UNIVERSITAT ROVIRA I VIRGILI

ESSAYS ON EGALITARIANISM-BASED SOLUTION CONCEPTS FOR COOPERATIVE TU-GAMES

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I STATE that the present study, entitled "Essays on egalitarianism-based solution concepts for cooperative TU-games", presented by Llúcia Mauri Masdeu for the award of the degree of Doctor, has been carried out under my supervision at the Department of Business Management of this university.

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Dr. Francesc Llerena Garrés
Doctoral Thesis Supervisor

"L'important és no deixar de fer-se preguntes"

Albert Einstein

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Abstract

For a class of reduced games satisfying a monotonicity property, we introduce a family of set-valued solution concepts based on egalitarian considerations and consistency principles, and study its relation with the core. Regardless of the reduction operation we consider, the intersection between both sets is either empty or a singleton containing the lexmax solution (Arin et al., 2003). This result induces a procedure for computing the lexmax solution for a class of games that contains games with large core (Sharkey, 1982). We extend the previous analysis by using the notion of the anti-dual game (Oishi and Nakayama, 2009). We find parallel results for the lexmin solution.

A class of balanced games, called exact partition games, is introduced. Within this class, it is shown that the egalitarian solution of Dutta and Ray (1989) behaves as in the class of convex games. Moreover, we provide two axiomatic characterizations by means of suitable properties such as consistency, rationality and Lorenz-fairness. As a by-product, alternative characterizations of the egalitarian solution over the class of convex games are obtained. Using the notion of anti-duality to axioms (Oishi et al., 2016), we obtain additional axiomatizations of the egalitarian solution on the domain of exact partition games but also on the domain of convex games. On the domain of balanced games, new axiomatic characterizations of the Lorenz maximal core are obtained.

We introduce the Lorenz stable set and provide an axiomatic characterization in terms of constrained egalitarianism and projection consistency. On the domain of all coalitional games, we find that this solution connects the weak constrained egalitarian solution (Dutta and Ray, 1989) with its strong counterpart (Dutta and Ray, 1991).

Contents

Introduction

This dissertation focuses on egalitarianism-based solution concepts in the framework of cooperative games with transferable utility (games hereafter). One of the main goals of game theory is to describe rules (or solution concepts) that will lead to binding agreements among a set of agents so that the output of a joint venture can be distributed. Nevertheless, in many real-life situations there is a tension between cooperation and private interests. Thus, the rules proposed must be supported by a set of properties (or axioms).

In this setting, and under the assumptions that agents believe in egalitarianism, as a social value, but their individual preferences dictate selfish behavior, Dutta and Ray (1989) introduce the weak constrained egalitarian solution (WCES). This solution concept is defined in a recursive manner and it selects the Lorenz maximal allocation within the Lorenz core, which is a proper extension of the core (Gillies, 1953). Although the WCES prescribes, at most, one allocation for the whole group of agents, in general this solution fails to satisfy existence and it is difficult to compute. However, for convex games (Shapley, 1971), Dutta and Ray (1989) provide an algorithm to determine the WCES and show that it exists, lies in the core and Lorenz dominates every other core element. From an axiomatic viewpoint, Dutta (1990) was the first to characterize the WCES using two properties: consistency (or reduced game property), with respect to the max reduced game (Davis and Maschler, 1965) and the self reduced game (Hart and

Mas-Colell, 1989), and constrained egalitarianism. Consistency is an outstanding property widely used in the axiomatic approach that relates the solution of a game to the solution of the reduced game that results from some players leaving. Constrained egalitarianism is a prescriptive property that fixes the solution for two person games. Alternative characterizations of the WCES over the domain of convex games can be found, among others, in Klijn et al. (2000), Hougaard et al. (2001) and Arin et al. (2003). Recently, Oishi et al. (2016) obtained new axiomatic characterizations of the WCES on the domain of convex games, by applying the notion of anti-dual axiom.

As we have mentioned above, the WCES lacks general existence properties. To overcome this drawback, the Lorenz maximal allocations can be used from a set of payoff vectors satisfying some minimal requirements. Although the output of this approach is not necessarily a unique distribution, all of them are on the same Lorenz curve. On the domain of balanced games, this approach was suggested by Dutta and Ray (1989), and latter assumed by Arin and Iñarra (2001) and Hougaard et al. (2001). To deal with the question of uniqueness, Arin and Iñarra (2001) and Yanovskaya (1995) introduced the lexmin solution, and Arin and Iñarra (2001) introduced the lexmax solution. Both are single-valued solutions, dual to each other, based on the lexicographical order. On the domain of weak superadditive games, Dutta and Ray (1991) introduced the strong constrained egalitarian solution (SCES), a solution concept that selects the Lorenz maximal allocations within the equal division core (Selten, 1972). On the domain of all games, and inspired by the algorithm of Dutta and Ray (1989), Branzei et al. (2006) introduced the equal split-off set, a discrete set-valued solution concept that coincides with the WCES on the domain of convex games.

The present dissertation aims to contribute to the study of egalitarianism in the framework of games from a theoretical point of view as follows. First, on the domain of all games, we introduce a family of discrete set-valued solutions that extends some of the aforementioned well-established egalitarian rules to certain domain of games. These solutions are defined sequentially and, at each step of the process, the payoffs to the players are determined by applying principles of fairness and consistency. We analyze the intersection between these solutions and the core, and we show that, for a kind of reduction operation satisfying monotonicity in payments, it is either the empty set or a singleton containing the lexmax solution. This result induces a procedure for finding the lexmax solution on a domain that includes games with large core. We extend the previous analysis by making use of the notion of anti-dual game (Oishi and Nakayama, 2009), and we find parallel results for the lexmin solution. All these results are collected in Chapter 1.

Second, we extend Dutta and Ray's-analysis (1989) by introducing the class of exact partition games, rich enough to include convex games, dominant diagonal assignment games (Solymosi and Raghavan, 2001) and also non-superadditive games, where the WCES behaves as it does in convex games. Within this class, we provide axiomatic characterizations of the WCES that can be extended to the class of convex games. One of this axiomatizations can be applied to the class of balanced games characterizing the Lorenz maximal core. Using the notion of anti-duality to axioms (Oishi et al. 2016), we obtain additional axiomatizations of the WCES on the domain of exact partition games but also on the domain of convex games. On the domain of balanced games, new axiomatic characterizations of the Lorenz maximal core are obtained. The first part of Chapter 2 contains these results.

Third, in the last part of Chapter 2 we focus on the axiomatic approach of the Lorenz maximal allocations in the imputation set. Within the domain of essential games, we observe that this solution is single-valued and admits a characterization of the WCES similar to that given by Dutta (1990), but using the projected reduced game (Funaki, 1998). We call this solution the Lorenz stable set. The reason is that it can be interpreted as a sort of stable set à la von Neumann-Morgenstern (1944) although the usual order in R is replaced by the Lorenz order. Finally, we connect the WCES and the SCES by means of the Lorenz stable set.

All the chapters contain a section with concluding remarks that highlight our main contributions and give some directions about possible future research.

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Chapter 1

Reduced games and egalitarian solutions[1]

1.1 Introduction

Transferable utility coalitional games (games, for short) describe situations in which a group of agents (or players) can get benefits from joint efforts. The question is how to distribute all the gains from cooperation among the players by making use of suitable properties. A solution is a mapping that assigns a set of feasible payoff vectors to each game. In this context, several solution concepts have been defined with the aim of accommodating egalitarianism and some particular interests. That is, to allocate the total worth of a coalition as equally as possible among its agents, while satisfying some individual requirements. One of the best known concepts is the *weak constrained egalitarian solution* (Dutta and Ray, 1989). For convex games, Dutta and Ray (1989) devised an algorithm to

[1]Some results of this chapter have been published at International Journal of Game Theory. Reference: Llerena, F. and Mauri, Ll. (2016) Reduced games and egalitarian solutions, International Journal of Game Theory, 45: 1053-1069.

find their egalitarian allocation and show that it belongs to the core and Lorenz dominates every other core element. Hokari (2002) generalizes their algorithm by defining non-symmetric extensions of this solution. Unfortunately, the class of convex games is the only standard class of games for which existence is guaranteed. In order to widen the domain of games for which egalitarian solutions exist, Dutta and Ray (1991) introduced the *strong constrained egalitarian solution*, a parallel concept that selects the Lorenz-maximal imputations in the *equal division core* (Selten, 1972). Related studies are Arin and Iñarra (2001), Hougaard et al. (2001) and Arin et al. (2003, 2008), who introduced other egalitarian solutions based on the notion of the *core*. Inspired by the Dutta and Ray (1989) algorithm, Branzei et al. (2006) introduced the *equal split-off set*, a non-empty set-valued solution that is well defined for all games.

Consistency (or the reduced game property) is an outstanding property that plays an important role in the axiomatization of a considerable number of solutions. Informally, a solution is consistent if it makes coherent choices in both the original game and the reduced game.[2] In this chapter, we introduce a family of solution concepts based on egalitarian considerations and consistency principles, and study its relation with the core. The central idea is that agents in a coalition that maximizes average worth share this value equally among them and leave the game. Then, the remaining agents play a suitable reduced game, in which agents in a coalition with the highest average worth again divide it equally among its members. The process stops when all agents have been paid. The output of this sequencial procedure is a finite set of efficient allocations that can be supported by an egalitarian criterion and a weak consistency property.

The chapter is organized as follows. Section 1.2 contains notation and terminology. In Section 1.3 we introduce the concept of *admissible subgroup corre-*

[2]See Thomson (2011) for an essay of consistency.

spondence α and the associated $\alpha-$*max reduced game*. For a given α, we define the $\alpha-$*reduced equal split-off set*. This set and the core have different qualitative properties. For instance, the $\alpha-$reduced equal split off set is always non-empty and finite, while the core is convex and its non-emptiness is not granted, except in balanced games. However, the intersection between them provides surprising results. For any admissible subgroup correspondence α satisfying a monotonicity property, weaker than the transitivity of the reduction operation, we find that when the intersection between both sets is non-empty, it becomes a singleton containing the *lexmax* solution of Arin et al. (2003). In Section 1.4, for a class of games that includes games with a large core (Sharkey, 1982), we show that the reduced equal split-off set à la Davis and Maschler (1965) turns out to be a singleton and it coincides with the lexmax solution. We also provide a procedure for finding the lexmax solution on this domain. To end this section, we connect the Davis and Maschler reduced equal split-off set with the weak constrained egalitarian solution on the domain of convex games. Section 1.5 complements the previous analysis by introducing, for a given α, the anti-dual solution of the $\alpha-$reduced equal split-off set, and studying its relationship with the core.

1.2 Notation and terminology

The set of natural numbers $\mathbb{N}$ denotes the universe of potential players. A **coalition** is a non-empty finite subset of $\mathbb{N}$ and let $\mathcal{N} := \{N \mid \emptyset \neq N \subseteq \mathbb{N}, |N| < \infty\}$ denote the set of all coalitions of $\mathbb{N}$. A **transferable utility coalitional game (a game)** is a pair (N, v) where $N \in \mathcal{N}$ is the set of players and $v : 2^N \longrightarrow \mathbb{R}$ is the characteristic function that assigns to each coalition $S \subseteq N$ a real number $v(S)$, with the convention that $v(\emptyset) = 0$. Given $S, T \in \mathcal{N}$, we use $S \subset T$ to indicate strict inclusion, that is, $S \subseteq T$ but $S \neq T$. By $|S|$ we denote the cardinality of the coalition $S \in \mathcal{N}$. By Γ we denote the class of all games.

Given $N \in \mathcal{N}$, let $\mathbb{R}^N$ stand for the space of real-valued vectors indexed by N, $x = (x_i)_{i\in N}$, and for all $S \subseteq N$, $x(S) = \sum_{i\in S} x_i$, with the convention $x(\emptyset) = 0$. For each $x \in \mathbb{R}^N$ and $T \subseteq N$, $x_{|T}$ denotes the restriction of x to T: $x_{|T} = (x_i)_{i\in T} \in \mathbb{R}^T$. Given two vectors $x, y \in \mathbb{R}^N$, $x \geq y$ if $x_i \geq y_i$, for all $i \in N$. We say that $x > y$ if $x \geq y$ and for some $j \in N$, $x_j > y_j$. Given N, a set $\pi = (P_1, \ldots, P_m)$, where $P_i \subseteq N$ for all $i \in \{1, \ldots, m\}$, with $m \leq |N|$, is a **partition** of N if the following conditions hold: (i) $P_i \neq \emptyset$ for all $i \in \{1, \ldots, m\}$, (ii) $\cup_{i=1}^m P_i = N$ and (iii)$P_i \cap P_j = \emptyset$, for all $i, j \in \{1, \ldots, m\}$, $i \neq j$.

The set of **feasible payoff vectors** of a game (N, v) is defined by $X^*(N, v) := \{x \in \mathbb{R}^N \,|\, x(N) \leq v(N)\}$. A **solution** on a class of games $\Gamma' \subseteq \Gamma$ is a mapping σ which associates with each game $(N, v) \in \Gamma'$ a subset $\sigma(N, v)$ of $X^*(N, v)$. Notice that σ is allowed to be empty. A solution on a class of games $\Gamma' \subseteq \Gamma$ is said to be **single-valued** if $|\sigma(N, v)| = 1$ for all $(N, v) \in \Gamma'$. The **pre-imputation set** of (N, v) is defined by $X(N, v) := \{x \in \mathbb{R}^N \,|\, x(N) = v(N)\}$, and the set of **imputations** by $I(N, v) := \{x \in X(N, v) \,|\, x_i \geq v(\{i\}),$ for all $i \in N\}$. A game is **essential** if it has a non-empty imputation set. By Γ_{Ess} we denote the class of essential games. The core of (N, v) is the set of those imputations where each coalition gets at least its worth, that is $C(N, v) = \{x \in X(N, v) \mid x(S) \geq v(S)$ for all $S \subseteq N\}$. A game (N, v) is **balanced** if it has a non-empty core. By Γ_{Bal} we denote the class of balanced games. A game is **superadditive**, if $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$. A game (N, v) is **convex** (Shapley, 1971) if, for every $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$. The class of convex games is denoted by Γ_{Con}. Recall that $\Gamma_{Con} \subset \Gamma_{Bal} \subset \Gamma_{Ess}$.

Given $N \in \mathcal{N}$, for any $x \in \mathbb{R}^N$, denote by $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ the vector obtained from x by rearranging its coordinates in a non-increasing order, that is, $\hat{x}_1 \geq \hat{x}_2 \geq \ldots \geq \hat{x}_n$. In a similar way, for $\emptyset \neq T \subseteq N$, $\widehat{x_{|T}}$ denotes the vector obtained from the restriction of x to T by ordering its coordinates in a non-increasing way:

$\widehat{x_{|T}}_1 \geq \widehat{x_{|T}}_2 \geq \ldots \geq \widehat{x_{|T}}_t$, where $t = |T|$. In addition, denote by $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ the vector obtained from x by rearranging its coordinates in a non-decreasing order, that is, $\bar{x}_1 \leq \bar{x}_2 \leq \ldots \leq \bar{x}_n$. For any two vectors $y, x \in \mathbb{R}^N$, we say that y **Lorenz dominates** x, denoted by $y \succ_{\mathcal{L}} x$, if $\sum_{j=1}^{k} \bar{y}_j \geq \sum_{j=1}^{k} \bar{x}_j$, for all $k \in \{1, \ldots, |N|\}$ with at least one strict inequality. If $y(N) = x(N)$, Lorenz domination can be defined equivalently as follows: $y \succ_{\mathcal{L}} x$ if $\sum_{j=1}^{k} \hat{y}_j \leq \sum_{j=1}^{k} \hat{x}_j$, for all $k \in \{1, \ldots, |N|\}$ with at least one strict inequality.

1.3 Reduced equal split-off set and the core

The egalitarian solution of Dutta and Ray (1989) is the output of a sequential procedure where the game is reduced each time the payoffs to players in a coalition maximizing average worth are assigned. Then, a reduced game is defined by only taking into account the whole group of players outside the game. Following this idea, but taking into account other notions of reduced games that allow for more coalitional options, we define a family of solutions and study its relation with the core. The equal split-off set of Branzei et al. (2006) turns out to be a particular case when we reduced the game à la Moulin (1985).

A single-valued egalitarian solution that will play an important role in our analysis is the lexmax solution of Arin et al. (2003). For any two vectors $x, y \in \mathbb{R}^N$, we say that $x \preceq_{lex} y$ if $x = y$ or $x_1 < y_1$ or there exists $k \in \{2, \ldots, |N|\}$ such that $x_i = y_i$ for $1 \leq i \leq k-1$ and $x_k < y_k$. For a balanced game (N, v), the **lexmax** solution is defined as $Lmax(N, v) = \{x \in C(N, v) \,|\, \hat{x} \preceq_{lex} \hat{y} \text{ for all } y \in C(N, v)\}$. For any balanced game (N, v), the lexmax solution is a singleton and it is Lorenz undominated within the core.

Next we introduce the concept of **admissible subgroup correspondence** inspired by the work of Thomson (1990) and also used by Izquierdo et al. (2005).

Definition 1. *An admissible subgroup correspondence* $\alpha : \mathcal{N} \to \mathcal{N}$ *is a correspondence that associates with each* $N \in \mathcal{N}$ *a non-empty list* $\alpha(N)$ *of coalitions of* N*.*

We denote by $\mathcal{A}$ the set of all admissible subgroup correspondences. Given $\alpha, \alpha' \in \mathcal{A}$, we write $\alpha \leq \alpha'$ if for all $N \in \mathcal{N}$, $\alpha(N) \subseteq \alpha'(N)$.

Now we introduce the $\alpha-$**max reduced game** by using the notion of admissible subgroup correspondence α. This game is defined over a set of agents where each subgroup evaluates its worth by considering the coalitional restrictions determined by α. Examples of admissible subgroup correspondences α can be given by taking into account several aspects of coordination between players: communication, hierarchies, geographical areas, law requirements, or the size of the subgroups.

Definition 2. *Let* (N, v) *be a game,* $\alpha \in \mathcal{A}$, $\emptyset \neq N' \subset N$ *and* $x \in \mathbb{R}^K$ *where* $N \setminus N' \subseteq K \subseteq N$*. The* $\alpha-$*max reduced game relative to* N' *at* x *is the game* $\left(N', r^{N'}_{\alpha,x}(v)\right)$ *defined by*

$$r^{N'}_{\alpha,x}(v)(S) = \begin{cases} 0 & \text{if } S = \emptyset, \\ \max\limits_{Q \in \alpha(N \setminus N')} \{v(S \cup Q) - x(Q)\} & \text{if } \emptyset \neq S \subset N', \\ v(N) - x(N \setminus N') & \text{if } S = N'. \end{cases} \tag{1.1}$$

The interpretation of the $\alpha-$max reduced game is as in Davis and Maschler (1965) but here the options of members in N' to cooperate with members in $N \setminus N'$ are restricted by the admissible subgroup correspondence α. The **Davis and Maschler reduced game** is a particular case when $\alpha(N) = 2^N$ for all $N \in \mathcal{N}$. Other well-known reduced games can also be obtained by taking a suitable admissible subgroup correspondence. For instance, the **complement reduced game** proposed by Moulin (1985) is defined by $\alpha(N) = \{N\}$ for all $N \in \mathcal{N}$, or the **projected reduced game** (Funaki, 1998) by $\alpha(N) = \{\emptyset\}$ for all

$N \in \mathcal{N}$. Another example is $\alpha(N) = \{\emptyset, N\}$, for all $N \in \mathcal{N}$. This correspondence formalizes a dichotomous situation where each coalition may stand alone or join the whole group of players. The above reduction operations will be denoted by α_{DM}, α_M, α_P and α_D, respectively.

A well-known property related with the notion of reduced game is **consistency**.

Definition 3. *Let σ be a solution on $\Gamma' \subseteq \Gamma$. Given $\alpha \in \mathcal{A}$, we say that σ satisfies $\alpha-$consistency on Γ' if for all $N \in \mathcal{N}$, all $(N,v) \in \Gamma'$, all $N' \subset N$, $N' \neq \emptyset$, and all $x \in \sigma(N,v)$, then $\left(N', r^{N'}_{\alpha,x}(v)\right) \in \Gamma'$ and $x_{|N'} \in \sigma\left(N', r^{N'}_{\alpha,x}(v)\right)$.*

On the domain of convex games, the weak constrained egalitarian solution of Dutta and Ray (1989) satisfies $\alpha_{DM}-$*consistency* (Dutta, 1990). On the domain of balanced games, the core also satisfies $\alpha_{DM}-$*consistency* (Peleg, 1986). Using the same proof as Peleg (1986), it can be easily shown that the core satisfies $\alpha-$*consistency* for all $\alpha \in \mathcal{A}$.

Proposition 1. *On the domain of balanced games, the core satisfies α-consistency, for all $\alpha \in \mathcal{A}$.*

Proof. Let (N,v) be a balanced game, $\alpha \in \mathcal{A}$ and $x \in C(N,v)$. Take $\emptyset \neq N' \subset N$. For every $S \subset N'$, there exists $Q^* \in \alpha(N \setminus N')$ such that

$$\begin{aligned} r^{N'}_{\alpha,x}(v)(S) &= \max_{Q \in \alpha(N \setminus N')} \{v(S \cup Q) - x(Q)\} = v(S \cup Q^*) - x(Q^*) \\ &\leq x(S \cup Q^*) - x(Q^*) = x(S). \end{aligned}$$

Moreover, $r^{N'}_{\alpha,x}(v)(N') = v(N) - x(N \setminus N') = x(N) - x(N \setminus N') = x(N')$. Then, $x_{|N'} \in C(N', r^{N'}_{\alpha,x}(v))$ and $(N', r^{N'}_{\alpha,x}(v))$ is balanced. $\square$

It is quite straightforward to see that the lexmax solution is α_{DM}-*consistent* on the domain of balanced games. Let (N,v) be a balanced game and $x = Lmax(N,v)$. Take $\emptyset \neq N' \subset N$ and suppose $x_{|N'} \neq Lmax\left(N', r^{N'}_{\alpha_{DM},x}(v)\right)$. By

$\alpha_{DM}-$*consistency* of the core, $x_{|N'} \in C\left(N', r^{N'}_{\alpha_{DM},x}(v)\right)$, and thus it holds $\hat{y} \preceq_{lex} \widehat{x_{|N'}}$, where $y = Lmax\left(N', r^{N'}_{\alpha_{DM},x}(v)\right)$. Notice that $z = \left(y, x_{|N\setminus N'}\right) \in C(N,v)$. But $\hat{z} \preceq_{lex} \hat{x}$, which leads a contradiction.[3] This completes the proof of the following proposition.

Proposition 2. *On the domain of balanced games, the lexmax solution satisfies* α_{DM}*-consistency.*

However, as shown Example 1 bellow, the lexmax solution is not α-*consistent* for any $\alpha \in \{\alpha_M, \alpha_P, \alpha_D\}$.

Example 1. *(Dutta and Ray, 1989) Let (N,v) be a balanced game with set of players $N = \{1,2,3,4\}$ and characteristic function as follows,*

S	$v(S)$	S	$v(S)$	S	$v(S)$	S	$v(S)$
$\{1\}$	0	$\{12\}$	0	$\{123\}$	1.05	$\{1234\}$	2
$\{2\}$	0	$\{13\}$	0	$\{124\}$	0		
$\{3\}$	0	$\{14\}$	0	$\{134\}$	1.9		
$\{4\}$	0	$\{23\}$	1.05	$\{234\}$	1.9		
		$\{24\}$	0				
		$\{34\}$	1.9				

Let us first show that $x = (0, 0.1, 0.95, 0.95)$ is the lexmax solution. Notice that $x \in C(N,v)$ and $\hat{x} = (0.95, 0.95, 0.1, 0)$. Suppose that $x \neq Lmax(N,v) = y$. Then, $\hat{y} \preceq_{lex} \hat{x}$. Since $y \in C(N,v)$, $y_3 + y_4 \geq v(\{34\}) = 1.9$. If $y_3 > \frac{1.9}{2} = 0.95$ or $y_4 > \frac{1.9}{2} = 0.95$, then $\hat{y}_1 > 0.95$ in contradiction with $\hat{y}_1 \leq \hat{x}_1 = 0.95$. Thus, $y_3 = y_4 = 0.95$, $\hat{y}_1 = \hat{y}_2 = 0.95$ and $\hat{y}_3 \leq \hat{x}_3 = 0.1$. Since $y_2 + y_3 \geq v(\{23\}) = 1.05$

[3]The following property is well known (see, for instance, Potters and Tijs, 1992). For any $n \in \mathbb{N}$ we define the map $\theta : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which arranges the coordinates of a point in $\mathbb{R}^n$ in non-increasing order. Take $x, y \in \mathbb{R}^n$ such that $\theta(x)$ is lexicographically not greater than $\theta(y)$. Take now any $z \in \mathbb{R}^p$ and consider the vectors $(x,z), (y,z) \in \mathbb{R}^{n+p}$. Then, $\theta(x,z)$ is lexicographically not greater than $\theta(y,z)$.

and $y_3 = 0.95$, *we have that* $y_2 \geq 0.1$, *which implies* $\hat{y}_3 \geq 0.1$. *But then* $\hat{y}_3 = 0.1$ *and, by efficiency,* $\hat{y} = \hat{x}$. *Finally, taking into account that the lexmax solution is a singleton we conclude that* $x = Lmax(N, v)$.

Next we show that the lexmax solution is not α*-consistent for any* $\alpha \in \mathcal{A}^* = \{\alpha_M, \alpha_P, \alpha_D\}$. *Indeed, consider the* α*-max reduced game* $\left(\{12\}, r^{\{12\}}_{\alpha,x}(v)\right)$. *It can be checked that, for any* $\alpha \in \mathcal{A}^*$,

$r^{\{12\}}_{\alpha,x}(v)(\{1\}) = r^{\{12\}}_{\alpha,x}(v)(\{2\}) = 0$ *and* $r^{\{12\}}_{\alpha,x}(v)(\{12\}) = 0.1$.

Hence, $Lmax\left(\{12\}, r^{\{12\}}_{\alpha,x}(v)\right) = (0.05, 0.05) \neq (0, 0.1) = x_{|\{12\}}$.

Associated with $\alpha \in \mathcal{A}$ we introduce the $\alpha-$reduced equal split-off set.

Definition 4. *Let* (N, v) *be a game and* $\alpha \in \mathcal{A}$. *We say that* $\pi = (T_1, \ldots, T_t)$ *is an* $\alpha-$**ordered partition** *of* N *if*

$$T_1 \in \arg \max_{\emptyset \neq S \subseteq N} \left\{ \frac{v(S)}{|S|} \right\} \text{ and } T_k \in \arg \max_{\emptyset \neq S \subseteq N \setminus T_1 \cup \ldots \cup T_{k-1}} \left\{ \frac{r^{N \setminus T_1 \cup \ldots \cup T_{k-1}}_{\alpha, x_{k-1}}(v)(S)}{|S|} \right\}$$

for each $k = 2, \ldots, t$, *where*

- $x_1 = \left(\frac{v(T_1)}{|T_1|}, \ldots, \frac{v(T_1)}{|T_1|} \right) \in \mathbb{R}^{T_1}$ *and*
- $x_k \in \mathbb{R}^{T_1 \cup \ldots \cup T_k}$ *is recursively defined as follows:*

$$x_{k,i} = \begin{cases} x_{k-1,i} & \text{if } i \in T_1 \cup \ldots \cup T_{k-1}, \\ \\ \dfrac{r^{N \setminus T_1 \cup \ldots \cup T_{k-1}}_{\alpha, x_{k-1}}(v)(T_k)}{|T_k|} & \text{if } i \in T_k. \end{cases} \tag{1.2}$$

We call the payoff vector $x_t \in \mathbb{R}^N$ *as the* $\alpha-$**reduced equal split-off allocation** *generated by* π.

Definition 5. *Let* (N, v) *be a game and* $\alpha \in \mathcal{A}$. *The* $\alpha-$**reduced equal split-off set** *of a game* (N, v), *denoted by* $\phi^\alpha(N, v)$, *is the set of all* $\alpha-$*reduced equal split-off allocations.*

For $\alpha = \alpha_M$ we recover the equal split-off set of Branzei et al. (2006). The next example illustrates the above procedure.

