.5. *For any family $\mathcal{G}$ composed by cycle graphs on a common odd-sized vertex set V, $\mathrm{Sd}_l(\mathcal{G}) = 2$ and any pair of vertices of V is a simultaneous local metric basis for $\mathcal{G}$.*

Proof. For any cycle $C_i \in \mathcal{G}$, the set $\{v\}$, $v \in V$, is not a local metric generator, as the adjacent vertices $v_{j+\lfloor\frac{|V|}{2}\rfloor}$ and $v_{j-\lfloor\frac{|V|}{2}\rfloor}$ (subscripts taken modulo $|V|$) are not distinguished by v, so item (i) of Remark 7.1 leads to $\mathrm{Sd}_l(\mathcal{G}) \geq \max_{i \in \{1,\ldots,k\}} \{\dim_l(G_i)\} \geq 2$. Moreover, any set $\{v, v'\}$ is a local metric generator for every $C_i \in \mathcal{G}$, as the single pair of adjacent vertices not distinguished by v is distinguished by v', so that $\mathrm{Sd}_l(\mathcal{G}) \leq 2$. □

The following result allows us to study the simultaneous local metric dimension of a family $\mathcal{G}$ from the family of graphs composed by all non-bipartite graphs belonging to $\mathcal{G}$.

Theorem 7.6. *Let $\mathcal{G}$ be a family of graphs on a common vertex set V, not all of them bipartite. If $\mathcal{H}$ is the subfamily of $\mathcal{G}$ composed by all non-bipartite graphs belonging to $\mathcal{G}$, then*

$$\mathrm{Sd}_l(\mathcal{G}) = \mathrm{Sd}_l(\mathcal{H}).$$

Proof. Since $\mathcal{H}$ is a non-empty subfamily of $\mathcal{G}$ we conclude that $\mathrm{Sd}_l(\mathcal{G}) \geq \mathrm{Sd}_l(\mathcal{H})$. Since any vertex of a bipartite graph G is a local metric generator for G, if $B \subseteq V$ is a simultaneous local metric basis of $\mathcal{H}$, then B is a simultaneous local metric generator for $\mathcal{G}$ and, as a result, $\mathrm{Sd}_l(\mathcal{G}) \leq |B| = \mathrm{Sd}_l(\mathcal{H})$. □

Some interesting situations may be observed regarding the simultaneous local metric dimension of some graph families versus its standard counterpart. In particular, the fact that false twin vertices need not be distinguished in the local variant leads to some cases where both parameters differ greatly. For instance, consider any family $\mathcal{G}$ composed by three or more star graphs having different centres. It was shown in [49] that any such family satisfies $\mathrm{Sd}(\mathcal{G}) = |V| - 1$, yet by Theorem 7.4 we have that $\mathrm{Sd}_l(\mathcal{G}) = 1$.

Given a family $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ of graphs $G_i = (V, E_i)$ on a common vertex set V, we define a *simultaneous vertex cover* for $\mathcal{G}$ as a set $S \subseteq V$ such that S is simultaneously a vertex cover for each G_i. The minimum cardinality among all simultaneous vertex covers for $\mathcal{G}$ is the *simultaneous vertex cover number* of $\mathcal{G}$, denoted by $\beta(\mathcal{G})$.

Theorem 7.7. *For any family $\mathcal{G}$ of connected graphs with common vertex set V,*

$$\mathrm{Sd}_l(\mathcal{G}) \leq \beta(\mathcal{G}).$$

Furthermore, if for every $uv \in \cup_{G\in\mathcal{G}} E(G)$ there exists $G' \in \mathcal{G}$ such that u and v are true twins in G', then $\mathrm{Sd}_l(\mathcal{G}) = \beta(\mathcal{G})$.

Proof. Let $B \subseteq V$ be a simultaneous vertex cover for $\mathcal{G}$. Since $V - B$ is a simultaneous independent set for $\mathcal{G}$, we conclude that $\mathrm{Sd}_l(\mathcal{G}) \leq \beta(\mathcal{G})$.

We assume that for every $uv \in \cup_{G\in\mathcal{G}} E(G)$ there exists $G' \in \mathcal{G}$ such that u and v are true twins in G' and suppose that $\mathrm{Sd}_l(\mathcal{G}) < \beta(\mathcal{G})$. In such a case, there exists a simultaneous local metric basis $C \subseteq V$ which is not a simultaneous vertex cover for $\mathcal{G}$. Hence, there exist $u, v \in V - C$ and $G \in \mathcal{G}$ such that $uv \in E(G)$. So that, u and v are true twins in G', for some $G' \in \mathcal{G}$, which contradicts the fact that C is a simultaneous local metric basis. Therefore, the result follows. □

7.3 Families obtained by small changes on a graph

Consider a graph G whose local metric dimension is known. In this section we address two related questions:

- *If a series of small changes is repeatedly performed on $E(G)$, thus producing a family $\mathcal{G}$ of consecutive versions of G, what is the behaviour of $\mathrm{Sd}_l(\mathcal{G})$ with respect to $\dim_l G$?*
- *If several small changes are performed on $E(G)$ in parallel, thus producing a family $\mathcal{G}$ of alternative versions of G, what is the behaviour of $\mathrm{Sd}_l(\mathcal{G})$ with respect to $\dim_l G$?*

Addressing this issue in the general case is hard, so we will analyse a number of particular cases. First, we will specify three operators that describe some types of changes that may be performed on a graph G:

- *Edge addition*: We say that a graph G' is obtained from a graph G by an edge addition if there is an edge $e \in E(\overline{G})$ such that $G' = (V(G), E(G) \cup \{e\})$. We will use the notation $G' = \mathrm{add}_e(G)$.
- *Edge removal*: We say that a graph G' is obtained from a graph G by an edge removal if there is an edge $e \in E(G)$ such that $G' = (V(G), E(G) - \{e\})$. We will use the notation $G' = \mathrm{rmv}_e(G)$.
- *Edge exchange*: We say that a graph G' is obtained from a graph G by an edge exchange if there is an edge $e \in E(G)$ and an edge $f \in E(\overline{G})$ such that $G' = (V(G), (E(G) - \{e\}) \cup \{f\})$. We will use the notation $G' = \mathrm{xch}_{e,f}(G)$.

Now, consider a graph G, and an ordered k-tuple of operations $O_k = (\mathrm{op}_1, \mathrm{op}_2, \ldots, \mathrm{op}_k)$, where $\mathrm{op}_i \in \{\mathrm{add}_{e_i}, \mathrm{rmv}_{e_i}, \mathrm{xch}_{e_i,f_i}\}$. We define the class $\mathcal{C}_{O_k}G$ containing all graph families of the form $\mathcal{G} = \{G, G'_1, G'_2, \ldots, G'_k\}$, composed by connected graphs on the common vertex set $V(G)$, where $G'_i = \mathrm{op}_i(G'_{i-1})$ for every $i \in \{1, \ldots, k\}$. Likewise, we define the class $\mathcal{P}_{O_k}G$ containing all graph families of the form $\mathcal{G} = \{G'_1, G'_2, \ldots, G'_k\}$, composed by connected graphs on the common vertex set $V(G)$, where $G'_i = \mathrm{op}_i(G)$ for every $i \in \{1, \ldots, k\}$. In particular, if $\mathrm{op}_i = \mathrm{add}_{e_i}$ ($\mathrm{op}_i = \mathrm{rmv}_{e_i}$, $\mathrm{op}_i = \mathrm{xch}_{e_i,f_i}$)

for every $i \in \{1, \ldots, k\}$, we will write $\mathcal{C}_{A_k}G$ ($\mathcal{C}_{R_k}G$, $\mathcal{C}_{X_k}G$) and $\mathcal{P}_{A_k}G$ ($\mathcal{P}_{R_k}G$, $\mathcal{P}_{X_k}G$).

We have that performing an edge exchange on any tree T (path graphs included) either produces another tree or a disconnected graph. Thus, the following result is a direct consequence of this fact and Theorem 7.4.

Remark 7.8. *For any tree T, any $k \geq 1$, and any graph family $\mathcal{T} \in \mathcal{C}_{X_k}(T) \cup \mathcal{P}_{X_k}(T)$,*

$$\mathrm{Sd}_l(\mathcal{T}) = 1.$$

Our next result covers a large class of families composed by unicyclic graphs that can be obtained by adding edges, in parallel, to a path graph.

Remark 7.9. *For any path graph P_n, $n \geq 4$, any $k \geq 1$, and any graph family $\mathcal{G} \in \mathcal{P}_{A_k}(P_n)$,*

$$1 \leq \mathrm{Sd}_l(\mathcal{G}) \leq 2.$$

Proof. Every graph $G \in \mathcal{G}$ is either a cycle or a unicyclic graph. If the cycle subgraphs of every graph in the family have even order, then $\mathrm{Sd}_l(\mathcal{G}) = 1$ by Theorem 7.4. If $\mathcal{G}$ contains at least one non-bipartite graph, then $\mathrm{Sd}_l(\mathcal{G}) \geq 2$. We now proceed to show that in this case $\mathrm{Sd}_l(\mathcal{G}) \leq 2$. To this end, we denote by $V = \{v_1, \ldots, v_n\}$ the vertex set of P_n, where $v_i v_{i+1} \in E(P_n)$ for every $i \in \{1, \ldots, n-1\}$. We claim that $\{v_1, v_n\}$ is a simultaneous local metric generator for the subfamily $\mathcal{G}' \subset \mathcal{G}$ composed by all non-bipartite graphs of $\mathcal{G}$. In order to prove this claim, consider an arbitrary graph $G \in \mathcal{G}'$, and let $e = v_p v_q$, $1 \leq p < q \leq n$ be the edge added to $E(P_n)$ to obtain G. We differentiate the following cases:

(1) $e = v_1 v_n$. In this case, G is an odd-order cycle graph, so $\{v_1, v_n\}$ is a local metric generator.

(2) $1 < p < q = n$. In this case, G is a unicyclic graph where v_p has degree three, v_1 has degree one and the remaining vertices have degree two. Consider two adjacent vertices $u, v \in V - \{v_1, v_n\}$. If u or v belong to the path from v_1 to v_p, then v_1 distinguishes them. If both, u and v, belong to the cycle subgraph of G, then $d(u, v_1) = d(u, v_p) + d(v_p, v_1)$ and $d(v, v_1) = d(v, v_p) + d(v_p, v_1)$. Thus, if v_p distinguishes u and v so does v_1, otherwise v_n does.

(3) $1 = p < q < n$. This case is analogous to case 2.

(4) $1 < p < q < n$. In this case, G is a unicyclic graph where v_p and v_q have degree three, v_1 and v_n have degree one and the remaining vertices have degree two. Consider two adjacent vertices $u, v \in V - \{v_1, v_n\}$. If u or v belong to the path from v_1 to v_p (or to the path from v_q to v_n), then v_1 (or v_n) distinguishes them. If both u and v belong to the cycle, then $d(u, v_1) = d(u, v_p) + d(v_p, v_1)$, $d(v, v_1) = d(v, v_p) + d(v_p, v_1)$, $d(u, v_n) = d(u, v_q) + d(v_q, v_n)$ and $d(v, v_n) = d(v, v_q) + d(v_q, v_n)$. Thus, if v_p distinguishes u and v so does v_1, otherwise v_q distinguishes them, which means that v_n also does.

