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On the (adjacency) metric dimension of corona and strong product graphs and their local variants: Combinatorial and computational results

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ABSTRACT

The metric dimension is quite a well-studied graph parameter. Recently, the adjacency metric dimension and the local metric dimension have been introduced. We combine these variants and introduce the local adjacency metric dimension. We show that the (local) metric dimension of the corona product of a graph of order n and some non-trivial graph H equals n times the (local) adjacency metric dimension of H. This strong relation also enables us to infer computational hardness results for computing the (local) metric dimension, based on according hardness results for (local) adjacency metric dimension that we also provide. We also study combinatorial properties of the strong product of graphs and emphasize the role of different types of twins play in determining in particular the adjacency metric dimension of a graph.

1. Introduction

Locating or resolving sets have been introduced as a graph-theoretic model of robot navigation, but for certain variants, also connections to coding theory have been established. This has led to graph parameters like metric dimension and adjacency metric dimension. As detailed below, we continue the study of these notions, focussing on three aspects: (1) We also include the investigation of local variants of the mentioned notions, which corresponds to a more myopic nature of robot sensors, a feature often encountered in practice. This also leads us to the introduction of a new graph parameter, called local adjacency metric dimension. (2) We show that the (local) metric dimension of the corona product of a graph of order n and some non-trivial graph H equals n times the (local) adjacency metric dimension of H. Also, tight formula involving the classical domination number are established. (3) These purely combinatorial identities are used to prove NP-hardness results for computing the (local) metric dimension, which seems to be a new methodology to derive such results.

Throughout this paper, we only consider undirected simple loop-free graphs. We collect the standard graph-theoretic terminology at the end of this section, as well as some notions on metric spaces.

A preliminary version of this paper, containing barely any proofs, was presented at *Computer Science in Russia* in Moscow 2014, see [15].

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1.1. Four notions of dimension in graphs

Let $\mathbb{N}$ denote the set of non-negative integers. Given a connected graph $G = (V, E)$, we consider the function $d_G : V \times V \to \mathbb{N}$, where $d_G(x, y)$ is the length of a shortest path between u and v. Clearly, (V, d_G) is a metric space. The diameter of a graph is understood in this metric. Alternatively, the diameter can be defined via the notion of eccentricity of a vertex, which is defined as $\varepsilon(v) = \sup\{d_G(v, u) : u \in V - \{v\}\}$. Namely, $\mathrm{diam}(G) = \max\{\varepsilon(v) : v \in V\}$. Similarly, the *radius* of a graph is defined as $r(G) = \min\{\varepsilon(v) : v \in V\}$.

A vertex set $S \subseteq V$ is said to be a *metric generator* for G if it is a generator of the metric space (V, d_G), *i.e.*, every point of the space is uniquely determined by its distances from the elements of S. A minimum metric generator is called a *metric basis*, and its cardinality the *metric dimension* of G, denoted by $\dim(G)$. Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension of a graph was introduced by Slater in [45], where the metric generators were called *locating sets*. The concept of metric dimension of a graph was also introduced by Harary and Melter in [23], where metric generators were called *resolving sets*. Applications of this invariant to the navigation of robots in networks are discussed in [35] and applications to chemistry in [32,33]. This graph parameter was studied further in a number of other papers including, for instance [1,6,9,14,21,26,30,39,43,47]. Several variations of metric generators including resolving dominating sets [4], independent resolving sets [10], local metric sets [40], strong resolving sets [44], etc. have since been introduced and studied.

A set S of vertices in a connected graph G is a *local metric generator* for G (also called local metric set for G [40]) if every two adjacent vertices of G are distinguished by some vertex of S. A minimum local metric generator is called a *local metric basis* for G and its cardinality, the *local metric dimension* of G, is denoted by $\dim_l(G)$.

A set S of vertices in a graph G is an *adjacency generator* for G (also adjacency resolving set for G [31]) if for every pair of different vertices $x, y \in V(G) - S$ there exists $s \in S$ such that $|N_G(s) \cap \{x, y\}| = 1$, where $N_G(s)$ denotes the neighborhood of s in G. This concept is very much related to that of a 1-locating dominating set [8]. A minimum adjacency generator is called an *adjacency basis* for G and its cardinality, the *adjacency dimension* of G, is denoted by $\dim_A(G)$. These concepts were introduced in [31] with the aim of study the metric dimension of the lexicographic product of graphs in terms of the adjacency dimension of graphs. Observe that an adjacency generator of a graph $G = (V, E)$ is also a generator in a suitably chosen metric space, namely by considering $(V, d_{G,2})$, with $d_{G,2}(x, y) = \min\{d_G(x, y), 2\}$, and vice versa.

Now, we combine the two variants of metric dimension defined so far and introduce the local adjacency dimension of a graph. We say that a set S of vertices in a graph G is a *local adjacency generator* for G if for every two adjacent vertices $x, y \in V(G) - S$ there exists $s \in S$ such that $|N_G(s) \cap \{x, y\}| = 1$. A minimum local adjacency generator is called a *local adjacency basis* for G and its cardinality, the *local adjacency dimension* of G, is denoted by $\dim_{A,l}(G)$.

1.2. Our main results

In this paper we study the (local) metric dimension of corona product graphs via the (local) adjacency dimension of a graph. We show that the (local) metric dimension of the corona product of a graph of order n and some non-trivial graph H equals n times the (local) adjacency metric dimension of H. This relation is much stronger and under weaker conditions compared to the results of Jannesari and Omoomi [31] concerning the lexicographic product of graphs. This also enables us to infer NP-hardness results for computing the (local) metric dimension, based on corresponding NP-hardness results for (local) adjacency metric dimension that we also provide. To our knowledge, this is the first time combinatorial results on this particular form of graph product have been used to deduce computational hardness results. The obtained reductions are relatively simple and also allow us to conclude hardness results based on the Exponential Time Hypothesis. We also discuss NP-hardness results for planar graphs, which seem to be of some particular importance to the sketched applications. This also shows the limitations of using corona products to obtain hardness results. In addition, we study combinatorial properties of the strong product of graphs and emphasize the role of different types of twins play in determining in particular the adjacency metric dimension of a graph. Finally, we indicate why computing the (local) adjacency metric dimension is in FPT (under the standard parameterization), contrasting what is known for computing the metric dimension.

1.3. Some notions from graph theory and metric spaces

In this paragraph, we collect some standard graph-theoretic terminology that we employ. As usual, graphs are specified like $G = (V, E)$, where V is the set of vertices and E is the set of edges of the graph G. $|V|$ is also known as the *order* of G. A graph of order greater than one is also called *non-trivial*. Two vertices $u, v \in V$ with an edge between them, i.e., $uv \in E$, are also called *adjacent* or *neighbors*, and this is also written as $u \sim v$. For a vertex v of G, $N_G(v)$ denotes the set of neighbors that v has in G, i.e., $N_G(v) = \{u \in V : u \sim v\}$. The set $N_G(v)$ is called the *open neighborhood of* v in G and $N_G[v] = N_G(v) \cup \{v\}$ is called the *closed neighborhood of* v in G. A vertex set $D \subseteq V$ is called a *dominating set* if $\bigcup_{v \in D} N_G[v] = V$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets in G. A vertex set $I \subseteq V$ is called an *independent set* if for all $u, v \in I$, $uv \notin E$. The *independent set number* of G, denoted by $\alpha(G)$, is the maximum cardinality among all independent sets in G. The difference between the order and the independent set number of a graph G is also known as the *vertex cover number* of G, written $\beta(G)$, as the complement of an independent set is called a *vertex cover*.

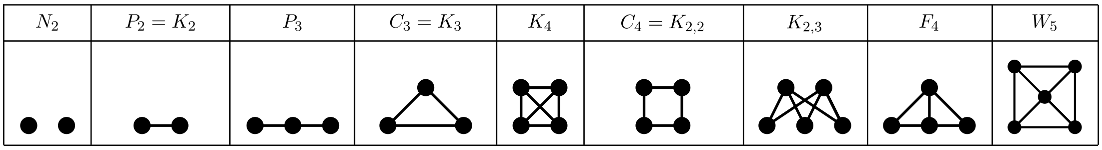

Fig. 1. Small examples of typical graphs.

Given a set $S \subseteq V$, we denote by $\langle S \rangle_G$ the subgraph of G induced by S, omitting the subscript G if clear from the context. In particular, if $S = \{x\}$ we will use the notation $\langle x \rangle$ instead of $\langle \{x\} \rangle$. A graph is *empty* if it contains no edges. A graph $G = (V, E)$ is *bipartite* if V can be partitioned into two sets V_1 and V_2 such that both $\langle V_1 \rangle_G$ and $\langle V_2 \rangle_G$ are empty graphs. Two vertices u, v are *connected* if there is a sequence of vertices

$$u = v_1, v_2, v_3, \ldots, v_r = v$$

such that $v_i \in N_G[v_{i+1}]$ for all $i = 1, \ldots, r-1$. Connectedness defines an equivalence relation on V, and the equivalence classes are known as the *connected component* s of G. Mostly, they are identified with the graphs they induce. A graph is *connected* if it has only one connected component.