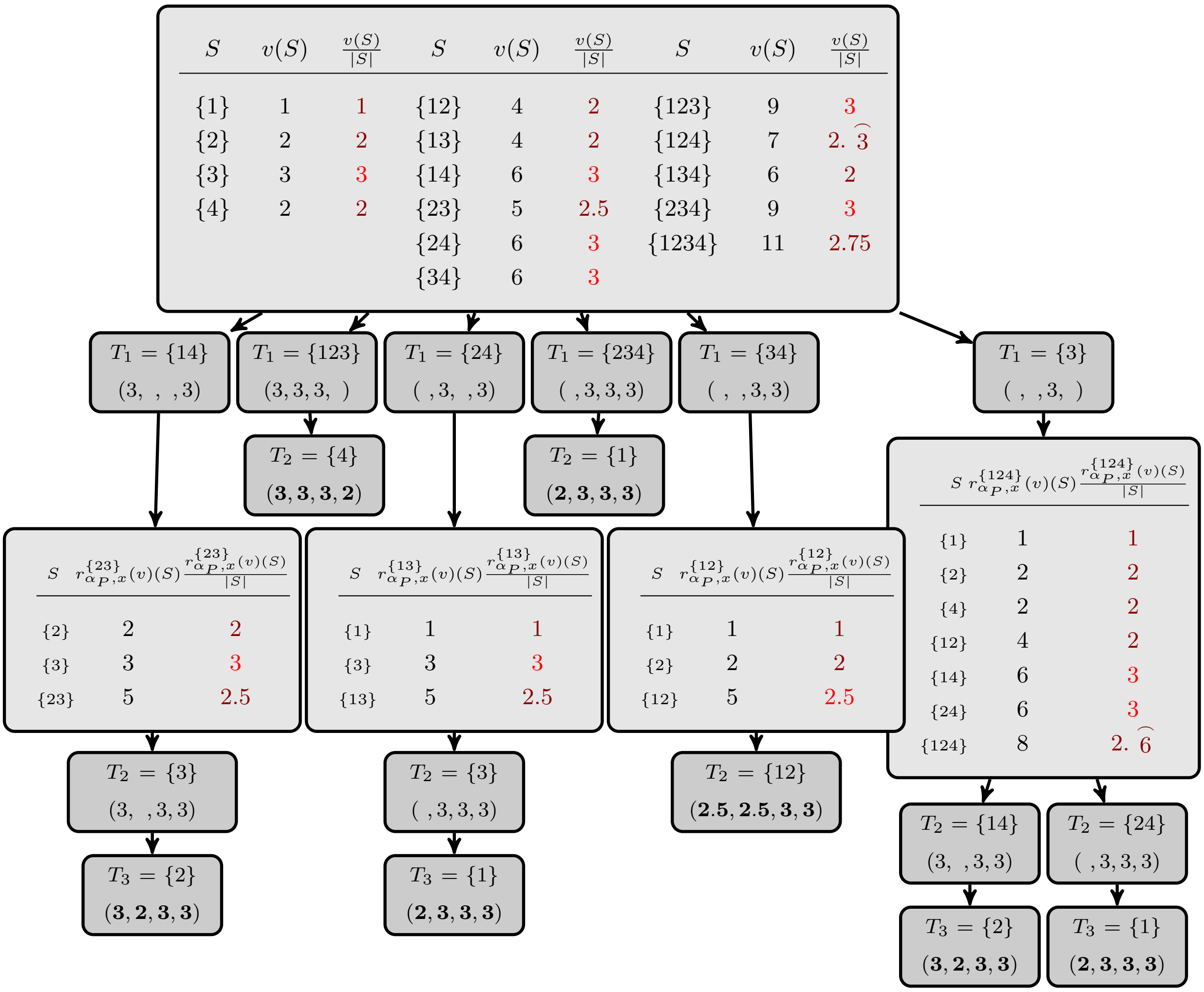
Example 2. *Let (N, v) be a game with set of players $N = \{1, 2, 3, 4\}$ and characteristic function:*

S	$v(S)$	S	$v(S)$	S	$v(S)$	S	$v(S)$
$\{1\}$	1	$\{12\}$	4	$\{123\}$	9	$\{1234\}$	11
$\{2\}$	2	$\{13\}$	4	$\{124\}$	7		
$\{3\}$	3	$\{14\}$	6	$\{134\}$	6		
$\{4\}$	2	$\{23\}$	5	$\{234\}$	9		
		$\{24\}$	6				
		$\{34\}$	6				

The procedure to obtain an $\alpha-$reduced equal split-off allocation, $\alpha \in \mathcal{A}$, is as follows. In the first step of the process we take an arbitrary coalition T_1 that maximizes the average worth of the game (N, v). Then, every player $i \in T_1$ receives $x_i = \frac{v(T_1)}{|T_1|}$. If $T_1 \neq N$, we consider the $\alpha-$max reduced game relative to $N \setminus T_1$ at $x_{|T_1} \in \mathbb{R}^{T_1}$. Again, we choose an arbitrary coalition $T_2 \subseteq N \setminus T_1$ that maximizes the average worth in this reduced game and every player $i \in T_2$ receives $x_i = \frac{r_{\alpha,x}^{N \setminus T_1}(T_2)}{|T_2|}$. The process stops when a partition of N of the form $\pi = (T_1, T_2, \ldots, T_t)$, for some $1 \leq t \leq |N|$, is reached. Repeating this process for each of the coalitions that maximize the average worth of the original game and the successive reduced games, we obtain the set of $\alpha-$reduced equal split-off allocations.

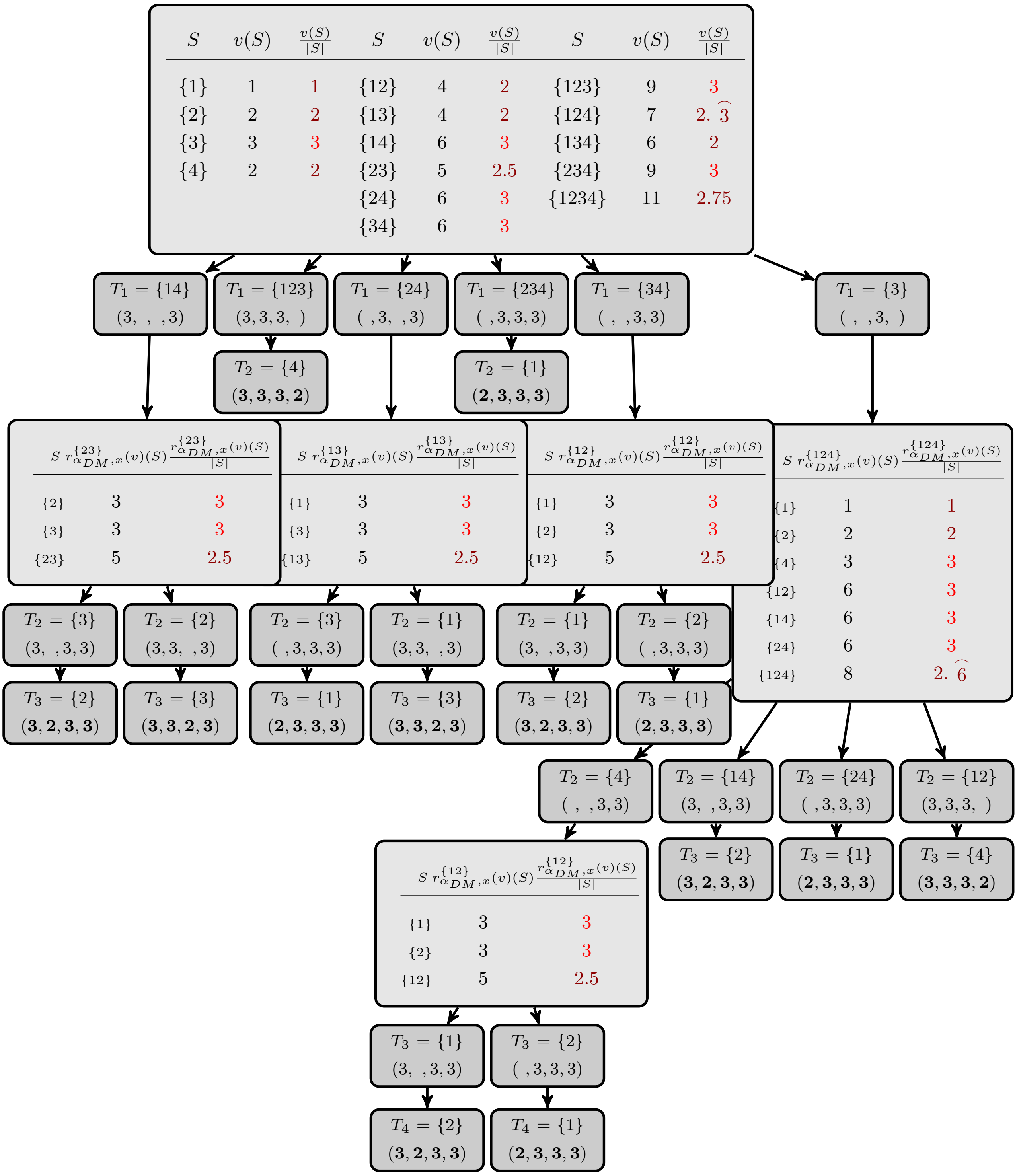
The following diagrams show how are obtained both $\phi^{\alpha_P}(N, v)$ and $\phi^{\alpha_{DM}}(N, v)$.

α_P−reduced equal split-off allocations:



Thus, the α_P−ordered partitions of N, $\pi_1 = (\{14\},\{3\},\{2\})$ and $\pi_2 = (\{3\},\{14\},\{2\})$, generate the payoff vector $(3,2,3,3)$. The α_P−ordered partitions $\pi_3 = (\{24\},\{3\},\{1\})$, $\pi_4 = (\{234\},\{1\})$ and $\pi_5 = (\{3\},\{24\},\{1\})$ generate the payoff vector $(2,3,3,3)$. In addition, the allocation $(2.5,2.5,3,3)$ is generated by $\pi_6 = (\{34\},\{12\})$ and the allocation $(3,3,3,2)$ by $\pi_7 = (\{123\},\{4\})$.

$\alpha_{DM}-$reduced equal split-off allocations:



The allocation $(3,2,3,3)$ *is generated by the* α_{DM}*-ordered partitions* $\pi'_1=(\{14\},\{3\},\{2\})$, $\pi'_2=(\{3\},\{4\},\{1\},\{2\})$, $\pi'_3=(\{3\},\{14\},\{2\})$ *and* $\pi'_4=(\{34\},\{1\},\{2\})$. *The partitions* $\pi'_5=(\{234\},\{1\})$, $\pi'_6=(\{24\},\{3\},\{1\})$, $\pi'_7=(\{34\},\{2\},\{1\})$, $\pi'_8=(\{3\},\{4\},\{2\},\{1\})$ *and* $\pi'_9=(\{3\},\{24\},\{1\})$ *produce* $(2,3,3,3)$. *Moreover,* $\pi'_{10}=(\{14\},\{2\},\{3\})$ *and* $\pi'_{11}=(\{24\},\{1\},\{3\})$ *produce the allocation* $(3,3,2,3)$*; and both* $\pi'_{12}=(\{123\},\{4\})$ *and* $\pi'_{13}=(\{3\},\{12\},\{4\})$ *provide* $(3,3,3,2)$.

Similarly, we can calculate the $\alpha-$*reduced equal split-off set for* $\alpha\in\{\alpha_M,\alpha_D\}$. *As we can see in table below, the different* $\alpha-$*reduced equal split-off sets are finite and different from each other.*

Allocation	$\phi^{\alpha_M}(N,v)$	$\phi^{\alpha_P}(N,v)$	$\phi^{\alpha_{DM}}(N,v)$	$\phi^{\alpha_D}(N,v)$
$(2,3,3,3)$	X	X	X	X
$(3,3,3,2)$	X	X	X	X
$(3,2,3,3)$		X	X	X
$(3,2.5,2.5,3)$	X			
$(3,3,2,3)$			X	
$(2.5,2.5,3,3)$		X		

In order to analyze the relation between the $\alpha-$reduced equal split-off set and the core of a game, we consider a family of admissible subgroup correspondences that satisfies a monotonicity property.

Definition 6. *Let* $\alpha\in\mathcal{A}$. *We say that* α *satisfies* **monotonicity in payments** *if for all* $N\in\mathcal{N}$, *all* $(N,v)\in\Gamma$, *and all* $x\in\phi^{\alpha}(N,v)$ *generated by* $\pi=(T_1,\ldots,T_t)$, *it holds that for all* $k<h\leq t$, $x_i\geq x_j$ *for all* $i\in T_k$ *and all* $j\in T_h$.

We denote by $\mathcal{A}_{mon}$ the set of admissible subgroup correspondences satisfying *monotonicity in payments.*

A natural requirement on $\alpha \in \mathcal{A}$ is that the associated $\alpha-$max reduced game should be transitive, in the sense that the repeated use of the reduced game does not depend on the order that players leave the game.

Definition 7. *Let $\alpha \in \mathcal{A}$. The α-max reduced game is said to be transitive if $r^{N''}_{\alpha,x_{|N'}}\left(r^{N'}_{\alpha,x}(v)\right) = r^{N''}_{\alpha,x}(v)$, for all $N \in \mathcal{N}$, all $(N,v) \in \Gamma$, all coalitions $\emptyset \neq N'' \subset N' \subset N$ and all payoff vector $x \in \mathbb{R}^K$ with $N \setminus N'' \subseteq K \subseteq N$.*

We denote by $\mathcal{A}_t$ the set of admissible subgroup correspondences such that the associated $\alpha-$max reduced game is transitive. It can be easily checked that $\alpha_P, \alpha_M \in \mathcal{A}_t$. To show that $\alpha_{DM} \in \mathcal{A}_t$ see, for instance, Chang and Hu (2007). The next lemma states a sufficient condition on $\alpha \in \mathcal{A}$ to guarantee transitivity of the associated $\alpha-$max reduced game.

Lemma 1. *Let $\alpha \in \mathcal{A}$. If for all $N \in \mathcal{N}$ and all $\emptyset \neq N'' \subset N' \subset N$ it holds $\alpha(N \setminus N'') \supseteq \{Q_1 \cup Q_2 \mid Q_1 \in \alpha(N' \setminus N'') \text{ and } Q_2 \in \alpha(N \setminus N')\}$, then the associated $\alpha-$max reduced game is transitive.*

Proof. Let $N \in \mathcal{N}$, (N,v) a game, $\emptyset \neq N'' \subset N' \subset N$ and $x \in \mathbb{R}^K$, where $N \setminus N'' \subseteq K \subseteq N$.

For $T = \emptyset$ or $T = N''$, the equality $r^{N''}_{\alpha,x_{|N'}}\left(r^{N'}_{\alpha,x}(v)\right)(T) = r^{N''}_{\alpha,x}(v)(T)$ follows straightforwardly.

For every $\emptyset \neq T \subset N''$, there is $Q_1 \in \alpha(N' \setminus N'')$ and $Q_2 \in \alpha(N \setminus N')$ such that

$$\begin{aligned}
r^{N''}_{\alpha,x_{|N'}}\left(r^{N'}_{\alpha,x}(v)\right)(T) &= \max_{Q \in \alpha(N' \setminus N'')} \left\{ r^{N'}_{\alpha,x}(v)(T \cup Q) - x(Q) \right\} \\
&= r^{N'}_{\alpha,x}(v)(T \cup Q_1) - x(Q_1) \\
&= \max_{Q \in \alpha(N \setminus N')} \{ v(T \cup Q_1 \cup Q) - x(Q) \} - x(Q_1) \\
&= v(T \cup Q_1 \cup Q_2) - x(Q_2) - x(Q_1) \\
&\leq \max_{Q \in \alpha(N \setminus N'')} \{ v(T \cup Q) - x(Q) \} \\
&= r^{N''}_{\alpha,x}(v)(T).
\end{aligned}$$

On the other hand, there is $Q^* \in \alpha(N \setminus N'')$, $Q^* = Q_1^* \cup Q_2^*$ with $Q_1^* \in \alpha(N' \setminus N'')$ and $Q_2^* \in \alpha(N \setminus N')$, such that

$$\begin{aligned} r_{\alpha,x}^{N''}(v)(T) &= v(T \cup Q^*) - x(Q^*) \\ &= v(T \cup Q_1^* \cup Q_2^*) - x(Q_1^* \cup Q_2^*) \\ &\leq \max_{R \in \alpha(N \setminus N')} \{v(T \cup Q_1^* \cup R) - x(R)\} - x(Q_1^*) \\ &= r_{\alpha,x}^{N'}(v)(T \cup Q_1^*) - x(Q_1^*) \\ &\leq \max_{S \in \alpha(N' \setminus N'')} \left\{ r_{\alpha,x}^{N'}(v)(T \cup S) - x(S) \right\} \\ &= r_{\alpha,x_{|N'}}^{N''} \left(r_{\alpha,x}^{N'}(v) \right)(T). \end{aligned}$$

Hence, $r_{\alpha,x_{|N'}}^{N''} \left(r_{\alpha,x}^{N'}(v) \right)(T) = r_{\alpha,x}^{N''}(v)(T)$, which concludes the proof. □

The next two propositions state that transitivity is a sufficient but not necessary condition to satisfy *monotonicity in payments*.

Proposition 3. $\mathcal{A}_t \subset \mathcal{A}_{mon}$.

Proof. Let (N, v) be a game, $\alpha \in \mathcal{A}_t$ and $x \in \phi^{\alpha}(N, v)$ generated by $\pi = (T_1, \ldots, T_t)$, with $t > 1$. For $k \in \{1, \ldots, t-1\}$, let us denote $N_k = N \setminus T_1 \cup \ldots \cup T_k$. Let $N_0 = N$ and $v = r_{\alpha,x}^{N_0}(v)$. For $k \leq t-1$, $i \in T_k$ and $j \in T_{k+1}$, we have

$$x_i = \frac{r_{\alpha,x}^{N_{k-1}}(v)(T_k)}{|T_k|} \text{ and } x_j = \frac{r_{\alpha,x}^{N_k}(v)(T_{k+1})}{|T_{k+1}|} = \frac{r_{\alpha,x_{|N_{k-1}}}^{N_k} \left(r_{\alpha,x}^{N_{k-1}}(v) \right)(T_{k+1})}{|T_{k+1}|}. \tag{1.3}$$

We distinguish two cases:

- **Case 1:** $T_{k+1} = N_k$. In this situation, for $j \in T_{k+1}$,

$$x_j = \frac{r_{\alpha,x}^{N_{k-1}}(v)(N_{k-1}) - r_{\alpha,x}^{N_{k-1}}(v)(T_k)}{|N_k|}. \tag{1.4}$$

  Suppose $x_j > x_i$, for $i \in T_k$ and $j \in T_{k+1}$. Then, combining (1.3) and (1.4) we obtain

$$r_{\alpha,x}^{N_{k-1}}(v)(N_{k-1}) > \frac{r_{\alpha,x}^{N_{k-1}}(v)(T_k)}{|T_k|}(|N_k| + |T_k|)$$

or, equivalently,

$$\frac{r_{\alpha,x}^{N_{k-1}}(v)(N_{k-1})}{|N_{k-1}|} > \frac{r_{\alpha,x}^{N_{k-1}}(v)(T_k)}{|T_k|}$$

in contradiction with the fact that $T_k \in \arg \max_{\emptyset \neq T \subseteq N_{k-1}} \left\{ \frac{r_{\alpha,x}^{N_{k-1}}(v)(T)}{|T|} \right\}$.

- **Case 2:** $T_{k+1} \subset N_k$. In this case, there is $Q^* \in \alpha(T_k)$ such that, for all $j \in T_{k+1}$,

$$x_j = \frac{r_{\alpha,x}^{N_{k-1}}(v)(T_{k+1} \cup Q^*) - |Q^*| \frac{r_{\alpha,x}^{N_{k-1}}(v)(T_k)}{|T_k|}}{|T_{k+1}|}. \tag{1.5}$$

If $x_j > x_i$ for $i \in T_k$, then combining (1.3) and (1.5) we have

$$r_{\alpha,x}^{N_{k-1}}(v)(T_{k+1} \cup Q^*) > \frac{r_{\alpha,x}^{N_{k-1}}(v)(T_k)}{|T_k|}(|T_{k+1}| + |Q^*|)$$

or equivalently,

$$\frac{r_{\alpha,x}^{N_{k-1}}(v)(T_{k+1} \cup Q^*)}{|T_{k+1} \cup Q^*|} > \frac{r_{\alpha,x}^{N_{k-1}}(v)(T_k)}{|T_k|},$$

in contradiction with the fact that $T_k \in \arg \max_{\emptyset \neq T \subseteq N_{k-1}} \left\{ \frac{r_{\alpha,x}^{N_{k-1}}(v)(T)}{|T|} \right\}$. Hence, $x_j \leq x_i$ for all $i \in T_k$, all $j \in T_{k+1}$ and all $k \in \{1, \ldots, t-1\}$, which concludes the proof. □

Proposition 4. $\alpha_D \in \mathcal{A}_{mon}$ *but* $\alpha_D \notin \mathcal{A}_t$.

Proof. Let (N, v) be a game and $x \in \phi^{\alpha_D}(N, v)$ generated by $\pi = (T_1, \ldots, T_t)$, with $t > 1$. For $k \in \{1, \ldots, t-1\}$ let us denote $N_k = N \setminus T_1 \cup \ldots \cup T_k$. Let $N_0 = N$ and $v = r_{\alpha_D,x}^{N_0}(v)$. For $k \leq t-1$, $i \in T_k$ and $j \in T_{k+1}$ we have

$$x_i = \frac{r_{\alpha_D,x}^{N_{k-1}}(v)(T_k)}{|T_k|} \text{ and } x_j = \frac{r_{\alpha_D,x}^{N_k}(v)(T_{k+1})}{|T_{k+1}|}.$$

If $k < t-1$, we distinguish two cases:

- **Case 1:** $x_j = \frac{v(T_{k+1})}{|T_{k+1}|}$.

  In this situation,

$$x_j = \frac{v(T_{k+1})}{|T_{k+1}|} \leq \frac{r^{N_{k-1}}_{\alpha_D,x}(v)(T_{k+1})}{|T_{k+1}|} \leq \frac{r^{N_{k-1}}_{\alpha_D,x}(v)(T_k)}{|T_k|} = x_i,$$

  where the first inequality follows from the definition of α_D and the second one from the fact that $T_k \in \arg \max_{\emptyset \neq T \subseteq N_{k-1}} \left\{ \frac{r^{N_{k-1}}_{\alpha_D,x}(v)(T)}{|T|} \right\}$.

- **Case 2:** $x_j = \frac{v(T_1 \cup \ldots \cup T_k \cup T_{k+1}) - x(T_1 \cup \ldots \cup T_k)}{|T_{k+1}|}$.

  Notice first that $x(T_k) = r^{N_{k-1}}_{\alpha_D,x}(v)(T_k)$. Then,

$$\begin{aligned}
x_j &= \frac{v(T_1 \cup \ldots \cup T_k \cup T_{k+1}) - x(T_1 \cup \ldots \cup T_{k-1}) - x(T_k)}{|T_{k+1}|} \\
&\leq \frac{r^{N_{k-1}}_{\alpha_D,x}(v)(T_k \cup T_{k+1}) - x(T_k)}{|T_{k+1}|} \\
&= \frac{r^{N_{k-1}}_{\alpha_D,x}(v)(T_k \cup T_{k+1}) - r^{N_{k-1}}_{\alpha_D,x}(v)(T_k)}{|T_{k+1}|} \\
&\leq \frac{r^{N_{k-1}}_{\alpha_D,x}(v)(T_k)}{|T_k|} = x_i,
\end{aligned}$$

  where the first inequality follows from the definition of α_D and the second one from the fact that $T_k \in \arg \max_{\emptyset \neq T \subseteq N_{k-1}} \left\{ \frac{r^{N_{k-1}}_{\alpha_D,x}(v)(T)}{|T|} \right\}$.

  If $i \in T_{t-1}$ and $j \in T_t$, then $x_j = \frac{v(T_1 \cup \ldots \cup T_{t-1} \cup T_t) - x(T_1 \cup \ldots \cup T_{t-1})}{|T_t|}$. Thus, as in the above Case 2 it can be shown that $x_j \leq x_i$.

To see that $\alpha_D \notin \mathcal{A}_t$, consider the game (N, v) with set of players $N = \{1, 2, 3, 4, 5\}$ and characteristic function as follows:

$$v(\{4\}) = 0.95\ ,\ v(\{14\}) = v(\{134\}) = 1.9\ ,\ v(\{23\}) = v(\{123\}) = 1.05,$$

$$v(\{34\}) = 1\ ,\ v(\{234\}) = 2.8\ ,\ v(\{1234\}) = 2\ ,\ v(\{12345\}) = 3.8$$

$$\text{and } v(S) = 0, \text{otherwise.}$$

Take $x = \left(0.95, 0.6\,\widehat{3}, 0.6\,\widehat{3}, 0.95, 0.6\,\widehat{3}\right)$. Routine verification shows that

$$r^{\{235\}}_{\alpha_D, x_{|\{1235\}}}\left(r^{\{1235\}}_{\alpha_D,x}(v)\right)(\{23\}) = 1.85 > r^{\{235\}}_{\alpha_D,x}(v)(\{23\}) = 1.05.$$

$\square$

Our main result in this section (Theorem 1) states that for any $\alpha \in \mathcal{A}_{mon}$, the intersection $\phi^\alpha(N,v) \cap C(N,v)$ is either the empty set or the lexmax solution. Before doing this, we need some preliminary results. The first one states that if the grand coalition N is a coalition maximizing average worth, then any $\alpha-$reduced equal split-off set, $\alpha \in \mathcal{A}_{mon}$, is a singleton containing the equal split-off allocation.

Proposition 5. *Let (N,v) be a game and $\alpha \in \mathcal{A}_{mon}$. If $N \in \arg\max_{\emptyset \neq S \subseteq N} \left\{\frac{v(S)}{|S|}\right\}$, then $\phi^\alpha(N,v) = \left\{\left(\frac{v(N)}{|N|}, \dots, \frac{v(N)}{|N|}\right)\right\}$.*

Proof. Let (N,v) be a game and $\alpha \in \mathcal{A}_{mon}$. If $N \in \arg\max_{\emptyset \neq S \subseteq N}\left\{\frac{v(S)}{|S|}\right\}$, then $x = \left(\frac{v(N)}{|N|}, \dots, \frac{v(N)}{|N|}\right) \in \phi^\alpha(N,v)$. Suppose there is $y \in \phi^\alpha(N,v)$, $y \neq x$, generated by $\pi_y = (S_1, \dots, S_s)$. For all $i \in S_1$, $y_i = x_i = \frac{v(N)}{|N|}$. By efficiency, $y(N) = x(N)$, and thus $y(N \setminus S_1) = x(N \setminus S_1)$. Moreover, since $\alpha \in \mathcal{A}_{mon}$, we have $y_i \leq \frac{v(S_1)}{|S_1|} = \frac{v(N)}{|N|} = x_i$, for all $i \in N \setminus S_1$. This inequality, together with $y(N \setminus S_1) = x(N \setminus S_1)$, imply $x_i = y_i$, for all $i \in N$. $\square$

Combining *monotonicity in payments* with Proposition 5, we obtain an inclusion between the intersections of the $\alpha-$reduced equal split-off sets with the core, depending on the order of the admissible subgroup correspondences $\alpha \in \mathcal{A}_{mon}$.

Proposition 6. *Let $\alpha, \alpha' \in \mathcal{A}_{mon}$ such that $\alpha \leq \alpha'$. Let (N,v) be a balanced game and $x \in \phi^\alpha(N,v) \cap C(N,v)$. Then, $x \in \phi^{\alpha'}(N,v) \cap C(N,v)$.*

Proof. Let (N,v) be a balanced game and $\alpha, \alpha' \in \mathcal{A}_{mon}$ with $\alpha \leq \alpha'$. Notice first that for all $\emptyset \neq N' \subset N$ and all $y \in \mathbb{R}^N$, it holds

$$r^{N'}_{\alpha',y}(v)(R) \geq r^{N'}_{\alpha,y}(v)(R), \tag{1.6}$$

for all $R \subseteq N'$.