According to the four cases above, we conclude that $\{v_1, v_n\}$ is a local metric generator for G, so it is a simultaneous local metric generator for $\mathcal{G}'$. Thus, by Theorem 7.6, $\mathrm{Sd}_l(\mathcal{G}) = \mathrm{Sd}_l(\mathcal{G}') \leq 2$. □

Remark 7.10. *Let C_n, $n \geq 4$, be a cycle graph and let e be an edge of its complement. If n is odd, then*

$$\dim_l(\mathrm{add}_e(C_n)) = 2.$$

Otherwise,

$$1 \leq \dim_l(\mathrm{add}_e(C_n)) \leq 2.$$

Proof. Consider $e = v_i v_j$. We have that C_n is bipartite for n even. If, additionally, $d_{C_n}(v_i, v_j)$ is odd, then the graph $\mathrm{add}_e(C_n)$ is also bipartite, so $\dim_l(\mathrm{add}_e(C_n)) = 1$. For every other case, $\dim_l(\mathrm{add}_e(C_n)) \geq 2$. From now on we assume that $n \geq 5$, and proceed to show that $\dim_l(\mathrm{add}_e(C_n)) \leq 2$. Note that $\mathrm{add}_e(C_n)$ is a bicyclic graph where v_i and v_j are vertices of degree three and the remaining vertices have degree two. We denote by C_{n_1} and C_{n-n_1+2} the two graphs obtained as induced subgraphs of $\mathrm{add}_e(C_n)$ which are isomorphic to a cycle of order n_1 and a cycle of order $n - n_1 + 2$, respectively. Since $n \geq 5$, we have that $n_1 > 3$ or $n - n_1 + 2 > 3$. We assume, without loss of generality, that $n_1 > 3$. Let $a, b \in V(C_{n_1})$ be two vertices such that:

- if n_1 is even, $ab \in E(C_{n_1})$ and $d(v_i, a) = d(v_j, b)$,
- if n_1 is odd, $ax, xb \in E(C_{n_1})$, where $x \in V(C_{n_1})$ is the only vertex such that $d(x, v_i) = d(x, v_j)$.

We claim that $\{a, b\}$ is a local metric generator for $\mathrm{add}_e(C_n)$. Consider two adjacent vertices $u, v \in V(\mathrm{add}_e(C_n)) - \{a, b\}$. We differentiate the following cases, where the distances are taken in $\mathrm{add}_e(C_n)$:

(1) $u, v \in V(C_{n_1})$. It is simple to verify that $\{a, b\}$ is a local metric generator for C_{n_1}, hence $d(u, a) \neq d(v, a)$ or $d(u, b) \neq d(v, b)$.

(2) $u \in V(C_{n_1})$ and $v \in V(C_{n-n_1+2}) - \{v_i, v_j\}$. In this case, $u \in \{v_i, v_j\}$ and $d(u, a) < d(v, a)$ or $d(u, b) < d(v, b)$.

(3) $u, v \in V(C_{n-n_1+2}) - \{v_i, v_j\}$. In this case, if $d(u, a) = d(v, a)$, then $d(u, v_i) = d(v, v_i)$, so $d(u, v_j) \neq d(v, v_j)$ and, consequently, $d(u, b) \neq d(v, b)$.

According to the three cases above, $\{a, b\}$ is a local metric generator for $\text{add}_e(C_n)$ and, as a result, the proof is complete. □

The next result is a direct consequence of Remarks 7.1 and 7.10.

Remark 7.11. *Let C_n, $n \geq 4$, be a cycle graph. If e, e' are two different edges of the complement of C_n, then*

$$1 \leq \text{Sd}_l(\text{add}_e(C_n), \text{add}_{e'}(C_n)) = \text{Sd}_l(C_n, \text{add}_e(C_n), \text{add}_{e'}(C_n)) \leq 4.$$

7.4 Families of corona product graphs

Several results presented in Chapter 5 describe the behaviour of the local metric dimension on corona product graphs. We now analyse how this behaviour extends to the simultaneous local metric dimension of families composed by corona product graphs.

Given a graph family $\mathcal{G} = \{G_1, \ldots, G_k\}$ on a common vertex set and a graph H, we define the graph family

$$\mathcal{G} \odot H = \{G_1 \odot H, \ldots, G_k \odot H\}.$$

Several results presented in [23] describe the behaviour of the local metric dimension on corona product graphs. We now analyse how this behaviour extends to the simultaneous local metric dimension of families composed by corona product graphs.

Theorem 7.12. [23] *Let G be a connected graph of order $n \geq 2$. For any nonempty graph H,*

$$\dim_l(G \odot H) = n \cdot \text{adim}_l(H).$$

As we can expect, if we review the proof of the result above, we check that if A is a local metric basis of $G \odot H$, then A does not contain elements in $V(G)$. Therefore, any local metric basis of $G \odot H$ is a simultaneous local metric basis of $\mathcal{G} \odot H$. This fact and the result above allow us to state the following theorem.

Theorem 7.13. *Let $\mathcal{G}$ be a family of connected nontrivial graphs on a common vertex set V. For any nonempty graph H,*

$$\mathrm{Sd}_l(\mathcal{G} \odot H) = |V| \, \mathrm{adim}_l(H).$$

Given a graph family $\mathcal{G}$ on a common vertex set and a graph family $\mathcal{H}$ on a common vertex set, we define the graph family

$$\mathcal{G} \odot \mathcal{H} = \{G \odot H : G \in \mathcal{G} \text{ and } H \in \mathcal{H}\}.$$

The following result generalizes Theorem 7.13.

Theorem 7.14. *For any family $\mathcal{G}$ of connected non-trivial graphs on a common vertex set V and any family $\mathcal{H}$ of nonempty graphs on a common vertex set,*

$$\mathrm{Sd}_l(\mathcal{G} \odot \mathcal{H}) = |V| \, \mathrm{Sad}_l(\mathcal{H}).$$

Proof. Let $n = |V|$ and let V' be the vertex set of the graphs in $\mathcal{H}$, V'_i the copy of V' corresponding to $v_i \in V$, $\mathcal{H}_i$ the i^{th}-copy of $\mathcal{H}$ and $H_i \in \mathcal{H}_i$ be i^{th}-copy of $H \in \mathcal{H}$.

We first need to prove that $\mathrm{Sd}_l(G \odot \mathcal{H}) = n \cdot \mathrm{Sad}_l(\mathcal{H})$. For any $i \in \{1, \ldots, n\}$, let S_i be a simultaneous local adjacency basis of $\mathcal{H}_i$. In order to show that $X = \bigcup_{i=1}^{n} S_i$ is a simultaneous local metric generator for $\mathcal{G} \odot \mathcal{H}$, we will show that X is a is a local metric generator for $G \odot H$, for any $G \in \mathcal{G}$ and $H \in \mathcal{H}$. To this end, we differentiate the following four cases for two adjacent vertices $x, y \in V(G \odot H) - X$.

Case 1. $x, y \in V'_i$. Since S_i is an adjacency generator of H_i, there exists a vertex $u \in S_i$ such that $|N_{H_i}(u) \cap \{x, y\}| = 1$. Hence,

$$d_{G \odot H}(x, u) = d_{\langle v_i \rangle + H_i}(x, u) \neq d_{\langle v_i \rangle + H_i}(y, u) = d_{G \odot H}(y, u).$$

Case 2. $x \in V'_i$ and $y \in V$. If $y = v_i$, then for $u \in S_j$, $j \neq i$, we have

$$d_{G \odot H}(x, u) = d_{G \odot H}(x, y) + d_{G \odot H}(y, u) > d_{G \odot H}(y, u).$$

Now, if $y = v_j$, $j \neq i$, then we also take $u \in S_j$ and we proceed as above.
Case 3. $x = v_i$ and $y = v_j$. For $u \in S_j$, we find that

$$d_{G\odot H}(x,u) = d_{G\odot H}(x,y) + d_{G\odot H}(y,u) > d_{G\odot H}(y,u).$$

Case 4. $x \in V_i'$ and $y \in V_j'$, $j \neq i$. In this case, for $u \in S_i$ we have

$$d_{G\odot H}(x,u) \leq 2 < 3 \leq d_{G\odot H}(u,y).$$

Hence, X is a local metric generator for $G \odot H$ and, since $G \in \mathcal{G}$ and $H \in \mathcal{H}$ are arbitrary graphs, X is a simultaneous local metric generator for $\mathcal{G} \odot \mathcal{H}$, which implies that

$$\mathrm{Sd}_l(G \odot \mathcal{H}) \leq \sum_{i=1}^{n} |S_i| = n \cdot \mathrm{Sad}_l(\mathcal{H}).$$

It remains to prove that $\mathrm{Sd}_l(\mathcal{G}\odot\mathcal{H}) \geq n\cdot\mathrm{Sad}_l(\mathcal{H})$. To do this, let W be a simultaneous local metric basis for $\mathcal{G}\odot\mathcal{H}$ and, for any $i \in \{1,\dots,n\}$, let $W_i = V_i' \cap W$. Let us show that W_i is a simultaneous adjacency generator for $\mathcal{H}_i$. To do this, consider two different vertices $x, y \in V_i' - W_i$ which are adjacent in $G \odot H$, for some $H \in \mathcal{H}$. Since no vertex $a \in V(G\odot H) - V_i'$ distinguishes the pair x, y, there exists some $u \in W_i$ such that $d_{G\odot H}(x,u) \neq d_{G\odot H}(y,u)$. Now, since $d_{G\odot H}(x,u) \in \{1,2\}$ and $d_{G\odot H}(y,u) \in \{1,2\}$, we conclude that $|N_{H_i}(u) \cap \{x,y\}| = 1$ and consequently, W_i must be an adjacency generator for H_i and, since $H \in \mathcal{H}$ is arbitrary, W_i is a simultaneous local adjacency generator for $\mathcal{H}_i$. Hence, for any $i \in \{1,\dots,n\}$, $|W_i| \geq \mathrm{Sad}_l(H_i)$. Therefore,

$$\mathrm{Sd}_l(\mathcal{G} \odot \mathcal{H} = |W| \geq \sum_{i=1}^{n} |W_i| \geq \sum_{i=1}^{n} \mathrm{Sad}_l(\mathcal{H}_i) = n \cdot \mathrm{Sad}_l(\mathcal{H}).$$

This completes the proof. □

The following result is a direct consequence of Theorem 7.14.