Two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic*, written $G \cong G'$ for short, if there exists a bijection $\phi : V \to V'$ such that $uv \in E$ if and only if $\phi(u)\phi(v) \in E'$.

For building examples, we also make use of well-known abbreviations for typical graphs, as described in the following. Some concrete drawings can be found in Fig. 1.

- P_n: the *path* on n vertices, with $n - 1$ edges;
- C_n: the *cycle* on n vertices (where $n \geq 3$), with n edges;
- K_n: the *complete graph* on n vertices, with $\binom{n}{2}$ edges.
- $K_{r,s}$ is the *complete bipartite graph* with r vertices on one side and s vertices on the other, with $r \cdot s$ edges.
- W_n is the *wheel graph* that can be described as $K_1 + C_{n-1}$ (where $n \geq 4$), with $2n - 2$ edges.
- F_n is the *fan graph* that can be described as $K_1 + P_{n-1}$ (with $n \geq 3$), with $2n - 3$ edges.
- N_n is the *null graph* (or empty graph) that can be described as the complement of K_n, *i.e.*, N_n consists of n isolated nodes with no edges.
- $K_1 = P_1 = N_1$ is also known as the *trivial graph*.

Let $\mathbb{R}_{\geq 0}$ denote the set of non-negative real numbers. A *metric space* is a pair (X, d), where X is a set of points and $d : X \times X \to \mathbb{R}_{\geq 0}$ satisfies $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$ for all $x, y \in X$ and $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$. The *diameter* of a point set $S \subseteq X$ is defined as $\text{diam}(S) = \sup\{d(x, y) : x, y \in S\}$. A *generator* of a metric space (X, d) is a set S of points in the space with the property that every point of the space is uniquely determined by the distances from the elements of S. A point $v \in X$ is said to *distinguish* two points x and y of X if $d(v, x) \neq d(v, y)$. Hence, S is a generator if and only if any pair of points of X is distinguished by some element of S.

We conclude this section by giving the definitions of the graph operations that we examine, starting with the better known operations and moving on to the less known ones that are yet more important in this paper. More details can be found in the textbooks of Harary [22] and Imrich and Klavžar [29]. Let G and H be two graphs of order n and n', respectively. Examples are given with the help of Fig. 1.

- The *complement (graph)* $\overline{G}$ of G has the same vertex set as G, but an edge between two distinct vertices x, y if and only if $x \notin N_G(y)$.
  Example: $\overline{N_2} = K_2$.
- The *graph union* $G \cup H$ is defined if the vertex sets $V(G)$ and $V(H)$ are disjoint and then refers to the graph $(V(G) \cup V(H), E(G) \cup E(H))$.
  Example: $N_{i+j} = N_i \cup N_j$ for any positive integers i, j.
- The *join (graph)* $G + H$ is defined as the graph obtained from vertex-disjoint graphs G and H by taking one copy of G and one copy of H and joining by an edge each vertex of G with each vertex of H.
  Example: The complete graph K_n, $n > 1$, can be recursively described as $K_1 + K_{n-1}$.
- The *corona product (graph)* $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G and n copies of H and joining by an edge each vertex from the ith copy of H with the ith vertex of G [19]. We will denote by $V = \{v_1, v_2, \ldots, v_n\}$ the set of vertices of G and by $H_i = (V_i, E_i)$ the ith copy of H such that $v_i \sim x$ for every $x \in V_i$.
  The corona graph $K_1 \odot H$ is isomorphic to the join graph $K_1 + H$. The graph $P_4 \odot P_5$ is depicted in Fig. 2.
- The *strong product (graph)* $G \boxtimes H$ of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph with vertex set $V(G \boxtimes H) = V_1 \times V_2$, where two distinct vertices $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$ are adjacent in $G \boxtimes H$ if and only if one of the following holds.

- $x_1 = y_1$ and $x_2 \sim y_2$, or
- $x_1 \sim y_1$ and $x_2 = y_2$, or
- $x_1 \sim y_1$ and $x_2 \sim y_2$.

Alternatively, two distinct vertices $(x_1, x_2), (y_1, y_2)$ of $G \boxtimes H$ are adjacent if and only if $x_1 \in N_G[y_1]$ and $x_2 \in N_H[y_2]$. Example: $K_{n^2} \cong K_n \boxtimes K_n$; $N_n \boxtimes G \cong G \cup G \cup \cdots \cup G$ (take n copies of G).

For our computational complexity results, it is important but easy to observe that all these graph operations can be performed in polynomial time, given one or two input graphs.

1.4. Simple facts

In this subsection, we summarize some simple observations that basically follow directly by the definitions of the graph parameters that we study in this paper. We will make use of these properties later on without further notice.

By definition of the different variants of generators, we can observe:

- Each adjacency generator is a metric generator.
- Each metric generator is a local metric generator.
- Each local adjacency generator is a local generator.
- Each adjacency generator is a local adjacency generator.

This shows that the following inequalities hold for any graph G:

- $\dim(G) \leq \dim_A(G)$;
- $\dim_l(G) \leq \dim(G)$;
- $\dim_l(G) \leq \dim_{A,l}(G)$;
- $\dim_{A,l}(G) \leq \dim_A(G)$.

Moreover, if S is an adjacency generator, then at most one vertex is not dominated by S, so that

$$\gamma(G) \leq \dim_A(G) + 1.$$

Namely, if x, y are not dominated by S, then no element in S distinguishes them.

We also observe that

$$\dim_{A,l}(G) \leq \beta(G),$$

because each vertex cover is a local adjacency generator.

However, all mentioned inequalities could be either equalities or quite weak bounds. Consider the following examples:

1. $\dim_l(P_n) = \dim(P_n) = 1 \leq \left\lfloor \frac{n}{4} \right\rfloor \leq \dim_{A,l}(P_n) \leq \left\lceil \frac{n}{4} \right\rceil \leq \left\lfloor \frac{2n+2}{5} \right\rfloor = \dim_A(P_n)$, $n \geq 4$;
2. $\dim_l(K_{1,n}) = \dim_{A,l}(K_{1,n}) = 1 \leq n - 1 = \dim(K_{1,n}) = \dim_A(K_{1,n})$, $n \geq 2$;
3. $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil \leq \left\lfloor \frac{2n+2}{5} \right\rfloor = \dim_A(P_n)$, $n \geq 7$;
4. $\left\lfloor \frac{n}{4} \right\rfloor \leq \dim_{A,l}(P_n) \leq \left\lceil \frac{n}{4} \right\rceil \leq \left\lfloor \frac{n}{2} \right\rfloor = \beta(P_n)$, $n \geq 2$.

The mentioned figures for the metric dimension can be found in [9] and those for the local metric dimension in [40]. Concerning the figures of the adjacency dimension, we refer to [31]. We are explicitly mentioning the local adjacency dimension of paths in Proposition 18. The remaining claims are straightforward.

2. The metric dimension of corona product graphs versus the adjacency dimension of a graph

The following is the first main combinatorial result of this paper and provides a strong link between the metric dimension of the corona product of two graphs and the adjacency dimension of the second graph involved in the product operation. A seemingly similar formula was derived in [30,47], but there, only the notion of metric dimension was involved (which makes it impossible to use the formula to obtain computational hardness results as we will do), and also, special conditions were placed on the second argument graph of the corona product.

Theorem 1. *For any connected graph G of order $n \geq 2$ and any non-trivial graph H,*

$$\dim(G \odot H) = n \cdot \dim_A(H).$$

Proof. We first need to prove that $\dim(G \odot H) \leq n \cdot \dim_A(H)$. For any $i \in \{1, \ldots, n\}$, let S_i be an adjacency basis of H_i, the ith-copy of H. In order to show that $X := \bigcup_{i=1}^{n} S_i$ is a metric generator for $G \odot H$, we differentiate the following four cases for two vertices $x, y \in V(G \odot H) - X$.