Let $x \in \phi^{\alpha}(N,v) \cap C(N,v)$ be generated by $\pi_x = (T_1, T_2, T_3, \ldots, T_t)$ and $z^1 \in \phi^{\alpha'}(N,v)$ be generated by $\pi_{z^1} = (T_1, S_2, \ldots, S_s)$. If $t = 1$ then, by Proposition 5, $\phi^{\alpha}(N,v) = \phi^{\alpha'}(N,v) = \left\{\left(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|}\right)\right\}$. Assume $t > 1$. For all $i \in T_1$, $x_i = z^1_i$, and by $\alpha'-$*consistency* of the core

$$x_{|N\setminus T_1} \in C\left(N \setminus T_1, r^{N\setminus T_1}_{\alpha',z^1}(v)\right). \tag{1.7}$$

By *monotonicity in payments*, for all $i \in T_2$ and all $j \in S_2$, it follows $x_i \geq x_j$. Since $x_i = x_k$ for all $i, k \in T_2$, we have $x_i = \frac{x(T_2)}{|T_2|} \geq \max_{j \in S_2}\{x_j\} \geq \frac{x(S_2)}{|S_2|}$. Thus, taking all this into account together with (1.6) and (1.7), we obtain the chain of inequalities

$$\begin{aligned}
\frac{r^{N\setminus T_1}_{\alpha,x}(v)(T_2)}{|T_2|} = \frac{x(T_2)}{|T_2|} \geq \frac{x(S_2)}{|S_2|} \quad &\geq \quad \frac{r^{N\setminus T_1}_{\alpha',z^1}(v)(S_2)}{|S_2|} \geq \frac{r^{N\setminus T_1}_{\alpha',z^1}(v)(T_2)}{|T_2|} \\
&\geq \quad \frac{r^{N\setminus T_1}_{\alpha,z^1}(v)(T_2)}{|T_2|} = \frac{r^{N\setminus T_1}_{\alpha,x}(v)(T_2)}{|T_2|},
\end{aligned}$$

which implies

$$\frac{r^{N\setminus T_1}_{\alpha,x}(v)(T_2)}{|T_2|} = \frac{r^{N\setminus T_1}_{\alpha',z^1}(v)(S_2)}{|S_2|} = \frac{r^{N\setminus T_1}_{\alpha',z^1}(v)(T_2)}{|T_2|}.$$

Thus, there is $z^2 \in \phi^{\alpha'}(N,v)$ generated by $\pi_{z^2} = (T_1, T_2, R_3, \ldots, R_r)$ and such that $z^2_i = x_i$ for all $i \in T_1 \cup T_2$. Again by $\alpha'-$*consistency* of the core we have $x_{|N\setminus T_1 \cup T_2} \in C\left(N \setminus T_1 \cup T_2, r^{N\setminus T_1 \cup T_2}_{\alpha',z^2}(v)\right)$, and by *monotonicity in payments* $x_i \geq x_j$, for all $i \in T_3$, $j \in R_3$. Since $x_i = x_k$ for all $i, k \in T_3$, we have $x_i = \frac{x(T_3)}{|T_3|} \geq \max_{j \in R_3}\{x_j\} \geq \frac{x(R_3)}{|R_3|}$. Thus, as before, we have that

$$\frac{r^{N\setminus T_1 \cup T_2}_{\alpha,x}(v)(T_3)}{|T_3|} = \frac{r^{N\setminus T_1 \cup T_2}_{\alpha',z^2}(v)(R_3)}{|R_3|} = \frac{r^{N\setminus T_1 \cup T_2}_{\alpha',z^2}(v)(T_3)}{|T_3|}.$$

Hence, there is $z^3 \in \phi^{\alpha'}(N,v)$ generated by $\pi_{z^3} = (T_1, T_2, T_3, P_4, \ldots, P_p)$ and such that $z^3_i = x_i$ for all $i \in T_1 \cup T_2 \cup T_3$.

Following this process step by step we find that $x \in \phi^{\alpha'}(N,v) \cap C(N,v)$. $\square$

Remark 1. *Observe that* $\phi^{\alpha}(N,v) \cap C(N,v) \subseteq \phi^{\alpha'}(N,v) \cap C(N,v)$, *whenever* $\alpha \leq \alpha'$. *However, in general,* $\phi^{\alpha}(N,v) \nsubseteq \phi^{\alpha'}(N,v)$, *as shown the next example.*

Example 3. *Let* (N,v) *be a balanced game with set of players* $N = \{1,2,3,4\}$ *and characteristic function:*

S	$v(S)$	S	$v(S)$	S	$v(S)$	S	$v(S)$
$\{1\}$	0	$\{1,2\}$	10	$\{1,2,3\}$	13	$\{1,2,3,4\}$	15
$\{2\}$	5	$\{1,3\}$	8	$\{1,2,4\}$	11		
$\{3\}$	3	$\{1,4\}$	6	$\{1,3,4\}$	10		
$\{4\}$	2	$\{2,3\}$	8	$\{2,3,4\}$	10		
		$\{2,4\}$	6				
		$\{3,4\}$	5				

It is not difficult to verify that $\phi^{\alpha_P}(N,v) = \{x = (5,5,3,2), y = (4,5,4,2)\}$, *where* x *is generated by* $\pi_x = (\{1,2\},\{3\},\{4\})$ *and* y *by* $\pi_y = (\{2\},\{1,3\},\{4\})$. *Moreover,* $\phi^{\alpha_M}(N,v) = \phi^{\alpha_D}(N,v) = \phi^{\alpha_{DM}}(N,v) = \{(5,5,3,2)\}$.

Now we have all the tools to state the main result of this section.

Theorem 1. *Let* (N,v) *be a balanced game,* $\alpha \in \mathcal{A}_{mon}$ *and* $x \in \phi^{\alpha}(N,v) \cap C(N,v)$. *Then,* $Lmax(N,v) = \{x\}$.

Proof. Let (N,v) be a balanced game, $\alpha \in \mathcal{A}_{mon}$ and $x \in \phi^{\alpha}(N,v) \cap C(N,v)$. Since $\alpha \leq \alpha_{DM}$, from Proposition 6 we know that $x \in \phi^{\alpha_{DM}}(N,v)$. Let $\pi = (S_1, S_2, \ldots, S_s)$ be an $\alpha_{DM}-$ordered partition of N generating x. If $s = 1$, then, by Proposition 5, $x = \left(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|}\right) = Lmax(N,v)$. If $s > 1$ suppose, on the contrary, $x \neq Lmax(N,v)$. Let $y = Lmax(N,v)$. As $\alpha_{DM} \in \mathcal{A}_{mon}$, we know that for all $i \in S_1$, $x_i = \frac{v(S_1)}{|S_1|} \geq x_j$ for all $j \in N$. Since $y \in C(N,v)$, there is $i_1 \in S_1$ such that $y_{i_1} \geq \frac{v(S_1)}{|S_1|}$, and thus $\hat{y}_1 \geq y_{i_1} \geq \frac{v(S_1)}{|S_1|}$. This inequality together with the fact that $\hat{y} \preceq_{lex} \hat{x}$ imply $y_{i_1} = x_{i_1}$. If $S_1 \setminus \{i_1\} \neq \emptyset$, then $y(S_1 \setminus \{i_1\}) = y(S_1) - \frac{v(S_1)}{|S_1|} \geq v(S_1) - \frac{v(S_1)}{|S_1|} = |S_1 \setminus \{i_1\}| \frac{v(S_1)}{|S_1|}$. Hence, there exists at least some player $i_2 \in S_1 \setminus \{i_1\}$ such that $y_{i_2} \geq \frac{v(S_1)}{|S_1|} = x_{i_2}$. Since $\widehat{y_{|N \setminus \{i_1\}}} \preceq_{lex} \widehat{x_{|N \setminus \{i_1\}}}$, we conclude that $y_{i_2} = x_{i_2}$. Following this process we can check that $y_k = x_k$ for all $k \in S_1$, and so $\widehat{y_{|N'}} \preceq_{lex} \widehat{x_{|N'}}$ where $N' = N \setminus S_1$. Now consider the reduced game $\left(N', r^{N'}_{\alpha_{DM},y}(v)\right)$. Since $y_{|S_1} = x_{|S_1}$, by $\alpha_{DM}-$*consistency* of the core, $x_{|N'}, y_{|N'} \in C\left(N', r^{N'}_{\alpha_{DM},y}(v)\right)$. Moreover, as $\alpha_{DM} \in \mathcal{A}_t$, $x_{|N'} \in \phi^{\alpha_{DM}}\left(N', r^{N'}_{\alpha_{DM},y}(v)\right)$ being $\pi_{|N'} = (S_2, \ldots, S_s)$ an $\alpha_{DM}-$ordered partition of N' generating $x_{|N'}$. On the other hand, by $\alpha_{DM}-$*consistency* of the lexmax solution $y_{|N'} = Lmax\left(N', r^{N'}_{\alpha_{DM},y}(v)\right)$. Now from the reasoning above we can see that $y_k = x_k$ for all $k \in S_2$. Following this line of argument we conclude that $x = y$. □

Remark 2. *It is worth to point out that monotonicity in payments is a necessary condition to guarantee that the intersection between the core and the $\alpha-$reduced equal split-off set, whenever non-empty, coincides with the lexmax solution. Indeed, let $\alpha \in \mathcal{A}$ be defined as follows: for each $N \in \mathcal{N}$*

$$\alpha(N) := \{\emptyset, N, S \subseteq N \text{ such that } |S| = 2\}.$$

Consider the balanced game (N,v) where $N = \{1,2,3,4,5,6\}$ and the characteristic function is given by: $v(\{1,2\}) = 6$, $v(\{3,4\}) = 5$, $v(\{1,3,5\}) = 8.5$, $v(N) = 14$ and $v(S) = 0$ for any other $S \subseteq N$. It can be checked that $\phi^{\alpha}(N,v) = \{x = (3,3,2.5,2.5,3,0)\}$ where $x \in C(N,v)$ and it is generated by the α-ordered partition $\pi_x = (\{1,2\},\{3,4\},\{5\},\{6\})$. Notice that $x_3 = x_4 = 2.5 < 3 = x_5$, which shows that $\alpha \notin \mathcal{A}_{mon}$. Moreover, $\phi^{\alpha_{DM}}(N,v) = \{y = (3,3,2.75,2.25,2.75,0.25)\}$. Since $y \in C(N,v)$ and $\alpha_{DM} \in \mathcal{A}_{mon}$, from Theorem 1 we know that $y = Lmax(N,v)$, being $y \neq x$.

From Theorem 1, a natural question arises: given a balanced game (N, v), is there some $\alpha \in \mathcal{A}_{mon}$ such that $Lmax(N, v) \in \phi^{\alpha}(N, v)$? Although in general this fact is not true (see Example 4 below), in Section 1.4 we will see that for some classes of games the lexmax solution can be interpreted as an α_{DM}−reduced equal split-off allocation.

Example 4. *Let (N, v) be a balanced game with set of players $N = \{1, 2, 3\}$ and characteristic function:*

S	$v(S)$	S	$v(S)$	S	$v(S)$
$\{1\}$	0	$\{1,2\}$	1	$\{1,2,3\}$	1
$\{2\}$	0	$\{1,3\}$	1		
$\{3\}$	0	$\{2,3\}$	0		

For all $\alpha \in \mathcal{A}$, $\phi^{\alpha}(N, v) = \{(0.5, 0.5, 0), (0.5, 0, 0.5)\}$ and $Lmax(N, v) = (1, 0, 0)$.

1.4 Davis and Maschler reduced equal split-off set and the lexmax solution

In this section, we show that on a class of games that includes games with large core (Sharkey, 1982) the lexmax solution turns out to be the unique α_{DM}-reduced equal split-off allocation. We first show that the Davis and Maschler reduced equal split-off set becomes a singleton when intersects with the core. Before proving it, we need a technical lemma.

Lemma 2. *Let (N, v) be a game, $M_1 = \arg\max_{\emptyset \neq T \subseteq N} \left\{ \frac{v(T)}{|T|} \right\}$, $N_1 = \{i \in S \mid S \in M_1\}$ and $x \in \phi^{\alpha_{DM}}(N, v)$ generated by the α_{DM}−ordered partition $\pi_x = (T_1, \ldots, T_t)$. Let $T_1 \cup \ldots \cup T_{q^*} = \{i \in N \mid x_i \geq x_j$ for all $j \in N\}$. If $N_1 \neq N$, then $N_1 = T_1 \cup \ldots \cup T_{q^*}$.*[4]

[4]As shown Example 4, Lemma 2 does not hold if $N_1 = N$.

Proof. Let (N,v) be a game and $x \in \phi^{\alpha_{DM}}(N,v)$ generated by $\pi_x = (T_1,\dots,T_t)$. Let $T_1 \cup \dots \cup T_{q^*} = \{i \in N \,|\, x_i \geq x_j \text{ for all } j \in N\}$. Notice first that $q^* < t$ since, otherwise, $x = \left(\frac{v(N)}{|N|},\dots,\frac{v(N)}{|N|}\right)$ which implies $N \in \arg\max_{\emptyset \neq T \subseteq N} \left\{\frac{v(T)}{|T|}\right\}$, in contradiction with $N_1 \neq N$.

First we show that $T_1 \cup \dots \cup T_{q^*} \subseteq N_1$. Let $i \in T_1 \cup \dots \cup T_{q^*}$. If $i \in T_1$, clearly $i \in N_1$. If $i \in T_h$ for some $h \in \{2,\dots,q^*\}$, then there is $Q^* \subseteq T_1 \cup \dots \cup T_{h-1}$ such that

$$x_i = \frac{v(T_1)}{|T_1|} = \frac{r^{N\setminus T_1\cup\dots\cup T_{h-1}}_{\alpha_{DM},x}(v)(T_h)}{|T_h|} = \frac{v(T_h \cup Q^*) - |Q^*|\frac{v(T_1)}{|T_1|}}{|T_h|}. \tag{1.8}$$

Reordering terms in (1.8), we have that $\frac{v(T_1)}{|T_1|} = \frac{v(T_h \cup Q^*)}{|T_h \cup Q^*|}$, which implies $T_h \cup Q^* \in \arg\max_{\emptyset \neq T \subseteq N} \left\{\frac{v(T)}{|T|}\right\}$, and thus $i \in N_1$.

To show the reverse inclusion, take $i \in N_1$ and suppose $i \notin T_1 \cup \dots \cup T_{q^*}$. Then, there is $R^* \in M_1$ such that $i \in R^*$. Next we show that $R^* \setminus T_1 \cup \dots \cup T_{q^*} \neq T_{q^*+1} \cup \dots \cup T_t$. Indeed, if $R^* \setminus T_1 \cup \dots \cup T_{q^*} = T_{q^*+1} \cup \dots \cup T_t$, then $N = T_1 \cup \dots \cup T_{q^*} \cup R^*$. As we have seen before, $T_1 \cup \dots \cup T_{q^*} \subseteq N_1$. This inclusion, together with $R^* \in \arg\max_{\emptyset \neq \subseteq N} \left\{\frac{v(T)}{|T|}\right\}$, imply $N_1 = N$, a contradiction. Hence,

$$\begin{aligned} \frac{v(T_1)}{|T_1|} &> \frac{r^{N\setminus T_1\cup\dots\cup T_{q^*}}_{\alpha_{DM},x}(v)(T_{q^*+1})}{|T_{q^*+1}|} \geq \frac{r^{N\setminus T_1\cup\dots\cup T_{q^*}}_{\alpha_{DM},x}(v)(R^*\setminus T_1\cup\dots\cup T_{q^*})}{|R^*\setminus T_1\cup\dots\cup T_{q^*}|} \\ &\geq \frac{v(R^*) - x(R^* \cap \{T_1\cup\dots\cup T_{q^*}\}}{|R^*\setminus T_1\cup\dots\cup T_{q^*}|} = \frac{v(R^*) - |R^* \cap \{T_1\cup\dots\cup T_{q^*}\}|\frac{v(T_1)}{|T_1|}}{|R^*\setminus T_1\cup\dots\cup T_{q^*}|}, \end{aligned} \tag{1.9}$$

where the first inequality follows from the definition of $T_1 \cup \dots \cup T_{q^*}$, the second one from $T_{q^*+1} \in \arg\max_{\emptyset \neq T \subseteq N\setminus T_1\cup\dots\cup T_{q^*}} \left\{\frac{r^{N\setminus T_1\cup\dots\cup T_{q^*}}_{\alpha_{DM},x}(v)(T)}{|T|}\right\}$, and the last one from the definition of the $\alpha_{DM}-$max reduced game and the fact that $R^* \setminus T_1 \cup \dots \cup T_{q^*} \neq T_{q^*+1} \cup \dots \cup T_t$. From (1.9) it follows $\frac{v(T_1)}{|T_1|} > \frac{v(R^*)}{|R^*|}$, in contradiction with $R^* \in \arg\max_{\emptyset \neq T \subseteq N} \left\{\frac{v(T)}{|T|}\right\}$. Hence, $i \in T_1 \cup \dots \cup T_{q^*}$ and $N_1 = T_1 \cup \dots \cup T_{q^*}$. $\square$

Theorem 2. *Let (N, v) be a game. If $x \in \phi^{\alpha_{DM}}(N, v) \cap C(N, v)$, then $\phi^{\alpha_{DM}}(N, v) = Lmax(N, v) = \{x\}$.*[5]

Proof. Let $x \in \phi^{\alpha_{DM}}(N, v) \cap C(N, v)$. From Theorem 1 we know that $Lmax(N, v) = \{x\}$. Suppose there is $y \in \phi^{\alpha_{DM}}(N, v) \setminus C(N, v)$. Let $\pi_x = (T_1, \ldots, T_t)$ and $\pi_y = (S_1, \ldots, S_s)$ be two $\alpha_{DM}-$ordered partitions of N generating x and y, respectively. Let

$$\begin{aligned} T_1 \cup \ldots \cup T_{q^*} &= \{i \in N \mid x_i \geq x_j \text{ for all } j \in N\} \\ S_1 \cup \ldots \cup S_{p^*} &= \{i \in N \mid y_i \geq y_j \text{ for all } j \in N\}. \end{aligned} \tag{1.10}$$

Denote $M_1 = \arg \max_{\emptyset \neq T \subseteq N} \left\{ \dfrac{v(T)}{|T|} \right\}$ and $N_1 = \{i \in S \,|\, S \in M_1\}$. We distinguish two cases.

- **Case 1**: $N_1 = N$

  If $T_1 = N$ then, by Proposition 5, $\phi^{\alpha_{DM}}(N, v) = \{x\}$. If $T_1 \neq N$, then for all $i \in T_1$, $x_i = \frac{v(T_1)}{|T_1|}$. Let $k \in \{2, \ldots, t\}$ and $i \in T_k$. Since $N = N_1$, there is $R \in M_1$ such that $i \in R$. As $x \in C(N, v)$, $x(R) = x(R \setminus T_1) + x(R \cap T_1) \geq v(R)$ or, equivalently, $x(R \setminus T_1) \geq v(R) - x(R \cap T_1) = v(R) - |R \cap T_1| \frac{v(T_1)}{|T_1|} = v(R)\left(1 - \frac{|R \cap T_1|}{|R|}\right) = \frac{v(R)}{|R|}|R \setminus T_1| = \frac{v(T_1)}{|T_1|}|R \setminus T_1|$. Since $\alpha_{DM} \in \mathcal{A}_{mon}$, for all $i \in R \setminus T_1$, we have $x_i \leq \frac{v(T_1)}{|T_1|}$. Combining both inequalities we obtain, for all $i \in R \setminus T_1$, $x_i = \frac{v(T_1)}{|T_1|}$. Therefore, for all $i, j \in N$, $x_i = x_j$. Finally, by efficiency, $x = \left(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|}\right)$ and, by Proposition 5, we conclude $\phi^{\alpha_{DM}}(N, v) = \{x\}$.

[5] Yanovskaya (2009) provides a similar result proving that if the equal split-off set of Branzei et al. (2006) intersects with the core, then it is single-valued and coincides with the weak constrained egalitarian solution of Dutta Ray (1989). However, for arbitrary $\alpha \in \mathcal{A}_{mon}$ this statement is not true. For instance, in Example 3, $\phi^{\alpha_P}(N, v) = \{(5, 5, 3, 2), (4, 5, 4, 2)\}$ and $(5, 5, 3, 2) \in C(N, v)$.

- **Case 2**: $N_1 \neq N$

  Let q^* and p^* as defined in (1.10). Notice that $q^* < t$ and $p^* < s$ since, otherwise, $N \in M_1$ contradicting $N_1 \neq N$. From Lemma 2, $N_1 = T_1 \cup \ldots \cup T_{q^*} = S_1 \cup \ldots \cup S_{p^*}$, which implies $x_i = y_i$ for all $i \in N_1$. Thus, the reduced games $\left(N \setminus N_1, r^{N\setminus N_1}_{\alpha_{DM},x}(v)\right)$ and $\left(N \setminus N_1, r^{N\setminus N_1}_{\alpha_{DM},y}(v)\right)$ coincide. By $\alpha_{DM}-$*consistency* of the core, $x_{|N\setminus N_1} \in C\left(N \setminus N_1, r^{N\setminus N_1}_{\alpha_{DM},x}(v)\right)$. Since $\alpha_{DM} \in \mathcal{A}_t$, $x_{|N\setminus N_1}, y_{|N\setminus N_1} \in \phi^{\alpha_{DM}}\left(N \setminus N_1, r^{N\setminus N_1}_{\alpha_{DM},x}(v)\right)$.

  Now define

  $$T_{q^*+1} \cup \ldots \cup T_k = \{i \in N \setminus N_1 \mid x_i \geq x_j \text{ for all } j \in N \setminus N_1\}$$
  $$S_{p^*+1} \cup \ldots \cup S_h = \{i \in N \setminus N_1 \mid y_i \geq y_j \text{ for all } j \in N \setminus N_1\}.$$

  Denote $M_2 = \arg \max\limits_{\emptyset \neq T \subseteq N\setminus N_1} \left\{ \dfrac{r^{N\setminus N_1}_{\alpha_{DM},x}(v)(T)}{|T|} \right\}$ and $N_2 = \{i \in S \mid S \in M_2\}$. If $N_2 = N \setminus N_1$ then, as in Case 1, $x_i = y_i$ for all $i \in N \setminus N_1$, and thus $x = y$. Otherwise, again from Lemma 2, we have that $N_2 = T_{q^*+1} \cup \ldots \cup T_k = S_{p^*+1} \cup \ldots \cup S_h$ and $x_i = y_i$ for all $i \in N_2$. Repeating this argument we conclude that $x = y$. □

Let us denote by $\Gamma_{\alpha_{DM}}$ the subclass of balanced games such that $(N, v) \in \Gamma_{\alpha_{DM}}$ if and only if $\phi^{\alpha_{DM}}(N, v) \cap C(N, v) \neq \emptyset$.

Remark 3. *It is worth to mention that there are games (N, v) and admissible subgroup correspondences $\alpha \in \mathcal{A}_{mon}$ such that $\phi^{\alpha}(N, v) \cap C(N, v) \neq \emptyset$ with $\alpha \neq \alpha_{DM}$. For instance, the class of games under the assumption of Proposition 5 satisfies the condition above for all $\alpha \in \mathcal{A}_{mon}$. Another example is the class of convex games. As we have commented before, on the domain of all games the equal split-off set of Branzei et al. (2006) coincides with ϕ^{α_M}. For convex games, these authors show that ϕ^{α_M} reduces to a singleton containing the weak constrained egalitarian solution of Dutta and Ray (1989), which is a core element.*

This fact, together with Proposition 6, means that $\phi^{\alpha}(N,v) \cap C(N,v) \neq \emptyset$ *for all* $\alpha \geq \alpha_M$, $\alpha \in \mathcal{A}_{mon}$, *and all convex games* (N,v).

Next we show that the class of games with large core is strictly included in $\Gamma_{\alpha_{DM}}$.

The concept of large core is based on the notion of aspiration. An **aspiration** of the game (N,v) is a vector $x \in \mathbb{R}^N$ such that $x(S) \geq v(S)$ for all $S \subseteq N$. We denote by $A(N,v)$ the set of aspirations of the game (N,v).

Definition 8. *The core of a game* (N,v) *is* **large** *if for all* $y \in A(N,v)$, *there exists* $x \in C(N,v)$ *such that* $x \leq y$.

By Γ_{lc} we denote the class of games with large core.

Theorem 3. $\Gamma_{lc} \subset \Gamma_{\alpha_{DM}}$

Proof. Let (N,v) be a game with large core and $x \in \phi^{\alpha_{DM}}(N,v)$ generated by $\pi = (T_1,\ldots,T_t)$. We will see that $x = Lmax(N,v)$.

Denote $M_1 = \arg \max\limits_{\emptyset \neq S \subseteq N} \left\{ \dfrac{v(S)}{|S|} \right\}$ and $N_1 = \{i \in S \,|\, S \in M_1\}$.
We distinguish two cases.

- **Case 1**: $N_1 = N$
  From Arin et al. (2003) it follows that $Lmax(N,v) = \left\{ \left(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|} \right) \right\}$.
  Hence, $N \in M_1$ and, by Proposition 5, $\phi^{\alpha_{DM}}(N,v) = Lmax(N,v)$.

- **Case 2**: $N_1 \neq N$
  Take $S \in M_1$ and define $y^1 \in \mathbb{R}^{N_1}$ as follows:

$$y_i^1 := \frac{v(S)}{|S|}, \text{ for all } i \in N_1. \tag{1.11}$$

  Let $T_1 \cup \ldots \cup T_{q^*} = \{i \in N \mid x_i \geq x_j \text{ for all } j \in N\}$. Notice that $q^* < t$ since, otherwise, $N \in M_1$, in contradiction with $N_1 \neq N$. Since $T_1 \cup \ldots \cup T_{q^*} = N_1$ (Lemma 2), we have that $x_{|N_1} = y^1$.

Let $(N \setminus N_1, w^1)$ be the reduced game relative to $N \setminus N_1$ at y^1 defined as follows:

$$w^1(\emptyset) = 0 \text{ and } w^1(R) = \max_{Q \subseteq N_1}\{v(R \cup Q) - y^1(Q)\}, \text{ for all } R \subseteq N \setminus N_1. \tag{1.12}$$

Denote $M_2 = \arg \max_{\emptyset \neq S \subseteq N} \left\{ \dfrac{w^1(S)}{|S|} \right\}$ and $N_2 = \{i \in S \mid S \in M_2\}$.
Take $S \in M_2$ and define $y^2 \in \mathbb{R}^{N_1 \cup N_2}$ as follows:

$$y_i^2 := y_i^1 \text{ if } i \in N_1, \text{ and } y_i^2 := \frac{w^1(S)}{|S|}, \text{ if } i \in N_2, \tag{1.13}$$

where y^1 is defined in (1.11).

- **Case 2.1**: If $N_2 = N \setminus N_1$, from (1.12) and (1.13) it is not difficult to verify that (a): $y^2 \in A(N,v)$ and (b): for a given $i \in N$, there is $R^i \subseteq N$ such that $i \in R^i$ and $y^2(R^i) = v(R^i)$. Since (N,v) has a large core, there is $z \in C(N,v)$ such that $z \leq y^2$. This inequality, together with both conditions (a) and (b), imply $y_i^2 \leq z_i$ for all $i \in N$. Hence, $y^2 = z \in C(N,v)$. From the efficiency of y^2, it follows that

$$M_2 = \arg \max_{\emptyset \neq S \subseteq N} \left\{ \frac{w^1(S)}{|S|} \right\} = \arg \max_{\emptyset \neq S \subseteq N} \left\{ \frac{r_{\alpha_{DM},x}^{N \setminus N_1}(v)(S)}{|S|} \right\}, \tag{1.14}$$

$$\text{and } r_{\alpha_{DM},x}^{N \setminus N_1}(v)(S) = w^1(S), \text{ for all } S \in M_2.$$

We claim that $N_2 = N \setminus N_1 \in M_2$. Indeed, suppose that $N_2 \notin M_2$. For all $i \in N_2$, $y_i^2 = \frac{w^1(S)}{|S|}$, where $S \in M_2$. Since y^2 is efficient, $v(N) = y^2(N_1) + y^2(N_2) = y^1(N_1) + |N_2|\frac{w^1(S)}{|S|} > y^1(N_1) + w^1(N_2) \geq y^1(N_1) + v(N_1 \cup N_2) - y^1(N_1) = v(N)$, getting a contradiction. Thus, $N_2 \in M_2$. By Proposition 5, and taking into account (1.14), we have that $\phi^{\alpha_{DM}}\left(N \setminus N_1, r_{\alpha_{DM},x}^{N \setminus N_1}(v)\right) = \left\{y^2_{|N \setminus N_1}\right\}$. By definition, and considering that $N_1 = T_1 \cup \ldots \cup T_{q^*}$ and $\alpha_{DM} \in \mathcal{A}_t$, we get $x_{|N \setminus N_1} \in \phi^{\alpha_{DM}}\left(N \setminus N_1, r_{\alpha_{DM},x}^{N \setminus N_1}(v)\right)$. Thus, $x_{|N \setminus N_1} =$

$y^2_{|N\setminus N_1}$. Since $x_{|N_1} = y^1$, we have that $x = y^2$. As $y^2 \in C(N, v)$, from Theorem 2 we conclude that $\phi^{\alpha_{DM}}(N, v) = Lmax(N, v) = \{x\}$.