Corollary 7.15. *For any family $\mathcal{G}$ of connected graphs on a common vertex set V, $|V| \geq 2$, and any family $\mathcal{H}$ of nonempty graphs on a common vertex set,*

$$\mathrm{Sd}_l(G \odot \mathcal{H}) \geq |V|\,\mathrm{Sd}_l(\mathcal{H}).$$

Furthermore, if every graph in $\mathcal{H}$ has diameter two, then

$$\mathrm{Sd}_l(G \odot \mathcal{H}) = |V|\,\mathrm{Sd}_l(\mathcal{H}).$$

Now, we give another result, which is a direct consequence of Theorem 7.14 and shows the general bounds of $\mathrm{Sd}_l(\mathcal{G} \odot \mathcal{H})$.

Corollary 7.16. *For any family $\mathcal{G}$ of connected non-trivial graphs on a common vertex set V and any family $\mathcal{H}$ of nonempty graphs on a common vertex set V',*

$$|V| \leq \mathrm{Sd}_l(G \odot \mathcal{H}) \leq |V|(|V'| - 1).$$

We now consider the case in which the graph H is empty.

Theorem 7.17. *Let $\mathcal{G}$ be a family of connected nontrivial graphs on a common vertex set. For any empty graph H,*

$$\mathrm{Sd}_l(\mathcal{G} \odot H) = \mathrm{Sd}_l(\mathcal{G}).$$

Proof. Let B be a simultaneous local metric basis of $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$. Since H is empty, any local metric generator $B' \subseteq B$ of G_i is a local metric generator for $G_i \odot H$, so B is a simultaneous local metric generator for $\mathcal{G} \odot H$. In consequence, $\mathrm{Sd}_l(\mathcal{G} \odot H) \leq \mathrm{Sd}_l(\mathcal{G})$.

Suppose that A is a simultaneous local metric basis for $\mathcal{G} \odot H$ and $|A| < |B|$. If there exists $x \in A \cap V_{ij}$ for the j-th copy of H in any graph $G_i \odot H$, then the pairs of vertices of $G_i \odot H$ which are distinguished by x can also be distinguished by v_i. In consequence, the set A' obtained from A by replacing by v_i each vertex $x \in A \cap V_{ij}, i \in \{1, \ldots, k\}, j \in \{1, \ldots, n\}$ is a simultaneous local metric generator for $\mathcal{G}$ such that $|A'| \leq |A| < \mathrm{Sd}_l(\mathcal{G})$, which is a contradiction, so $\mathrm{Sd}_l(\mathcal{G} \odot H) \geq \mathrm{Sd}_l(\mathcal{G})$. □

As for the previous case, Theorem 5.3 is extensible to the simultaneous setting.

Theorem 7.18. *Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set V and let $\mathcal{H}$ be a family of non-empty graphs on a common vertex set. The following assertions hold.*

(i) *If the vertex of K_1 does not belong to any simultaneous local metric basis of $K_1 + \mathcal{H}$, then*

$$\mathrm{Sd}_l(\mathcal{G} \odot \mathcal{H}) = n \cdot \mathrm{Sd}_l(K_1 + \mathcal{H}).$$

(ii) *If the vertex of K_1 belongs to a simultaneous local metric basis of $K_1 + \mathcal{H}$, then*

$$\mathrm{Sd}_l(\mathcal{G} \odot \mathcal{H}) = n \cdot (\mathrm{Sd}_l(K_1 + \mathcal{H}) - 1).$$

Proof. As above, let $n = |V|$ and let V' be the vertex set of the graphs in $\mathcal{H}$, V'_i the copy of V' corresponding to $v_i \in V$, $\mathcal{H}_i$ the i^{th}-copy of $\mathcal{H}$ and $H_i \in \mathcal{H}_i$ be i^{th}-copy of $H \in \mathcal{H}$.

We will apply a reasoning analogous to the one used for the proof of Theorem 5.3. If $n = 1$, then $\mathcal{G} \odot \mathcal{H} \cong K_1 + \mathcal{H}$, so the result holds. Assume that $n \geq 2$, Let S_i be a simultaneous local metric basis for $\langle v_i \rangle + \mathcal{H}_i$ and let $S'_i = S_i - \{v_i\}$. Note that $S'_i \neq \emptyset$ because $\mathcal{H}_i$ is family of nonempty graphs and v_i does not distinguish any pair of adjacent vertices belonging to V'_i. In order to show that $X = \cup_{i=1}^{n} S'_i$ is a simultaneous local metric generator for $\mathcal{G} \odot \mathcal{H}$ we differentiate the following cases for two vertices x, y which are adjacent in an arbitrary graph $G \odot H$.

Case 1. $x, y \in V'_i$. Since v_i does not distinguish x, y, there exists $u \in S'_i$ such that $d_{G \odot H}(x, u) = d_{\langle v_i \rangle + H_i}(x, u) \neq d_{\langle v_i \rangle + H_i}(y, u) = d_{G \odot H}(y, u)$.

Case 2. $x \in V'_i$ and $y = v_i$. For $u \in S'_j$, $j \neq i$, we have $d_{G \odot H}(x, u) = 1 + d_{G \odot H}(y, u) > d_{G \odot H}(y, u)$.

Case 3. $x = v_i$ and $y = v_j$. For $u \in S'_j$, we have $d_{G \odot H}(x, u) = 2 = d_{G \odot H}(x, y) + 1 > 1 = d_{G \odot H}(y, u)$.

Hence, X is a local metric generator for $G \odot H$ and, since $G \in \mathcal{G}$ and $H \in \mathcal{H}$ are arbitrary graphs, X is a simultaneous local metric generator for $\mathcal{G} \odot \mathcal{H}$.

Now we shall prove (i). If the vertex of K_1 does not belong to any simultaneous local metric basis for $K_1 + \mathcal{H}$, then $v_i \notin S_i$ for every $i \in \{1, ..., n\}$ and, as a consequence,

$$\mathrm{Sd}_l(\mathcal{G} \odot \mathcal{H}) \leq |X| = \sum_{i=1}^{n} |S'_i| = \sum_{i=1}^{n} \mathrm{Sd}_l(\langle v_i \rangle + \mathcal{H}_i) = n \cdot \mathrm{Sd}_l(K_1 + \mathcal{H}).$$

Now we need to prove that $\mathrm{Sd}_l(\mathcal{G} \odot \mathcal{H}) \geq n \cdot \mathrm{Sd}_l(K_1 + \mathcal{H})$. In order to do this, let W be a simultaneous local metric basis for $\mathcal{G} \odot \mathcal{H}$ and let $W_i = V'_i \cap W$. Consider two adjacent vertices $x, y \in V'_i - W_i$ in $G \odot H$. Since no vertex $a \in W - W_i$ distinguishes the pair x, y, there exists $u \in W_i$ such that $d_{\langle v_i \rangle + H_i}(x, u) = d_{G \odot H}(x, u) \neq d_{G \odot H}(y, u) = d_{\langle v_i \rangle + H_i}(y, u)$. So we conclude that $W_i \cup \{v_i\}$ is a simultaneous local metric generator for $\langle v_i \rangle + \mathcal{H}_i$. Now, since v_i does not belong to any simultaneous local metric basis for $\langle v_i \rangle + \mathcal{H}_i$, we have that $|W_i| + 1 = |W_i \cup \{v_i\}| > \mathrm{Sd}_l(\langle v_i \rangle + \mathcal{H}_i)$ and, as a consequence,

$|W_i| \geq \mathrm{Sd}_l(\langle v_i\rangle + \mathcal{H}_i)$. Therefore,

$$\mathrm{Sd}_l(\mathcal{G} \odot \mathcal{H}) = |W| \geq \sum_{i=1}^{n} |W_i| \geq \sum_{i=1}^{n} \mathrm{Sd}_l(\langle v_i\rangle + \mathcal{H}_i) = n \cdot \mathrm{Sd}_l(K_1 + \mathcal{H}),$$

and the proof of (i) is complete.

Finally, we shall prove (ii). If the vertex of K_1 belongs to a simultaneous local metric basis for $K_1 + \mathcal{H}$, then we assume that $v_i \in S_i$ for every $i \in \{1, ..., n\}$. Suppose that there exists B such that B is a simultaneous local metric basis for $\mathcal{G} \odot \mathcal{H}$ and $|B| < |X|$. In such a case, there exists $i \in \{1, ..., n\}$ such that the set $B_i = B \cap V_i'$ satisfies $|B_i| < |S_i'|$. Now, since no vertex of $B - B_i$ distinguishes the pairs of adjacent vertices belonging to V_i', the set $B_i \cup \{v_i\}$ must be a simultaneous local metric generator for $\langle v_i\rangle + \mathcal{H}_i$. So, $\mathrm{Sd}_l(\langle v_i\rangle + \mathcal{H}_i) \leq |B_i| + 1 < |S_i'| + 1 = |S_i| = \mathrm{Sd}_l(\langle v_i\rangle + \mathcal{H}_i)$, which is a contradiction. Hence, X is a simultaneous local metric basis for $\mathcal{G} \odot \mathcal{H}$ and, as a consequence,

$$\mathrm{Sd}_l(\mathcal{G} \odot \mathcal{H}) = |X| = \sum_{i=1}^{n} |S_i'| = \sum_{i=1}^{n} (\mathrm{Sd}_l(\langle v_i\rangle + \mathcal{H}_i) - 1) = n(\mathrm{Sd}_l(K_1 + \mathcal{H}) - 1).$$

The proof of (ii) is now complete.

□

Corollary 7.19. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{K_{r_1,n'-r_1}, K_{r_2,n'-r_2}, \ldots, K_{r_k,n'-r_k}\}$, $1 \leq r_i \leq n' - 1$, be a family composed by complete bipartite graphs on a common vertex set V'. Then,*

$$\mathrm{Sd}_l(G \odot \mathcal{H}) = n.$$

Proof. For every $x \in V'$, the set $\{v, x\}$ is a simultaneous local metric basis of $\langle v\rangle + \mathcal{H}$, so $\mathrm{Sd}(G \odot \mathcal{H}) = n \cdot (\mathrm{Sd}(K_1 + \mathcal{H}) - 1) = n$. □

Lemma 7.20. *Let $\mathcal{H}$ be a graph family on a common vertex set V such that $r(H) \geq 4$ for every $H \in \mathcal{H}$. Then the vertex of K_1 does not belong to any simultaneous local metric basis of $K_1 + \mathcal{H}$.*

Proof. Let B be a simultaneous local metric basis of $\{K_1 + H_1, \ldots, K_1 + H_k\}$. We suppose that the vertex v of K_1 belongs to B. Note that $v \in B$ if and only if there exists $u \in V - B$ such that $B \subseteq N_{K_1+H_i}(u)$ for some $H_i \in \mathcal{H}$. If $r(H_i) \geq 4$, proceeding in a manner analogous to that of the proof of Lemma 5.7, we take $u' \in V$ such that $d_{H_i}(u, u') = 4$ and a shortest path $uu_1u_2u_3u'$. In such a case, for every $b \in B - \{v\}$, we will have that $d_{K_1+H_i}(b, u_3) = d_{K_1+H_i}(b, u') = 2$, which is a contradiction. Hence, v does not belong to any simultaneous local metric basis of $K_1 + \mathcal{H}$. □

As a direct consequence of item (i) of Theorem 7.18 and Lemma 7.20, we obtain the following result.