1. $x, y \in V_i$. Since S_i is an adjacency basis of H_i, there exists a vertex $u \in S_i$ such that $|N_{H_i}(u) \cap \{x, y\}| = 1$. Hence,
$$d_{G\odot H}(x, u) = d_{\langle v_i\rangle+H_i}(x, u) \neq d_{\langle v_i\rangle+H_i}(y, u) = d_{G\odot H}(y, u).$$
2. $x \in V_i$ and $y \in V$. If $y = v_i$, then for $u \in S_j, j \neq i$, we have
$$d_{G\odot H}(x, u) = d_{G\odot H}(x, y) + d_{G\odot H}(y, u) > d_{G\odot H}(y, u).$$
Now, if $y = v_j, j \neq i$, then we also take $u \in S_j$ and we proceed as above.
3. $x = v_i$ and $y = v_j$. For $u \in S_j$, we find that
$$d_{G\odot H}(x, u) = d_{G\odot H}(x, y) + d_{G\odot H}(y, u) > d_{G\odot H}(y, u).$$
4. $x \in V_i$ and $y \in V_j, j \neq i$. In this case, for $u \in S_i$ we have
$$d_{G\odot H}(x, u) \leq 2 < 3 \leq d_{G\odot H}(u, y).$$

Hence, X is a metric generator for $G \odot H$ and, as a consequence,
$$\dim(G \odot H) \leq \sum_{i=1}^{n} |S_i| = n \cdot \dim_A(H).$$

It remains to prove that $\dim(G\odot H) \geq n\cdot\dim_A(H)$. To do this, let W be a metric basis for $G\odot H$ and, for any $i \in \{1, \dots, n\}$, let $W_i := V_i \cap W$. Let us show that W_i is an adjacency metric generator for H_i. To do this, consider two different vertices $x, y \in V_i - W_i$. Since no vertex $a \in V(G \odot H) - V_i$ distinguishes the pair x, y, there exists some $u \in W_i$ such that $d_{G\odot H}(x, u) \neq d_{G\odot H}(y, u)$. Now, since $d_{G\odot H}(x, u) \in \{1, 2\}$ and $d_{G\odot H}(y, u) \in \{1, 2\}$, we conclude that $|N_{H_i}(u) \cap \{x, y\}| = 1$ and consequently, W_i must be an adjacency generator for H_i. Hence, for any $i \in \{1, \dots, n\}$, $|W_i| \geq \dim_A(H_i)$. Therefore,
$$\dim(G \odot H) = |W| \geq \sum_{i=1}^{n} |W_i| \geq \sum_{i=1}^{n} \dim_A(H_i) = n \cdot \dim_A(H).$$
This completes the proof. □

2.1. Consequences of Theorem 1

Theorem 1 allows us to investigate $\dim(G \odot H)$ through the study of $\dim_A(H)$, and vice versa. For previous results on $\dim(G \odot H)$ we refer to [30,47] and for previous results on $\dim_A(H)$ we refer to [31].

Theorem 2 ([47]). *Let G be a connected graph of order n and let H be some graph.*

(i) *If* $\operatorname{diam}(H) \leq 2$, *then* $\dim(G \odot H) = n \cdot \dim(H)$.
(ii) *If* $\operatorname{diam}(H) \geq 6$ *or H is a cycle graph of order at least* 7, *then*
$$\dim(G \odot H) = n \cdot \dim(K_1 + H).$$

As a direct consequence of Theorems 1 and 2 we obtain the following result.

Proposition 3. *Let H be a graph.*

(i) *If* $\operatorname{diam}(H) \leq 2$, *then* $\dim(H) = \dim_A(H)$.
(ii) *If* $\operatorname{diam}(H) \geq 6$ *or H is a cycle graph of order at least* 7, *then*
$$\dim(K_1 + H) = \dim_A(H).$$

In particular, it was shown in [5] that for any wheel graph W_{r+1} and any fan graph F_{r+1}, $r \geq 7$, it holds that $\dim(W_{r+1}) = \dim(F_{r+1}) = \left\lfloor \frac{2r+2}{5} \right\rfloor$. As $W_{r+1} = K_1 + C_r$ and $F_{r+1} = K_1 + P_r$, it holds that $\dim_A(C_r) = \dim_A(P_r) = \left\lfloor \frac{2r+2}{5} \right\rfloor$ for any $r \geq 7$.

Theorem 4 ([31]). *For any graph H of order $n' \geq 2$,*

(i) $\dim_A(H) = \dim_A(\overline{H})$.
(ii) $\dim_A(H) = 1$ *if and only if* $H \in \{P_2, P_3, \overline{P_2}, \overline{P_3}\}$.
(iii) $\dim_A(H) = n' - 1$ *if and only if* $H \cong K_{n'}$ *or* $H \cong \overline{K}_{n'}$.

The following result is a direct consequence of Theorems 1 and 4.

Proposition 5. *For any connected graph G of order $n \geq 2$ and any graph H of order $n' \geq 2$,*

(i) $\dim(G \odot H) = \dim(G \odot \overline{H})$.
(ii) $\dim(G \odot H) = n$ *if and only if* $H \in \{P_2, P_3, \overline{P_2}, \overline{P_3}\}$.
(iii) $\dim(G \odot H) = n(n' - 1)$ *if and only if* $H \cong K_{n'}$ *or* $H \cong \overline{K}_{n'}$.

2.2. A detailed analysis of the adjacency dimension of the corona product via the adjacency dimension of the second operand

We now analyze the adjacency dimension of the corona product $G \odot H$ in terms of the adjacency dimension of H. In particular, we show that for any connected graph G of order $n \geq 2$ and any non-trivial graph H,

$$n - 1 \geq \dim_A(G \odot H) - n \cdot \dim_A(H) \geq 0.$$

The bounds in the inequalities are attained in very specific situations which we are going to characterize.

Theorem 6. *Let G be a connected graph of order $n \geq 2$ and let H be a non-trivial graph. If there exists an adjacency basis S for H, which is also a dominating set, and if for every $v \in V(H) - S$, it is satisfied that $S \not\subseteq N_H(v)$, then*

$$\dim_A(G \odot H) = n \cdot \dim_A(H).$$

Proof. Suppose that S is an adjacency basis for H which is also a dominating set. Let S_i be the copy of S in the ith copy of H in $G \odot H$. First of all, note that by Theorem 1 we have

$$\dim_A(G \odot H) \geq \dim(G \odot H) = n \cdot \dim_A(H).$$

Suppose that for every $v \in V(H) - S$ it is satisfied that $S \not\subseteq N_H(v)$. We claim that $\dim_A(G \odot H) \leq n|S|$. To see this, let $S' = \bigcup_{i=1}^{n} S_i$ and let us prove that S' is an adjacency generator for $G \odot H$. So we differentiate the following cases for any pair x, y of vertices of $G \odot H$ not belonging to S'.

1. $x, y \in V_i$. Since S_i is an adjacency basis of H_i, there exists $u_i \in S_i$ such that either $u_i \sim x$ and $u_i \not\sim y$ or $u_i \not\sim x$ and $u_i \sim y$.
2. $x \in V_i, y \in V_j, j \neq i$. As S_i is a dominating set of H_i, there exists $u \in S_i$ such that $u \sim x$ and, obviously, $u \not\sim y$.
3. $x \in V_i, y = v_i \in V$. By assumption, we have that $S_i \not\subseteq N_{H_i}(x)$, so for every $u \in S_i - N_{H_i}(x)$, we find that $u \sim y$.
4. $x \in V_i, y = v_l \in V, i \neq l$. In this case for every $u \in S_l$, we have $u \sim y$ and $u \not\sim x$.
5. $x = v_i, y = v_j \in V, i \neq j$. Taking $u \in S_i$, we have $u \sim x$ and $u \not\sim y$.

From the cases above, we conclude that S' is an adjacency generator for $G \odot H$ and, as a consequence, $\dim_A(G \odot H) \leq |S'| = n \cdot |S| = n \cdot \dim_A(H)$. $\square$

Corollary 7. *Let $r \geq 7$ be an integer such that $r \not\equiv 1 \bmod 5$ and $r \not\equiv 3 \bmod 5$. For any connected graph G of order $n \geq 2$,*

$$\dim_A(G \odot C_r) = \dim_A(G \odot P_r) = n \cdot \left\lfloor \frac{2r+2}{5} \right\rfloor.$$

Proof. We shall construct an adjacency basis of C_r (and also of P_r), say S_r, which must satisfy the premisses of Theorem 6. Notice that as a consequence of Proposition 3 we previously showed that $\dim_A(C_r) = \dim_A(P_r) = \left\lfloor \frac{2r+2}{5} \right\rfloor$. So, the cardinality of S_r must be $\left\lfloor \frac{2r+2}{5} \right\rfloor$. Let $V_r = \{0, \ldots, r-1\}$ be the set of vertices of the cycle C_r (or of the path P_r, respectively). Define

$$S_r = \{j \in V_r \mid 1 \equiv j \bmod 5 \vee 3 \equiv j \bmod 5\}.$$

It is easy to verify that for $r \not\equiv 1 \bmod 5$ and $r \not\equiv 3 \bmod 5$ the set S_r is an adjacency generator for C_r (and of P_r) that is also a dominating set. Finally, it is clear that since $r \geq 7$, for every vertex of $H \in \{C_r, P_r\}$ we have $S_r \not\subseteq N_H(v)$, as $|N_H(v)| \leq 2$ and $|S_r| \geq 3$. $\square$

It is instructive to notice that S_r (as defined in the previous proof) is also an adjacency generator of the cycle C_r and also of the path P_r if $r \equiv 1 \bmod 5$, but in that case, it fails to be a dominating set, while S_r is a dominating set of C_r and of P_r that fails to be an adjacency generator if $r \equiv 3 \bmod 5$.