• **Case 2.2**: If $N_2 \neq N \setminus N_1$, first observe that expression (1.14) holds. Let $T_{q^*+1} \cup \ldots \cup T_{p^*} = \{i \in N \setminus N_1 \,|\, x_i \geq x_j \text{ for all } j \in N \setminus N_1\}$. From Lemma 2 we know that $N_2 = T_{q^*+1} \cup \ldots \cup T_{p^*}$.

Let $(N \setminus N_1 \cup N_2, w^2)$ be the reduced game relative to $N \setminus N_1 \cup N_2$ at y^2 defined as follows:

$$w^2(\emptyset) = 0 \text{ and } w^2(R) = \max_{Q \subseteq N_1 \cup N_2} \{v(R \cup Q) - y^2(Q)\}, \text{ for all } R \subseteq N \setminus N_1 \cup N_2. \tag{1.15}$$

Denote $M_3 = \arg \max_{\emptyset \neq S \subseteq N} \left\{ \dfrac{w^2(S)}{|S|} \right\}$ and $N_3 = \{i \in S \,|\, S \in M_3\}$.

Take $S \in M_3$ and define $y^3 \in \mathbb{R}^{N_1 \cup N_2 \cup N_3}$ as follows:

$$y^3_i := y^2_i \text{ if } i \in N_1 \cup N_2, \text{ and } y^3_i := \frac{w^2(S)}{|S|}, \text{ if } i \in N_3, \tag{1.16}$$

where y^2 is defined in (1.13).

• **Case 2.2.1**: If $N_3 = N \setminus N_1 \cup N_2$, following the arguments above, we obtain that $x = y^3 \in C(N, v)$ and $\phi^{\alpha_{DM}}(N, v) = Lmax(N, v) = \{x\}$.

• **Case 2.2.2**: If $N_3 \neq N \setminus N_1 \cup N_2$, repeating the same procedure, in a finite number of steps we will get the result.

To see that the set of games with large core is strictly included in $\Gamma_{\alpha_{DM}}$, consider the game (N, v) with set of players $N = \{1, 2, 3\}$ and characteristic function $v(\{1\}) = v(\{2\}) = 0$, $v(\{3\}) = -1$, $v(\{1, 2\}) = v(\{1, 2, 3\}) = 1$ and $v(\{1, 3\}) = v(\{2, 3\}) = 0$. The core is $C(N, v) = \{(\alpha, 1 - \alpha, 0) \in \mathbb{R}^3 \text{ s.t. } \alpha \in [0, 1]\}$ and $\phi^{\alpha_{DM}}(N, v) = \left\{(\frac{1}{2}, \frac{1}{2}, 0)\right\}$. Hence, $(N, v) \in \Gamma_{\alpha_{DM}}$. Let $y = (1, 1, -1) \in A(N, v)$. Clearly, there is no $x \in C(N, v)$ such that $x \leq y$. This concludes the proof. □

Combining Theorem 2 and 3 we obtain the next result.

Corollary 1. *Let (N,v) be a game with large core. Then, $\phi^{\alpha_{DM}}(N,v) = Lmax(N,v)$.*

The proof of Theorem 2 provides a procedure for calculating the lexmax solution on $\Gamma_{\alpha_{DM}}$ by using the Davis and Maschler reduced game.

PROCEDURE 1: The input is a game $(N,v) \in \Gamma_{\alpha_{DM}}$ and the output is a payoff vector $\mathbf{F}^v \in \mathbb{R}^N$.

- **Step 1**: Let $M_1 = \arg \max\limits_{\emptyset \neq S \subseteq N} \left\{ \dfrac{v(S)}{|S|} \right\}$ and $N_1 = \{i \in S \,|\, S \in M_1\}$. Every player in N_1 receives $\frac{v(T_1)}{|T_1|}$, where $T_1 \in \arg \max\limits_{\emptyset \neq S \subseteq N} \left\{ \dfrac{v(S)}{|S|} \right\}$. That is, $\mathbf{F}^v_i = \frac{v(T_1)}{|T_1|}$ for all $i \in N_1$.

- **Step 2**: If $N_1 \neq N$, let us denote $w = r^{N \setminus N_1}_{\alpha_{DM}, x^1}(v)$, being $x^1 = \left(\frac{v(T_1)}{|T_1|}, \ldots, \frac{v(T_1)}{|T_1|} \right) \in \mathbb{R}^{N_1}$. Let $M_2 = \arg \max\limits_{\emptyset \neq S \subseteq N \setminus N_1} \left\{ \dfrac{w(S)}{|S|} \right\}$ and $N_2 = \{i \in S \,|\, S \in M_2\}$. Every player in N_2 receives $\frac{w(T_2)}{|T_2|}$, where $T_2 \in M_2$. That is, $\mathbf{F}^v_i = \frac{w(T_2)}{|T_2|}$ for all $i \in N_2$.

- The process stops when an ordered partition $(N_1, N_2, \ldots N_t)$ of N, for some $1 \leq t \leq |N|$, is reached.

Interestingly, the above algorithm can be applied in a more general setting: if the input is an arbitrary game, not necessarily belonging to $\Gamma_{\alpha_{DM}}$, and the output is a core element, then it coincides with the lexmax solution.

Let us denote by $\mathbf{F}^v_*$ the allocation generated by Procedure 1 when the input is the game (N,v).

Theorem 4. *Let (N,v) be a balanced game. If $\mathbf{F}^v_* \in C(N,v)$, then $Lmax(N,v) = \mathbf{F}^v_*$.*

Proof. Let (N,v) be a balanced game and $\pi = (N_1, \ldots, N_t)$ be the ordered partition of N generating $\mathbf{F}^v_*$. Recall that $N_1 = \{i \in S \mid S \in M_1\}$, where $M_1 = \arg \max\limits_{\emptyset \neq S \subseteq N} \left\{ \dfrac{v(S)}{|S|} \right\}$. If $N_1 = N$ then, by efficiency, $\mathbf{F}^v_* = \left(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|} \right)$, and thus $Lmax(N, v) = \mathbf{F}^v_*$. If $N_1 \neq N$, let $Lmax(N, v) = y$ and suppose $y \neq \mathbf{F}^v_*$. Let $S \in M_1$. Since $\alpha_{DM} \in \mathcal{A}_{mon}$, the same argument used in the proof of Theorem 1 leads to $\mathbf{F}^v_{*i} = y_i$ for all $i \in S$. Consequently, $\mathbf{F}^v_{*i} = y_i$ for all $i \in N_1$. Hence, $\widehat{y_{|N\setminus N_1}} \preceq_{lex} \widehat{\mathbf{F}^v_{*|N\setminus N_1}}$. Now consider the $\alpha_{DM}-$max reduced game $(N \setminus N_1, w)$, where $w = r^{N\setminus N_1}_{\alpha_{DM},\mathbf{F}^v_*}(v)$. Since $y_{|N_1} = \mathbf{F}^v_{*|N_1}$, by $\alpha_{DM}-$*consistency* of the core $\mathbf{F}^v_{*|N\setminus N_1}, y_{|N\setminus N_1} \in C(N \setminus N_1, w)$. Moreover, by $\alpha_{DM}-$*consistency* of the lexmax solution $y_{|N\setminus N_1} = Lmax(N \setminus N_1, w)$. Since $\mathbf{F}^v_{*|N\setminus N_1} = \mathbf{F}^w_*$, as before we can check that $y_i = \mathbf{F}^v_{*i}$ for all $i \in N_2$. Following this process step by step, and considering that $\alpha_{DM} \in \mathcal{A}_t$, we conclude that $\mathbf{F}^v_* = Lmax(N, v)$. □

In order to identify other kind of balanced games where the allocation $\mathbf{F}^v_*$ belongs to the core, let us introduce exact games (Schmeidler, 1972).

Definition 9. *A balanced game (N, v) is* **exact** *if for every coalition $S \subseteq N$ there is $x \in C(N, v)$ such that $v(S) = x(S)$.*

The exactification (N, v^E) of an arbitrary balanced game (N, v) is the unique exact game with the same core as the original game (N, v), that is, for each $S \subseteq N$, $v^E(S) = \min\{x(S) \mid x \in C(N, v)\}$.

Note that if for two balanced games (N, v_1), (N, v_2) we have $C(N, v_1) = C(N, v_2) \neq \emptyset$, then $Lmax(N, v_1) = Lmax(N, v_2)$. Thus, the lexmax solution is invariant with respect to exactification. This open a natural question: on the domain of exact games, can Theorem 4 be applied? It was noted by Biswas et al. (1999) that any exact game (N, v) with $|N| \leq 4$ has a large core, and therefore $Lmax(N, v) = \mathbf{F}^v_*$. By contrast, if $|N| \geq 5$, there are exact games (N, v) not having a large core. Unfortunately, as shown Example 5 for exact games with more than four players our procedure for finding the lexmax solution does not work.

Example 5. *(Biswas et al. 1999) Let (N, v) be an exact game with $|N| = 5$, where the characteristic function is defined by $v(S) = \min\{x(S), y(S)\}$, being*

$x = (1, -1, -1, 0, 0)$ *and* $y = (0, 1, 0, -1, -1)$.

S	$v(S)$	S	$v(S)$	S	$v(S)$	S	$v(S)$
$\{1\}$	0	$\{12\}$	0	$\{123\}$	-1	$\{1234\}$	-1
$\{2\}$	-1	$\{13\}$	0	$\{124\}$	0	$\{1235\}$	-1
$\{3\}$	-1	$\{14\}$	-1	$\{125\}$	0	$\{1245\}$	-1
$\{4\}$	-1	$\{15\}$	-1	$\{134\}$	-1	$\{1345\}$	-2
$\{5\}$	-1	$\{23\}$	-2	$\{135\}$	-1	$\{2345\}$	-2
		$\{24\}$	-1	$\{145\}$	-2	$\{12345\}$	-1
		$\{25\}$	-1	$\{234\}$	-2		
		$\{34\}$	-1	$\{235\}$	-2		
		$\{35\}$	-1	$\{245\}$	-1		
		$\{45\}$	-2	$\{345\}$	-2		

We can check that $M_1 = \arg \max_{\emptyset \neq S \subseteq N} \left\{ \frac{v(S)}{|S|} \right\} = \{\{1\}, \{12\}, \{13\}, \{124\}, \{125\}\}$ *and* $N_1 = N$. *Then,* $\mathbf{F}^v_* = (0, 0, 0, 0, 0)$ *and* $\mathbf{F}^v_* \notin C(N, v)$.

We end this section linking the $\alpha_{DM}-$reduced equal split-off set with the egalitarian solution of Dutta and Ray (1989).

On the domain of convex games, Dutta and Ray (1989) show that the weak constrained egalitarian solution, denoted by EL, is the unique Lorenz maximal allocation in the core, and hence it coincides with the lexmax solution. This, together with the fact that convex games have large core, leads to the following corollary.

Corollary 2. *Let* (N, v) *be a convex game. Then,* $\phi^{\alpha_{DM}}(N, v) = EL(N, v)$.

1.5 Anti-dual reduced equal split-off set

To complement the analysis provided in the previous sections, for a given admissible subgroup correspondence $\alpha \in \mathcal{A}$ we introduce the anti-dual solution of ϕ^α, denoted by $(\phi^{\alpha})^{ad}$, and study its relation with the core.

The notion of duality applied to solutions and properties has been very successful in the axiomatic approach of bankruptcy problems (see, for instance, Aumann and Maschler, 1985; Herrero and Villar, 2001; Thomson and Yeh, 2008), and it has also play a role in the general framework of games connecting different economic models and solutions. However, its applicability is limited because of some well-established domains of games are not closed under the dual operator: for instance, the dual game of a balanced game is not a balanced game, and the dual game of a convex game is not a convex game. To overcome this difficulty, Oishi and Nakayama (2009) define the *anti-dual* game to be the dual game with opposite sign. They show that both balancedness and convexity of a game are preserved under the anti-dual operator, and solutions such as the core, the nucleolus or the Shapley value of the anti-dual game are obtained by multiplying the corresponding solution in the original game by -1. In this section, we show that the anti-dual solution of the lexmax is the **lexmin** solution (Arin and Iñarra, 2001; Yanovskaya, 1995). Making use of this relation, we provide a characterization of the lexmin solution. Finally, we prove that for any monotonic admissible subgroup correspondence $\alpha \in \mathcal{A}_{mon}$, the intersection between $(\phi^{\alpha})^{ad}$ and the core is either the empty set or the lexmin solution. Additionally, we find out that $(\phi^{\alpha_{DM}})^{ad}$ becomes a singleton containing the lexmin allocation when intersects with the core.

Let us first introduce some additional definitions.

Given a game (N, v), the **dual game** (N, v^d) is defined by setting, for all $S \subseteq N$, $v^d(S) = v(N) - v(N \setminus S)$. Let Γ^* be a class of games such that, for all $N \in \mathcal{N}$, it holds $(N, v), (N, v^d) \in \Gamma^*$. Given a solution σ on Γ^*, the **dual solution of** σ, denoted by σ^d, is defined by setting $\sigma^d(N, v) = \sigma(N, v^d)$. A solution σ on Γ^* is **self-dual** if for all $(N, v) \in \Gamma^*$, $\sigma(N, v) = \sigma^d(N, v)$.

Given a game (N, v), the **anti-dual game** is $(N, -v^d)$. Let Γ^{**} be a class of games such that, for all $N \in \mathcal{N}$, it holds $(N, v), (N, -v^d) \in \Gamma^{**}$. The class of balanced games and the class of convex games are examples of Γ^{**}. Given a solution σ on Γ^{**}, the **anti-dual solution of** σ, denoted by σ^{ad}, is defined by setting $\sigma^{ad}(N, v) = -\sigma(N, -v^d)$. A solution σ on Γ^{**} is **self-anti-dual** if for all $(N, v) \in \Gamma^{**}$, $\sigma(N, v) = \sigma^{ad}(N, v)$. Some well-known self-anti-dual solutions are, among others, the core (on the domain of balanced games) and the weak constrained egalitarian solution (on the domain of convex games).[6]

Definition 10. *For a balanced game* (N, v)*, the* **lexmin** *solution is defined as* $Lmin(N, v) = \left\{x \in C(N, v) \,|\, \widehat{-x} \preceq_{lex} \widehat{-y} \text{ for all } y \in C(N, v)\right\}$.

For any balanced game (N, v), the lexmin solution is a singleton and then sometimes we write $x = Lmin(N, v)$. It is quite straightforward to see that the lexmin is the anti-dual of the lexmax solution.

Proposition 7. *Let* (N, v) *be a balanced game. Then,*

$$Lmin(N, v) = L^{ad}max(N, v).$$

Proof. Let (N, v) be a balanced game and $x = Lmin(N, v)$. We must check that $-x = Lmax(N, -v^d)$, that is, $\widehat{-x} \preceq_{lex} \widehat{y}$ for all $y \in C(N, -v^d)$. By the definition of lexmin solution, $\widehat{-x} \preceq_{lex} \widehat{-y}$ for all $y \in C(N, v)$. Now, taking into account that the core is self-anti-dual, $C(N, v) = -C(N, -v^d)$, the above inequality is equivalent to $\widehat{-x} \preceq_{lex} \widehat{y}$ for all $y \in C(N, -v^d)$. This concludes the proof. $\square$

Due to the anti-dual relation between the lexmax and the lexmin solutions, and a characterization of the former provided by Arin et al. (2008), we get a similar one for the lexmin solution.

Given a game (N, v) and an allocation $x \in \mathbb{R}^N$, we say that a coalition $S \subseteq N$ is **tight** at x if $x(S) = v(S)$. Let us denote by $\mathcal{T}(v, x)$ the set of all tight coalitions

[6] See Oishi et al. (2016) for others examples of self-anti-dual solutions.

at x. By $\mathcal{T}^c(v,x)$ we denote the set of all tight complement coalitions at x, that is, $\mathcal{T}^c(v,x) = \{S \subseteq N \mid x(N \setminus S) = v(N \setminus S)\}$.

Definition 11. *(Arin et al., 2008) Given $N \in \mathcal{N}$, let $\mathcal{I} = (I_1, \ldots, I_p)$ be an ordered partition of N and let $\mathcal{C}$ be a collection of subsets of N. We say that the pair $(\mathcal{I}, \mathcal{C})$ has property* **I** *if $z = (0, \ldots, 0) \in \mathbb{R}^N$ is the unique solution to the following system of (in)equalities:*

(1) $z(N) = 0$,

(2) $z(S) \geq 0$ *for all* $S \in \mathcal{C}$,

(3) for all $k \in \{1, \ldots, p\}$, if $z_i = 0$ for all $i \in I_1 \cup \ldots \cup I_{k-1}$, then $z_i \leq 0$ for all $i \in I_k$.

Definition 12. *Let $N = \{1, \ldots, n\}$ be a finite set of players and $x \in \mathbb{R}^N$. We define the ordered partition of N induced by x, $\mathcal{I}(x) = (I_1, \ldots, I_p)$, as follows:*

$$\begin{aligned} I_1 &= \{i \in N \,|\, x_i \geq x_k \text{ for all } k \in N\}, \\ I_2 &= \{i \in N \setminus N_1 \,|\, x_i \geq x_k \text{ for all } k \in N \setminus I_1\}, \\ &\vdots \\ I_p &= \{i \in N \setminus I_1 \cup \ldots \cup I_{p-1} \,|\, x_i \geq x_k \text{ for all } k \in N \setminus I_1 \cup \ldots \cup I_{p-1}\}. \end{aligned}$$

Given $\mathcal{I}(x) = (I_1, \ldots, I_p)$, we define the complement ordered partition of N induced by x, $\mathcal{I}^c(x) = (I_1^c, \ldots, I_p^c)$, as $I_1^c = I_p, \ldots, I_p^c = I_1$.

Lemma 1 in Arin et al. (2008) says that for a given balanced game (N, v), $x = Lmax(N, v)$ if and only if $(\mathcal{I}(x), \mathcal{T}(v,x))$ has property **I**.[7] Here, the anti-dual operator bring us a parallel characterization for the lexmin solution in a direct way.

[7]The characterization of the lexmax solution provided by Arin et al. (2008) (Lemma 1) has some resemblance with the characterization of the nucleolus (Schmeider, 1966) enunciated by Kohlberg (1971) (Theorem 2).

Lemma 3. *Let (N,v) be a balanced game and $x \in C(N,v)$. Then, $x = Lmin(N,v)$ if and only if $(\mathcal{I}^c(x), \mathcal{T}^c(v,x))$ has property* **I**.

Proof. Let (N,v) be a balanced game and $x = Lmin(N,v)$. By Proposition 7, $Lmin(N,v) = L^{ad}max(N,v)$, and from Lemma 1 in Arin et al. (2008) we know that $-x = Lmax(N,-v^d)$ if and only if $(\mathcal{I}(-x), \mathcal{T}(-v^d,-x))$ has property **I**. Taking into account that $\mathcal{I}(-x) = \mathcal{I}^c(x)$ and $v(N) = x(N)$, we obtain

$$\begin{aligned}
\mathcal{T}(-v^d,-x) &= \left\{S \subseteq N \,|\, -x(S) = -v^d(S)\right\}, \\
&= \left\{S \subseteq N \,|\, x(N \setminus S) = v(N \setminus S)\right\}, \\
&= \mathcal{T}^c(v,x).
\end{aligned}$$

Hence, $x = Lmin(N,v)$ if and only if $(\mathcal{I}^c(x), \mathcal{T}^c(v,x))$ has property **I**. $\square$

Finally, we state the anti-dual results of Theorems 1 and 2.

Theorem 5. *Let (N,v) be a balanced game, $\alpha \in \mathcal{A}_{mon}$ and $x \in (\phi^\alpha)^{ad}(N,v) \cap C(N,v)$. Then, $Lmin(N,v) = \{x\}$.*

Proof. Let $\alpha \in \mathcal{A}_{mon}$ and $x \in (\phi^\alpha)^{ad}(N,v) \cap C(N,v)$. Then, $x \in -\phi^\alpha(N,-v^d) \cap C(N,v)$ or, equivalently, $-x \in \phi^\alpha(N,-v^d) \cap C(N,-v^d)$. From Theorem 1 we know that $Lmax(N,-v^d) = \{-x\}$. Since $Lmin(N,v) = -Lmax(N,-v^d)$, we conclude that $Lmin(N,v) = \{x\}$. $\square$

Theorem 6. *Let (N,v) be a balanced game. If $x \in (\phi^{\alpha_{DM}})^{ad}(N,v) \cap C(N,v)$, then $(\phi^{\alpha_{DM}})^{ad}(N,v) = Lmin(N,v) = \{x\}$.*

Proof. Let $x \in (\phi^{\alpha_{DM}})^{ad}(N,v) \cap C(N,v)$. Then, $x \in -\phi^{\alpha_{DM}}(N,-v^d) \cap C(N,v)$ or, equivalently, $-x \in \phi^{\alpha_{DM}}(N,-v^d) \cap C(N,-v^d)$. From Theorem 2 we know that $\phi^{\alpha_{DM}}(N,-v^d) = Lmax(N,-v^d) = \{-x\}$. Since $Lmin(N,v) = -Lmax(N,-v^d)$, we conclude that

$$(\phi^{\alpha_{DM}})^{ad}(N,v) = -\phi^{\alpha_{DM}}(N,-v^d) = -Lmax(N,-v^d) = Lmin(N,v) = \{x\}. \quad \square$$

The next example shows that Theorem 6 does not hold on the domain of games with large core, unlike what happens for the lexmax solution (see Corollary 1). This example also illustrates that largeness of the core is not preserved by the anti-dual operator.

Example 6. *Let (N, v) be a game with set of players $N = \{1, 2, 3\}$ and characteristic function $v(\{1\}) = -1$, $v(\{2\}) = v(\{3\}) = 0$ and $v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = v(\{1, 2, 3\}) = -1$. The core of this game is $C(N, v) = \{(-1, 0, 0)\}$ and clearly it has large core since $(-1, 0, 0) \leq x$ for any aspiration $x \in A(N, v)$. However, for any $\alpha \in \mathcal{A}$,*

$$(\phi^{\alpha})^{ad}(N, v) = -\phi^{\alpha}(N, -v^d) = \{(-0.5, -0.5, 0), (-0.5, 0, -0.5)\},$$

where $Lmin(N, v) = \{(-1, 0, 0)\}$.

Notice that $\phi^{\alpha_{DM}}(N, v) = Lmax(N, v) = \{(-1, 0, 0)\}$.

We end this part stating sufficient conditions to guarantee when the output of Procedure 1 is the lexmin solution.

Theorem 7. *Let (N, v) be a balanced game and $\mathbf{F}_*^{-v^d}$ the allocation generated by Procedure 1 when the input is the anti-dual game $(N, -v^d)$. If $\mathbf{F}_*^{-v^d} \in C(N, -v^d)$, then $Lmin(N, v) = -\mathbf{F}_*^{-v^d}$.*

Proof. By Theorem 4, $Lmax(N, -v^d) = \mathbf{F}_*^{-v^d}$. Since $Lmin(N, v) = -Lmax(N, -v^d)$, we conclude that $Lmin(N, v) = -\mathbf{F}_*^{-v^d}$. □

1.6 Conclusions

For each admissible subgroup correspondence $\alpha \in \mathcal{A}$ (Thomson, 1990), we have introduced the $\alpha-$reduced equal split-off set, ϕ^{α}, a discrete set-valued solution concept based on egalitarian and consistency principles. Surprisingly, when we consider any $\alpha \in \mathcal{A}$ satisfying a monotonicity property in payments, weaker than the transitivity of the reduction operation, the intersection between ϕ^{α} and the core is either the empty set or a singleton containing the lexmax solution (Arin et al., 2003). Moreover, if we make use of the Davis and Maschler (1965) admissible subgroup correspondence, $\alpha_{DM} \in \mathcal{A}$, and $\phi^{\alpha_{DM}}$ intersects with the core, then it coincides with the lexmax solution. We have identified a subclass of balanced games, that includes games with large core (Sharkey, 1982), where this occurs. Within this subclass, we have provided a procedure to calculate the lexmax solution. Although on the full domain of balanced games this procedure does not work, if the output is a core element, then it matches the lexmax solution.

Finally, we find parallel results for the lexmin solution (Arin and Iñarra, 2001; Yanovskaya, 1995) by considering the anti-dual solution of ϕ^{α}, $(\phi^{\alpha})^{ad}$. Interestingly, for a given balanced game (N, v), if the input in the procedure defined to calculate the lexmax solution is its anti-dual game $(N, -v^{d})$ and the final output is a core element x, then x coincides with the lexmin solution. Unfortunately, unlike what happens for the lexmax solution, for games with large core this "anti-dual procedure" does not work. Thus, in future research it could be interesting to design mechanisms to find the lexmin solution in this domain. Although the lexmax solution has been axiomatized on the domain of games with large core (Arin et al, 2003), as far as we know, there is no proper characterization for the lexmin solution in this domain. This, together with axiomatic characterizations of ϕ^{α}, could be interesting topics for future investigations. Further studies should also examine the relation between ϕ^{α} and its anti-dual counterpart $(\phi^{\alpha})^{ad}$ with other egalitarian solutions such as the equal division core (Selten, 1991). Recall that Dutta and Ray's strong constrained egalitarian solution (1991) selects the Lorenz-maximal allocations within the equal division core.

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Chapter 2

On the weak constrained egalitarian solution and other Lorenz maximal imputations[1]

2.1 Introduction

As we have commented in Chapter 1, on the domain of transferable utility coalitional games (games, for short), different solution concepts have been motivated by the idea of egalitarianism. One of the most prominent is the weak constrained egalitarian solution (WCES), introduced by Dutta and Ray (1989). This solution is defined in a setting where agents believe in equality as a desirable social goal, but their individual preferences dictate selfish behavior. The WCES yields, whenever it exists, the unique Lorenz maximal imputation within the Lorenz

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core, which is a proper extension of the core. Although this is a sharp result because the Lorenz domination generates a partial ranking, this solution lacks general existence properties. In fact, the class of convex games (Shapley, 1971) is the only standard class of games in which its existence is guaranteed. On this domain, Dutta and Ray (1989) describe an algorithm for finding their egalitarian allocation and show that it belongs to the core and Lorenz dominates every other core element. Unfortunately, several examples in the same paper show that, in a general domain, these assertions are not true: there are games with a nonempty core where the WCES does not exist, and vice-versa, games where both the core and the WCES exist but the latter does not lie in the core, or games where the WCES belongs to the core but does not Lorenz dominate every other core element. On the domain of balanced games, an alternative route, already suggested by Dutta and Ray (1989) and latter adopted by Arin and Iñarra (2001) and Hougaard et al. (2001), is to focus on the Lorenz maximal allocations within the core. A problem with this solution concept is that it is not single-valued. To overcome this drawback, Arin and Iñarra (2001) and Arin et al. (2003) propose single-valued solutions which are derived from the application of the Rawlsian criterion on the core. On the domain of convex games all these solution concepts produce the same outcome.