Proposition 7.21. *For any family $\mathcal{G}$ of connected graphs on a commong vertex set V and any graph family $\mathcal{H}$ on a common vertex set V' such that $r(H) \geq 4$ for every $H \in \mathcal{H}$,*

$$\mathrm{Sd}_l(\mathcal{G} \odot \mathcal{H}) = |V| \cdot \mathrm{Sd}_l(K_1 + \mathcal{H}).$$

7.5 Families of lexicographic product graphs

Let $\mathcal{G} = \{G_1, \ldots, G_r\}$ be a family of connected graphs with common vertex set $V = \{u_1, \ldots, u_n\}$. For each $u_i \in V$ let $\mathcal{H}^i = \{H_{i1}, \ldots H_{is_i}\}$ be a family of graphs with common vertex set V_i. For each $i = 1, \ldots, n$ choose $H_{ij} \in \mathcal{H}^i$ and consider the family $\mathcal{H}_j = \{H_{1j}, H_{2j}, \ldots, H_{nj}\}$. Notice that the families $\mathcal{H}^i$ can be represented in the following scheme where the columns correspond to the families $\mathcal{H}_j$.

$$\begin{array}{llllllll}
\mathcal{H}^1 = & \{H_{11}, & \ldots & H_{1j}, & \ldots & H_{1s_1}\} & \text{defined on } V_1 \\
\vdots & \vdots & & \vdots & & \vdots & \\
\mathcal{H}^i = & \{H_{i1}, & \ldots & H_{ij}, & \ldots & H_{is_i}\} & \text{defined on } V_i \\
\vdots & \vdots & & \vdots & & \vdots & \\
\mathcal{H}^n = & \{H_{n1}, & \ldots & H_{nj}, & \ldots & H_{ns_n}\} & \text{defined on } V_n
\end{array}$$

For a graph $G_k \in \mathcal{G}$ and the family $\mathcal{H}_j$ we define the *lexicographic product* of G_k and $\mathcal{H}_j$ as the graph $G_k \circ \mathcal{H}_j$ such that $V(G_k \circ \mathcal{H}_j) = \bigcup_{u_i \in V}(\{u_i\} \times V_i)$ and $(u_{i_1}, v)(u_{i_2}, w) \in E(G_k \circ \mathcal{H}_j)$ if and only if $u_{i_1}u_{i_2} \in E(G_k)$ or $i_1 = i_2$ and $vw \in E(H_{i_1j})$. Let $\mathcal{H} = \{\mathcal{H}_1, \mathcal{H}_2, \ldots \mathcal{H}_s\}$. We are interested in the simultaneous local metric dimension of the family:

$$\mathcal{G} \circ \mathcal{H} = \{G_k \circ \mathcal{H}_j : G_k \in \mathcal{G}, \mathcal{H}_j \in \mathcal{H}\}.$$

The relation between distances in a lexicographic product graph and those in its factors is presented in the following remark.

Remark 7.22. *If (u,v) and (u',v') are vertices of $G \circ H$, then*

$$d_{G\circ\mathcal{H}}((u,v),(u',v')) = \begin{cases} d_G(u,u'), & \text{if } u \neq u', \\ \\ \min\{d_H(v,v'),2\}, & \text{if } u = u'. \end{cases}$$

We would point out that the remark above was stated in [29, 32] for the case where $H_{ij} \cong H$ for all $H_{ij} \in \mathcal{H}_j$.

By Remark 7.22 we deduce that if $u \in V - \{u_i\}$, then two adjacent vertices $(u_i,w),(u_i,y)$ are not distinguished by $(u,v) \in V(\mathcal{G}\circ\mathcal{H})$. Therefore, we can state the following remark.

Remark 7.23. *If B is a simultaneous local metric generator for the family of lexicographic product graphs $\mathcal{G}\circ\mathcal{H}$, then $B_i = \{v : (u_i,v) \in B\}$ is a simultaneous local adjacency generator for $\mathcal{H}^i$.*

In order to state our main result (Theorem 7.35) we need to introduce some additional notation. Let B be a simultaneous local adjacency generator for a family of nontrivial connected graphs $\mathcal{H}^i = \{H_{i1},\ldots,H_{is}\}$ on a common vertex set V_i and let $\mathcal{G}\circ\mathcal{H}$ be family of lexicographic product graphs defined as above.

- $D[\mathcal{H}^i,B] = \{v \in V_i : B \subseteq N_{H_{ij}}(v)$ for some $H_{ij} \in \mathcal{H}^i\}$.
- If $D[\mathcal{H}^i,B] \neq \emptyset$, then we define the graph $\mathcal{D}[\mathcal{H}^i,B]$ in the following way. The vertex set of $\mathcal{D}[\mathcal{H}^i,B]$ is $D[\mathcal{H}^i,B]$ and two vertices v,w are adjacent in $\mathcal{D}[\mathcal{H}^i,B]$ if and only if for for every $H_{ij} \in \mathcal{H}^i$, $vw \notin E(H_{ij})$.
- If $D[\mathcal{H}^i,B] = \emptyset$, then define $\Psi(B) = |B|$, otherwise $\Psi(B) = \gamma(\mathcal{D}[\mathcal{H}^i,B]) + |B|$.
- $\Gamma(\mathcal{H}^i) = \{C \subseteq V_i : C$ is a simultaneous local adjacency generator for $\mathcal{H}^i\}$
- $\Psi(\mathcal{H}^i) = \min\{\Psi(B) : B \in \Gamma(\mathcal{H}^i)\}$.
- $\mathcal{S}_0$ is the family of empty graphs.
- $\Phi(V,\mathcal{H}) = \{u_i \in V : \mathcal{H}^i \subset \mathcal{S}_0\}$
- $I(V,\mathcal{H}) = \{u_i \in V : \Psi(\mathcal{H}^i) > \mathrm{Sad}_l(\mathcal{H}^i)\}$. Notice that $\Phi(V,\mathcal{H}) \subseteq I(V,\mathcal{H})$.

- $\Upsilon(V,\mathcal{H})$ is the family of subsets of $I(V,\mathcal{H})$ as follows. We say that $A \in \Upsilon(V,\mathcal{H})$ if for every $u', u'' \in I(V,\mathcal{H}) - A$ such that $u'u'' \in E(G_k)$, for some $G_k \in \mathcal{G}$, there exists $u \in (A \cup (V - \Phi(V,\mathcal{H}))) - \{u', u''\}$ such that $d_{G_k}(u,u') \neq d_{G_k}(u,u'')$.

- $\mathbf{G}(\mathcal{G}, I(V,\mathcal{H}))$ is the graph with vertex set $I(V,\mathcal{H})$ and two vertices u_i, u_j are adjacent in $\mathbf{G}(\mathcal{G}, I(V,\mathcal{H}))$ if and only if there exists $G_k \in \mathcal{G}$ such that $u_i u_j \in E(G_k)$.

Remark 7.24. *$\Psi(\mathcal{H}^i) = 1$ if and only if $H_{i,j} \cong N_{|V_i|}$ for every $H_{i,j} \in \mathcal{H}^i$.*

Proof. If $H_{i,j} \cong N_{|V_i|}$ for every $H_{i,j} \in \mathcal{H}^i$, then $B = \emptyset$ is the only simultaneous local adjacency basis for $\mathcal{H}^i$, $\mathcal{D}[\mathcal{H}^i, \emptyset] \cong K_{|V_i|}$ and then $\Psi(\mathcal{H}^i) = \gamma(K_{|V_i|}) = 1$. On the other side, suppose that $H_{i,j} \ncong N_{|V_i|}$ for some $H_{i,j} \in \mathcal{H}^i$. In this case, $\mathrm{Sad}_l(\mathcal{H}^i) \geq 1$. If $\mathrm{Sad}_l(\mathcal{H}^i) > 1$, then we are done. Suppose that $\mathrm{Sad}_l(\mathcal{H}^i) = 1$. For any simultaneous local adjacency basis $B = \{v_1\}$ of $\mathcal{H}^i$ there exists $v_2 \in N_{H_{ij}}(v_1)$ for some H_{ij}, which implies that $D[\mathcal{H}^i, \{v_2\}] \neq \emptyset$ and so $|\gamma(\mathcal{D}[\mathcal{H}^i, \{v_2\}])| \geq 1$. Therefore, $\Psi(\mathcal{H}^i) \geq 2$ and the result follows. □

As we will show in the next example, in order to get the value of $\Psi(\mathcal{H}^i)$, it is interesting to remark the necessity of consider the family $\Gamma(\mathcal{H}^i)$ of all simultaneous local adjacency generators and not just the family of simultaneous local adjacency bases of $\mathcal{H}^i$.

Example 7.25. *Let $H_1 \cong H_2 \cong P_5$ be two copies of the path on five vertices. $V(H_1) = V(H_2) = \{v_1, v_2, \ldots, v_5\}$ but with different edge sets $E(H_1) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5\}$ and $E(H_2) = \{v_2v_1, v_1v_3, v_3v_5, v_5v_4\}$. Consider the family $\mathcal{H} = \{H_1, H_2\}$. $B_1 = \{v_3\}$ is a simultaneous local adjacency basis for $\mathcal{H}$ and $B_2 = \{v_1, v_4\}$ is a simultaneous local adjacency generator for $\mathcal{H}$. Then $D[\mathcal{H}, B_1] = \{v_1, v_2, v_4, v_5\}$, $E(\mathcal{D}[\mathcal{H}, B_1]) = \{v_1v_4, v_4v_2, v_2v_5, v_5v_1\}$, $\gamma(\mathcal{D}[\mathcal{G}, B_1]) = 2$, $\Psi(B_1) = 2+1 = 3$. However, $D[\mathcal{H}, B_2] = \emptyset$ and $\Psi(B_2) = 2$.*

We define the following families of graphs.

- $\mathcal{S}_1$ is the family of graphs having at least two non trivial components.

- $\mathcal{S}_2$ is the family of graphs having at least one component of radius at least four.

- $\mathcal{S}_3$ is the family of graphs having at least one component of girth at least seven.

- $\mathcal{S}_4$ is the family of graphs having at least two non singleton true twin classes U_1, U_2 such that $d(U_1, U_2) \geq 3$.

Lemma 7.26. *Let $\mathcal{H} \not\subset \mathcal{S}_0$ be a family of graphs on a common vertex set V. If $\mathcal{H} \subset \bigcup_{i=0}^{4} \mathcal{S}_i$, then*

$$\Psi(\mathcal{H}) = \mathrm{Sad}_l(\mathcal{H}).$$

Proof. Let B be a simultaneous local adjacency generator for $\mathcal{H}$ and $v \in V$. We claim that, $B \not\subseteq N_H(v)$. To see this, we differentiate the following cases for $H \in \mathcal{H}$.