Theorem 8. *Let G be a connected graph of order $n \geq 2$ and let H be a non-trivial graph. If there exists an adjacency basis for H, which is also a dominating set and if, for any adjacency basis S for H, there exists $v \in V(H) - S$ such that $S \subseteq N_H(v)$, then*

$$\dim_A(G \odot H) = n \cdot \dim_A(H) + \gamma(G).$$

Proof. Let W be an adjacency basis for $G \odot H$ and let $W_i = W \cap V_i$ and $U = W \cap V$. Since two vertices belonging to V_i are not distinguished by any $u \in W - V_i$, the set W_i must be an adjacency generator for H_i. Now consider the partition $\{V', V''\}$ of V defined as follows:

$$V' = \{v_i \in V : \quad |W_i| = \dim_A(H)\} \text{ and } V'' = \{v_j \in V : \quad |W_j| \geq \dim_A(H) + 1\}.$$

Note that, if $v_i \in V'$, then W_i is an adjacency basis for H_i, thus in this case there exists $u_i \in V_i$ such that $W_i \subseteq N_{H_i}(u_i)$. Then the pair u_i, v_i is not distinguished by the elements of W_i and, as a consequence, either $v_i \in U$ or there exists $v_j \in U$ such that $v_j \sim v_i$. Hence, $U \cup V''$ must be a dominating set of G and, as a result,

$$|U \cup V''| \geq \gamma(G).$$

So we obtain the following:

$$\begin{aligned}\dim_A(G \odot H) &= |W| \\ &= \bigcup_{v_i \in V'} |W_i| + \bigcup_{v_j \in V''} |W_j| + |U| \\ &\geq \sum_{v_i \in V'} \dim_A(H) + \sum_{v_j \in V''} (\dim_A(H) + 1) + |U| \\ &= n \cdot \dim_A(H) + |V''| + |U| \\ &\geq n \cdot \dim_A(H) + |V'' \cup U| \\ &\geq n \cdot \dim_A(H) + \gamma(G).\end{aligned}$$

To conclude the proof, we consider an adjacency basis S for H which is also a dominating set, and we denote by S_i the copy of S corresponding to H_i. We claim that for any dominating set D of G of minimum cardinality $|D| = \gamma(G)$, the set $D \cup (\bigcup_{i=1}^{n} S_i)$ is an adjacency generator for $G \odot H$ and, as a result,

$$\dim_A(G \odot H) \leq \left| D \cup \left(\bigcup_{i=1}^{n} S_i \right) \right| = n \cdot \dim_A(H) + \gamma(G).$$

To see this, we differentiate the same cases as in the proof of Theorem 6 with the only difference that now in Case 3 either $y = v_i \in D$ or there exists some $v_j \in D$ such that $v_j \sim y$. Of course, if $y = v_i \notin D$, then v_j distinguishes the pair x, y. Therefore, the result follows. □

Corollary 9. *Let $r \geq 2$. For any connected graph G of order $n \geq 2$,*

$$\dim_A(G \odot K_r) = n(r-1) + \gamma(G).$$

Theorem 10. *Let G be a connected graph of order $n \geq 2$ and let H be a non-trivial graph. If no adjacency basis for H is a dominating set, then*

$$\dim_A(G \odot H) = n \cdot \dim_A(H) + n - 1.$$

Proof. We assume that no adjacency basis for H is a dominating set. As explained in Section 1.4, if B is an adjacency basis for H which is not a dominating set, then there exists exactly one vertex of H which is not dominated by B.

As in the proof of the previous theorem, we take W as an adjacency basis for $G \odot H$ and we deduce that every $W_i = W \cap V_i$ must be an adjacency generator for H_i. So, for any W_i which is not an adjacency basis for H_i we have $|W_i| \geq \dim_A(H) + 1$. Also, for any pair W_i, W_j which are adjacency bases for H_i and H_j, there exist two vertices $w_i \in V_i - W_i$ and $w_j \in V_j - W_j$ which are not dominated by the elements of W_i and W_j, respectively. Then, v_i or v_j must belong to W. Hence, if $W_{l_1}, W_{l_2}, \ldots, W_{l_k}$ are adjacency bases for $H_{l_1}, H_{l_2}, \ldots, H_{l_k}$, respectively, then $|\{v_{l_1}, v_{l_2}, \ldots, v_{l_k}\} \cap W| \geq k - 1$, and, as a consequence,

$$\begin{aligned}\dim_A(G \odot H) &= |W| \\ &= |V \cap W| + \sum_{i=1}^{k} |W_{l_i}| + \sum_{j \notin \{l_1, \ldots, l_k\}} |W_{l_j}| \\ &\geq (k-1) + k \cdot \dim_A(H) + (n-k)(\dim_A(H) + 1) \\ &= n \cdot \dim_A(H) + n - 1.\end{aligned}$$

Now we claim that for any adjacency basis B of H the set $B' = (V - \{v_n\}) \cup (\bigcup_{i=1}^{n} B_i)$ is an adjacency generator for $G \odot H$, where B_i is the copy of B corresponding to the graph H_i. To see this we differentiate some cases for $x, y \notin B'$. If $x, y \in V_i$, then there exists $b_i \in B_i$ which distinguishes them. If $x \in V_i$ and $y \in V_j$, for $i < j$, then $v_i \in B'$ satisfies $v_i \sim x$ and $v_i \not\sim y$. Finally, if $x = v_n$, then the pair x, y is distinguished by $v \in N_G(v_n) \subset B'$, when $y \in V_n$, and by $b_n \in B_n \subset S$, when $y \notin V_n$. Hence, B' is an adjacency generator for $G \odot H$ and, as a result,

$$\dim_A(G \odot H) \leq |B'| = n \cdot \dim_A(H) + n - 1.$$

Therefore, the proof is complete. □

It is easy to check that any adjacency basis of a star graph $K_{1,r}$ is composed of $r - 1$ leaves. This will leave the last leaf non-dominated. Thus, Theorem 10 leads to the following result.

Corollary 11. *For any connected graph G of order $n \geq 2$,*

$$\dim_A(G \odot K_{1,r}) = n \cdot r - 1.$$

Given a vertex $v \in V$ we denote by $G - v$ the subgraph obtained from G by removing v and the edges incident with it. We define the following auxiliary domination parameter:

$$\gamma'(G) := \min_{v \in V(G)} \{\gamma(G - v)\}.$$

Theorem 12. *Let H be a non-trivial graph such that some of its adjacency bases are also dominating sets, and some are not. If there exists an adjacency basis S' for H such that for every $v \in V(H) - S'$ it is satisfied that $S' \not\subseteq N_H(v)$, and for any adjacency basis S for H, which is also a dominating set, there exists some $v \in V(H) - S$ such that $S \subseteq N_H(v)$, then for any connected graph G of order $n \geq 2$,*

$$\dim_A(G \odot H) = n \cdot \dim_A(H) + \gamma'(G).$$

Proof. Assume that for any adjacency basis S for H which is also a dominating set, there exists $v \in V(H) - S$ such that $S \subseteq N_H(v)$. Also, assume that there exists an adjacency basis S' for H such that for every $v \in V(H) - S'$ it is satisfied that $S' \not\subseteq N_H(v)$. Let S_i be the copy of S corresponding to H_i and, analogously, let S'_j be the copy of S' corresponding to H_j.

Let $V(G) = \{v_1, \ldots, v_n\}$. We suppose, without loss of generality, that $\gamma'(G) = \gamma(G - v_n) = |D|$, where D is a dominating set of $G - v_n$. We claim that $X = D \cup S'_n \cup \left(\bigcup_{i=1}^{n-1} S_i\right)$ is an adjacency generator for $G \odot H$. To show it, we differentiate the following cases for any pair x, y of vertices of $G \odot H$ not belonging to X.