The characterization of the non-emptyness of the WCES on the full domain of games is still an open problem and, in our opinion, a nontrivial task. A little step in this direction is to observe that the statements of both Theorem 1 and Theorem 2 in Dutta and Ray (1989) hold under weaker conditions than convexity. With this objective, in Section 2.3 we introduce a subclass of balanced games called exact partition games. This class of games is rich enough to include convex games and dominant diagonal assignment games (Solymosi and Raghavan, 2001), but also nonsuperadditive games. On the domain of exact partition games, in Section

2.4 we use Lorenz order to provide two axiomatic characterizations of the WCES by means of suitable properties such as consistency (à la Davis and Maschler, 1965), rationality and two new properties inspired by von Neumann and Morgensten's notion of stable sets (1944). As particular cases, we obtain alternative characterizations of the WCES over the domain of convex games, and of the set of Lorenz maximal allocations within the core over the domain of balanced games. In Section 2.5 we obtain additional axiomatizations of these solutions by making use of the anti-duality notion for linking solutions and properties as introduced in Oishi et al. (2016).

In the second part of this chapter, we consider the domain of essential games. In particular, we interpret the Lorenz maximal allocations in the imputation set as a kind of stable set à la von Neumann-Morgenstern. There, a *stable set* is defined as a subset of imputations satisfying *internal stability* and *external stability*, where the notion of stability is defined by means of a domination relation that uses the standard order in $\mathbb{R}$. Unfortunately, finding stable sets is a difficult task and neither existence nor uniqueness are guaranteed. In Section 2.6, we propose combining the idea of internal and external stability with the Lorenz order. In this way, a set of imputations $\mathcal{V}$ is said to be *Lorenz stable* if it satisfies *internal Lorenz stability* (no element in $\mathcal{V}$ is Lorenz dominated by other element in $\mathcal{V}$) and *external Lorenz stability* (every element outside $\mathcal{V}$ is Lorenz dominated by some element in $\mathcal{V}$). Clearly, this definition leads to selecting the Lorenz maximal allocations in the imputation set. We find that the Lorenz stable set is a singleton and can be computed with a simple formula. We also provide an axiomatic characterization similar to the ones given by Dutta (1990) to characterize the WCES. Finally, in Section 2.7 we connect the Lorenz stable set with the WCES and Dutta and Ray's *strong constrained egalitarian* solution (1991) (SCES). Some final remarks conclude the chapter. We begin with some preliminaries.

2.2 Notation and terminology

Together with the notation and terminology introduced in Chapter 1, here we will use the additional one.

Two games (N, v) and (N, v') are **strategically equivalent** if there is a vector $(d_1, \dots, d_n) \in \mathbb{R}^N$ and $\alpha > 0$ such that for all coalitions $S \subseteq N$, $v'(S) = \alpha\, v(S) + \sum_{i \in S} d_i$. A solution σ on $\Gamma' \subseteq \Gamma$ satisfies **covariance** if for all two strategically equivalent games $(N, v), (N, v') \in \Gamma'$, $\sigma(N, v') = \alpha\, \sigma(N, v) + \sum_{i \in N} d_i$.

A coalition S is an **equity coalition** of (N, v) if $S \in \arg\max_{\emptyset \neq R \subseteq N} \left\{ \frac{v(R)}{|R|} \right\}$. In addition, S is a **maximal** (w.r.t. inclusion) **equity coalition** of (N, v) if $S \in \arg\max_{\emptyset \neq R \subseteq N} \left\{ \frac{v(R)}{|R|} \right\}$ and there is no $T \in \arg\max_{\emptyset \neq R \subseteq N} \left\{ \frac{v(R)}{|R|} \right\}$ such that $S \subset T$. Given a coalition $S \in \mathcal{N}$ and a set $A \subseteq \mathbb{R}^S$, EA denotes the set of allocations that are Lorenz undominated within A. That is, $EA := \{x \in A \mid \nexists\, y \in A \text{ such that } y \succ_{\mathcal{L}} x\}$. Given a game (N, v), the **Lorenz core** is defined in a recursive way as follows. The Lorenz core of a singleton coalition is $L(\{i\}, v) = \{v(\{i\})\}$. Now suppose that the Lorenz core for all coalitions of cardinality k or less have been defined, where $1 < k < |N|$. The Lorenz core of a coalition $S \subseteq N$ of size $(k + 1)$ is defined by

$$L(S, v) = \left\{ x \in \mathbb{R}^S \mid x(S) = v(S) \text{ and } \nexists\, T \subset S \text{ and } y \in EL(T, v) \text{ such that } y > x_{|T} \right\}.$$

Note that, for all $S \subseteq N$, $C(S, v) \subseteq L(S, v)$. The **WCES**, denoted by EL, selects the vectors that are Lorenz undominated within the Lorenz core. For all $(N, v) \in \Gamma$, $|EL(N, v)| \leq 1$ (Dutta and Ray, 1989). The **strong Lorenz core** (Dutta and Ray, 1991) is defined in a similar way, but replacing $>$ by $\gg$. Dutta and Ray (1991) show that the strong Lorenz core, denoted by L^*, coincides with the **equal division core** when the coalition structure is N and there are no restrictions on coalition formation (see Selten, 1972 for details). That is, given an essential game (N, v), $L^*(N, v) = \{x \in I(N, v) \mid \text{for all } \emptyset \neq$

$S \subset N$, there is $i \in S$ with $x_i \geq \frac{v(S)}{|S|}\}$. The **SCES**, denoted by EL^*, chooses the vectors Lorenz-undominated within the strong Lorenz core. The **constrained egalitarian solution**, denoted by CE, is a single-valued solution defined for two person games as follows: let (N, v) be a game with $N = \{i, j\}$ and suppose, without loss of generality, $v(i) \leq v(j)$, then $CE_j(N, v) = \max\left\{\frac{v(N)}{2}, v(j)\right\}$ and $CE_i(N, v) = v(N) - CE_j(N, v)$.

The next two observations will be useful to prove our results.

Remark 4. *(Hougaard et al. 2001 p. 153) Let N be a finite set of players, and let $S \subseteq N$, $S \neq \emptyset$. If x_S, $y_S \in \mathbb{R}^S$, $x_S(S) = y_S(S)$ and $z_{N\setminus S} \in \mathbb{R}^{N\setminus S}$, then x_S Lorenz dominates y_S if and only if $\left(x_S, z_{N\setminus S}\right)$ Lorenz dominates $\left(y_S, z_{N\setminus S}\right)$.*

Remark 5. *Let N be a finite set of players, $c \in \mathbb{R}$ and $(x_1, \ldots, x_n) \in \mathbb{R}^N$. It is well-known that if $\sum_{i\in N} x_i = nc$, then x is Lorenz dominated by $(c, \ldots, c) \in \mathbb{R}^N$. If $\sum_{i\in N} x_i > nc$, let $\epsilon = \sum_{i\in N} x_i - nc$ and define $x^\epsilon = (x_1 - \frac{\epsilon}{n}, \ldots, x_n - \frac{\epsilon}{n})$. Note that $\widehat{x^\epsilon}_i = \widehat{x}_i - \frac{\epsilon}{n} < \widehat{x}_i$, for all $i \in N$. Thus, x^ϵ is Lorenz dominated by $(c, \ldots, c)$ which implies, for all $k = 1, \ldots, n$, $\hat{x}_1 + \ldots + \hat{x}_k > \widehat{x^\epsilon}_1 + \ldots + \widehat{x^\epsilon}_k \geq kc$.*

2.3 Exact partition games

On the domain of convex games, Dutta and Ray (1989) show that the WCES picks the payoff vector that is obtained by the following algorithm.

Let (N, v) be a convex game and $EL(N, v) = \{x\}$.

Step 1: Define $v_1 = v$. Then find the unique coalition $T_1 \subseteq N$ such that for all $T \subseteq N$, (i) $\frac{v_1(T_1)}{|T_1|} \geq \frac{v_1(T)}{|T|}$, and (ii) if $\frac{v_1(T_1)}{|T_1|} = \frac{v_1(T)}{|T|}$ and $T \neq T_1$, then $|T_1| > |T|$. Uniqueness of such a coalition is guaranteed by convexity of (N, v). For all $i \in T_1$,

$$x_i = \frac{v_1(T_1)}{|T_1|}.$$

Step k: Suppose that $T_1, \ldots, T_{k-1}$ have been defined.

Let $N_k = N \setminus \{T_1 \cup \ldots \cup T_{k-1}\}$ and let (N_k, v_k) be the **marginal game** defined as follows:

$$v_k(S) := v(T_1 \cup \ldots \cup T_{k-1} \cup S) - v(T_1 \cup \ldots \cup T_{k-1}), \tag{2.1}$$

for all $S \subseteq N_k$.

It can be shown that (N_k, v_k) is convex. Then find the unique coalition $T_k \subseteq N_k$ such that for all $T \subseteq N_k$, (i) $\frac{v_k(T_k)}{|T_k|} \geq \frac{v_k(T)}{|T|}$, and (ii) if $\frac{v_k(T_k)}{|T_k|} = \frac{v_k(T)}{|T|}$ and $T \neq T_k$, then $|T_k| > |T|$. For all $i \in T_k$,

$$x_i = \frac{v_k(T_k)}{|T_k|} = \frac{v(T_1 \cup \ldots \cup T_k) - v(T_1 \cup \ldots \cup T_{k-1})}{|T_k|}.$$

By construction, the WCES satisfies the following conditions: if $\pi = (T_1, \ldots, T_t)$ is the ordered partition of N induced by $EL(N, v) = \{x\}$, then

- **(C1)**: $x_i = x_j$ for all $i, j \in T_k$ and $k = 1, \ldots, t$,
- **(C2)**: $x(T_1 \cup \ldots \cup T_k) = v(T_1 \cup \ldots \cup T_k)$, for all $k = 1, \ldots, t$,
- **(C3)**: $x_i > x_j$ if $i \in T_k$, $j \in T_h$, and $k < h \leq t$.

The idea underlying this procedure is that agents in the unique maximal (w.r.t. inclusion) coalition T_1 maximizing the average worth $\frac{v(T_1)}{|T_1|}$ share equally the amount $v(T_1)$ among them and leave the game. Then, the remaining agents $N \setminus T_1$ play a suitable reduced convex game where, again, agents in the unique maximal coalition with highest average worth divide its worth equally among its members. The process stops when all agents have been paid.

Theorem 2 in Dutta and Ray (1989) states that, on the domain of convex games, the output of this algorithm is the WCES and that it belongs to the core. Theorem 3 in the same paper tells us that, for convex games, the WCES Lorenz dominates every other core element. Nevertheless, an analysis of the proofs of the aforementioned results reveals that much weaker conditions than convexity are sufficient to guarantee the same results.

Definition 13. *Let $N = \{1, \ldots, n\}$ be a finite set of players and $x \in \mathbb{R}^N$. We define the ordered partition of N induced by x, $\pi = (N_1, \ldots, N_m)$, as follows:*

$$\begin{aligned} N_1 &= \{i \in N \,|\, x_i \geq x_k \text{ for all } k \in N\}, \\ N_2 &= \{i \in N \setminus N_1 \,|\, x_i \geq x_k \text{ for all } k \in N \setminus N_1\}, \\ &\vdots \\ N_m &= \{i \in N \setminus N_1 \cup \ldots \cup N_{m-1} \,|\, x_i \geq x_k \text{ for all } k \in N \setminus N_1 \cup \ldots \cup N_{m-1}\}. \end{aligned}$$

Theorem 8. *Let (N, v) be a balanced game, $x \in C(N, v)$ and let $\pi = (N_1, \ldots, N_m)$ be the ordered partition of N induced by x. If $x(N_1 \cup \ldots \cup N_k) = v(N_1 \cup \ldots \cup N_k)$, for all $k = 1, \ldots, m$, then $EL(N, v) = \{x\}$ and $x \succ_{\mathcal{L}} y$, for all $y \in C(N, v) \setminus \{x\}$.*

Proof. First we show that $x \succ_{\mathcal{L}} y$, for all $y \in C(N, v) \setminus \{x\}$.

Assume, without loss of generality, that $x_1 \geq x_2 \geq \ldots \geq x_n$. Then, the vector obtained from x by rearranging its coordinates in non-increasing order is $\widehat{x} = x$. Let us denote

$$c_k = \begin{cases} \frac{v(N_1)}{|N_1|} & \text{if } k = 1 \\ \\ \frac{v(N_1 \cup \ldots \cup N_{k-1} \cup N_k) - v(N_1 \cup \ldots \cup N_{k-1})}{|N_k|} & \text{if } k > 1 \end{cases}$$

for all $k = 1, \ldots, m$, $(m > 1)$.

Notice that $x_i = c_k$ for all $i \in N_k$ and $k = 1, \ldots, m$. Let $y \in C(N, v)$, $y \neq x$. From Remark 4 we may suppose, without loss of generality, $x_i \neq y_i$ for all $i \in N$. Since $y(N_1) \geq v(N_1) = x(N_1) = c_1|N_1|$, and by Remark 5, we have that for all $t = 1, \ldots, |N_1|$,

$$tc_1 \leq \widehat{y_{|N_1}}_1 + \ldots + \widehat{y_{|N_1}}_t, \tag{2.2}$$

with at least one strict inequality.

Next we are going to prove that, for all $t = 1, \ldots, |N_2|$,

$$x(N_1) + tc_2 \leq y(N_1) + \widehat{y_{|N_2}}_1 + \ldots + \widehat{y_{|N_2}}_t. \tag{2.3}$$

If $y(N_2) \geq x(N_2) = |N_2|c_2$, again by Remark 5, $tc_2 \leq \widehat{y_{|N_2}}_1 + \ldots + \widehat{y_{|N_2}}_t$, for all $t = 1, \ldots, |N_2|$. This set of inequalities, together with (2.2), lead to expression (2.3).

If $y(N_2) < x(N_2)$, let us denote $\varphi_1 = y(N_1) - x(N_1) \geq 0$ and $\beta_1 = x(N_2) - y(N_2) > 0$. Let $z \in \mathbb{R}^{N_2}$ defined as $z_i = y_i + \frac{\beta_1}{|N_2|}$ for all $i \in N_2$. Since $x(N_2) = y(N_2) + \beta_1 = z(N_2)$, by Remark 5 we have $c_2 \leq \hat{z}_1 = \widehat{y_{|N_2}}_1 + \frac{\beta_1}{|N_2|} \leq \widehat{y_{|N_2}}_1 + \beta_1$, which implies $\beta_1 \geq c_2 - \widehat{y_{|N_2}}_1$. This last inequality, together with $y(N_1 \cup N_2) \geq v(N_1 \cup N_2) = x(N_1 \cup N_2)$, lead to

$$\varphi_1 = y(N_1) - x(N_1) \geq x(N_2) - y(N_2) = \beta_1 \geq c_2 - \widehat{y_{|N_2}}_1. \tag{2.4}$$

Now from (2.4) it follows

$$x(N_1) + c_2 \leq y(N_1) + \widehat{y_{|N_2}}_1. \tag{2.5}$$

If $|N_2| \geq 2$ and $\sum_{i=2}^{|N_2|} \widehat{y_{|N_2}}_i \geq (|N_2| - 1)\, c_2$, then from Remark 5, $tc_2 \leq \widehat{y_{|N_2}}_2 + \ldots + \widehat{y_{|N_2}}_{t+1}$, for all $t = 1, \ldots, |N_2| - 1$, which leads, together with (2.5), to (2.3). Otherwise, if $|N_2| \geq 2$ and $\sum_{i=2}^{|N_2|} \widehat{y_{|N_2}}_i < (|N_2| - 1)\, c_2$, let us denote

$$\varphi_2 = y(N_1) + \widehat{y_{|N_2}}_1 - x(N_1) - c_2 \text{ and } \beta_2 = (|N_2| - 1)\, c_2 - \sum_{i=2}^{|N_2|} \widehat{y_{|N_2}}_i > 0.$$

From (2.4) it follows $\varphi_2 \geq \beta_2 > 0$. Next we show that $\beta_2 \geq c_2 - \widehat{y_{|N_2}}_2$. Choose $k \in N_2$ such that $y_k \geq y_i$ for all $i \in N_2$ and define $z \in \mathbb{R}^{N_2 \setminus \{k\}}$ as $z_i = y_i + \frac{\beta_2}{|N_2|-1}$ for all $i \in N_2 \setminus \{k\}$. Since $z(N_2 \setminus \{k\}) = y(N_2 \setminus \{k\}) + \beta_2 = x(N_2) - c_2$, by Remark 5 we have $c_2 \leq \hat{z}_1 = \widehat{y_{|N_2}}_2 + \frac{\beta_2}{|N_2|-1} \leq \widehat{y_{|N_2}}_2 + \beta_2$, which implies $\beta_2 \geq c_2 - \widehat{y_{|N_2}}_2$. Since $\varphi_2 \geq \beta_2$, we obtain

$$\varphi_2 \geq c_2 - \widehat{y_{|N_2}}_2. \tag{2.6}$$

Now from (2.6) it can be checked that $x(N_1) + 2c_2 \leq y(N_1) + \widehat{y_{|N_2}}_1 + \widehat{y_{|N_2}}_2$. Applying the same reasoning for $t = 3, \ldots, |N_2|$ we obtain (2.3).

Following the same line of argument it can be proved that, for all $k = 3, \ldots, m$ and all $t = 1, \ldots, |N_k|$,

$$x\left(N_1 \cup \ldots \cup N_{k-1}\right) + tc_k \quad \leq \quad y\left(N_1 \cup \ldots \cup N_{k-1}\right) + \sum_{j=1}^{t} \widehat{y_{|N_k}}_j. \tag{2.7}$$

Finally, combining (2.2), (2.3) and (2.7) we get

$$\begin{aligned}
x_1 = c_1 &\leq \widehat{y_{|N_1}}_1 \leq \widehat{y}_1 \\
x_1 + x_2 = 2c_1 &\leq \widehat{y_{|N_1}}_1 + \widehat{y_{|N_1}}_2 \leq \widehat{y}_1 + \widehat{y}_2 \\
&\vdots \\
x_1 + \ldots + x_{|N_1|} = x(N_1) &\leq y(N_1) \leq \widehat{y}_1 + \ldots \widehat{y}_{|N_1|} \\
x_1 + \ldots + x_{|N_1|+1} = x(N_1) + c_2 &\leq y(N_1) + \widehat{y_{|N_2}}_1 \leq \widehat{y}_1 + \ldots \widehat{y}_{|N_1|+1} \\
&\vdots \\
x_1 + \ldots + x_{|N_1|+|N_2|} = x\left(N_1 \cup N_2\right) &\leq y\left(N_1 \cup N_2\right) \leq \widehat{y}_1 + \ldots \widehat{y}_{|N_1|+|N_2|} \\
&\vdots \\
x_1 + \ldots + x_n = x\left(N_1 \cup \ldots \cup N_m\right) &= y\left(N_1 \cup \ldots \cup N_m\right) = \widehat{y}_1 + \ldots + \widehat{y}_n,
\end{aligned}$$

with at least one strict inequality,[2] which means that $x \succ_{\mathcal{L}} y$.

To see that $EL(N, v) = \{x\}$, we replicate the induction argument used by Dutta and Ray (1989) to prove their Theorem 2 (step 2).[3]

Note first that $EL\left(N_1, v\right) = \left\{x_{|N_1}\right\}$. Next we see that for all $t = 1, \ldots, m-1$, if $EL\left(N_1 \cup \ldots \cup N_t, v\right) = \left\{x_{|N_1 \cup \ldots \cup N_t}\right\}$, then $EL\left(N_1 \cup \ldots \cup N_{t+1}, v\right) = \left\{x_{|N_1 \cup \ldots \cup N_{t+1}}\right\}$.

Suppose that $EL\left(N_1 \cup \ldots \cup N_t, v\right) = \left\{x_{|N_1 \cup \ldots \cup N_t}\right\}$ but $EL\left(N_1 \cup \ldots \cup N_{t+1}, v\right) \neq \left\{x_{|N_1 \cup \ldots \cup N_{t+1}}\right\}$, for some t. Since $x\left(N_1 \cup \ldots \cup N_{t+1}\right) = v\left(N_1 \cup \ldots \cup N_{t+1}\right)$ and $x \in C\left(N, v\right)$, we have

$$x_{|N_1 \cup \ldots \cup N_{t+1}} \in C\left(N_1 \cup \ldots \cup N_{t+1}, v_{|N_1 \cup \ldots \cup N_{t+1}}\right) \subseteq L\left(N_1 \cup \ldots \cup N_{t+1}, v\right),$$

[2]This strict inequality follows from expression (2.2).

[3]We describe in detail the induction argument for the convenience of the reader.

and thus there exists $y \in L(N_1 \cup \ldots \cup N_{t+1}, v)$ with $y \succ_{\mathcal{L}} x_{|N_1 \cup \ldots \cup N_{t+1}}$. Then,

$$\begin{array}{lcl} \hat{y}_1 & \leq & x_1 \\ \hat{y}_1 + \hat{y}_2 & \leq & x_1 + x_2 \\ & \vdots & \\ \hat{y}_1 + \ldots + \hat{y}_{|N_1 \cup \ldots \cup N_{t+1}|} & = & x_1 + \ldots + x_{|N_1 \cup \ldots \cup N_{t+1}|} \end{array} \tag{2.8}$$

with at least one strict inequality.

Since $y(N_1 \cup \ldots \cup N_{t+1}) = x(N_1 \cup \ldots \cup N_{t+1})$, if $y_j \geq x_j$ for all $j \in N_1 \cup \ldots \cup N_{t+1}$ then we would have $y = x_{|N_1 \cup \ldots \cup N_{t+1}}$, in contradiction with $y \succ_{\mathcal{L}} x_{|N_1 \cup \ldots \cup N_{t+1}}$. As a consequence, the set $\mathcal{J} := \{j \in N_1 \cup \ldots \cup N_{t+1} \,|\, y_j < x_j\}$ must be non-empty. Take then $q^* = min\{k \in \{1, \ldots, t+1\} \,|\, \mathcal{J} \cap N_k \neq \emptyset\}$. We claim that,

$$y_i \leq x_i \text{ for all } i \in N_{q^*}.$$

Indeed, if $q^* = 1$, for all $i \in N_1$ it follows from (2.8) that $y_i \leq \hat{y}_1 \leq \widehat{x}_1 = x_i$. If $q^* > 1$, from $y_i \geq x_i$ for all $i \in N_1$ and expression (2.8) we have $y_i = x_i$ for all $i \in N_1$.Then, again from (2.8), we obtain $\hat{y}_{|N_1|+1} \leq x_{|N_1|+1}$. The repetition of the same argument leads to $y_i = x_i$ for all $i \in N_1 \cup \ldots \cup N_{q^*-1}$. Now, taking into account (2.8) and the definition of π we obtain, for all $i \in N_{q^*}$,

$$y_i \leq \hat{y}_{|N_1 \cup \ldots \cup N_{q^*-1}|+1} \leq \widehat{x}_{|N_1 \cup \ldots \cup N_{q^*-1}|+1} = x_i.$$

Note that $q^* \leq t$, since otherwise $y(N_1 \cup \ldots \cup N_{t+1}) < x(N_1 \cup \ldots \cup N_{t+1})$.

So, denote $T = N_1 \cup \ldots \cup N_{q^*}$. By hypothesis, $EL(T, v) = \{x_{|T}\}$. But then, since $y_i \leq x_i$ for all $i \in T$ and there exists $j^* \in N_{q^*}$ such that $y_{j^*} < x_{j^*}$, we conclude that $y \notin L(N_1 \cup \ldots \cup N_{t+1}, v)$, getting a contradiction. This means that $EL(N, v) = \{x\}$. □

Remark 6. *Under some conditions of positivity, a similar result was stated by Sánchez-Soriano et al. (2014). In that paper, Proposition 2 says the following: The vector* $a = (1_{n_1}a_1, 1_{n_2}a_2 \ldots, 1_{n_t}a_t)$ *such that* $a_1 \geq a_2 \geq \ldots \geq a_t > 0$ *and*

$\sum_{i=1}^{t} n_i = n$, where $1_{n_i} = (1,\ldots,1) \in \mathbb{R}^{n_i}$ for all $i = 1,\ldots,t$, Lorenz dominates each other element $x \in \mathbb{R}^n$ satisfying $\sum_{i=1}^{n_1} x_i \geq n_1 a_1$, $\sum_{i=1}^{n_1+n_2} x_i \geq \sum_{i=1}^{2} n_i a_i$, $\ldots$, $\sum_{i=1}^{n-n_t} x_i \geq \sum_{i=1}^{t-1} n_i a_i$, and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{t} n_i a_i$.

In our context, this implies $v(N_1 \cup \ldots \cup N_i) > 0$, for all $i = 1,\ldots,m$, being $(N_1,\ldots,N_m)$ a partition of N as described in Definition 13. At this point, it is important to point out that the WCES fails to satisfies covariance (see Dutta and Ray, 1989) and so the problem of existence of the WCES and the properties of Lorenz domination can not be solved just by looking at positive games.

Let us show an example to illustrate this point. Let (N,v) be a game with $N = \{1,2,3\}$ and $v(\{1\}) = 0.8$, $v(\{2\}) = -1$, $v(\{3\}) = -2$, $v(\{12\}) = -0.1$, $v(\{13\}) = -0.8$, $v(\{23\}) = -3.5$ and $v(\{123\}) = -1.5$. Let $x = (0.8,-0.9,-1.4) \in C(N,v)$. Then, the ordered partition of N induced by x is $\pi = (\{1\},\{2\},\{3\})$, with $x_1 = v(\{1\}) > 0$, $x_1 + x_2 = v(\{1\} \cup \{2\}) < 0$ and $x_1 + x_2 + x_3 = v(\{1\} \cup \{2\} \cup \{3\}) < 0$. From Theorem 8, $EL(N,v) = \{x\}$ and x Lorenz dominates every other core element. However, this last assertion can not be derived from Proposition 2 in Sánchez-Soriano et al. (2014).

Theorem 8 generalizes both Theorem 2 and Theorem 3 in Dutta and Ray (1989), and it can be useful to check that a core element is the WCES.

Let us introduce the class of games that satisfies the conditions stated in Theorem 8.

Definition 14. *A game (N,v) is an exact partition game if there exists a core element x such that the ordered partition of N induced by x, $\pi = (N_1,\ldots,N_m)$, satisfies $x(N_1 \cup \ldots \cup N_k) = v(N_1 \cup \ldots \cup N_k)$, for all $k = 1,\ldots,m$.*

Let Γ_{EP} denote the class of exact partition games. This class is large enough to include convex games and dominant diagonal assignment games,[4] but also

[4] Using different arguments, Llerena (2012) shows that on the class of dominant diagonal assignment games, the τ-value (Tijs, 1981) satisfies the requirements of Theorem 8.

nonsuperadditive games.

Example 7. *Let (N, v) be a balanced game with set of players $N = \{1, 2, 3\}$ and characteristic function:*

S	$v(S)$	S	$v(S)$	S	$v(S)$
$\{1\}$	1	$\{12\}$	0	$\{123\}$	9
$\{2\}$	1	$\{13\}$	7		
$\{3\}$	1	$\{23\}$	0		

This games is not supperadditive since $v(\{12\}) < v(\{1\}) + v(\{2\})$, but $(N, v) \in \Gamma_{EP}$. Indeed, take $x = (3.5, 2, 3.5) \in C(N, v)$. The ordered partition of N induced by x, $\pi = (\{13\}, \{2\})$, satisfies $x_1 + x_3 = v(\{13\})$ and $x(N) = v(N)$. Hence, $EL(N, v) = \{x\}$ and $(N, v) \in \Gamma_{EP}$.

2.4 Axiomatic characteritzations

The main concern of this section is to characterize the WCES over the domain of exact partition games, Γ_{EP}. As particular cases, we obtain new axiomatic characterizations over the class of convex games.

On the domain of convex games, the first characterization was provided by Dutta (1990) by means of *constrained egalitarianism* and *consistency* with respect to both the max reduced game (Davis and Maschler, 1965) and the self reduced game (Hart and Mas-Colell, 1989). *Constrained egalitarianism* is a prescriptive property that imposes to select, for two person games, the Lorenz maximal allocation within the core. Consistency is a sort of internal stability requirement that relates the solution of a game to the solution of an associated game when some players leave the original game.

A solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Constrained egalitarianism** if for all $N \in \mathcal{N}$ with $|N| = 2$, and all $(N, v) \in \Gamma'$, it holds $\sigma(N, v) = CE(N, v)$.