- H has two non trivial connected components J_1, J_2. In this case $B \cap J_1 \neq \emptyset$ and $B \cap J_2 \neq \emptyset$, which implies that $B \not\subseteq N_H(v)$.

- H has one non trivial component J such that $r(J) \geq 4$. If H has two non trivial components, then we are in the first case. So, we can assume that J is the only non trivial component of H. Suppose that $B \subseteq N_H(v)$ and get $v' \in V$ such that $d_H(v, v') = 4$. If $vv_1v_2v_3v'$ is a shortest path from v to v', then v_3 and v' are adjacent and they are not distinguished by the elements in B, which is a contradiction.

- H has one non trivial component J of girth $\mathtt{g}(J) \geq 7$. In this case, if H has two non trivial components, then we are in the first case. So we can assume that H has just one nontrivial component of girth $\mathtt{g}(J) \geq 7$. Suppose that $B \subseteq N_H(v)$. For each cycle $v_1v_2 \ldots v_nv_1$ there exists $v_iv_{i+1} \in E(J)$ such that $d_H(v, v_i) \geq 3$ and $d_H(v, v_{i+1}) \geq 3$, therefore for each $b \in B$ we have $d_H(b, v_i) \geq 2$ and $d_H(b, v_{i+1}) \geq 2$, which is a contradiction.

- H has two non singleton true twin classes U_1, U_2 such that $d_H(U_1, U_2) \geq 3$. Since $B \cap U_1 \neq \emptyset$ and $B \cap U_2 \neq \emptyset$, we can conclude that $B \not\subseteq N_H(v)$.

- $H \cong N_{|V|}$. Notice that $B \neq \emptyset$, as $\mathcal{H} \not\subseteq \mathcal{S}_0$, so that $B \not\subseteq \emptyset = N_H(v)$.

According to the five cases above, $\mathcal{H} \subset \cup_{i=0}^{4} \mathcal{S}_i$ leads to $D[\mathcal{H}, B] = \emptyset$, for any simultaneous local adjacency generator, which implies that $\Psi(\mathcal{H}) = \mathrm{Sad}_l(\mathcal{H})$. □

Remark 7.27. *If $A \in \Upsilon(V, \mathcal{H})$ then $A \cup (V - \Phi(V, \mathcal{H}))$ is a simultaneous local metric generator for $\mathcal{G}$. However, the converse is not true, as we can see in the following example.*

Example 7.28. *Consider the family of connected graphs* $\mathcal{G} = \{G_1, G_2, G_3\}$ *on a common vertex set* $V = \{u_1, \ldots, u_8\}$ *with* $E(G_i) = \{u_1u_2, u_1u_{2i+1}, u_2u_{2i+2}, u_ju_{2i+1}, u_ju_{2i+2}$, *for* $j \notin \{1, 2, 2i+1, 2i+2\}\}$. *Let* $\mathcal{H}^i$ *be the family consisting in only one graph* H_i, *as follow:* $H_1 \cong H_2 \cong K_2$, $H_3 \cong H_4 \cong \cdots \cong H_8 \cong N_2$. $\mathcal{G} \circ \mathcal{H} = \{G_i \circ \{H_1, \ldots, H_8\}, i = 1, 2, 3\}$. $I(V, \mathcal{H}) = V$. *If we take* $A = \emptyset$, *then* $A \cup (V - \Phi(V, \mathcal{H})) = \{u_1, u_2\} \subseteq I(V, \mathcal{H})$ *is a simultaneous local metric basis for* $\mathcal{G}$. *However,* $\emptyset \notin \Upsilon(V, \mathcal{H})$ *because* u_1 *is adjacent to* u_2 *in* G_i, $i \in \{1, 2, 3\}$, *and* $(V - \Phi(V, \mathcal{H})) - \{u_1, u_2\} = \emptyset$.

Lemma 7.29. *Let* $\mathcal{G} \circ \mathcal{H}$ *be a family of lexicographic product graphs. Let* $B \subseteq V$ *be a simultaneous local metric generator for* $\mathcal{G}$. *Then* $B \cap I(V, \mathcal{H}) \in \Upsilon(V, \mathcal{H})$.

Proof. Let $A = B \cap I(V, \mathcal{H})$ and $u_i, u_j \in I(V, \mathcal{H}) - A = I(V, \mathcal{H}) - B$. Since $B \subseteq V$ is a simultaneous local metric generator for $\mathcal{G}$, for each $G_k \in \mathcal{G}$ there exists $b \in B$ such that $d_{G_k}(b, u_i) \neq d_{G_k}(b, u_j)$. If $b \notin I(V, \mathcal{H})$ then necessarily $b \in (V - I(V, \mathcal{H})) \subseteq ((V - \Phi(V, \mathcal{H})) - \{u_i, u_j\})$ and if $b \in I(V, \mathcal{H})$ then $b \in A - \{u_i, u_j\}$ and we are done. □

Corollary 7.30. *If there exists a simultaneous local metric generator* B *for* $\mathcal{G}$ *such that* $B \subseteq V - I(V, \mathcal{H})$ *or the graph* $\boldsymbol{G}(\mathcal{G}, I(V, \mathcal{H}))$ *is empty, then* $\emptyset \in \Upsilon(V, \mathcal{H})$.

Remark 7.31. *If* B *is a vertex cover for* $\boldsymbol{G}(\mathcal{G}, I(V, \mathcal{H}))$, *then* $B \in \Upsilon(V, \mathcal{H})$.

Lemma 7.32. *Let* $\mathcal{G} \circ \mathcal{H}$ *be a family of lexicographic product graphs. For each* $u_i \in V$ *let* $B_i \subseteq V_i$ *be a simultaneous local adjacency generator for* $\mathcal{H}^i$ *and let* $C_i \subseteq V_i$ *be a dominating set for* $\mathcal{D}[\mathcal{H}^i, B_i]$. *Then, for any* $A \in \Upsilon(V, \mathcal{H})$, *the set* $B = (\cup_{u_i \in A}\{u_i\} \times (B_i \cup C_i)) \bigcup (\cup_{u_i \notin A}\{u_i\} \times B_i)$ *is a local metric generator for* $\mathcal{G} \circ \mathcal{H}$.

Proof. In order to prove the lemma let $G_k \in \mathcal{G}$, $\mathcal{H}_j \in \mathcal{H}$ and let $(u_{i_1}, v_1), (u_{i_2}, v_2)$ be a pair of adjacent vertices of $G_k \circ \mathcal{H}_j$. If $i_1 = i_2$, then there exists $v \in B_{i_1}$ such that (u_{i_1}, v) distinguishes the pair. Otherwise $i_1 \neq i_2$ and we consider the following cases:

- Case 1: $|\{u_{i_1}, u_{i_2}\} \cap I(V, \mathcal{H})| \leq 1$, say $u_{i_1} \notin I(V, \mathcal{H})$. In this case there exists $v \in B_{i_1}$ such that $vv_1 \notin E(H_{i_1j})$ and then (u_{i_1}, v) distinguishes the pair.

- Case 2: $u_{i_1}, u_{i_2} \in I(V,\mathcal{H})$ and $\{u_{i_1}, u_{i_2}\} \cap A = \emptyset$. In this case, by definition of A, there exists $u_{i_3} \in (A \cup (V - \Phi(V,\mathcal{H}))) - \{u_{i_1}, u_{i_2}\}$ such that $d_{G_k}(u_{i_3}, u_{i_1}) \neq d_{G_k}(u_{i_3}, u_{i_2})$. For any $v \in B_{i_3} \cup C_{i_3}$,

$$d_{G_k \circ \mathcal{H}_j}((u_{i_3}, v), (u_{i_1}, v_1)) = d_{G_k}(u_{i_3}, u_{i_1}) \neq$$

$$d_{G_k}(u_{i_3}, u_{i_2}) = d_{G_k \circ \mathcal{H}_j}((u_{i_3}, v), (u_{i_2}, v_2)).$$

- Case 3: $u_{i_1}, u_{i_2} \in I(V,\mathcal{H})$ and $|\{u_{i_1}, u_{i_2}\} \cap A| \geq 1$, say $u_{i_1} \in A$. In this case, if there exists $v \in B_{i_1}$ such that $vv_1 \notin E(H_{i_1 j})$ then (u_{i_1}, v) distinguishes the pair. Otherwise v_1 is a vertex of $\mathcal{D}[\mathcal{H}^{i_1}, B_{i_1}]$ and either $v_1 \in C_{i_1}$ and $(u_{i_1}, v_1) \in B$ distinguishes the pair or there exists $v \in C_{i_1}$ such that $vv_1 \in E(\mathcal{D}[\mathcal{H}^{i_1}, B_{i_1}])$, that means $vv_1 \notin E(H_{i_1 j})$ and then (u_{i_1}, v) distinguishes the pair.

□

Corollary 7.33. *Let $\mathcal{G} \circ \mathcal{H}$ be a family of lexicographic product graphs. Then*

$$\mathrm{Sd}_l(\mathcal{G} \circ \mathcal{H}) \leq \min_{A \in \Upsilon(V,\mathcal{H})} \left\{ \sum_{u_i \in A} \Psi(\mathcal{H}^i) + \sum_{u_i \notin A} \mathrm{Sad}_l(\mathcal{H}^i) \right\}.$$

Proof. Let $A \in \Upsilon(V,\mathcal{H})$. For each $u_i \notin A$, let $B_i \subseteq V_i$ be a simultaneous local adjacency basis for $\mathcal{H}^i$. For each $u_i \in A$, let B_i be a local adjacency generator for $\mathcal{H}^i$ and $C_i \subseteq V_i$ a dominating set for $\mathcal{D}(\mathcal{H}^i, B_i)$ such that $|B_i \cup C_i| = \Psi(\mathcal{H}^i)$. Let

$$B = (\cup_{u_j \in A} \{u_j\} \times (B_j \cup C_j)) \bigcup (\cup_{u_i \notin A} \{u_i\} \times B_i)$$

then, by Lemma 7.32, B is a simultaneous local metric generator for $\mathcal{G} \circ \mathcal{H}$ and

$$\mathrm{Sd}_l(\mathcal{G} \circ \mathcal{H}) \leq |B| = \sum_{u_i \in A} \Psi(\mathcal{H}^i) + \sum_{u_i \notin A} \mathrm{Sad}_l(\mathcal{H}^i)$$

As $A \in \Upsilon(V,\mathcal{H})$ is arbitrary

$$\mathrm{Sd}_l(\mathcal{G} \circ \mathcal{H}) \leq \min_{A \in \Upsilon(V,\mathcal{H})} \left\{ \sum_{u_i \in A} \Psi(\mathcal{H}^i) + \sum_{u_i \notin A} \mathrm{Sad}_l(\mathcal{H}^i) \right\}$$

and the result follows. □

Lemma 7.34. *Let F be a simultaneous local metric basis of $\mathcal{G} \circ \mathcal{H}$. Let $F_i = \{v \in V_i : (u_i, v) \in F\}$ and $X_F = \{u_i \in I(V, \mathcal{H}) : |F_i| \geq \Psi(\mathcal{H}^i)\}$. Then $X_F \in \Upsilon(V, \mathcal{H})$.*