1. $x, y \in V_i$. Suppose $i \neq n$. Since S_i is an adjacency basis of H_i, there exists $u_i \in S_i$ such that either $u_i \sim x$ and $u_i \not\sim y$ or $u_i \not\sim x$ and $u_i \sim y$. Analogously, for $i = n$ there exists $u_n \in S'_n$ which differentiates the pair x, y.
2. $x \in V_i, y \in V_j, j > i$. As S_i is a dominating set of H_i, there exists $u \in S_i$ such that $u \sim x$ and, obviously, $u \not\sim y$.
3. $x \in V_i, y = v_i \in V$. Let $i = n$. If x is dominated by S'_n, then by assumption we have that $S'_n \not\subseteq N_{H_n}(x)$, so every $u \in S'_n - N_{H_n}(x)$ distinguishes x and y. Also, if x is not dominated by S'_n, then for every $u \in S'_n$ we have $u \sim y = v_n$ and $u \not\sim x$. For $i \neq n$ we have that either $v_i \in D$ or $v_i \sim v_j$, for some $v_j \in D$. Obviously, if $v_i \notin D$, then v_j distinguishes the pair x, y.
4. $x \in V_i, y = v_l \in V, i \neq l$. If $l \neq n$, then for every $u \in S_l$ we have $u \sim y$ and $u \not\sim x$. Analogously, if $l = n$, then for every $u \in S'_n$ we have $u \sim y$ and $u \not\sim x$.
5. $x = v_i, y = v_j \in V, i < j$. Taking $u \in S_i$ we have $u \sim x$ and $u \not\sim y$.

From the cases above we conclude that X is an adjacency generator for $G \odot H$ and, as a consequence,

$$\dim_A(G \odot H) \leq |X| = n \cdot \dim_A(H) + \gamma'(G).$$

To conclude the proof we need to prove that $\dim_A(G \odot H) \geq \dim_A(H) + \gamma'(G)$. Let W be an adjacency basis for $G \odot H$ and let $W_i = W \cap V_i$ and $U = W \cap V$. We know that since two vertices belonging to V_i are not distinguished by any $u \in W - V_i$, the set W_i must be an adjacency generator for H_i. Now consider the partition $\{V', V'', V'''\}$ of V defined as follows: V' is composed of the vertices v_i of G such that W_i is an adjacency basis but it is not a dominating set of H_i, V'' is composed of the vertices v_i of G such that W_i is an adjacency basis and also it is dominating set of H_i and finally V''' is composed of the vertices v_i of G such that W_i is not an adjacency basis for H_i.

Note that, if $v_i, v_j \in V'$, then there exist two vertices $w_i \in V_i - W_i$ and $w_j \in V_j - W_j$ which are not dominated by the elements of W_i and W_j, respectively. Then, v_i or v_j must belong to U and, as a consequence, $|U \cap V'| \geq |V'| - 1$. Now, if $v_i \in V''$, then there exists $u_i \in V_i$ such that $W_i \subset N_{H_i}(u_i)$. Then the pair u_i, v_i is not distinguished by the elements of W_i and, as a consequence, either $v_i \in U$ or there exists $v_j \in U$ such that $v_j \sim v_i$. Hence, at most one vertex of G is not dominated by $U \cup V'''$ and, as a result,

$$|U \cup V'''| \geq \gamma'(G).$$

So we have the following:

$$\begin{aligned}
\dim_A(G \odot H) &= |W| \\
&= \bigcup_{v_i \in V' \cup V''} |W_i| + \bigcup_{v_j \in V'''} |W_j| + |U| \\
&\geq \sum_{v_i \in V' \cup V''} \dim_A(H) + \sum_{v_j \in V'''} (\dim_A(H) + 1) + |U| \\
&= n \cdot \dim_A(H) + |V'''| + |U| \\
&\geq n \cdot \dim_A(H) + |V''' \cup U| \\
&\geq n \cdot \dim_A(H) + \gamma'(G).
\end{aligned}$$

Therefore, the result follows. □

As indicated in Fig. 2, $H = P_5$ satisfies the premises of Theorem 12, as in particular there are adjacency bases that are also dominating set (see the leftmost copy of a P_5 in Fig. 2) as well as adjacency bases that are not dominating sets (see the rightmost copy of a P_5 in that drawing). Hence, we can conclude:

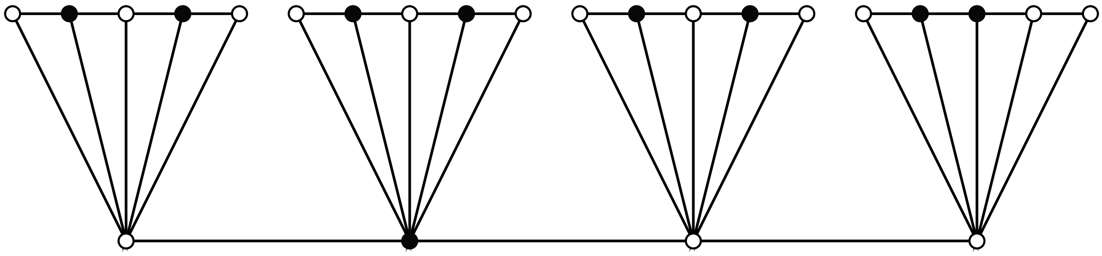

Fig. 2. The bold type indicates an adjacency basis for $P_4 \odot P_5$ which is not a dominating set. Since $H = P_5$ satisfies the premises of Theorem 12, we conclude that $\dim_A(P_4 \odot P_5) = n \cdot \dim_A(P_5) + \gamma'(P_4) = 4 \cdot 2 + 1 = 9$.

Corollary 13. *For any connected graph G of order $n \geq 2$,*

$$\dim_A(G \odot P_5) = 2n + \gamma'(G).$$

Since the assumptions of Theorems 6 8, 10 and 12 are complementary, we obtain the following result.

Remark 14. For any connected graph G of order $n \geq 2$ and any non-trivial graph H,

$$\dim_A(G \odot H) = n \cdot \dim_A(H)$$

or

$$\dim_A(G \odot H) = n \cdot \dim_A(H) + \gamma(G)$$

or

$$\dim_A(G \odot H) = n \cdot \dim_A(H) + \gamma'(G)$$

or

$$\dim_A(G \odot H) = n \cdot \dim_A(H) + n - 1.$$

Moreover, since the assumptions of Theorems 6 8, 10 and 12 are complementary and for any graph G of order $n \geq 3$ it holds that $0 < \gamma'(G) \leq \gamma(G) \leq \frac{n}{2} < n - 1$, we can conclude that in fact, Theorems 6 and 12 are equivalences for $n \geq 3$. Notice that for $n = 2$, Theorem 6 is also an equivalence. Therefore, we obtain the following two results.

Theorem 15. *Let G be a connected graph of order $n \geq 2$ and let H be a non-trivial graph. The following statements are equivalent:*

(i) *There exists an adjacency basis S for H, which is also a dominating set, such that for every $v \in V(H) - S$ it is satisfied that $S \not\subseteq N_H(v)$.*
(ii) $\dim_A(G \odot H) = n \cdot \dim_A(H)$.
(iii) $\dim_A(G \odot H) = \dim(G \odot H)$.

As an example of application of Theorem 15 we can take H as the cycle graphs C_r or the path graphs P_r, where $r \geq 7$, $r \not\equiv 1 \bmod 5$ and $r \not\equiv 3 \bmod 5$, as explained in Corollary 7.

Theorem 16. *Let G be a connected graph of order $n \geq 3$ and let H be a non-trivial graph. The following statements are equivalent:*

(i) *No adjacency basis for H is a dominating set.*
(ii) $\dim_A(G \odot H) = n \cdot \dim_A(H) + n - 1$.
(iii) $\dim_A(G \odot H) = \dim(G \odot H) + n - 1$.

An example of graph H where we can apply Theorem 16 is the star graph $K_{1,r}$ (see Cor. 11), $r \geq 2$, or the path graphs P_r, where $r \geq 7$, $r \equiv 1 \bmod 5$ or $r \equiv 3 \bmod 5$.

3. The local metric dimension of corona product graphs versus the local adjacency dimension of a graph

In the beginning, we consider some straightforward cases. If H is an empty graph, then $K_1 \odot H$ is a star graph and $\dim_l(K_1 \odot H) = 1$. Moreover, if H is a complete graph of order n, then $K_1 \odot H$ is a complete graph of order $n + 1$ and $\dim_l(K_1 \odot H) = n$. It was shown in [42] that for any connected nontrivial graph G and any empty graph H,

$$\dim_l(G \odot H) = \dim_l(G).$$

As this section is organized similar to the previous one, we refrain from structuring it by explicit subsections.
Our next result allow us to express $\dim_l(G \odot H)$ in terms of the order of G and $\dim_{A,l}(H)$.

Theorem 17. *For any connected graph G of order $n \geq 2$ and any non-empty graph H,*

$$\dim_l(G \odot H) = n \cdot \dim_{A,l}(H).$$

Proof. The result is deduced by analogy to the proof of Theorem 1 where the analysis is restricted to pairs of adjacent vertices. □

Now we summarize some more results obtained in [42].

Let H be a non-empty graph of order n' and let G be a connected graph of order $n \geq 2$. The following assertions hold.