Note that any two person exact partition game is convex. Thus, the WCES satisfies *constrained egalitarianism* on Γ_{EP}.

To define consistency, we need to introduce the notion of reduced game.

Definition 15. *(Davis and Maschler, 1965) Let (N, v) be a game, $\emptyset \neq N' \subset N$ and $x \in \mathbb{R}^N$. The max reduced game relative to N' at x is the game $\left(N', r^{N'}_{M,x}(v)\right)$ defined by*

$$r^{N'}_{M,x}(v)(S) = \begin{cases} 0 & \text{if } S = \emptyset, \\ \max\limits_{Q \subseteq N \setminus N'} \{v(S \cup Q) - x(Q)\} & \text{if } \emptyset \neq S \subset N', \\ v(N) - x(N \setminus N') & \text{if } S = N'. \end{cases} \tag{2.9}$$

Remark 7. *The max reduction operation is transitive (see, for instance, Chang and Hu, 2007). That is, $r^{N''}_{M,x_{|N'}}\left(r^{N'}_{M,x}(v)\right) = r^{N''}_{M,x}(v)$, for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma$, all coalitions $\emptyset \neq N'' \subset N' \subset N$ and all payoff vector $x \in \mathbb{R}^N$.*

In the max reduced game (relative to N' at x), the worth of a coalition $S \subset N'$ is determined under the assumption that S can choose the best partners in $N \setminus N'$, provided they are paid according to x. *Max consistency* says that in this max reduced game, the original agreement should be confirmed.

A solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Max consistency** if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$, all $N' \subset N$, $N' \neq \emptyset$, and all $x \in \sigma(N, v)$, then $\left(N', r^{N'}_{M,x}(v)\right) \in \Gamma'$ and $x_{|N'} \in \sigma\left(N', r^{N'}_{M,x}(v)\right)$.

- **Weak max consistency** if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$, all $N' \subset N$ with $1 \leq |N'| \leq 2$ and all $x \in \sigma(N, v)$, then $\left(N', r^{N'}_{M,x}(v)\right) \in \Gamma'$ and $x_{|N'} \in \sigma\left(N', r^{N'}_{M,x}(v)\right)$.

- **Rich player max consistency** if for all $N \in \mathcal{N}$, all $(N,v) \in \Gamma'$ and all $x \in \sigma(N,v)$, if $N_1 \subseteq N, N_1 \neq N$, is the set of players with highest payoff (w.r.t. x), then $\left(N \setminus N_1, r_{M,x}^{N\setminus N_1}(v)\right) \in \Gamma'$ and $x_{|N\setminus N_1} \in \sigma\left(N \setminus N_1, r_{M,x}^{N\setminus N_1}(v)\right)$.

Weak max consistency applies the condition of *max consistency* to reduced games with at most two players. *Rich player max consistency* weakens *max consistency* just requiring this condition when rich players leave the game. Clearly, *max consistency* implies both *weak* and *rich player max consistency.*

Proposition 8. *The WCES satisfies max consistency on* Γ_{EP}.

Proof. For two person games, *max consistency* clearly holds. Let $(N,v) \in \Gamma_{EP}$ and $x = EL(N,v)$ with $|N| > 2$. Since the max reduction operation is transitive (see Remark 7), it is enough to see that, for all $i \in N$, $\left(N \setminus \{i\}, r_{M,x}^{N\setminus\{i\}}(v)\right) \in \Gamma_{EP}$ and $x_{|N\setminus\{i\}} = EL\left(N \setminus \{i\}, r_{M,x}^{N\setminus\{i\}}(v)\right)$.

Let $\pi = (N_1, \ldots, N_m)$ be the ordered partition of N induced by x. We distinguish two cases:

1) If $m = 1$, then $x = \left(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|}\right) \in C(N,v)$. Let $i \in N$. By *max consistency* of the core (Peleg, 1986), $x_{|N\setminus\{i\}} \in C\left(N \setminus \{i\}, r_{M,x}^{N\setminus\{i\}}(v)\right)$. Hence, $\left(N \setminus \{i\}, r_{M,x}^{N\setminus\{i\}}(v)\right) \in \Gamma_{EP}$ and $x_{|N\setminus\{i\}} = EL\left(N \setminus \{i\}, r_{M,x}^{N\setminus\{i\}}(v)\right)$.

2) If $m > 1$, take $k \in \{1, \ldots, m\}$ and $i \in N_k$. The ordered partition of $N \setminus \{i\}$ induced by $x_{|N\setminus\{i\}}$ is either $\pi' = (N_1, \ldots, N_{k-1}, N_k \setminus \{i\}, N_{k+1}, \ldots, N_m)$, if $|N_k| > 1$, or $\pi' = (N_1, \ldots, N_{k-1}, N_{k+1}, \ldots, N_m)$, otherwise.

From the *max consistency* of the core, the definition of max reduced game and the fact that $x(N_1 \cup \ldots \cup N_k) = v(N_1 \cup \ldots \cup N_k)$ for all $k \in \{1, \ldots, m\}$, we have

- For $h \in \{1, \ldots, k-1\}$,

$$\begin{aligned} x(N_1 \cup \ldots \cup N_h) &\geq r_{M,x}^{N\setminus\{i\}}(v)(N_1 \cup \ldots \cup N_h) \\ &\geq v(N_1 \cup \ldots \cup N_h) \\ &= x(N_1 \cup \ldots \cup N_h), \end{aligned}$$

  which means that

$$x(N_1 \cup \ldots \cup N_h) = r_{M,x}^{N\setminus\{i\}}(v)(N_1 \cup \ldots \cup N_h). \tag{2.10}$$

- For $h \in \{k, \ldots, m\}$,

$$\begin{aligned} x(N_1 \cup \ldots \cup N_k \setminus \{i\} \cup \ldots \cup N_h) &\geq r_{M,x}^{N\setminus\{i\}}(v)(N_1 \cup \ldots \cup N_k \setminus \{i\} \cup \ldots \cup N_h) \\ &\geq v(N_1 \cup \ldots \cup N_k \cup \ldots \cup N_h) - x_i \\ &= x(N_1 \cup \ldots \cup N_k \cup \ldots \cup N_h) - x_i \\ &= x(N_1 \cup \ldots \cup N_k \setminus \{i\} \cup \ldots \cup N_h), \end{aligned}$$

  which means that

$$x(N_1 \cup \ldots \cup N_k \setminus \{i\} \cup \ldots \cup N_h) = r_{M,x}^{N\setminus\{i\}}(v)(N_1 \cup \ldots \cup N_k \setminus \{i\} \cup \ldots \cup N_h). \tag{2.11}$$

From (2.10) and (2.11) it follows that $x_{|N\setminus\{i\}}$ satisfies the conditions stated in Theorem 8 (w.r.t. π'). Hence, we conclude that $\left(N \setminus \{i\}, r_{M,x}^{N\setminus\{i\}}(v)\right) \in \Gamma_{EP}$ and $x_{|N\setminus\{i\}} = EL\left(N \setminus \{i\}, r_{M,x}^{N\setminus\{i\}}(v)\right)$. □

To prove that *max consistency* together with *constrained egalitarianism* characterize the WCES over the class of convex games, Dutta (1990) invokes *converse max consistency*, which is the dual property of *max consistency*. This property is crucial in his proof of uniqueness.

A solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Converse max consistency** if for all $N \in \mathcal{N}$ with $|N| \geq 3$, all $(N, v) \in \Gamma'$ and all $x \in \mathbb{R}^N$ with $x(N) = v(N)$, if for all $N' \subset N$ with $|N'| = 2$, $\left(N', r_{M,x}^{N'}(v)\right) \in \Gamma'$ and $x_{|N'} \in \sigma\left(N', r_{M,x}^{N'}(v)\right)$, then $x \in \sigma(N, v)$.

Converse max consistency says that if the projection of an efficient allocation x is chosen for every two player max reduced game, then x should be chosen for the original game.

Unfortunatly, Example 8 bellow reveals that the WCES is in conflict with *converse max consistency* on Γ_{EP}.

Example 8. *(Arin and Iñarra, 2001) Let (N,v) be a balanced game with set of players $N=\{1,2,3,4\}$ and characteristic function:*

S	$v(S)$	S	$v(S)$	S	$v(S)$	S	$v(S)$
$\{1\}$	0	$\{12\}$	0	$\{123\}$	0	$\{1234\}$	4
$\{2\}$	0	$\{13\}$	2	$\{124\}$	0		
$\{3\}$	0	$\{14\}$	2	$\{134\}$	0		
$\{4\}$	0	$\{23\}$	2	$\{234\}$	0		
		$\{24\}$	2				
		$\{34\}$	0				

Take $x=(1,1,1,1)\in C(N,v)$. The ordered partition of N induced by x is $\pi=(\{N\})$ and $x(N)=v(N)$. Hence, $EL(N,v)=\{x\}$ and $(N,v)\in\Gamma_{EP}$. Now choose $y=(2,2,0,0)\in C(N,v)$. Below, we describe the max reduced games $\left(N',r^{N'}_{M,y}\right)$ relative to $N'\subset N$ at y with $|N'|=2$,

S	$r^{\{12\}}_{M,y}(v)$	S	$r^{\{12\}}_{M,y}(v)$	S	$r^{\{13\}}_{M,y}(v)$	S	$r^{\{13\}}_{M,y}(v)$
$\{1\}$	2	$\{12\}$	4	$\{1\}$	2	$\{13\}$	2
$\{2\}$	2			$\{3\}$	0		

S	$r^{\{14\}}_{M,y}(v)$	S	$r^{\{14\}}_{M,y}(v)$	S	$r^{\{23\}}_{M,y}(v)$	S	$r^{\{23\}}_{M,y}(v)$
$\{1\}$	2	$\{14\}$	2	$\{2\}$	2	$\{23\}$	2
$\{4\}$	0			$\{3\}$	0		

S	$r^{\{24\}}_{M,y}(v)$	S	$r^{\{24\}}_{M,y}(v)$	S	$r^{\{34\}}_{M,y}(v)$	S	$r^{\{34\}}_{M,y}(v)$
$\{2\}$	2	$\{24\}$	2	$\{3\}$	0	$\{34\}$	0
$\{4\}$	0			$\{4\}$	0		

Routine verification shows that the corresponding constrained egalitarian solutions associated with the different max reduced game are:

$$CE\left(\{12\}, r_{M,y}^{\{12\}}(v)\right) = (2,2) = y_{|\{12\}} \qquad CE\left(\{13\}, r_{M,y}^{\{13\}}(v)\right) = (2,0) = y_{|\{13\}},$$

$$CE\left(\{14\}, r_{M,y}^{\{14\}}(v)\right) = (2,0) = y_{|\{14\}} \qquad CE\left(\{23\}, r_{M,y}^{\{23\}}(v)\right) = (2,0) = y_{|\{23\}},$$

$$CE\left(\{24\}, r_{M,y}^{\{24\}}(v)\right) = (2,0) = y_{|\{24\}} \qquad CE\left(\{34\}, r_{M,y}^{\{34\}}(v)\right) = (0,0) = y_{|\{34\}}.$$

However, $y \neq EL(N,v)$.

To be precise, Dutta (1990) only uses *bilateral max consistency*, that is, *max consistency* for only two person games, together with *constrained egalitarianism*, to characterize the WCES on Γ_{Con}. Let us see that on Γ_{EP}, these two properties do not characterize the WCES. To do this, we introduce the egalitarian core (Arin and Iñarra, 2001).

Definition 16. *The egalitarian core of a balanced game (N,v), denoted by E_gC, is the set $E_gC(N,v) = \{x \in C(N,v) \mid x_i > x_j \Rightarrow S_{ij}(x) = 0\}$, where $S_{ij}(x) = \max\{v(S) - x(S) \mid i \in S, j \notin S, S \subset N\}$.*

Arin and Iñarra (2001) show that the egalitarian core satisfies *max consistency* and *constrained egalitarianism* on Γ_{Bal}. Note that a two person balanced games is an exact partition game since the constrained egalitarian solution is a core element satisfying the conditions stated in Theorem 8. Thus, the egalitarian core satisfies *bilateral max consistency* and *constrained egalitarianism* on Γ_{EP}. In Example 8, $EL(N,v) = \{(1,1,1,1)\}$ and $(2,2,0,0) \in E_gC(N,v)$, which means that $EL(N,v) \neq E_gC(N,v)$. The same example also illustrates that the egalitarian core is not *max consistent* on Γ_{EP}. Indeed, consider the max reduced game $\left(N \setminus \{4\}, r_{M,y}^{N\setminus\{4\}}(v)\right)$ with $y = (2,2,0,0)$. As the reader can easily check, $E_gC\left(N \setminus \{4\}, r_{M,y}^{N\setminus\{4\}}(v)\right) = \{(2,2,0)\}$ and $\left(N \setminus \{4\}, r_{M,y}^{N\setminus\{4\}}(v)\right) \notin \Gamma_{EP}$.

The second characterization of the WCES provided by Dutta (1990) uses *self consistency* (Hart and Mas-Collel, 1989). This property is defined for single-valued solutions.

A single-valued solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Self consistency** if for all $N \in \mathcal{N}$, all $(N,v) \in \Gamma'$ and all $N' \subset N$, $N' \neq \emptyset$, then $\left(N', r^{N'}_{S,\sigma}(v)\right) \in \Gamma'$ and, for all $i \in N'$, $\sigma_i(N,v) = \sigma_i\left(N', r^{N'}_{S,\sigma}(v)\right)$, where $\left(N', r^{N'}_{S,\sigma}(v)\right)$ is the **self reduced game** of (N,v) relative to N' and σ definded as follows:

$$r^{N'}_{S,\sigma}(v)(R) = \begin{cases} 0 & \text{if } R = \emptyset, \\ v(R \cup (N \setminus N')) - \sum\limits_{i \in N \setminus N'} \sigma_i\left(R \cup (N \setminus N'), v_{|R \cup (N \setminus N')}\right) & \text{if } \emptyset \neq R \subseteq N'. \end{cases} \tag{2.12}$$

In the self reduced game (relative to N' at σ), the worth of a coalition $R \subseteq N'$ is the worth of $R \cup (N \setminus N')$ in the original game minus the sum of the payoffs that the solution assigns the members of $N \setminus N'$ for the subgame faced by the group $R \cup (N \setminus N')$. *Self consistency* states that in this self reduced game, the original agreement should be accepted. The next example shows that the WCES fails to satisfies *self consistency* on Γ_{EP}.

Example 9. *Let (N,v) be a balanced game with set of players $N = \{1,2,3\}$ and characteristic function:*

S	$v(S)$	S	$v(S)$	S	$v(S)$
$\{1\}$	2	$\{12\}$	4	$\{123\}$	4
$\{2\}$	1	$\{13\}$	2		
$\{3\}$	0	$\{23\}$	1.5		

Take $x = (2,2,0) \in C(N,v)$. The ordered partition of N induced by x, $\pi = (\{12\},\{3\})$, satisfies $x_1 + x_2 = v(\{12\})$ and $x(N) = v(N)$. Hence, from Theorem 8 we have that $EL(N,v) = \{x\}$ and $(N,v) \in \Gamma_{EP}$.

Let $N' = \{13\}$. Then,

$$\begin{aligned} r^{N'}_{S,EL}(v)(\{1\}) &= v(\{12\}) - EL_2\left(\{12\}, v_{|\{12\}}\right) = 4 - 2 = 2, \\ r^{N'}_{S,EL}(v)(\{3\}) &= v(\{23\}) - EL_2\left(\{23\}, v_{|\{23\}}\right) = 1.5 - 1 = 0.5 \;\; \textit{and} \\ r^{N'}_{S,EL}(v)(\{13\}) &= v(N) - EL_2(N, v) = 4 - 2 = 2. \end{aligned} \tag{2.13}$$

Note that $\left(N', r^{N'}_{S,EL}(v)\right)$ has no imputations. Thus, the WCES is not defined and $\left(N', r^{N'}_{S,EL}(v)\right) \notin \Gamma_{EP}$.

In order to characterize the WCES within the domain of exact partition games, together with consistency property, we will make use of the following properties.

A solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Nonemptiness** if for all $N \in \mathcal{N}$ and all $(N, v) \in \Gamma'$, it holds $\sigma(N, v) \neq \emptyset$.
- **Efficiency** if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$ and all $x \in \sigma(N, v)$, then $x(N) = v(N)$.
- **Individual rationality** if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$, all $x \in \sigma(N, v)$ and all $i \in N$, then $x_i \geq v(\{i\})$.
- **Core selection** if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$, all $x \in \sigma(N, v)$ and all $S \subseteq N$, then $x(S) \geq v(S)$.
- **Rich player feasibility** if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$ and all $x \in \sigma(N, v)$, it holds $x(N_1) \leq v(N_1)$, where N_1 denotes the set of players with highest payoff (w.r.t. x).
- **Internal Lorenz stability** if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$ and all $x, y \in \sigma(N, v)$, neither $x \succ_{\mathcal{L}} y$ nor $y \succ_{\mathcal{L}} x$.
- **External Lorenz stability (over the core)** if for all $N \in \mathcal{N}$ and all $(N, v) \in \Gamma'$, if $x \in C(N, v) \setminus \sigma(N, v)$, then there is $y \in \sigma(N, v)$ such that $y \succ_{\mathcal{L}} x$.

Efficiency says that all the gains from cooperation should be shared among the players. *Individual rationality* means that the proposed solution can not be improved upon by a single player, while *core selection* extends this impossibility to any coalition. Note that *core selection*, together with the feasibility assumption of a solution, imply *efficiency*. *Rich player feasibility* states that the total amount received by players with the highest payoff can not exceed what they can get for themselves. *Internal Lorenz stability* is a natural requirement in an egalitarian framework. *External Lorenz stability* (*over the core*) gives priority to the social goal of equality in front of particular interests, in the sense that if a core element is not an outcome of the solution is because there is an allocation in the solution which is more egalitarian (w.r.t. the Lorenz criterion).

Next, we state our first characterization result.

Theorem 9. *The WCES is the unique solution on Γ_{EP} that satisfies weak max consistency, individual rationality, internal Lorenz stability and external Lorenz stability (over the core).*

Proof. Proposition 8 implies *weak max consistency*, and *individual rationality* comes from the fact that the WCES selects a core element. *Internal Lorenz stability* is because the WCES is single-valued, and *external Lorenz stability* (*over the core*) follows from Theorem 8.

In order to show uniqueness, suppose there is a solution $\sigma \neq EL$ satisfying the above four properties. Let $(N, v) \in \Gamma_{EP}$. Note that *external Lorenz stability* (*over the core*) implies *nonemptiness*. If $|N| = 1$, by *nonemptiness* and *individual rationality* (and feasibility) $\sigma(N, v) = EL(N, v)$. Suppose $|N| \geq 2$. We first show that $\sigma(N, v) \subseteq C(N, v)$. Let $x \in \sigma(N, v)$ and $i \in N$. Then, *weak max consistency* and *efficiency* for one person game imply $x_i = r^{\{i\}}_{M,x}(v)(\{i\}) = v(N) - \sum_{j \in N\setminus\{i\}} x_j$, which proves *efficiency*. To check coalitional rationality, let $\emptyset \neq S \subset N$ and $i \in N\setminus S$. Chose $k \in S$ and consider the max reduced game $\left(\{ik\}, r^{\{ik\}}_{M,x}(v)\right)$. By *weak* *max consistency*, $x_{|\{ik\}} \in \sigma\left(\{ik\}, r_{M,x}^{\{ik\}}(v)\right)$ and, by *individual rationality*, $x_k \geq r_{M,x}^{\{ik\}}(v)(\{k\}) \geq v(S) - x(S \setminus \{k\})$, which implies $x(S) \geq v(S)$. Hence, $x \in C(N,v)$. Let us denote $x^* = EL(N,v)$. If $x^* \notin \sigma(N,v)$, by *external Lorenz domination* (*over the core*) there is $y \in \sigma(N,v)$ such that $y \succ_{\mathcal{L}} x^*$, a contradiction. Hence, $x^* \in \sigma(N,v)$. Finally, by *internal Lorenz stability* we conclude that $\sigma(N,v) = EL(N,v)$. □

To see that the properties in Theorem 9 are logically independent we introduce the following solutions:

- Let σ_1 defined as follows: $\sigma_1(N,v) = \nu^*(N,v)$, for each $(N,v) \in \Gamma_{EP}$, where ν^* denotes the **prenucleolus** (Schmeidler, 1969).[5] Then, σ_1 satisfies *weak max consistency*, *individual rationality*, *internal Lorenz stability*, but not *external Lorenz stability* (*over the core*).

- Let σ_2 defined as follows: $\sigma_2(N,v) = C(N,v)$, for each $(N,v) \in \Gamma_{EP}$. Then, σ_2 satisfies *weak max consistency*, *individual rationality*, *external Lorenz stability* (*over the core*), but not *internal Lorenz stability.*

- Let σ_3 defined as follows: $\sigma_3(N,v) = EI(N,v)$, for each $(N,v) \in \Gamma_{EP}$. That is, σ_3 chooses the Lorenz maximal allocations in the imputation set. Llerena and Mauri (2015) show that this solution is single-valued and Lorenz dominates all core elements. Then, σ_3 satisfies *individual rationality*, *internal Lorenz stability*, *external Lorenz stability* (*over the core*), but not *weak max consistency.*

[5]Given a game (N,v), the excess of a coalition $\emptyset \neq S \subseteq N$ at a payoff vector $x \in \mathbb{R}^N$ is $v(S) - x(S)$. The prenucleolus is the pre-imputation that minimizes, with respect to the lexicographic order, the vector of excesses over the set of pre-imputations.

- Let σ_4 defined as follows: for each $(N, v) \in \Gamma_{EP}$

$$\sigma_4(N,v) = \begin{cases} EL(N,v) & \text{if } |N| \geq 3 \text{ or } |N| = 1, \\ \{(x_i, x_j), (x_j, x_i)\} & \text{if } N = \{i,j\}, \end{cases}$$

  where $(x_i, x_j) = EL(\{i,j\}, v)$. Then, σ_4 satisfies *weak max consistency*, *internal Lorenz stability*, *external Lorenz stability* (*over the core*), but not *individual rationality.*

It is well-known that the max reduced game of a convex game relative to a core element is also convex (see, for instance, Hokari, 2002). Moreover, on this domain the WCES selects the unique Lorenz maximal allocation within the core (Dutta and Ray, 1989). Thus, Theorem 9 holds on the domain of convex games.

Theorem 10. *The WCES is the unique solution on Γ_{Con} that satisfies weak max consistency, individual rationality, internal Lorenz stability and external Lorenz stability (over the core).*

Defined on the domain of convex games, σ_1, σ_2, σ_3 and σ_4 show the independence of the properties in Theorem 10.

Although the WCES satisfies nice properties on the domain of convex games, and some of them are inherited on the domain of exact partition games, its existence is not linked to the nonemptiness of the core. On the domain of balanced games, an alternative track is to put the attention on the Lorenz maximal allocations within the core.

Definition 17. *The Lorenz maximal core of a balanced game (N, v), denoted by $EC(N, v)$, is the set* $EC(N,v) = \{x \in C(N,v) \mid \nexists\, y \in C(N,v) \text{ such that } y \succ_{\mathcal{L}} x\}$.

Example 4 in Dutta and Ray (1989) shows that the Lorenz maximal core is not single-valued. This instance also confirms that the WCES not always Lorenz dominates other core allocations when the game is not exact partition. On the domain of balanced games, the Lorenz maximal core is a proper subset of the egalitarian core (Arín and Iñarra, 2001), and Example 8 above illustrates that this feature also holds on the domain of exact partition games. Thus, on Γ_{EP}, the egalitarian core becomes a singleton if and only if it coincides with the WCES.

By definition, the Lorenz maximal core satisfies *individual rationality* and *internal Lorenz stability*. *External Lorenz stability* (*over the core*) follows by compactness of the core. Arin and Iñarra (2001) and also Hougaard et al. (2001), show that the Lorenz maximal core satisfies *max consistency*. Since *weak max consistency* and *individual rationality* imply *core selection*, uniqueness follows directly from *internal Lorenz stability* and *external Lorenz stability* (*over the core*). Thus, properties in Theorem 9 also characterize the Lorenz maximal core on the domain of balanced games.

Theorem 11. *The Lorenz maximal core is the unique solution on Γ_{Bal} that satisfies weak max consistency, individual rationality, internal Lorenz stability and external Lorenz stability (over the core).*

Solutions σ_1, σ_2 and σ_3 defined on Γ_{Bal}, together with solution σ_5 defined bellow, show that the properties in Theorem 11 are logically independent.

- Let σ_5 defined as follows: for each $(N,v) \in \Gamma_{Bal}$

$$\sigma_5(N,v) = \begin{cases} EC(N,v) & \text{if } |N| \geq 3 \text{ or } |N| = 1, \\ \{(x_i,x_j), (x_j,x_i)\} & \text{if } N = \{i,j\}, \end{cases}$$

  where $(x_i,x_j) = EC(\{i,j\},v)$. Then, σ_5 satisfies *weak max consistency*, *internal Lorenz stability*, *external Lorenz stability* (*over the core*), but not *individual rationality*.

Our second characterization is by means of *nonemptiness*, *rich player max consistency*, *core selection* and *rich payer feasibility*.

Theorem 12. *The WCES is the unique solution on Γ_{EP} that satisfies nonemptiness, rich player max consistency, core selection, and rich player feasibility.*

Proof. Proposition 8 implies *rich player max consistency*, *nonemptiness* and *core selection* follow from the fact that the WCES selects a core element, *rich player feasibility* comes from the structure of the WCES on Γ_{EP}.

In order to show uniqueness, suppose there is a solution $\sigma \neq EL$ satisfying the above four properties. Let $(N, v) \in \Gamma_{EP}$, $EL(N, v) = \{x\}$ and $\pi = (N_1, N_2, \ldots, N_m)$ be the ordered partition of N induced by x. First, we will see that N_1 is the unique maximal equity coalition of (N, v). Let $R \subseteq N$ be an equity coalition. Recall that $x_k = \frac{v(N_1)}{|N_1|}$, for all $k \in N_1$. Since $x \in C(N, v)$, there exists $i \in R$ such that $x_i \geq \frac{v(R)}{|R|}$. Thus, for each $k \in N_1$, it holds $x_k = \frac{v(N_1)}{|N_1|} \geq x_i \geq \frac{v(R)}{|R|} \geq \frac{v(N_1)}{|N_1|}$, which means that $\frac{v(R)}{|R|} = \frac{v(N_1)}{|N_1|}$. Hence, N_1 is an equity coalition. Suppose that $R \setminus N_1 \neq \emptyset$. Then,

$$
\begin{aligned}
x(R) &= \sum_{i \in N_1 \cap R} x_i + \sum_{i \in R \setminus N_1} x_i = |N_1 \cap R| \frac{v(N_1)}{|N_1|} + \sum_{i \in R \setminus N_1} x_i \\
&< |N_1 \cap R| \frac{v(N_1)}{|N_1|} + |R \setminus N_1| \frac{v(N_1)}{|N_1|} = \frac{v(N_1)}{|N_1|} |R| = v(R),
\end{aligned}
$$

contradicting $x \in C(N, v)$. Hence, $R \subseteq N_1$.