Proof. Suppose for contradiction, that $X_F \notin \Upsilon(V, \mathcal{H})$, that means that there exists $u_{i_1}, u_{i_2} \in I(V, \mathcal{H}) - X_F$ and $G_k \in \mathcal{G}$ such that $u_{i_1}u_{i_2} \in E(G_k)$, and $d_{G_k}(u, u_{i_1}) = d_{G_k}(u, u_{i_2})$ for every $u \in (X_F \cup (V - \Phi(V, \mathcal{H}))) - \{u_{i_1}, u_{i_2}\}$. As $u_{i_1}, u_{i_2} \in I(V, \mathcal{H}) - X_F$, $|F_{i_1}| < \Psi(\mathcal{H}^{i_1})$ and $|F_{i_2}| < \Psi(\mathcal{H}^{i_2})$, so that there exist $H_{i_1j_1} \in \mathcal{H}^{i_1}$ and $H_{i_2j_2} \in \mathcal{H}^{i_2}$ such that for some $v_1 \in V_{i_1}$, $v_2 \in V_{i_2}$, $F_{i_1} \subseteq N_{H_{i_1j_1}}(v_1)$ and $F_{i_2} \subseteq N_{H_{i_2j_2}}(v_2)$. Let $\mathcal{H}_j$ be such that $H_{i_1j_1}, H_{i_2j_2} \in \mathcal{H}_j$. Consider the pair of vertices $(u_{i_1}, v_1), (u_{i_2}, v_2)$ adjacent in $G_k \circ \mathcal{H}_j$. As F is a simultaneous local metric generator there exists $(u_{i_3}, v) \in F$ that resolves the pair, which implies that $F_{i_3} \neq \emptyset$. By hypothesis $u_{i_3} \in (\Phi(V, \mathcal{H}) - X_F) \cup \{u_{i_1}, u_{i_2}\}$, and so $u_{i_3} \in \{u_{i_1}, u_{i_2}\}$. Without loss of generality, we assume that $u_{i_3} = u_{i_1}$ and, in this case,

$$\begin{aligned} d_{G_k \circ \mathcal{H}_j}((u_{i_3}, v), (u_{i_1}, v_1)) &= d_{H_{i_1j_1},2}(v, v_1) \\ &= d_{G_k}(u_{i_3}, u_{i_2}) \\ &= d_{G_k \circ \mathcal{H}_j}((u_{i_3}, v), (u_{i_2}, v_2)), \end{aligned}$$

which is a contradiction. Therefore, $X_F \in \Upsilon(V, \mathcal{H})$. □

Theorem 7.35. *Let $\mathcal{G} \circ \mathcal{H}$ be a family of lexicographic product graphs.*

$$\mathrm{Sd}_l(\mathcal{G} \circ \mathcal{H}) = \min_{A \in \Upsilon(V, \mathcal{H})} \left\{ \sum_{u_i \in A} \Psi(\mathcal{H}^i) + \sum_{u_i \notin A} \mathrm{Sad}_l(\mathcal{H}^i) \right\}$$

Proof. Let B be a simultaneous local metric basis for $\mathcal{G} \circ \mathcal{H}$. Let $B_i = \{v \in V_i : (u_i, v) \in B\}$ and $X_B = \{u_i \in I(V, \mathcal{H}) : |B_i| \geq \Psi(\mathcal{H}^i)\}$. By Remark 7.23, $|B_i| \geq \mathrm{Sad}_l(\mathcal{H}^i)$ for every $u_i \in V$, so that Lemma 7.34 leads to

$$\min_{A \in \Upsilon(V, \mathcal{H})} \left\{ \sum_{u_i \in A} \Psi(\mathcal{H}^i) + \sum_{u_i \notin A} \mathrm{Sad}_l(\mathcal{H}^i) \right\} \leq \sum_{u_i \in X_B} \Psi(\mathcal{H}^i) + \sum_{u_i \notin X_B} \mathrm{Sad}_l(\mathcal{H}^i) \leq |B|$$

and the result follows by Corollary 7.33. □

Now we will show some cases where the calculation of $\mathrm{Sd}_l(\mathcal{G} \circ \mathcal{H})$ is easy. At first glance we have two main types of simplification: first to simplify the calculation of $\Psi(\mathcal{H}^i)$ and second the calculation of the $A \in \Upsilon(V, \mathcal{H})$ that makes that the sum achieves its minimum.

For the first type of simplification we have can apply Lemma 7.26 to deduce the following corollary.

Corollary 7.36. *If for each i, $\mathcal{H}^i \not\subset \mathcal{S}_0$ and $\mathcal{H}^i \subset \bigcup_{j=0}^{4} \mathcal{S}_j$, then*

$$\mathrm{Sd}_l(\mathcal{G} \circ \mathcal{H}) = \sum \mathrm{Sad}_l(\mathcal{H}^i)$$

.

Given a family $\mathcal{G}$ of graphs on a common vertex set V and a graph H we define the family of lexicographic product graphs

$$\mathcal{G} \circ H = \{G \circ H : \, G \in \mathcal{G}\}.$$

By Theorem 7.35 we deduce the following result.

Corollary 7.37. *Let $\mathcal{G}$ be a family of graphs on a common vertex set V. For any graph H such that $H \notin \Theta$,*

$$\mathrm{Sd}_l(\mathcal{G} \circ H) = |V| \, \mathrm{adim}_l(H).$$

By Corollary 7.30 and Theorem 7.35 we have the following result.

Proposition 7.38. *If $V - I(V, \mathcal{H})$ is a simultaneous local metric generator for $\mathcal{G}$ or the graph $\boldsymbol{G}(\mathcal{G}, I(V, \mathcal{H}))$ is empty, then*

$$\mathrm{Sd}_l(\mathcal{G} \circ \mathcal{H}) = \sum \mathrm{Sad}_l(\mathcal{H}^i)$$

For the second type of simplification we have the following remark.

Remark 7.39. *As $\mathrm{Sad}_l(\mathcal{H}^i) \le \Psi(\mathcal{H}^i)$, if $A \subseteq B \subseteq V$ then*

$$\sum_{u_i \in A} \Psi(\mathcal{H}^i) + \sum_{u_i \notin A} \mathrm{Sad}_l(\mathcal{H}^i) \le \sum_{u_i \in B} \Psi(\mathcal{H}^i) + \sum_{u_i \notin B} \mathrm{Sad}_l(\mathcal{H}^i)$$

From Remark 7.39 we can get some consequences of Theorem 7.35.

Proposition 7.40. *Let $\mathcal{G} \circ \mathcal{H}$ be a family of lexicographic product graphs. For any vertex cover B of $\boldsymbol{G}(\mathcal{G}, I(V, \mathcal{H}))$,*

$$\mathrm{Sd}_l(\mathcal{G} \circ \mathcal{H}) \le \sum_{u_i \in B} \Psi(\mathcal{H}^i) + \sum_{u_i \notin B} \mathrm{Sad}_l(\mathcal{H}^i)$$

Proposition 7.41. *Let $\mathcal{G}$ be a family of connected graphs with common vertex set V and let $\mathcal{G} \circ \mathcal{H}$ be a family of lexicographic product graphs. The following statements hold.*

(i) *If the subgraph of G_j induced by $I(V, \mathcal{H})$ is empty for every $G_j \in \mathcal{G}$, then*

$$\mathrm{Sd}_l(\mathcal{G} \circ \mathcal{H}) = \sum_{u_i \in V} \mathrm{Sad}_l(\mathcal{H}^i).$$

(ii) *Let $u_{i_0} \in I(V, \mathcal{H})$ be such that $\Psi(\mathcal{H}^{i_0}) = \max\{\Psi(u_i) : u_i \in I(V, \mathcal{H})\}$. If $\mathrm{Sd}_l(\mathcal{G}) = |V| - 1$ and $|I(V, \mathcal{H})| \geq 2$, then*

$$\mathrm{Sd}_l(\mathcal{G} \circ \mathcal{H}) = \sum_{u_i \notin I(V,\mathcal{H})} \mathrm{Sad}_l(\mathcal{H}^i) + \sum_{u_i \in I(V,\mathcal{H}) - \{u_{i_0}\}} \Psi(\mathcal{H}^i) + \mathrm{Sad}_l(\mathcal{H}^{i_0})$$

Proof. It is clear that if the subgraph of G_j induced by $I(V, \mathcal{H})$ is empty for every $G_j \in \mathcal{G}$, then $\emptyset \in \Upsilon(V, \mathcal{H})$, so that Theorem 7.35 leads to (i). On the other hand, let $\mathcal{G}$ be a family of connected graphs with common vertex set V such that $\mathrm{Sd}_l(\mathcal{G}) = |V| - 1$ and $|I(V, \mathcal{H})| \geq 2$. By Lemma 7.3, for every $u_i, u_j \in I(V, \mathcal{H})$ there exists $G_{ij} \in \mathcal{G}$ such that u_i, u_j are true twins in G_{ij}. Hence, no vertex $u \notin \{u_i, u_j\}$ resolves u_i and u_j. Therefore $A \in \Upsilon(V, \mathcal{H})$ implies $|A| = |I(V, \mathcal{H})| - 1$ and (ii) follows from Theorem 7.35 and Remark 7.39. □

Proposition 7.42. *Let $\mathcal{G}$ be a family of nontrivial connected graphs with common vertex set V. For any family of lexicographic product graphs $\mathcal{G} \circ \mathcal{H}$,*

$$\mathrm{Sd}_l(\mathcal{G} \circ \mathcal{H}) \geq \mathrm{Sd}_l(\mathcal{G}).$$

Furthermore, if $\mathcal{H} = \{N_{|V_1|}, \ldots, N_{|V_n|}\}$, then

$$\mathrm{Sd}_l(\mathcal{G} \circ \mathcal{H}) = \mathrm{Sd}_l(\mathcal{G}).$$

Proof. Let W be a simultaneous local metric basis of $\mathcal{G} \circ \mathcal{H}$ and $W_V = \{u \in V : (u, v) \in W\}$. We suppose that W_V is not a simultaneous local metric generator for $\mathcal{G}$. Let $u_i, u_j \notin W_V$ and $G \in \mathcal{G}$ such that $u_i u_j \in E(G)$ and $d_G(u_i, u) = d_G(u_j, u)$ for every $u \in W_V$. Thus, for any $v \in V_i$, $v' \in V_j$ and $(x, y) \in W$ we have

$$d_{G \circ H_i}((x, y), (u_i, v)) = d_G(x, u_i) = d_G(x, u_j) = d_{G \circ H_i}((x, y), (u_j, v')),$$

which is a contradiction. Therefore, W_V is a simultaneous local metric generator for $\mathcal{G}$ and, as a result, $\mathrm{Sd}_l(\mathcal{G}) \leq |W_V| \leq |W| = \mathrm{Sd}_l(\mathcal{G} \circ \mathcal{H})$.