(1) If the vertex of K_1 does not belong to any local metric basis for $K_1 + H$, then for any connected graph G of order n,

$$\dim_l(G \odot H) = n \cdot \dim_l(K_1 + H).$$

(2) If the vertex of K_1 belongs to a local metric basis for $K_1 + H$, then for any connected graph G of order $n \geq 2$,

$$\dim_l(G \odot H) = n(\dim_l(K_1 + H) - 1).$$

(3) Let $t \geq 4$ be an integer. If $t \equiv 1 \bmod 4$, then $\dim_l(G \odot P_t) = n \left\lfloor \frac{t}{4} \right\rfloor$, and if $t \not\equiv 1 \bmod 4$, then $\dim_l(G \odot P_t) = n \left\lceil \frac{t}{4} \right\rceil$.

(4) For any integer $t \geq 4$, $\dim_l(G \odot C_t) = n \left\lceil \frac{t}{4} \right\rceil$.

(5) If H has diameter two, then

$$\dim_l(G \odot H) = n \cdot \dim_l(H).$$

(6) If H has radius $r(H) \geq 4$, then

$$\dim_l(G \odot H) = n \cdot \dim_l(K_1 + H).$$

(7) $\dim_l(G \odot H) = n$ if and only if H is a bipartite graph having only one non-trivial connected component H^* and $r(H^*) \leq 2$.

(8) $\dim_l(G \odot H) = n(n' - 1)$ if and only if $H \cong K_{n'}$.

According to the results listed above, we can conclude the following three results.

For paths and cycles, we can conclude:

Proposition 18. *Let $t \geq 4$ be an integer.*

- $\dim_{A,l}(C_t) = \left\lceil \frac{t}{4} \right\rceil$.
- *If $t \equiv 1 \bmod 4$, then $\dim_{A,l}(P_t) = \left\lfloor \frac{t}{4} \right\rfloor$, and if $t \not\equiv 1 \bmod 4$, then $\dim_{A,l}(P_t) = \left\lceil \frac{t}{4} \right\rceil$.*

Next, we link adjacency and metric dimension.

Theorem 19. *Let H be a non-empty graph of order n'. The following assertions hold.*

(1) *If the vertex of K_1 does not belong to any local metric basis for $K_1 + H$, then*

$$\dim_{A,l}(H) = \dim_l(K_1 + H).$$

(2) *If the vertex of K_1 belongs to a local metric basis for $K_1 + H$, then*

$$\dim_{A,l}(H) = \dim_l(K_1 + H) - 1.$$

(3) *If H has radius $r(H) \geq 4$, then*

$$\dim_{A,l}(H) = \dim_l(K_1 + H).$$

(4) *If H has diameter two, then*

$$\dim_{A,l}(H) = \dim_l(H).$$

Finally, we can describe situations with very small or large dimensions.

Theorem 20. *Let H be a non-empty graph of order n'. The following assertions hold.*

(1) $\dim_{A,l}(H) = 1$ *if and only if H is a bipartite graph having only one non-trivial connected component H^* and $r(H^*) \leq 2$.*

(2) $\dim_{A,l}(H) = n' - 1$ *if and only if $H \cong K_{n'}$.*

Fortunately, the comparison of the local adjacency dimension of the corona product with the one of the second argument is much simpler in the local version as in the previously studied non-local version.

Theorem 21. *Let G be a connected graph of order $n \geq 2$ and let H be a non-trivial graph. If there exists a local adjacency basis S for H such that for every $v \in V(H) - S$ it is satisfied that $S \not\subseteq N_H(v)$, then*

$$\dim_{A,l}(G \odot H) = n \cdot \dim_{A,l}(H).$$

Proof. Suppose that S is a local adjacency basis for H. Let S_i be the copy of S in the ith copy of H in $G \odot H$. First of all, note that by Theorem 17 we have

$$\dim_{A,l}(G \odot H) \geq \dim_l(G \odot H) = n \cdot \dim_{A,l}(H).$$

Suppose that for every $v \in V(H) - S$ it is satisfied that $S \not\subseteq N_H(v)$. We claim that $\dim_{A,l}(G \odot H) \leq n|S|$. To see this, let $S' = \bigcup_{i=1}^{n} S_i$ and let us prove that S' is a local adjacency generator for $G \odot H$. So we differentiate the following cases for any pair x, y of adjacent vertices of $G \odot H$ not belonging to S'.

1. $x, y \in V_i$. Since S_i is a local adjacency basis of H_i, there exists some $u_i \in S_i$ such that either $u_i \sim x$ and $u_i \not\sim y$ or $u_i \not\sim x$ and $u_i \sim y$.
2. $x \in V_i$, $y = v_i \in V$. By assumption, we have that $S_i \not\subseteq N_{H_i}(x)$, so for every $u \in S_i - N_{H_i}(x)$, we have $u \sim y$.
3. $x = v_i$, $y = v_j \in V$, $i \neq j$. Taking $u \in S_i$, we have $u \sim x$ and $u \not\sim y$.

From the cases above, we conclude that S' is a local adjacency generator for $G \odot H$ and, as a consequence, $\dim_{A,l}(G \odot H) \leq |S'| = n|S| = n \cdot \dim_{A,l}(H)$. The proof is complete. □

Theorem 22. *Let G be a connected graph of order $n \geq 2$ and let H be a non-trivial graph. If for any local adjacency basis for H, there exists some $v \in V(H) - S$ which satisfies that $S \subseteq N_H(v)$, then*

$$\dim_{A,l}(G \odot H) = n \cdot \dim_{A,l}(H) + \gamma(G).$$

Proof. Let W be a local adjacency basis for $G \odot H$ and let $W_i = W \cap V_i$ and $U = W \cap V$. Since two adjacent vertices belonging to V_i are not distinguished by any $u \in W - V_i$, the set W_i must be a local adjacency generator for H_i. Now consider the partition $\{V', V''\}$ of V defined as follows:

$$V' = \{v_i \in V : \quad |W_i| = \dim_{A,l}(H)\} \text{ and } V'' = \{v_j \in V : \quad |W_j| \geq \dim_{A,l}(H) + 1\}.$$

Note that, if $v_i \in V'$, then W_i is a local adjacency basis for H_i, thus in this case there exists $u_i \in V_i$ such that $W_i \subset N_{H_i}(u_i)$. Then the pair u_i, v_i is not distinguished by the elements of W_i and, as a consequence, either $v_i \in U$ or there exists $v_j \in U$ such that $v_j \sim v_i$. Hence, $U \cup V''$ must be a dominating set and, as a result,

$$|U \cup V''| \geq \gamma(G).$$

So we obtain the following:

$$\begin{aligned} \dim_{A,l}(G \odot H) &= |W| \\ &= \bigcup_{v_i \in V'} |W_i| + \bigcup_{v_j \in V''} |W_j| + |U| \\ &\geq \sum_{v_i \in V'} \dim_{A,l}(H) + \sum_{v_j \in V''} (\dim_{A,l}(H) + 1) + |U| \\ &= n \cdot \dim_{A,l}(H) + |V''| + |U| \\ &\geq n \cdot \dim_{A,l}(H) + |V'' \cup U| \\ &\geq n \cdot \dim_{A,l}(H) + \gamma(G). \end{aligned}$$

To conclude the proof, we consider a local adjacency basis S for H and we denote by S_i the copy of S corresponding to H_i. We claim that for any dominating set D of G of minimum cardinality $|D| = \gamma(G)$, the set $D \cup (\bigcup_{i=1}^{n} S_i)$ is a local adjacency generator for $G \odot H$ and, as a result,

$$\dim_{A,l}(G \odot H) \leq \left| D \cup \left(\bigcup_{i=1}^{n} S_i \right) \right| = n \cdot \dim_{A,l}(H) + \gamma(G).$$

To see this, we differentiate the same cases as in the proof of Theorem 21 with the difference that now in Cases 2 either $v_i \in D$ or v_i is dominated by some element of D. Therefore, the result follows. □

Remark 23. As a concrete example for the previous theorem, consider $H = K_{n'}$. Clearly, $\dim_{A,l}(H) = n' - 1$, and the neighborhood of the only vertex that is not in the local adjacency basis coincides with the local adjacency basis. For any connected graph G of order $n \geq 2$, we can deduce that

$$\dim_{A,l}(G \odot K_{n'}) = n \cdot \dim_{A,l}(K_{n'}) + \gamma(G) = n(n' - 1) + \gamma(G).$$

Since the assumptions of Theorems 21 and 22 are complementary, we obtain the following property for $\dim_{A,l}(G \odot H)$.