By *nonemptiness*, $\sigma(N, v) \neq \emptyset$. Let $y \in \sigma(N, v)$ and $\pi' = (R_1, R_2, \ldots, R_k)$ be the ordered partition of N induced by y. By *core selection* and *rich player feasibility*, $y_i = \frac{v(R_1)}{|R_1|}$ for all $i \in R_1$. If $R_1 = N$, by *core selection* $y = x$. Otherwise, since $x \succ_{\mathcal{L}} y$, $\hat{x}_1 \leq \hat{y}_1$ which means that $y_i \geq \frac{v(N_1)}{|N_1|}$ for all $i \in R_1$. Hence, $\frac{v(R_1)}{|R_1|} \geq \frac{v(N_1)}{|N_1|}$. This, together with the fact that N_1 is the unique maximal equity coalition of (N, v), leads to $R_1 \subseteq N_1$. Suppose that $|R_1| < |N_1|$. Then,

$$\begin{array}{lcl} \hat{x}_1 & = & \hat{y}_1 \\ \hat{x}_1 + \hat{x}_2 & = & \hat{y}_1 + \hat{y}_2 \\ & \vdots & \\ \hat{x}_1 + \ldots + \hat{x}_{|R_1|} & = & \hat{y}_1 + \ldots + \hat{y}_{|R_1|} \\ \hat{x}_1 + \ldots + \hat{x}_{|R_1|} + \hat{x}_{|R_1|+1} & > & \hat{y}_1 + \ldots + \hat{y}_{|R_1|} + \hat{y}_{|R_1|+1} \end{array}$$

in contradiction with $x \succ_{\mathcal{L}} y$. Thus, $R_1 = N_1$ and $x_i = y_i$ for all $i \in N_1$, which imply $\left(N \setminus N_1, r_{M,x}^{N \setminus N_1}(v)\right) = \left(N \setminus R_1, r_{M,y}^{N \setminus R_1}(v)\right)$. By *rich player max consistency*, $y_{|N \setminus N_1} \in \sigma\left(N \setminus N_1, r_{M,x}^{N \setminus N_1}(v)\right)$ and $x_{|N \setminus N_1} = EL\left(N \setminus N_1, r_{M,x}^{N \setminus N_1}(v)\right)$, with $\left(N \setminus N_1, r_{M,x}^{N \setminus N_1}(v)\right) \in \Gamma_{EP}$. Applying the same arguments as before, it can be checked that $N_2 = R_2$ and $x_i = y_i$ for all $i \in N_2$. Following this reasoning step by step we reach $x = y$. Thus, we conclude that $\sigma = EL$. $\square$

To see that the properties in Theorem 12 are logically independent we introduce the following solutions:

- Let σ_6 defined as follows: $\sigma_6(N, v) = \left\{\left(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|}\right)\right\} \cap C(N, v)$, for each $(N, v) \in \Gamma_{EP}$. Then, σ_6 satisfies *rich player max consistency*, *core selection* and *rich player feasibility*, but not *nonemptiness*.

- Let σ_7 defined as follows: $\sigma_7(N, v) = \{x \in C(N, v) \mid x(N_1) = v(N_1)\}$, for each $(N, v) \in \Gamma_{EP}$, where N_1 denotes the set of players with highest payoff (w.r.t. x). Then, σ_7 satisfies *nonemptiness*, *core selection* and *rich player feasibility*, but not *rich player max consistency*.

- Let σ_8 defined as follows: $\sigma_8(N, v) = \left\{\left(\frac{v(N)}{|N|}, \ldots \frac{v(N)}{|N|}\right)\right\}$, for each $(N, v) \in \Gamma_{EP}$. Then, σ_8 satisfies *nonemptiness*, *rich player max consistency* and *rich player feasibility*, but not *core selection*.

- Let σ_9 defined as follows: $\sigma_9(N, v) = EL(N, v)$ if $|N| \geq 3$, and $\sigma_9(N, v) = C(N, v)$ if $|N| \leq 2$, for each $(N, v) \in \Gamma_{EP}$. Then, σ_9 satisfies *nonempti-*

ness, rich player max consistency and *core selection*, but not *rich player feasibility.*

Theorem 12 also holds on the domain of convex games.

Theorem 13. *The WCES is the unique solution on* Γ_{Con} *that satisfies nonemptiness, rich player max consistency, core selection and rich player feasibility.*

Definded on the domain of convex games, σ_6, σ_7, σ_8 and σ_9 show the independence of the properties in Theorem 13.

Finally, let us point out that on the domain of balanced games, the properties stated in Theorem 13 do not characterize the Lorenz maximal core since it fails to satisfy *rich player feasibility.* In fact, *nonemptiness*, *core selection* and *rich player feasibility* are incompatible on this domain. Indeed, suppose there is a solution σ on Γ_{Bal} that satisfies these three properties and consider Example 1 in Dutta and Ray (1989): let (N, v) with $N = \{1, 2, 3\}$ and $v(\{i\}) = 0$ for all $i \in N$, $v(\{12\}) = v(\{13\}) = v(\{123\}) = 1$ and $v(\{23\}) = 0$. Since $C(N, v) = \{(1, 0, 0)\}$, by *nonemptiness* and *core selection* $\sigma(N, v) = \{(1, 0, 0)\}$, in contradiction with *rich player feasibility.* As was noted by these authors, this game has no WCES. Nevertheless, Example 4 in the same paper describes a non exact partition balanced game where the WCES belongs to the core, satisfies *rich player feasibility* and its restriction to the complementary set of players with highest payoff coincides with the WCES of the corresponding max reduced game. Actually, in this example the WCES coincides with the lexmax solution (Arin et al., 2003). Let us recall the definition. For any two vectors $x, y \in \mathbb{R}^N$, we say that $x \preceq_{lex} y$ if $x = y$ or $x_1 < y_1$ or there exists $k \in \{2, \ldots, |N|\}$ such that $x_i = y_i$ for $1 \leq i \leq k-1$ and $x_k < y_k$. For a balanced game (N, v), the **lexmax** solution is defined as $Lmax(N, v) = \{x \in C(N, v) \,|\, \hat{x} \preceq_{lex} \hat{y} \text{ for all}\, y \in C(N, v)\}$. For any balanced game (N, v), the lexmax solution is a singleton and it is Lorenz undominated within the core.

Next we show that the compatibility of these properties leads to the lexmax solution.

Theorem 14. *Let $\Gamma' \subsetneq \Gamma_{Bal}$ be a subclass of balanced games such that there exists a solution σ that satisfies nonemptiness, rich player max consistency, core selection and rich player feasibility. Then, σ coincides with the lexmax solution.*

Proof. Let σ be a solution satisfying these four properties on Γ'. Let $(N, v) \in \Gamma'$, $x \in \sigma(N, v)$ and $\pi = (N_1, N_2, \ldots, N_m)$ be the ordered partition of N induced by x. If $m = 1$, by *core selection* $x = \left(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|}\right) = Lmax(N, v)$. If $m > 1$ suppose, on the contrary, $x \neq Lmax(N, v) = y$. By *core selection* and *rich player feasibility*, $x(N_1) = v(N_1)$. Moreover, for all $i \in N_1$, $x_i = \frac{v(N_1)}{|N_1|} > x_j$ for all $j \in N \backslash N_1$. Since $y \in C(N, v)$, there exists $i_1 \in N_1$ such that $y_{i_1} \geq \frac{v(N_1)}{|N_1|} = x_{i_1}$ and thus $\hat{y}_1 \geq y_{i_1} \geq x_{i_1} = \hat{x}_1$. This inequality, together with the fact that $\hat{y} \preceq_{lex} \hat{x}$, imply $\hat{y}_1 = \hat{x}_1$ and $y_{i_1} = x_{i_1}$. If $|N| > 1$, then $y(N_1 \setminus \{i_1\}) = y(N_1) - y_{i_1} = y(N_1) - \frac{v(N_1)}{|N_1|} \geq v(N_1) - \frac{v(N_1)}{|N_1|} = |N_1 \setminus \{i_1\}| \cdot \frac{v(N_1)}{|N_1|}$, which implies $\frac{y(N_1 \setminus \{i_1\})}{|N_1 \setminus \{i_1\}|} \geq \frac{v(N_1)}{|N_1|}$. Hence, there exists $i_2 \in N_1 \setminus \{i_1\}$ with $y_{i_2} \geq \frac{v(N_1)}{|N_1|} = x_{i_2}$. Since $\hat{y}_1 \geq y_{i_2} \geq x_{i_2} = \hat{x}_1 = \hat{y}_1$, we have $y_{i_2} = x_{i_2}$. Following this process we can check that $y_k = x_k$ for all $k \in N_1$, which means that $\left(N \setminus N_1, r_{M,x}^{N \setminus N_1}(v)\right) = \left(N \setminus N_1, r_{M,y}^{N \setminus N_1}(v)\right)$. Moreover, by *rich player max consistency*, $\left(N \setminus N_1, r_{M,x}^{N \setminus N_1}(v)\right) \in \Gamma'$ and $x_{|N \setminus N_1} \in \sigma\left(N \setminus N_1, r_{M,x}^{N \setminus N_1}(v)\right)$. In addition, by *max consistency* of the lexmax solution, $y_{|N \setminus N_1} = Lmax\left(N \setminus N_1, r_{M,y}^{N \setminus N_1}(v)\right)$. Now following the reasoning above we obtain $x_k = y_k$ for all $k \in N_2$. Repeating this line of argument, we can conclude that $\sigma(N, v) = Lmax(N, v)$. □

The class of exact partition games, and also the class of convex games, are instances of subdomains of balanced games satisfying the assumptions of Theorem 14 where the WCES coincides with the lexmax solution. However, there are other subdomains. Indeed, let Γ' be the domain of balanced games where the lexmax solution satisfies *rich player feasibility* and *rich player max consistency*. Note that $\Gamma' \neq \emptyset$ since $\Gamma_{EP} \subset \Gamma'$. Now consider the following example.

Example 10. *(Dutta and Ray, 1989) Let (N, v) be a balanced game with set of players $N = \{1, 2, 3, 4\}$ and characteristic function as follows:*

S	$v(S)$	S	$v(S)$	S	$v(S)$	S	$v(S)$
$\{1\}$	0	$\{12\}$	0	$\{123\}$	1.05	$\{1234\}$	2
$\{2\}$	0	$\{13\}$	0	$\{124\}$	0		
$\{3\}$	0	$\{14\}$	0	$\{134\}$	1.9		
$\{4\}$	0	$\{23\}$	1.05	$\{234\}$	1.9		
		$\{24\}$	0				
		$\{34\}$	1.9				

Here, $EL(N, v) = (0.05, 0.05, 0.95, 0.95) \notin C(N, v)$ which implies $(N, v) \notin \Gamma_{EP}$. Moreover, $x = Lmax(N, v) = (0, 0.1, 0.95, 0.95)$. Let us check that $(N, v) \in \Gamma'$. Note first that $x_3 + x_4 = v(\{34\}) = 1.9$. Now consider the max reduced game $\left(\{12\}, r^{\{12\}}_{M,x}(v)\right)$. Then, $r^{\{12\}}_{M,x}(v)(\{1\}) = 0$, $r^{\{12\}}_{M,x}(v)(\{2\}) = 0.1$ and $r^{\{12\}}_{M,x}(v)(\{12\}) = 2 - 1.9 = 0.1$. By *max consistency* of the lexmax solution, $Lmax\left(\{12\}, r^{\{12\}}_{M,x}(v)\right) = x_{|\{12\}} = (0, 0.1)$. Thus, $x_2 = r^{\{12\}}_{M,x}(v)(\{2\})$. Finally, $Lmax\left(\{1\}, r^{\{1\}}_{M,x}(v)\right) = x_1 = r^{\{1\}}_{M,x}(v)(\{1\})$. Hence, $(N, v) \in \Gamma'$.

2.5 Anti-dual axioms

Recently, Oishi et al. (2016) apply the notion of anti-duality to axioms in order to obtain new axiomatic characterizations of the WCES on the domain of convex games. In this section, we use this approach to provide additional axiomatizations of the WCES on the domain of exact partition games, that also hold on the domain of convex games, and of the Lorenz maximal core on the domain of balanced games.

Let us first remember some definitions on duality.

Given a game (N, v), the **dual game** (N, v^d) is defined by setting for all $S \subseteq N$, $v^d(S) = v(N) - v(N \setminus S)$. Let Γ^* be a class of games such that, for all $N \in \mathcal{N}$, it holds $(N, v), (N, v^d) \in \Gamma^*$. Given a solution σ on Γ^*, the **dual solution of** σ, denoted by σ^d, is defined by setting $\sigma^d(N, v) = \sigma(N, v^d)$. A solution σ on Γ^* is **self-dual** if for all $(N, v) \in \Gamma^*$, $\sigma(N, v) = \sigma^d(N, v)$.

Given a game (N, v), the **anti-dual game** is $(N, -v^d)$.

Let Γ^{**} be a class of games such that, for all $N \in \mathcal{N}$, it holds $(N, v), (N, -v^d) \in \Gamma^{**}$. The class of balanced games and the class of convex games are examples of Γ^{**}. Given a solution σ on Γ^{**}, the **anti-dual solution of** σ, denoted by σ^{ad}, is defined by setting $\sigma^{ad}(N, v) = -\sigma(N, -v^d)$. A solution σ on Γ^{**} is **self-anti-dual** if for all $(N, v) \in \Gamma^{**}$, $\sigma(N, v) = \sigma^{ad}(N, v)$. Some well-known self-anti-dual solutions are, among others, the core (on the domain of balanced games) and the WCES (on the domain of convex games).[6]

Making use of the anti-dual solution, Oishi et al. (2016) introduce the concept of anti-dual axioms.

Definition 18. *Given two axioms* **A** *and* **B**, *we say that*

- **A** *and* **B** *are* **anti-dual** *to each other if for all solution σ satisfying* **A** *it holds that σ^{ad} satisfies* **B**, *and conversely, for all solution σ satisfying* **B** *it holds that σ^{ad} satisfies* **A**.

- **A** *is* **self-anti-dual** *if for all solution σ satisfying* **A** *it holds that σ^{ad} also satisfies* **A**.

In order to apply the anti-duality approach on the domain of exact partition games Γ_{EP}, first we need to see if Γ_{EP} is closed under the anti-dual operator.

Proposition 9. *The class of exact partition games Γ_{EP} is preserved under the anti-dual operator.*

[6] See Oishi et al. (2016) for others examples of self-anti-dual solutions.

Proof. Let $(N,v) \in \Gamma_{EP}$. Then, there exists $x \in C(N,v)$ such that the ordered partition of N induced by x, $\pi = (N_1, \ldots, N_m)$, satisfies $x(N_1 \cup \ldots \cup N_k) = v(N_1 \cup \ldots \cup N_k)$ for all $k = 1, \ldots, m$. It is straightforward to check that $-x \in C(N, -v^d)$ and the ordered partition of N induced by $-x$ is $\pi' = (N_m, \ldots, N_1)$ satisfying $-x(N_m) = -v^d(N_m)$, $-x(N_m \cup N_{m-1}) = -v^d(N_m \cup N_{m-1})$, and so on. Hence, $(N, -v^d) \in \Gamma_{EP}$. $\square$

The following remark will be of help to prove that the WCES is self-anti-dual on the domain of exact partition games, and that the Lorenz maximal core is self-anti-dual on the domain of balanced. This fact will allow us to detect new axiomatic characterizations of these solutions by means of the anti-dual axioms of the ones involved in Theorems 9, 10, 11, 12 and 13.

Remark 8. *Given $N \in \mathcal{N}$, for any $x, y \in \mathbb{R}^N$ with $x(N) = y(N)$ it holds that*

$$x \succ_{\mathcal{L}} y \Longleftrightarrow -x \succ_{\mathcal{L}} -y. \tag{2.14}$$

For any $x \in \mathbb{R}^N$, denote by $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ the vector obtained from x by rearranging its coordinates in a non-decreasing order, that is, $\bar{x}_1 \leq \bar{x}_2 \leq \ldots \leq \bar{x}_n$. Let $x, y \in \mathbf{R}^N$ with $x(N) = y(N)$. We know that

$$x \succ_{\mathcal{L}} y \iff \sum_{j=1}^{k} \bar{x}_j \geq \sum_{j=1}^{k} \bar{y}_j, \tag{2.15}$$

for all $k \in \{1, \ldots, |N|\}$ with at least one strict inequality. Or equivalently,

$$x \succ_{\mathcal{L}} y \iff \sum_{j=1}^{k} \hat{x}_j \leq \sum_{j=1}^{k} \hat{y}_j, \tag{2.16}$$

for all $k \in \{1, \ldots, |N|\}$ with at least one strict inequality, where $\hat{x}_1 \geq \ldots \geq \hat{x}_n$, *and* $\hat{y}_1 \geq \ldots \geq \hat{y}_n$. *Combining (2.15) and (2.16) we obtain,*

$$\begin{aligned} x \succ_{\mathcal{L}} y \quad &\Longleftrightarrow \quad \textstyle\sum_{j=1}^{k} \bar{x}_j \geq \sum_{j=1}^{k} \bar{y}_j \\ &\Longleftrightarrow \quad -\textstyle\sum_{j=1}^{k} \bar{x}_j \leq -\sum_{j=1}^{k} \bar{y}_j \\ &\Longleftrightarrow \quad \textstyle\sum_{j=1}^{k} \widehat{-x}_j \leq \sum_{j=1}^{k} \widehat{-y}_j \\ &\Longleftrightarrow \quad -x \succ_{\mathcal{L}} -y \end{aligned} \tag{2.17}$$

for all $k \in \{1, \ldots, |N|\}$ *with at least one strict inequality.*

Now we have all the tools to state that both the WCES and the Lorenz maximal core are self-anti-dual.

Proposition 10. *On the domain of exact partition games* Γ_{EP}*, the WCES is self-anti-dual.*

Proof. Let $(N, v) \in \Gamma_{EP}$ and $x = EL^{ad}(N, v)$. Notice that $x \in C(N, v)$. Then, $x = -EL(N, -v^d) \iff -x \succ_{\mathcal{L}} y$ for all $y \in C(N, -v^d)$ (by Proposition 9 and Theorem 8) $\iff x \succ_{\mathcal{L}} -y$ for all $-y \in -C(N, -v^d) = C(N, v)$ (by Remark 8) $\iff x = EL(N, v)$. □

The same proof can be applied to show that the WCES is self-anti-dual on the domain of convex games.[7]

Proposition 11. *On the domain of balanced games, the Lorenz maximal core is self-anti-dual.*

Proof. Let (N, v) be a balanced game. We must show that $EC(N, v) = EC^{ad}(N, v)$. Take $x \in EC(N, v)$ and suppose that $x \notin EC^{ad}(N, v)$. Then, $-x \notin EC(N, -v^d)$ and so there is $y \in C(N, -v^d)$ such that $y \succ_{\mathcal{L}} -x$ or, equivalently, $-y \succ_{\mathcal{L}} x$ (by Remark 8) where $-y \in -C(N, -v^d) = C(N, v)$, in contradiction with $x \in EC(N, v)$. Similarly it can be checked that $EC^{ad}(N, v) \subseteq EC(N, v)$. □

[7] Oishi et al. (2016) use an alternative proof to claim that the WCES is self-anti-dual on the domain of convex games.

Now, we are going to determine the anti-dual axioms of the ones that appear in the above characterizations. It is quite natural to expect that the anti-dual axiom of *individual rationality* will recommend that the payoff assigned to a player should be bounded from above by his marginal contribution to the grand coalition.

Formally, a solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Upper boundedness** if for all $N \in \mathcal{N}$, all $(N,v) \in \Gamma'$, all $x \in \sigma(N,v)$ and all $i \in N$, then $x_i \leq v(N) - v(N \setminus \{i\})$.

From the Lorenz equivalence formulate in (2.14), it comes that both *internal Lorenz stability* and *external Lorenz stability (over the core)* are self-anti-dual.

In the next two propositions we assume that, if $x, y \in \sigma(N,v)$, then $x(N) = y(N)$, for all game (N,v) and all solution σ.

Proposition 12. *Let σ be a solution on a domain Γ^{**} such that, for all $(N,v) \in \Gamma^{**}$ and all $x, y \in \sigma(N,v)$, it holds that $x(N) = y(N)$. If σ satisfies internal Lorenz stability, then σ^{ad} also satisfies it.*

Proof. Let σ be a solution on Γ^{**} satisfying the above conditions. Let $(N,v) \in \Gamma^{**}$ and $x, y \in \sigma^{ad}(N,v)$. Then, $-x, -y \in \sigma(N, -v^d)$. By *internal Lorenz stability* of σ, neither $-x \succ_{\mathcal{L}} -y$ nor $-y \succ_{\mathcal{L}} -x$. From Remark 8 this is equivalent to neither $x \succ_{\mathcal{L}} y$ nor $y \succ_{\mathcal{L}} x$, which mean that σ^{ad} also satisfies *internal Lorenz stability*. $\square$

Proposition 13. *Let σ be a solution on a domain Γ^{**} such that, for all $(N,v) \in \Gamma^{**}$ and all $x, y \in \sigma(N,v)$, it holds that $x(N) = y(N)$. If σ satisfies external Lorenz stability (over the core), then σ^{ad} also satisfies it.*

Proof. Let σ be a solution on Γ^{**} satisfying the above conditions. Let $(N,v) \in \Gamma^{**}$ and $x \in C(N,v) \setminus \sigma^{ad}(N,v)$. That is, $x \in C(N,v) \setminus -\sigma(N,-v^d)$. Since the core is self-anti-dual, $x \in C^{ad}(N,v) = -C(N,-v^d)$. By *external Lorenz stability* *(over the core)* of σ, there is $y \in \sigma(N, -v^d)$ such that $y \succ_{\mathcal{L}} -x$. From Remark 8 this is equivalent to $-y \succ_{\mathcal{L}} x$, where $-y \in -\sigma(N, -v^d) = \sigma^{ad}(N, v)$. Thus, σ^{ad} also satisfies *external Lorenz stability (over the core).* $\square$

Oishi et al. (2016) show that max consistency is self-anti-dual, and that complement consistency (Moulin, 1985) and projected consistency (Funaki, 1998) are anti-dual to each other. With the aim to generalize these results, we make reference to the notion of **admissible subgroup correspondence** (Thomson, 1990).

Definition 19. *An admissible subgroup correspondence $\alpha : \mathcal{N} \to \mathcal{N}$ is a correspondence that associates with each $N \in \mathcal{N}$ a non-empty list $\alpha(N)$ of coalitions of N.*

We denote by $\mathcal{A}$ the set of all admissible subgroup correspondences. For a given $\alpha \in \mathcal{A}$, we define the α-**max reduced game**.

Definition 20. *Let (N, v) be a game, $\alpha \in \mathcal{A}$, $\emptyset \neq N' \subset N$ and $x \in \mathbb{R}^K$ where $N \setminus N' \subseteq K \subseteq N$. The $\alpha-$max reduced game relative to N' at x is the game $\left(N', r^{N'}_{\alpha,x}(v)\right)$ defined by*

$$r^{N'}_{\alpha,x}(v)(S) = \begin{cases} 0 & \text{if } S = \emptyset, \\ \max\limits_{Q \in \alpha(N \setminus N')} \{v(S \cup Q) - x(Q)\} & \text{if } \emptyset \neq S \subset N', \\ v(N) - x(N \setminus N') & \text{if } S = N'. \end{cases} \tag{2.18}$$

The max reduced game is a particular case when $\alpha(N) = 2^N$ for all $N \in \mathcal{N}$. The complement reduced game (Moulin, 1985) is defined by $\alpha(N) = \{N\}$ for all $N \in \mathcal{N}$, and the projected reduced game (Funaki, 1998) by $\alpha(N) = \{\emptyset\}$ for all $N \in \mathcal{N}$. The above reduction operations will be denoted by α_{DM}, α_M and α_P, respectively.

Given $\alpha \in \mathcal{A}$ and the corresponding α-max reduced game, we introduce α-**consistency**.

Definition 21. *Let σ be a solution on $\Gamma' \subseteq \Gamma$. Given $\alpha \in \mathcal{A}$, we say that σ satisfies α-consistency on Γ' if for all $N \in \mathcal{N}$, all $(N,v) \in \Gamma'$, all $N' \subset N$, $N' \neq \emptyset$, and all $x \in \sigma(N,v)$, then $\left(N', r^{N'}_{\alpha,x}(v)\right) \in \Gamma'$ and $x_{|N'} \in \sigma\left(N', r^{N'}_{\alpha,x}(v)\right)$.*

Given $\alpha \in \mathcal{A}$, the associated **complement admissible subgroup correspondence**, denoted by $\alpha^c \in \mathcal{A}$, is defined by setting, for all $N \in \mathcal{N}$,

$$\alpha^c(N) = \{N \setminus S \mid S \in \alpha(N)\}. \tag{2.19}$$

For a given $\alpha \in \mathcal{A}$, we will show that α-consistency and α^c-consistency are anti-dual to each other. From the observation that $\alpha_{DM} = \alpha^c_{DM}$ and $\alpha_M = \alpha^c_P$ (or $\alpha_P = \alpha^c_M$), the Oishi's results can be obtained as particular cases. To do this, we need a previous lemma.

Lemma 4. *Let (N,v) be a game, $\alpha \in \mathcal{A}$, $\emptyset \neq N' \subset N$ and $x \in \mathbb{R}^K$ where $N \setminus N' \subseteq K \subseteq N$. Then, for all $S \subseteq N'$,*

$$r^{N'}_{\alpha,-x}(-v^d)(S) = -\left(r^{N'}_{\alpha^c,x}(v)\right)^d(S). \tag{2.20}$$

Proof. We can distinguish two cases:

- If $\emptyset \neq S \subset N'$,

$$\begin{aligned}
r^{N'}_{\alpha,-x}(-v^d)(S) &= \max_{Q \in \alpha(N \setminus N')} \left\{-v^d(S \cup Q) + x(Q)\right\} \\
&= \max_{Q \in \alpha(N \setminus N')} \{-v(N) + v(N \setminus S \cup Q) + x(Q)\} \\
&= -v(N) + \max_{Q \in \alpha(N \setminus N')} \{v(N \setminus S \cup Q) + x(Q)\} \\
&= -v(N) + x(N \setminus N') + \max_{Q \in \alpha(N \setminus N')} \{v(N \setminus S \cup Q) + x(Q) - x(N \setminus N')\} \\
&= -v(N) + x(N \setminus N') \\
&\quad + \max_{Q \in \alpha(N \setminus N')} \{v((N' \setminus S) \cup (N \setminus N' \cup Q)) - x(N \setminus N' \cup Q)\} \\
&= -\left(v(N) - x(N \setminus N') - \max_{Q \in \alpha^c(N \setminus N')} \{v((N' \setminus S) \cup Q) - x(Q)\}\right) \\
&= -\left(r^{N'}_{\alpha^c,x}(v)(N') - r^{N'}_{\alpha^c,x}(v)(N' \setminus S)\right) \\
&= -\left(r^{N'}_{\alpha^c,x}(v)\right)^d(S).
\end{aligned}$$

- If $S = N'$,

$$\begin{aligned} r^{N'}_{\alpha,-x}(-v^d)(N') &= -v^d(N) + x(N \setminus N') \\ &= -(v(N) - x(N \setminus N')) \\ &= -\left(r^{N'}_{\alpha^c,x}(v)\right)^d (N'). \end{aligned}$$

□

Proposition 14. *On a domain Γ^{**}, and for a given $\alpha \in \mathcal{A}$, α-consistency and α^c-consistency are anti-dual to each other.*

Proof. Let σ be a solution on a domain Γ^{**} satisfying α-consistency, with $\alpha \in \mathcal{A}$. Let $N \in \mathcal{N}$, $(N,v) \in \Gamma^{**}$, $x \in \sigma^{ad}(N,v)$ and $\emptyset \neq N' \subset N$. By definition, $x \in -\sigma(N,-v^d)$. Since σ satisfies α-consistency, $-x_{|N'} \in \sigma\left(N', r^{N'}_{\alpha,-x}(-v^d)\right)$. By Lemma 4, $r^{N'}_{\alpha,-x}(-v^d) = -\left(r^{N'}_{\alpha^c,x}(v)\right)^d$, and thus $-x_{|N'} \in \sigma\left(N', -\left(r^{N'}_{\alpha^c,x}(v)\right)^d\right)$. Hence, $x_{|N'} \in -\sigma\left(N', -\left(r^{N'}_{\alpha^c,x}(v)\right)^d\right) = \sigma^{ad}\left(N', r^{N'}_{\alpha^c,x}(v)\right)$, which prove that σ^{ad} satisfies α^c-consistency. In a similar way it can be showed that if a solution σ satisfies α^c-consistency, then its anti-dual σ^{ad} satisfies α-consistency. □

From the fact that *max consistency* is self-anti-dual it comes directly that *weak max consistency* also does. Hence, by replacing *individual rationality* by *upper boundedness* in Theorems 9, 10 and 11 we obtain the following additional axiomatizations.

Theorem 15. *On the domain of exact partition games Γ_{EP}, the WCES is the unique solution that satisfies weak max consistency, upper boundedness, internal Lorenz stability and external Lorenz stability (over the core).*

Theorem 16. *On the domain of convex games Γ_{Con}, the WCES is the unique solution that satisfies weak max consistency, upper boundedness, internal Lorenz stability and external Lorenz stability (over the core).*

Theorem 17. *The Lorenz maximal core is the unique solution on Γ_{Bal} that satisfies weak max consistency, upper boundedness, internal Lorenz stability and external Lorenz stability (over the core).*

To check that properties in Theorems 15, 16 and 17 are independent it is enough to consider the anti-dual solutions of the ones used to prove the independence of the properties in Theorems 9, 10 and 11.