On the other hand, if $\mathcal{H} = \{N_{|V_1|}, \ldots, N_{|V_n|}\}$, then $V = I(V, \mathcal{H}) = \Phi(V, \mathcal{H})$. Let $B \subseteq V$ be a simultaneous local metric basis for $\mathcal{G}$. Now, for each $u_i \in B$ we choose $v_i \in V_i$ and, by Remark 7.27, we claim that $B' = \{(u_i, v_i) : u_i \in B\}$ is a simultaneous local metric generator for $\mathcal{G} \circ \mathcal{H}$. Thus, $\mathrm{Sd}_l(\mathcal{G} \circ \mathcal{H}) \leq |B'| = |B| = \mathrm{Sd}_l(\mathcal{G})$. □

Proposition 7.43. *Let $\mathcal{G} \neq \{K_2\}$ be a family of nontrivial connected bipartite graphs with common vertex set V and $\mathcal{H} \neq \{\mathcal{H}_1, \ldots, \mathcal{H}_n\}$ such that $\mathcal{H}_j \not\subset \mathcal{S}_0$, for some j. If $V = I(V, \mathcal{H})$ and there exist $u_1, u_2 \in V$ and $G_k \in \mathcal{G}$ such that $V - \Phi(V, \mathcal{H}) = \{u_1, u_2\}$ and $u_1u_2 \in E(G_k)$, then*

$$\mathrm{Sd}_l(\mathcal{G} \circ \mathcal{H}) = \sum \mathrm{Sad}_l(\mathcal{H}^i) + 1,$$

otherwise,

$$\mathrm{Sd}_l(\mathcal{G} \circ \mathcal{H}) = \sum \mathrm{Sad}_l(\mathcal{H}^i).$$

Proof. If $V = I(V, \mathcal{H})$ and there exist $u_1, u_2 \in V$ and $G_k \in \mathcal{G}$ such that $V - \Phi(V, \mathcal{H}) = \{u_1, u_2\}$ and $u_1u_2 \in E(G_k)$, then $\emptyset \notin \Upsilon(V, \mathcal{H})$ because no vertex in $(V - \Phi(V, \mathcal{H})) - \{u_1, u_2\} = \emptyset$ distinguishes u_1 and u_2. Let $x, y \in I(V, \mathcal{H})$ such that $xy \in \cup_{G \in \mathcal{G}} E(G)$. Since any $u_i \in \Phi(V, \mathcal{H})$ distinguishes x and y, we can conclude that $\{u_i\} \in \Upsilon(V, \mathcal{H})$, and by Remark 7.24, $\Psi(\mathcal{H}^i) = 1$. Therefore, Theorem 7.35 leads to $\mathrm{Sd}_l(\mathcal{G} \circ \mathcal{H}) = \sum \mathrm{Sad}_l(\mathcal{H}^i) + 1$.

Assume that there exists $u_i \in V - I(V, \mathcal{H})$ or $V - \Phi(V, \mathcal{H}) = \{u_i\}$ or $V - \Phi(V, \mathcal{H}) = \{u_i, u_j\}$ and, for every $G_k \in \mathcal{G}$, $u_iu_j \notin E(G_k)$ or $\{u_i, u_j, u_k\} \subseteq V - \Phi(V, \mathcal{H})$. In any one of these cases $\{u_i\}$ is a simultaneous local metric basis for $\mathcal{G}$ and, for every pair u_1, u_2 of adjacent vertices in some $G_k \in \mathcal{G}$ such that $u_i \notin \{u_1, u_2\}$, u_i distinguishes the pair. Since $u_i \in V - \Phi(V, \mathcal{H})$, we can claim that $\emptyset \in \Upsilon(V, \mathcal{H})$ and, by Theorem 7.35, $\mathrm{Sd}_l(\mathcal{G} \circ \mathcal{H}) = \sum \mathrm{Sad}_l(\mathcal{H}^i)$. □

Families of join graphs

For two graph families $\mathcal{G} = \{G_1, \ldots, G_{k_1}\}$ and $\mathcal{H} = \{H_1, \ldots, H_{k_2}\}$, defined on common vertex sets V_1 and V_2, respectively, such that $V_1 \cap V_2 = \emptyset$, we define the family

$$\mathcal{G} + \mathcal{H} = \{G_i + H_j : \ 1 \leq i \leq k_1, 1 \leq j \leq k_2\}.$$

Notice that, since for any $G_i \in \mathcal{G}$ and $H_j \in \mathcal{H}$ the graph $G_i + H_j$ has diameter two,

$$\mathrm{Sd}_l(\mathcal{G} + \mathcal{H}) = \mathrm{Sad}_l(\mathcal{G} + \mathcal{H}).$$

The following result is a direct consequence of Theorem 7.35.

Corollary 7.44. *For any pair of families $\mathcal{G}$ and $\mathcal{H}$ of non-trivial graphs on common vertex sets V_1 and V_2, respectively,*

$$\mathrm{Sd}_l(\mathcal{G} + \mathcal{H}) = \min\{Sd_{A,l}(\mathcal{G}) + \Psi(\mathcal{H}), Sd_{A,l}(\mathcal{H}) + \Psi(\mathcal{G})\}$$

Remark 7.45. *Let $\mathcal{G}$ a family of graphs defined on a common vertex set V_1. If there exists B a simultaneous local adjacency basis of $\mathcal{G}$ such that $D[\mathcal{G}, B] = \emptyset$, then for every $\mathcal{H}$ family of graphs defined on a common vertex set V_2 we have*

$$\mathrm{Sd}_l(\mathcal{G} + \mathcal{H}) = \mathrm{Sad}_l(\mathcal{G}) + \mathrm{Sad}_l(\mathcal{H})$$

By Lemma 7.26 and Remark 7.45 we deduce the following result.

Proposition 7.46. *Let $\mathcal{G}$ and $\mathcal{H}$ be two families of nontrivial connected graphs on a common vertex set V_1 and V_2, respectively. If $\mathcal{G} \subset \cup_{i=1}^{4} \mathcal{S}_i$, then*

$$\mathrm{Sd}_l(\mathcal{G} + \mathcal{H}) = \mathrm{Sad}_l(\mathcal{G}) + \mathrm{Sad}_l(\mathcal{H}).$$

7.6 Computability of the simultaneous local metric dimension

In previous sections, we have seen that there is a large number of classes of graph families for which the simultaneous local metric dimension is well determined. This includes some cases of graph families whose simultaneous metric dimension is hard to compute, e.g. families composed by trees [49], yet the simultaneous local metric dimension is constant. However, as proven in [23], the problem of finding the local metric dimension of a graph is NP-hard in the general case, which trivially leads to the fact that finding the simultaneous local metric dimension of a graph family is also NP-hard in the general case.

Here, we will focus on a different aspect, namely that of showing that the requirement of simultaneity adds on the computational difficulty of the original problem. To that end, we will show that there exist families composed by graphs whose individual local metric dimensions are constant, yet it is hard to compute their simultaneous local metric dimension.

To begin with, we will formally define the decision problems associated to the computation of the local metric dimension of one graph and the simultaneous local metric dimension of a graph family.

Local Metric Dimension (LDIM)
INSTANCE: A graph $G = (V, E)$ and an integer p, $1 \leq p \leq |V(G)| - 1$.
QUESTION: Is $\dim_l G \leq p$?

As we mentioned above, this problem was proven to be computationally difficult.

Lemma 7.47. [23] *The Local Metric Dimension Problem (LDIM) is NP-complete.*

Simultaneous Local Metric Dimension (SLD)
INSTANCE: A graph family $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ on a common vertex set V and an integer p, $1 \leq p \leq |V| - 1$.
QUESTION: Is $\mathrm{Sd}_l(\mathcal{G}) \leq p$?

With these definitions in mind, it is straightforward to see that SLD is NP-complete.

Remark 7.48. *The Simultaneous Local Metric Dimension Problem (SLD) is NP-complete.*

Proof. It is simple to see that determining whether a vertex set $S \subset V$, $|S| \leq p$, is a simultaneous local metric generator can be done in polynomial time, so SLD is in NP. Moreover, for any graph $G = (V, E)$ and any integer $1 \leq p \leq |V(G)| - 1$, the corresponding instance of LDIM can be transformed into an instance of SLD in polynomial time by making $\mathcal{G} = \{G\}$, so SLD is NP-complete. □

Now, we will address the issue of the complexity added by the simultaneity requirement. To this end, we will consider families composed by the so-called *tadpole graphs* [40]. An (h, t)-tadpole graph (or (h, t)-*tadpole* for short) is the graph obtained from a cycle graph C_h and a path graph P_t by joining with an edge a leaf of P_t to an arbitrary vertex of C_h. We will use the notation $T_{h,t}$ for (h, t)-tadpoles. Since $(2q, t)$-tadpoles are bipartite, we have that $\dim_l(T_{2q,t}) = 1$. In the case of $(2q+1,t)$-tadpoles, we have that $\dim_l(T_{2q+1,t}) = 2$, as they are not bipartite (so, $\dim_l(T_{2q+1,t}) \geq 2$) and any set composed by two vertices of the subgraph C_{2q+1} is a local metric generator (so, $\dim_l(T_{2q+1,t}) \leq 2$). Additionally, consider the sole vertex v of degree 3 in $T_{2q+1,t}$ and a local metric generator for $T_{2q+1,t}$ of the form $\{v, x\}$, $x \in V(C_{2q+1}) - \{v\}$. It is simple to verify that for any vertex $y \in V(P_t)$ the set $\{y, x\}$ is also a local metric generator for $T_{2q+1,t}$.

Consider a family $\mathcal{T} = \{T_{h_1,t_1}, T_{h_2,t_2}, \ldots, T_{h_k,t_k}\}$ composed by tadpole graphs on a common vertex set V. By Theorem 7.6, we have that $\mathrm{Sd}_l(\mathcal{T}) = \mathrm{Sd}_l(\mathcal{T}')$, where $\mathcal{T}'$ is composed by $(2q+1,t)$-tadpoles. As we discussed previously, $\dim_l(T_{2q+1,t}) = 2$. However, by Remark 7.1 and Theorem 7.3, we have that $2 \leq \mathrm{Sd}_l(\mathcal{T}') \leq |V| - 1$, being both bounds tight[1]. Moreover, as we will show, the problem of computing $\mathrm{Sd}_l(\mathcal{T}')$ is NP-hard, as its associated decision problem is NP-complete. We will do so by showing a transformation from the **Hitting set Problem**, which was shown to be NP-complete by Karp [37]. The Hitting Set Problem is defined as follows:

Hitting Set Problem (HSP)
INSTANCE: A collection $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ of non-empty subsets of a finite set S and a positive integer $p \leq |S|$.
QUESTION: Is there a subset $S' \subseteq S$ with $|S'| \leq p$ such that S' contains at least one element from each subset in $\mathcal{C}$?