Theorem 24 (*Dichotomy*). *For any connected graph G of order $n \geq 2$ and any non-trivial graph H either*

$$\dim_{A,l}(G \odot H) = n \cdot \dim_{A,l}(H)$$

or

$$\dim_{A,l}(G \odot H) = n \cdot \dim_{A,l}(H) + \gamma(G).$$

Now, since for any graph H it is satisfied that $0 < \gamma(H)$ and the assumptions of Theorems 21 and 22 are complementary, we conclude that, in fact, Theorems 21 and 22 are equivalences.

Theorem 25. *Let G be a connected graph of order $n \geq 2$ and let H be a non-trivial graph. Then the following assertions are equivalent.*

(i) *There exists a local adjacency basis S for H such that for every $v \in V(H) - S$ it is satisfied that $S \not\subseteq N_H(v)$.*
(ii) $\dim_{A,l}(G \odot H) = n \cdot \dim_{A,l}(H)$.
(iii) $\dim_l(G \odot H) = \dim_{A,l}(G \odot H)$.

An example of graph H where we can apply the above result is the path P_r, $r \geq 6$. In this case, for any connected graph G of order $n \geq 2$, the following is true:

$$\dim_{A,l}(G \odot P_r) = \begin{cases} n\left\lfloor \frac{r}{4} \right\rfloor & \text{if } r \equiv 1 \bmod 4; \\ n\left\lceil \frac{r}{4} \right\rceil & \text{if } r \not\equiv 1 \bmod 4. \end{cases}$$

Theorem 26. *Let G be a connected graph of order $n \geq 2$ and let H be a non-trivial graph. Then the following assertions are equivalent.*

(i) *For any local adjacency basis S for H, there exists some $v \in V(H) - S$ which satisfies that $S \subseteq N_H(v)$.*
(ii) $\dim_{A,l}(G \odot H) = n \cdot \dim_{A,l}(H) + \gamma(G)$.
(iii) $\dim_l(G \odot H) = \dim_{A,l}(G \odot H) - \gamma(G)$.

As a concrete example of graph H where we can apply the above result is the star $K_{1,r}$, $r \geq 2$. In this case, for any connected graph G of order $n \geq 2$, we find that

$$\dim_{A,l}(G \odot K_{1,r}) = n \cdot \dim_{A,l}(K_{1,r}) + \gamma(G) = n + \gamma(G).$$

4. Twins and strong products of graphs

We define the *twin equivalence relation* $\mathcal{R}$ on $V(G)$ as follows:

$$x\mathcal{R}y \longleftrightarrow N_G[x] = N_G[y] \text{ or } N_G(x) = N_G(y).$$

We have three possibilities for each twin equivalence class U:

(a) U is a singleton set, or
(b) $|U| > 1$ and $N_G(x) = N_G(y)$ for any $x, y \in U$, or
(c) $|U| > 1$ and $N_G[x] = N_G[y]$ for any $x, y \in U$.

We will refer to the type (c) classes as the *true twin equivalence classes*.

Let us see three different examples where every vertex is twin. An example of a graph where every equivalence class is a true twin equivalence class is $K_r + (K_s \cup K_t)$, $r, s, t \geq 2$. In this case, there are three equivalence classes composed of r, s and t true twin vertices, respectively. As an example where no class is composed of true twin vertices, we take the complete bipartite graph $K_{r,s}$, $r, s \geq 2$. Finally, the graph $K_r + N_s$, $r, s \geq 2$, has two equivalence classes and one of them is composed of r true twin vertices. On the other hand, $K_1 + (K_r \cup N_s)$, $r, s \geq 2$, is an example where one class is singleton, one class is composed of true twin vertices and the other one is composed of false twin vertices.

If U is a twin equivalence class in a connected graph G with $|U| = r \geq 2$, then every metric generator for G contains at least $r - 1$ elements from U. Thus, we point out the following remark stated in [10].

Remark 27 (*[10]*). Let G be a connected graph of order n. If G has t twin equivalence classes, then

$$\dim(G) \geq n - t.$$

Theorem 28. *Let G be a connected graph of order n having t twin equivalence classes. If G does not have singleton twin equivalence classes, then*

$$\dim_A(G) = \dim(G) = n - t.$$

Proof. Let $U_1, U_2, \ldots, U_t$ be the twin equivalence classes of G. Let u_i be an arbitrary element of U_i, for each $i \in \{1, \ldots, t\}$. We claim that $W := \bigcup_{i=1}^{t}(U_i - \{u_i\})$ is an adjacency generator for G. For any $i, j \in \{1, \ldots, t\}$, $i \neq j$, we have that u_i and u_j are not twins and, as a consequence, there exists $v \in N_G(u_i) - N_G(u_j)$, $v \neq u_j$ or $v \in N_G(u_j) - N_G(u_i)$, $v \neq u_i$. If $v \in W$, it distinguishes u_i and u_j, and if not, any other vertex from the equivalence class of v does. We conclude that W is an adjacency generator for G and that $\dim(G) \leq \dim_A(G) \leq |W| = n - t$.

Moreover, by Remark 27 we have $\dim_A(G) \geq \dim(G) \geq n - t$. Therefore, the proof is complete. □

Lemma 29. *Let G and H be two connected graphs. If G and H have t and t' true twin equivalence classes, and they have n_1 and n_1' vertices not belonging to any true twin equivalence class, respectively, then $G \boxtimes H$ has $n_1t' + n_1't + tt'$ true twin equivalence classes and the remaining twin equivalence classes (if any) are singleton.*

Proof. Let $U_1, \ldots, U_t$ and $U_1', \ldots, U_{t'}'$ be the true twin equivalence classes of G and H, respectively.

For any two vertices $a, c \in U_i$ and $b \in V(H)$,

$$\begin{aligned} N_{G\boxtimes H}[(a, b)] &= \{(x, y) : x \in N_G[a], y \in N_H[b]\} \\ &= \{(x, y) : x \in N_G[c], y \in N_H[b]\} \\ &= N_{G\boxtimes H}[(c, b)]. \end{aligned}$$

Thus, (a, b) and (c, b) are true twin vertices. By analogy we check that for any two vertices $b, c \in U_j'$ and $a \in V(G)$, it follows that (a, b) and (a, c) are true twin vertices.

Then we have that the sets of the form $U_i \times U_j$ are composed of true twin vertices. To conclude that $U_i \times U_j$ is a true twin equivalence class, we take $(a, b) \in U_i \times U_j$ and we differentiate two cases for any $(x, y) \notin U_i \times U_j$.

1. $x \notin U_i$. Since x and a are not true twin in G, either there exists $a_x \in N_G(a) - N_G[x]$ or there exists $x_a \in N_G(x) - N_G[a]$. Thus, we have two possibilities in $G \boxtimes H$: either $(a, b) \sim (a_x, b) \not\sim (x, y)$ or $(x, y) \sim (x_a, y) \not\sim (a, b)$.
2. $y \notin U_i'$. Now we proceed by analogy to Case 1. Since y and b are not true twin in H, either there exists $b_y \in N_H(b) - N_H[y]$ or there exists $y_b \in N_H(y) - N_H[b]$. Thus, we have two possibilities in $G \boxtimes H$: either $(a, b) \sim (a, b_y) \not\sim (x, y)$ or $(x, y) \sim (x, b_y) \not\sim (a, b)$.

In both cases, Case 1 and Case 2, (a, b) and (x, y) are not true twin vertices in $G \boxtimes H$ and, as a result, $U_i \times U_j$ is a true twin equivalence class in $G \boxtimes H$.

By a similar process we conclude that for every $a, x \in V(G) - \bigcup_{i=1}^{t} U_i$ and every $b, y \in V(H) - \bigcup_{i=1}^{t'} U_i'$ the sets of the form $\{a\} \times U_i'$ or $U_i \times \{b\}$ are true twin equivalence classes and the vertices of the form $(a, b), (x, y)$ are neither true twins nor false twins in $G \boxtimes H$. □

Note that according to the lemma above, if G and H are connected bipartite graphs different from K_2, then all the twin equivalence classes of $G \boxtimes H$ are singleton sets. A more general result is stated by the next corollary.

Corollary 30. *Let G and H be two connected graphs. If G and H do not have true twin vertices, then all the twin equivalence classes of $G \boxtimes H$ are singleton sets.*

Another interesting consequence of Lemma 29 is that for any connected bipartite graph H of order n', and any integer $n \geq 2$, $V(K_n \boxtimes H)$ is partitioned into n' true twin classes. This fact is generalized by the following result.

Corollary 31. *Let G and H be two connected graphs. If $V(G)$ is partitioned into t true twin equivalence classes and H does not have true twin vertices, then $V(G \boxtimes H)$ is partitioned into tn' true twin classes.*

Now we would point out some direct consequence of combining Remark 27 and Theorem 28 with Lemma 29 and its consequences.