Finally, to obtain the anti-dual results of Theorems 12 and 13, notice first that *nonemptiness* and *core selection* are clearly self-anti-dual. The anti-dual axioms of *rich player max consistency* and *rich player feasibility* are, respectively, the following.

A solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Poor player max consistency** if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$ and all $x \in \sigma(N, v)$, if $N_1 \subseteq N, N_1 \neq N$, is the set of players with lowest payoff (w.r.t. x), then $\left(N \setminus N_1, r_{M,x}^{N\setminus N_1}(v)\right) \in \Gamma'$ and $x_{|N\setminus N_1} \in \sigma\left(N \setminus N_1, r_{M,x}^{N\setminus N_1}(v)\right)$.
- **Bounded minimum payoff property** if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$ and all $x \in \sigma(N, v)$, it holds $x(N_1) \geq v(N) - v(N \setminus N_1)$, where N_1 denotes the set of players with lowest payoff (w.r.t. x).

Poor player max consistency is a weaker version of *max consistency* that only applies when players with the lowest payoffs leave the game. *Bounded minimum payoff property* simple says that the total amount received by the set of players with lowest payoff should be, at least, his marginal contribution to the grand coalition.

Indeed, let σ be a solution satisfying *rich player max consistency* on a domain Γ^{**}. Let $(N, v) \in \Gamma^{**}$ and $x = EL^{ad}(N, v)$. Then, $x = -EL(N, -v^d)$. If N_1 denotes the set of players with lowest payoff w.r.t. x, then N_1 contains the players with highest payoff w.r.t. $-x$. Thus, by *rich max consistency* of σ, we have that

$-x_{|N_1} = EL\left(N \setminus N_1, r^{N\setminus N_1}_{M,-x}(-v^d)\right)$. By Lemma 4, and taking into account that *max consistency* is self-anti-dual, $r^{N\setminus N_1}_{M,-x}(-v^d) = -\left(r^{N\setminus N_1}_{M,x}(v)\right)^d$. Hence, $x_{|N_1} = -EL\left(N \setminus N_1, -\left(r^{N\setminus N_1}_{M,x}(v)\right)^d\right) = EL^{ad}\left(N \setminus N_1, r^{N\setminus N_1}_{M,x}(v)\right)$, which proves that the anti-dual of the WCES satisfies *poor player max consistency*.

To check that *bounded minimum payoff property* is the anti-dual of *rich player feasibility*, consider a solution σ satisfying *rich player feasibility* on a domain Γ^{**}. Let $(N, v) \in \Gamma^{**}$ and $x = EL^{ad}(N, v)$. Then, $x = -EL(N, -v^d)$. As before, if N_1 denotes the set of players with lowest payoff w.r.t. x, then N_1 contains the players with highest payoff w.r.t. $-x$. By *rich player feasibility* of σ, $-x(N_1) \leq -v^d(N_1) = -v(N) + v(N \setminus N_1)$ or, equivalently, $x(N_1) \geq v(N) - v(N \setminus N_1)$.

Now, by replacing *rich player max consistency* and *rich player feasibility* in Theorems 12 and 13 by *poor player max consistency* and *bounded minimum payoff property*, respectively, we can state the following characterizations.

Theorem 18. *On the domain of exact partition games Γ_{EP}, the WCES is the unique solution that satisfies nonemptiness, poor player max consistency, core selection and bounded minimum payoff property.*

Theorem 19. *On the domain of convex games Γ_{Con}, the WCES is the unique solution that satisfies nonemptiness, poor player max consistency, core selection and bounded minimum payoff property.*

To check that properties in Theorems 18 and 19 are independent it is enough to consider the anti-dual solutions of the ones used to prove the independence of the properties in Theorems 12 and 13.

2.6 Lorenz stable set

Until now, we have considered the core as the reference set from which to select Lorenz maximal allocations. On the domain of exact partition games (an also on the domain of convex games) this leads to the WCES. However, on the whole domain of balanced games, the set of Lorenz maximal core allocations is not a singleton and the WCES may not exist. To overcome this drawback, Dutta and Ray (1991) introduce the SCES, a solution concept that chooses the Lorenz maximal allocations in the equal division core (Selten, 1972). In this section, we focus on the axiomatic approach of the Lorenz maximal allocation in the imputation set. We call this solution the Lorenz stable set. The reason is that it can be interpreted as a kind of stable set à la von Neumann-Morgenstern where the domination relation is base on the Lorenz order. Finally, we observe that the WCES and the SCES are connected by the Lorenz stable set.

Given an essential game (N, v), for $X \subseteq I(N, v)$ we denote by $\mathcal{L}^v(X)$ the set of all imputations Lorenz dominated by some imputation of the set X. Formally, $\mathcal{L}^v(X) = \{y \in I(N,v) \,|\, \exists\, x \in X,\, x \succ_{\mathcal{L}} y\}$. A non-empty set of imputations $\mathcal{V} \subseteq I(N, v)$ is a **Lorenz stable set** for the game (N, v) if it satisfies the next two conditions:

1. $\mathcal{V}$ is *internally Lorenz stable*: no imputation in $\mathcal{V}$ Lorenz dominates another imputation in $\mathcal{V}$. Formally, $\mathcal{V} \cap \mathcal{L}^v(\mathcal{V}) = \emptyset$.

2. $\mathcal{V}$ is *externally Lorenz stable*: any imputation outside the set $\mathcal{V}$ is dominated by some imputation in $\mathcal{V}$. Formally, $\mathcal{V} \cup \mathcal{L}^v(\mathcal{V}) = I(N, v)$.

On the domain of essential games, we find that the Lorenz stable set is a singleton and admits a formula that has the same flavor of the *constrained equal awards rule* for bankruptcy problems. On this domain, we provide an axiomatic characterization of this solution, similar to the ones provided by Dutta (1991) by only changing the notion of reduced game.

Definition 22. *Let (N, v) be an essential game. The vector $\mathbf{I}^v \in \mathbb{R}^N$ is defined as*

$$\mathbf{I}^v_i := \max\{v(i), \lambda\}, \tag{2.21}$$

for all $i \in N$, where λ is chosen so as to achieve efficiency.

Theorem 20. *Let (N, v) be an essential game. Then, there is a unique Lorenz stable set $\mathcal{V}$. Moreover, $\mathcal{V} = \{\mathbf{I}^v\}$.*

Proof. Let (N, v) be an essential game with $N = \{1, \ldots, n\}$. Define the game (N, v^*) as follows: $v^*(S) = \sum_{i \in S} v(i)$ for all $S \subset N$, and $v^*(N) = v(N)$. Notice that (N, v^*) is convex and $C(N, v^*) = I(N, v)$. Since for convex games the WCES Lorenz dominates every other point in the core, we only need to check that $EL(N, v^*) = \{\mathbf{I}^v\}$. Assume, w.l.o.g, $v(1) \geq \ldots \geq v(n)$. If $v(1) \leq \frac{v^*(N)}{n}$, then $EL(N, v^*) = \left\{\mathbf{I}^v = \left(\frac{v^*(N)}{n}, \ldots, \frac{v^*(N)}{n}\right)\right\}$. Otherwise, take $k \in \{1, \ldots, n-1\}$, $n \geq 2$, and define the vector

$$y^k := \left(v(1), \ldots, v(k), \frac{v(N) - (v(1) + \ldots + v(k))}{n-k}, \ldots, \frac{v(N) - (v(1) + \ldots + v(k))}{n-k}\right).$$

Observe that $\mathbf{I}^v = y^{k^*}$, where $k^* = \min\{k \in \{1, \ldots, n-1\} \mid y_i^k \geq v(i) \text{ for all } i \in N\}$. Let $\mathcal{P} = \{S_1, \ldots, S_m\}$ be the partition of N generated by the Dutta and Ray (1989) algorithm to compute $EL(N, v^*)$. Denote $EL(N, v^*) = \{z\}$. Notice that $m \geq 2$ because $v(1) > \frac{v^*(N)}{n}$. It can be easily checked that $z_i = v(i)$ for all $i \in S_h$ and all $h \in \{1, \ldots, m-1\}$, and $z_i = \frac{v(N) - \sum_{i \in N \setminus S_m} v(i)}{|S_m|}$ for all $i \in S_m$. Hence, $z = y^k$ where $k = |S_1 \cup \ldots \cup S_{m-1}|$. Suppose $k > k^*$. By the minimality of k^*, we have $z_i \leq y_i^{k^*}$ for all $i \in \{1, \ldots, k^*, \ldots, k\}$. Moreover, for all $i > k$, since $i \in S_m$ and $k \in S_{m-1}$, we have $z_i < z_k = v(k) \leq y_k^{k^*} = y_i^{k^*}$. Then, $z(N) < y^{k^*}(N) = v(N)$, a contradiction. Hence, $k = k^*$ and $EL(N, v^*) = \{\mathbf{I}^v\}$. □

From Theorem 20 and the characterization of Lorenz domination given by Hardy et al. (1934),[8] it follows that the Lorenz stable solution selects the allocation in the imputation set that minimize the Euclidean distance to the equal

[8]If x and y are two vectors in $\mathbb{R}^n$ with $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, the following statements are equivalent: (a) x Lorenz dominates y; (b) for any strictly concave function $U : \mathbb{R} \longrightarrow \mathbb{R}$, we have $\sum_{i=1}^n U(x_i) > \sum_{i=1}^n U(y_i)$.

division payoff vector. Formally, for all essential game (N, v),

$$\mathbf{I}^v = \arg \min_{x \in I(N,v)} \sum_{i \in N} \left(x_i - \frac{v(N)}{|N|} \right)^2 . \tag{2.22}$$

As we have mentioned before, on the domain of convex games Dutta (1990) characterizes the weak constrained egalitarian solution (Dutta and Ray, 1989) by means of *constrained egalitarianism* and either *max consistency* or *self consistency*. Interestingly, on the domain of essential games, replacing *max consistency* or *self consistency* by *projected consistency* (Funaki, 1998) we characterize the Lorenz stable set.[9]

Definition 23. *(Funaki, 1998) Let* (N, v) *be a game,* $\emptyset \neq N' \subset N$ *and* $x \in \mathbb{R}^K$ *where* $N \setminus N' \subseteq K \subseteq N$*. The projected reduced game relative to* N' *at* x *is the game* $\left(N', r^{N'}_{P,x}(v)\right)$ *defined by*

$$r^{N'}_{P,x}(v)(S) = \begin{cases} v(S) & \text{if } \emptyset \neq S \subset N', \\ v(N) - x(N \setminus N') & \text{if } S = N'. \end{cases} \tag{2.23}$$

In the projected reduced game (relative to N' at x), when players in $N \setminus N'$ leave the game, for a proper subcoalition $S \subset N'$ cooperation is no longer possible with them. *Projected consistency* tell us that in the projected reduced game the initial agreement should be adopted.

A solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Projected consistency** if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$, all $N' \subset N$, $N' \neq \emptyset$, and all $x \in \sigma(N, v)$, then $\left(N', r^{N'}_{P,x}(v)\right) \in \Gamma'$ and $x_{|N'} \in \sigma\left(N', r^{N'}_{P,x}(v)\right)$.

The next result connects consistency and *constrained egalitarianism* with *efficiency*.

[9] *Projected consistency has been used to characterize, am*ong others, the equal division core (Bhattacharya, 2004) or the undominated core (Llerena and Rafels, 2007).

Lemma 5. *Let σ be a single-valued solution on Γ_{Ess} that satisfies constrained egalitarianism and either projected consistency, max consistency or self consistency. Then, σ satisfies efficiency.*

Proof. Let σ be a single-valued solution on Γ_{Ess} that satisfies *constrained egalitarianism* and *projected consistency.* Let $(\{i\}, v)$ be a one-person game and for some $j \in \mathbb{N}\setminus\{i\}$ consider the essential game $(\{i,j\}, v')$ defined by $v'(i) = v'(ij) = v(i)$ and $v'(j) = 0$. By *constrained egalitarianism*, $\sigma(\{i,j\}, v') = (v(i), 0)$. It is easy to check that $(\{i\}, v) = \left(\{i\}, r^{\{i\}}_{P,x}(v')\right)$ where $x = \sigma(\{i,j\}, v')$. By *projected consistency*, $\sigma(\{i\}, v) = \sigma\left(\{i\}, r^{\{i\}}_{P,x}(v')\right) = \sigma_i(\{i,j\}, v') = v(i)$, which implies efficiency for one-person game. Let $N \in \mathcal{N}$ with $|N| \geq 2$, $(N, v) \in \Gamma_{Ess}$ and $i \in N$. Let us denote $x = \sigma(N, v)$. By *projected consistency* and efficiency for one-person game, $\sigma_i(N, v) = \sigma_i\left(\{i\}, r^{\{i\}}_{P,x}(v)\right) = v(N) - \sum_{j\in N\setminus\{i\}} \sigma_j(N, v)$, which proves *efficiency.*

The same reasoning holds replacing *projected consistency* by either *max consistency* or *self consistency.* □

Theorem 21. *On the domain of essential games, the only single-valued solution satisfying projected consistency and constrained egalitarianism is the Lorenz stable set.*

Proof. *Constrained egalitarianism* is obvious. Next we prove *projected consistency.* Let $N \in \mathcal{N}$, $(N, v) \in \Gamma_{Ess}$, $x = \mathbf{I}^v$ and $(T, r^T_{P,x}(v))$ be the projected reduced game relative to $\emptyset \neq T \subset N$ at x. Since $x_{|T} \in I(T, r^T_{P,x}(v))$, we have $(T, r^T_{P,x}(v)) \in \Gamma_{Ess}$. Let $y = \mathbf{I}^{r^T_{P,x}(v)}$ be the Lorenz stable set of $(T, r^T_{P,x}(v))$ and suppose $y \neq x_{|T}$. Then, $y \succ_{\mathcal{L}} x_{|T}$. Now consider the vector $z = (x_{|N\setminus T}, y) \in \mathbb{R}^N$. Since $z \in I(N, v)$, $x \succ_{\mathcal{L}} z$, which implies $x_{|T} \succ_{\mathcal{L}} y$, a contradiction.[10] Hence, $x_{|T} = y$. To prove uniqueness, let σ be a single-valued solution on Γ_{Ess} satisfying *constrained egalitarianism* and *projected consistency.* From Lemma 5 we

[10]See Remark 4.

know that these two properties together imply *efficiency*. For $|N| = 1$ and $|N| = 2$ uniqueness follows from *efficiency* and *constrained egalitarianism*, respectively. Let $(N, v) \in \Gamma_{Ess}$ with $N = \{1, 2, \ldots, n\}$, $n \geq 3$, and $x = \sigma(N, v)$. Let $T = \{i, j\} \subset N$. By *constrained egalitarianism* and *projected consistency*, $x_{|T} = \sigma(T, r^T_{P,x}(v)) = CE(T, r^T_{P,x}(v))$. Thus, $x \in I(N, v)$. If $x_1 = \ldots = x_n$, then $x = \mathbf{I}^v$. Otherwise, suppose, w.l.o.g., $x_1 > \ldots > x_{k+1} = \ldots = x_n$, for some $k \in \{1, \ldots, n-1\}$. For $i \in \{1, \ldots, k\}$, let $T = \{i, i+1\}$. By *projected consistency*, $x_{|T} = CE(T, r^T_{P,x}(v))$. Since $x_i > x_{i+1}$, $x_i = v(i)$ for all $i \in \{1, \ldots, k\}$. Now, by *efficiency* we obtain $x_i = \frac{v(N)-(v(1)+\ldots+v(k))}{n-k}$ for all $i \in \{k+1, \ldots, n\}$. Thus, for all $i \in N$, $x_i = \max\{v(i), \lambda\}$ being $\lambda = \frac{v(N)-(v(1)+\ldots+v(k))}{n-k}$, and $x = \mathbf{I}^v$. $\square$

The axioms in Theorem 21 are independent. For instance, the single-valued solution σ_1 defined, for all $N \in \mathcal{N}$ and all $(N, v) \in \Gamma_{Ess}$, as $\sigma_1(N, v) = \left(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|}\right)$, satisfies *projected consistency* but not *constrained egalitarianism*. The single-valued solution σ_2 defined, for all $N \in \mathcal{N}$ and all $(N, v) \in \Gamma_{Ess}$, as $\sigma_2(N, v) = CE(N, v)$ if $|N| = 2$, and $\sigma_2(N, v) = (v(i))_{i \in N}$ otherwise, satisfies *constrained egalitarianism* but not *projected consistency*.

The above characterization opens an interesting question: on the domain of essential games, which is the set of rules that emerges from substituting in Theorem 21 *projected consistency* by either *max consistency* or *self consistency*. As we will see, the combination of these properties leads to imposibility results.

Theorem 22. *There is no single-valued solution on Γ_{Ess} that satisfies*

1. *max consistency and constrained egalitarianism.*

2. *self consistency and constrained egalitarianism.*

Proof.

1. Suppose, on the contrary, there is a single-valued solution on Γ_{Ess} that satisfies *max consistency* and *constrained egalitarianism*. By Lemma 5,

σ satisfies *efficiency*. Next we see that it also satisfies *individual rationality*. Let $(N, v) \in \Gamma_{Ess}$. If $|N| = 1$ or $|N| = 2$, by *efficiency* and *constrained egalitarianism* we have that $\sigma(N, v) \in I(N, v)$. If $|N| \geq 3$, choose two arbitrary players $i, j \in N$ and consider the max reduced game $\left(\{i, j\}, r^{\{i,j\}}_{M,x}(v)\right)$, being $x = \sigma(N, v)$. By *max consistency* and *constrained egalitarianism*, $\sigma_{|\{i,j\}}(N, v) = CE\left(\{i, j\}, r^{\{i,j\}}_{M,x}(v)\right) \geq (v(i), v(j))$. Hence, $\sigma(N, v) \in I(N, v)$. Let $(N, v) \in \Gamma_{Ess}$ with $N = \{1, 2, 3\}$ and characteristic function as follows: $v(i) = 0$ for all $i \in N$, and $v(S) = 1$ for any other coalition $S \subseteq N$. Denote $\sigma(N, v) = x$. By *individual rationality* $0 \leq x_i \leq 1$, for all $i \in N$. Now consider the max reduced game $\left(\{1, 2\}, r^{\{1,2\}}_{M,x}(v)\right)$. Note that $r^{\{1,2\}}_{M,x}(v)(1) = r^{\{1,2\}}_{M,x}(v)(2) = \max\{0, 1 - x_3\} = 1 - x_3$, and $r^{\{1,2\}}_{M,x}(v)(12) = 1-x_3$. By *max consistency*, $\left(\{1, 2\}, r^{\{1,2\}}_{M,x}(v)\right) \in \Gamma_{Ess}$, which means that $2(1 - x_3) \leq 1 - x_3$ or, equivalently, $1 \leq x_3$. This, together with the fact that $x_3 \leq 1$, imply $x_3 = 1$. In a similar way, it can be checked that $x_1 = x_2 = 1$, contradicting *efficiency*.

2. Suppose, on the contrary, there is a single-valued solution on Γ_{Ess} that satisfies *self consistency* and *constrained egalitarianism*. By Lemma 5, σ satisfies *efficiency*. Let $(N, v) \in \Gamma_{Ess}$ with $N = \{1, 2, 3\}$ and characteristic function as follows: $v(i) = 0$ for all $i \in N$, and $v(S) = 1$ for any other coalition $S \subseteq N$. Now consider the self reduced game $\left(\{1, 2\}, r^{\{1,2\}}_{S,\sigma}(v)\right)$. Recall that $r^{\{1,2\}}_{S,\sigma}(v)(1) = v(13) - \sigma_3\left(\{1, 3\}, v_{|\{1,3\}}\right)$, $r^{\{1,2\}}_{S,\sigma}(v)(2) = v(23)-\sigma_3\left(\{2, 3\}, v_{|\{2,3\}}\right)$ and $r^{\{1,2\}}_{S,\sigma}(v)(12) = v(N)-\sigma_3(N, v)$. By *constrained egalitarianism*, $\sigma_3\left(\{1, 3\}, v_{|\{1,3\}}\right) = \sigma_3\left(\{2, 3\}, v_{|\{2,3\}}\right) = \frac{1}{2}$. Hence, $r^{\{1,2\}}_{S,\sigma}(v)(1) = r^{\{1,2\}}_{S,\sigma}(v)(2) = 1 - \frac{1}{2} = \frac{1}{2}$. By *self consistency*, $\left(\{1, 2\}, r^{\{1,2\}}_{S,\sigma}(v)\right) \in \Gamma_{Ess}$, which implies $\frac{1}{2} + \frac{1}{2} \leq v(N) - \sigma_3(N, v) = 1 - \sigma_3(N, v)$ or, equivalently, $\sigma_3(N, v) \leq 0$. In a similar way, it can be checked that $\sigma_1(N, v) \leq 0$ and $\sigma_2(N, v) \leq 0$, contradicting *efficiency*. $\square$

2.7 Connecting the weak and the strong constrained egalitarian solutions

Dutta and Ray (1991) characterize the class of superadditive games in which WCES and SCES coincide. Here we show that, on the domain of all games, the unique weak constrained egalitarian allocation happens to be a strong if and only if the two set of allocations are singleton containing the Lorenz stable allocation. Consequently, for superadditive games we find an easy way to check when coincidence occurs.

Theorem 23. *Let (N, v) be an game. Then, the following statements are equivalent:*

(i) $EL(N, v) \cap EL^*(N, v) \neq \emptyset$.

(ii) $EL(N, v) = \{\mathbf{I}^v\}$.

(iii) $EL(N, v) = EL^*(N, v) \neq \emptyset$.

Proof. (i) $\Rightarrow$ (ii): Let $EL(N, v) \cap EL^*(N, v) = \{y\}$ and let us assume, w.l.o.g., that $y_1 \geq y_2 \geq \ldots \geq y_n$. If $y_1 = y_n$, then $y = \left(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|}\right)$ and so $y = \mathbf{I}^v$. If $y_1 > y_n$, then $T = \{i \in N \,|\, y_i > y_n\} \neq \emptyset$. Let $j^* \in T$, by Lemma 2 of Dutta and Ray (1991)[11] there exists an equity coalition R containing j^* and such that $\frac{v(R)}{|R|} = y_{j^*}$ and $R \subset \{i \in N \,|\, y_i < y_{j^*}\} \cup \{j^*\}$. If $|R| = 1$, then $y_{j^*} = v(j^*)$. Otherwise, if $|R| \geq 2$, then $EL(R, v) = \left\{\left(\frac{v(R)}{|R|}, \ldots, \frac{v(R)}{|R|}\right)\right\}$. Since $y \in EL(N, v)$ there exists $i^* \in R$ such that $y_{i^*} > \frac{v(R)}{|R|} = y_{j^*}$, getting a contradiction. Then $R = \{j^*\}$. Thus, $y_i = v(i)$ for all $i \in T$ and, by *efficiency*, $y_i = \frac{v(N) - \sum_{j \in T} v(j)}{|N| - |T|}$, for all $i \in N \backslash T$. We know that $\mathbf{I}^v = \left(v(1), \ldots, v(k), \frac{v(N) - \sum_{i=1}^{k} v(i)}{n-k}, \ldots, \frac{v(N) - \sum_{i=1}^{k} v(i)}{n-k}\right)$

[11] Lemma 2 in Dutta and Ray (1991) states the following: *For some $S \subseteq N$, let $y \in EL^*(S, v)$. For any $i \in S$, if $y_i > \min_{j \in S} y_j$, then there exists an equity coalition T containing i and satisfying: (i) $\frac{v(T)}{|T|} = y_i$ and (ii) $T \subset \{k \in S \,|\, y_k < y_i\} \cup \{i\}$.*

where $k = \min\left\{ j \in N \mid \frac{v(N)-\sum_{i=1}^{j} v(i)}{n-j} \geq v(j+1) \right\}$. Since $y \in I(N,v)$, $|T| = t \geq k$. Suppose $t > k$. For all $i \in \{1,\ldots,k\}$, $\mathbf{I}^v_i = y_i = v(i)$, for all $i \in \{k+1,\ldots,t\}$, $\mathbf{I}^v_i \geq v(i) = y_i$, and for all $i \in \{t+1,\ldots,n\}$, $\mathbf{I}^v_i = \mathbf{I}^v_t \geq v(t) = y(t) > y_i$. But then, $v(N) = \mathbf{I}^v(N) > y(N)$ in contradiction with $y(N) = v(N)$. Hence, $k = t$ and $y = \mathbf{I}^v$.

The implication (ii) $\Rightarrow$ (iii) follows from $L(N,v) \subseteq L^*(N,v) \subseteq I(N,v)$ and the fact that $\mathbf{I}^v$ Lorenz dominates every other point in the imputation set. Obviously (iii) $\Rightarrow$ (i). $\square$

As a consequence of Theorem 23 we obtain the following corollary for superadditive games.

Corollary 3. *Let (N,v) be a superadditive game. Then, the following statements are equivalent:*

(i) $EL(N,v) = EL^*(N,v)$.

(ii) $\mathbf{I}^v \in C(N,v)$.

Proof. Notice first that for superadditive games, $EL^*(N,v) \neq \emptyset$. From Theorem 23, $EL(N,v) = EL^*(N,v) \neq \emptyset$ implies $EL(N,v) = EL^*(N,v) = \{\mathbf{I}^v\}$. On this domain, both solution coincide when the unique strong constrained egalitarian allocation belongs to the core (Dutta and Ray, 1991), thus $\mathbf{I}^v \in C(N,v)$. Conversely, since $C(N,v) \subseteq L(N,v) \subseteq L^*(N,v)$ and $\mathbf{I}^v$ Lorenz dominates every other point in the imputation set, we have $EL(N,v) = EL^*(N,v) = \{\mathbf{I}^v\}$. $\square$

2.8 Conclusions

We have introduced a subclass of balanced games, called exact partition games Γ_{EP}. This class is large enough to include convex games and dominant diagonal assignment games, but also nonsuperadditive games. On Γ_{EP}, we have shown that the WCES behaves as it does in convex games, that is, it exists, belongs to the core and Lorenz dominates every other core element. We have also provided two axiomatic characterizations. The former uses *weak max consistency*, *individual rationality*, *internal Lorenz stability* and *external Lorenz stability (over the core).* The second characterization uses *nonemptiness*, *rich player max consistency*, *core selection* and *rich player feasibility.* Interestingly, both characterizations hold over the domain of convex games. The first one can also be extended to balanced games characterizing the Lorenz maximal core on this domain. On the full domain of games, *nonemptiness*, *core selection* and *rich player feasibility* are incompatible. Nevertheless, we observe that if these properties, together with *rich player max consistency*, can be reconciled on an admissible subdomain of balanced games, then they determine the lexman solution. Although we have not reached any definitive conclusion, the above characterization leads us to conjecture that if the WCES exists and belongs to the core, then it coincides with the lexmax solution. Since the WCES is self-anti-dual (see Oishi et al., 2016), in Section 2.5 we studied the anti-dual axioms of the ones used in the above characteritzations and found new axiomatizations.

For future research, it could be worthwhile studying whether the characterizations of the WCES given by Klijn et al. (2000), Hougaard et al. (2001) and Arin et al. (2003) over the domain of convex games can be extended to Γ_{EP}. It could also be interesting to analyze what sort of solutions emerge if the properties used in these axiomatizations are compatible on the whole domain of balanced games.

On the domain of essential games, we have introduced the Lorenz stable set and shown that it is single-valued and selects the unique Lorenz maximal allocation in the imputation set. Dutta (1990) characterizes the WCES on the domain of convex games by using *constrained egalitarianism* and either *max consistency* or *self consistency.* On the domain of essential games, we have shown that these properties are incompatible together. However, by replacing *max consistency* and *self consistency* with *projected consistency* we obtain a parallel characterization of the Lorenz stable set. Finally, we find that this solution connects the WCES with the SCES. Another interesting issue for future research would be to investigate an alternative axiomatic characterization of the Lorenz stable set by replacing the prescriptive property of *constrained egalitarianism* with other suitable properties.

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