Theorem 7.49. *The Simultaneous Local Metric Dimension Problem (SLD) is NP-complete for families of* $(2q+1,t)$*-tadpoles.*

Proof. As we discussed previously, determining whether a vertex set $S \subset V$, $|S| \leq p$, is a simultaneous local metric generator for a graph family $\mathcal{G}$ can be done in polynomial time, so SLD is in NP.

Now, we will show a polynomial time transformation of HSP into SLD. Let $S = \{v_1, v_2, \ldots, v_n\}$ be a finite set and let $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$, where every $C_i \in \mathcal{C}$ satisfies $C_i \subseteq S$. Let p be a positive integer such that $p \leq |S|$. Let $A = \{w_1, w_2, \ldots, w_k\}$ such that $A \cap S = \emptyset$. We construct the family $\mathcal{T} = \{T_{2q_1+1,t_1}, T_{2q_2+1,t_2}, \ldots, T_{2q_k+1,t_k}\}$ composed by $(2q+1,t)$-tadpoles on

[1]The lower bound is trivially satisfied by unitary families, whereas the upper bound is reached, for instance, by any family composed by all different labelled graphs isomorphic to an arbitrary $(3,t)$-tadpole, as it satisfies the premises of Theorem 7.3.

the common vertex set $V = S \cup A \cup \{u\}$, $u \notin S \cup A$, by performing one of the two following actions, as appropriate, for every $r \in \{1, \ldots, k\}$:

- If $|C_r|$ is even, let C_{2q_r+1} be a cycle graph on the vertices of $C_r \cup \{u\}$, let P_{t_r} be a path graph on the vertices of $(S - C_r) \cup A$, and let T_{2q_r+1,t_r} be the tadpole graph obtained from C_{2q_r+1} and P_{t_r} by joining with an edge a leaf of P_{t_r} to a vertex of C_{2q_r+1} different from u.

- If $|C_r|$ is odd, let C_{2q_r+1} be a cycle graph on the vertices of $C_r \cup \{u, w_r\}$, let P_{t_r} be a path graph on the vertices of $(S - C_r) \cup (A - \{w_r\})$, and let T_{2q_r+1,t_r} be the tadpole graph obtained from C_{2q_r+1} and P_{t_r} by joining with an edge the vertex w_r to a leaf of P_{t_r}.

Figure 7.4 shows an example of this construction.

In order to prove the validity of this transformation, we claim that there exists a subset $S'' \subseteq S$ of cardinality $|S''| \leq p$ that contains at least one element from each $C_r \in \mathcal{C}$ if and only if $\mathrm{Sd}_l(\mathcal{T}) \leq p + 1$.

To prove this claim, first assume that there exists a set $S'' \subseteq S$ which contains at least one element from each $C_r \in \mathcal{C}$ and satisfies $|S''| \leq p$. Recall that any set composed by two vertices of C_{2q_r+1} is a local metric generator for T_{2q_r+1,t_r}, so $S'' \cup \{u\}$ is a simultaneous local metric generator for $\mathcal{T}$. Thus, $\mathrm{Sd}_l(\mathcal{T}) \leq p + 1$.

Now, assume that $\mathrm{Sd}_l(\mathcal{T}) \leq p + 1$ and let W be a simultaneous local metric generator for $\mathcal{T}$ such that $|W| = p + 1$. For every $T_{2q_r+1,t_r} \in \mathcal{T}$, we have that $u \in V(C_{2q_r+1})$ and $\delta_{T_{2q_r+1,t_r}}(u) = 2$, so $|((W - \{x\}) \cup \{u\}) \cap V(C_{2q_r+1})| \geq |W \cap V(C_{2q_r+1})|$ for any $x \in W$. In consequence, if $u \notin W$, any set $(W - \{x\}) \cup \{u\}$, $x \in W$, is also a simultaneous local metric generator for $\mathcal{T}$, so we can assume that $u \in W$. Moreover, applying an analogous reasoning for every set $C_r \in \mathcal{C}$ such that $W \cap C_r = \emptyset$, we have that, firstly, there is at least one vertex $v_{r_i} \in C_r$ such that $v_{r_i} \in V(C_{2q_r+1}) - \{u\}$ and $\delta_{T_{2q_r+1,t_r}}(v_{r_i}) = 2$, and secondly, there is at least one vertex $x_r \in W \cap (\{w_r\} \cup V(P_{t_r}))$, which can be replaced by v_{r_i}. Then, the set

$$W' = \bigcup_{W \cap C_r = \emptyset} ((W - \{x_r\}) \cup \{v_{r_i}\})$$

is also a simultaneous local metric generator for $\mathcal{T}$ of cardinality $|W'| = p+1$ such that $u \in W'$ and $(W' - \{u\}) \cap C_r \neq \emptyset$ for every $C_r \in \mathcal{C}$. Thus the set $S'' = W' - \{u\}$ satisfies $|S''| \leq p$ and contains at least one element from each $C_r \in \mathcal{C}$.

To conclude our proof, it is simple to verify that the transformation of HSP into SLD described above can be done in polynomial time. □

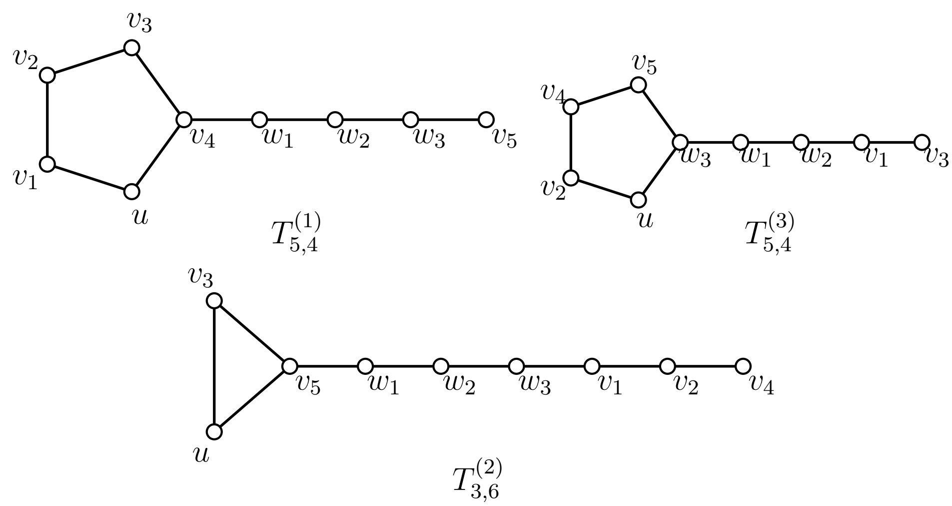

Figure 7.4: The family $\mathcal{T} = \{T^{(1)}_{5,4}, T^{(2)}_{3,6}, T^{(3)}_{5,4}\}$ is constructed for transforming an instance of HSP, where $S = \{v_1, v_2, v_3, v_4, v_5\}$ and $\mathcal{C} = \{\{v_1, v_2, v_3, v_4\}, \{v_3, v_5\}, \{v_2, v_4, v_5\}\}$, into an instance of SLD for families of $(2q + 1, t)$-tadpoles.

Conclusions

In this thesis we have studied the problem of computing the local metric dimension of graphs. We first reported on the state of the art on the local metric dimension, and we presented some original results in which we have characterized all graphs that reach some known bounds. Secondly, we obtained closed formulas and tight bounds on the local metric dimension of several families of graphs, including strong product graphs, corona product graphs, rooted product graphs and lexicographic product graphs. Finally, we introduced the study of simultaneous local metric dimension, we gave some general results on this new research line and we obtained the formula for the simultaneous metric dimension of specific families of graphs.

Contributions of the thesis

The results presented in this work have been published, or are in the process of being published, in several venues. Three papers have been published and one is submitted to ISI-JCR journals, while some of the principal results have been presented in conferences.

Publications in ISI-JCR journals

- Barragán-Ramírez, G.A., Rodríguez-Velázquez, J.A., The local metric dimension of strong product graphs. Graphs and Combinatorics **32** (2016) 1263–1278.

- Rodríguez-Velázquez, J. A., Barragán-Ramírez, G. A., García-Gómez, C., On the local metric dimension of corona product graphs. Bulletin of the Malaysian Mathematical Sciences Society **39** (2016) 157 – 173.

- Rodríguez-Velázquez, J. A.,García-Gómez, C., Barragán-Ramírez, G.

A., Computing the local metric dimension of a graph from the local metric dimension of primary subgraphs. International Journal of Computer Mathematics **92** (2015) 686 – 693.

Submitted papers

- Barragán-Ramírez, G. A, Ramírez-Cruz, Y. Estrada-Moreno, A., Rodríguez-Velázquez, J. A., The simultaneous local metric dimension of graph families. Bulletin of the Malaysian Mathematical Sciences Society, submitted.

Contributions to conferences

- Barragán-Ramírez, G. A., The local metric dimension of the lexicographic product of graphs. S. Gómez and A. Valls-Mateu (Eds.), *3rd URV Doctoral Workshop in Computer Science and Mathematics*, Tarragona, Spain, 2016. Actas 3–6. ISBN: 978-84-8424-495-0

- Barragán-Ramírez, G.A., The local metric dimension of a graph from its primary subgraphs. A. Valls-Mateu and M. Sánchez-Artigas (Eds.), *2nd URV Doctoral Workshop in Computer Science and Mathematics*, Tarragona, Spain, 2015. Actas 43–46. ISBN: 978-84-8424-399-1

- Barragán-Ramírez, G. A.; García-Gómez, C.; Rodríguez-Velázquez, J. A. Closed formulae for the local metric dimension of corona product graphs. *IX Jornadas de Matemática Discreta y Algorítmica*, Tarragona, 2014. Electronic Notes in Discrete Mathematics **46** (2014) 27–34.

Future Works

- Closed formulae or lower bounds on the local metric dimension provide lower bounds on the metric dimension, as $\dim_l(G) \leq \dim(G)$. For instance, using this fact, Theorem 3.16 gives the solution of a conjecture proposed in [52] on the value of the metric dimension of $P_r \boxtimes P_s$. We propose the study of graphs with $\dim_l(G) = \dim(G)$.

- We propose the study of the local metric dimension of graphs for which the metric dimension has been previously studied. For instance, we propose the families of circulant graphs, direct product graphs, Sierpiński graphs, Cartesian sum graphs, amalgamation graphs, among others.

- It is known that $\dim_l(G) = \operatorname{adim}_l(G)$ for graphs of diameter at most two, for graphs obtained from the lexicographic product of non-empty graphs, and also for graphs of order n with $\dim_l(G) = n - \alpha(G)$. The question is if there are other families of graphs satisfying this strong relationship.

- Up to now, the study of the local metric dimension has been restricted to the case of the geodetic distance. We propose the study of other metrics defined on the graph. For instance, we can use the metric $d_{G,t}(u,v) = \min\{d_G(u,v), t\}$. In such a study, the case $t = 2$ corresponds to the local adjacency dimension and the case $t \geq D(G)$ corresponds to the local metric dimension.

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