Theorem 32. *Let G and H be two connected graphs of order n and n', respectively. If G and H have t and t' true twin equivalence classes, and n_1 and n_1' vertices not belonging to any true twin equivalence class, respectively, then*

$$\dim(G \boxtimes H) \geq nn' - n_1t' - n_1't - tt' - n_1n_1'.$$

Moreover, if $V(G)$ is partitioned into t true twin equivalence classes, then

$$\dim_A(G \boxtimes H) = \dim(G \boxtimes H) = nn' - n_1't - tt'.$$

Corollary 33. *Let G and H be two connected graphs of order n and n', respectively. If $V(G)$ is partitioned into t true twin equivalence classes and H does not have true twin vertices, then*

$$\dim_A(G \boxtimes H) = \dim(G \boxtimes H) = n'(n - t).$$

Given a family $H_1, H_2, \ldots, H_k$ of graphs we denote

$$\prod_{i=1}^{k}{}_{\boxtimes} H_i = H_1 \boxtimes H_2 \boxtimes \cdots \boxtimes H_k.$$

(Notice that the strong product is associative, see [29], so that this setting is well-defined.)

We emphasize the following particular case of Corollary 33, which is also derived from Theorem 28 and Corollary 30.

Remark 34. Let $n \geq 2$ be an integer. For any family $H_1, H_2, \ldots, H_k$ of connected bipartite graphs of order $n_1, n_2, \ldots, n_k$, respectively,

$$\dim_A\left(K_n \boxtimes \left(\prod_{i=1}^{k}{}_{\boxtimes} H_i\right)\right) = \dim\left(K_n \boxtimes \left(\prod_{i=1}^{k}{}_{\boxtimes} H_i\right)\right) = (n-1)\prod_{i=1}^{k} n_i.$$

5. The computational complexity of the four dimension variants

In this section, we not only prove NP-hardness of all dimension variants, but also show that the problems (viewed as minimization problems) cannot be solved in time $O(pol(n+m)2^{o(n)})$ on any graph of order n (and size m). Yet, it is straightforward to see that each of our computational problems can be solved in time $O(pol(n+m)2^n)$, simply by cycling through all vertex subsets by increasing cardinality and then checking if the considered vertex set forms an appropriate basis. More specifically, based on our reductions we can conclude that these trivial brute-force algorithms are in a sense optimal, assuming the validity of the Exponential Time Hypothesis (ETH). A direct consequence of ETH (using the sparsification lemma) is the hypothesis that 3-SAT instances cannot be solved in time $O(pol(n+m)2^{o(n+m)})$ on instances with n variables and m clauses, see [7,28].

From a mathematical point of view, the most interesting fact is that most of our computational results are based on the combinatorial results on the dimensional graph parameters on corona and strong products of graphs that are derived earlier in this paper.

Due to the practical motivations (for instance, robot navigation and coding) of the parameters, we also study their computational complexity on planar graph instances.

We are going to study the following problems:

Dim: Given a graph G and an integer k, decide if $\dim(G) \leq k$ or not.
LocDim: Given a graph G and an integer k, decide if $\dim_l(G) \leq k$ or not.
AdjDim: Given a graph G and an integer k, decide if $\dim_A(G) \leq k$ or not.
LocAdjDim: Given a graph G and an integer k, decide if $\dim_{A,l}(G) \leq k$ or not.

As auxiliary problems, we will also consider:

VC: Given a graph G and an integer k, decide if $vc(G) \leq k$ or not.
Dom: Given a graph G and an integer k, decide if $\gamma(G) \leq k$ or not.
1-LocDom: Given a graph G and an integer k, decide if there exists a 1-locating dominating set of G with at most k vertices or not. Recall that a dominating set $D \subseteq V$ in a graph $G = (V, E)$ is called a 1-locating dominating set if for every two vertices $u, v \in V \setminus D$, the symmetric difference of $N(u) \cap D$ and $N(v) \cap D$ is non-empty.

We first recall the following result first mentioned in the textbook of Garey and Johnson [20], with a proof first (for general graphs) published in [35, Theorem A.1].

Theorem 35. Dim *is NP-complete, even when restricted to planar graphs.*

Remark 36. Different proofs of this type of hardness result appeared in the literature. For planar instances, we refer to [12]. In fact, we can offer a further one, based upon Theorem 1 and the following result. Namely, if there were a polynomial-time algorithm for computing $\dim(G)$, then we could compute $\dim_A(H)$ for any (non-trivial) graph H by first computing $d := \dim(K_2 \odot H)$ with the assumed polynomial-time algorithm and then $\dim_A(H) = \frac{d}{2}$.

Theorem 37. AdjDim *is NP-complete, even when restricted to planar graphs.*

Proof. Membership in NP is easy to see. We reduce from 1-LocDom, see [8,11] for the NP-hardness, and also Theorem 40 below. Clearly, any 1-locating dominating set is also an adjacency generator, but the converse need not be true, as an adjacency generator need not be a dominating set. However, if an adjacency generator is not a dominating set, then there is exactly one vertex which is not dominated. Hence, we propose the following reduction: From an instance $G = (V, E)$ and k

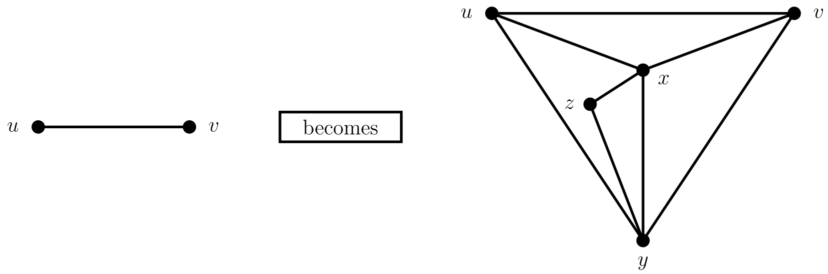

Fig. 3. A simple edge gadget, or how to replace an edge preserving planarity.

of 1-LocDom, produce an instance (G', k) of AdjDim by obtaining G' from G by adding a new isolated vertex $x \notin V$ to G. We claim that G has a 1-locating dominating set of size at most k if and only if $\dim_A(G') \leq k$. Firstly, every 1-locating dominating set D of G is also an adjacency generator of G', as x is the only vertex that is not contained in $N[D]$ in G'. Secondly, let S be an adjacency generator of G' of size at most k. If there is no vertex v with $v \notin N[S]$, then $S \cap V$ is a 1-locating dominating set of size at most k for G. Otherwise, there is a vertex v with $v \notin N[S]$. If $v \in V$, then $x \in S$, as otherwise x and v cannot be differentiated. As $x \in S$ does not help distinguish any two vertices $u, w \in V$, $S' = (S \setminus \{x\}) \cup \{v\}$ is another adjacency generator for G' of size at most k. Hence, we can assume that for an adjacency generator S of G' with a vertex v with $v \notin N[S]$, $v \notin V$ holds, *i.e.*, $v = x$. Then, S is also a 1-locating dominating set. □

As we like to exploit further properties of the reduction, we provide a reduction for NP-hardness of 1-LocDom in the following. We need some further auxiliary observations.

Lemma 38. *Assuming ETH, there is no $O(pol(n+m)2^{o(n)})$ algorithm solving* VC *on graphs of order n and size m.*

Proof. The textbook reduction [20] shows just this, as it produces, starting from a 3-SAT formula with n variables and m clauses, a graph of order $3m + 2n$ and size $6m + n$. □

Lemma 39. *Assuming ETH, there is no $O(pol(n+m)2^{o(n)})$ algorithm solving* Dom *on graphs of order n and size m.*

Proof. The textbook reduction of [25, Theorem 1.7] takes a 3-SAT formula with n variables and m clauses and produces a graph of order $3n + m$ and size $3n + 3m$.

An alternative well-known reduction works as follows: It takes a VC instance G of order n and size m and produces a graph of order n' and size m' by replacing any edge of G by a triangle, so that $n' = n + m$ and $m' = 3m$. Then, any minimum vertex cover C of G is also a dominating set in G', and there are always minimum dominating sets D in G' that only contain vertices of G and that also form a vertex cover in G. Hence, the claim follows by the previous lemma. □

We will call the second construction of the previous proof *triangle construction* in the following.

Notice that the two proofs of the preceding lemmas preserve planarity. This means that if the clause-and-variable graph associated to a Boolean formula (as introduced by Lichtenstein in [38]) is planar, then the three graphs resulting from the construction sketched in the preceding two lemmas are also planar. This is important to notice, as this fact gives the inspiration for our proof of the next theorem. Notice that the NP-hardness itself already follows from the statement given in [11], but that proof (starting out again from 3-SAT) does not preserve planarity, as the variable gadget alone already contains a $K_{2,3}$ subgraph that inhibits non-crossing interconnections with the clause gadgets.

Theorem 40. 1-LocDom *is NP-hard, even when restricted to planar graphs of bounded degree. Moreover, assuming ETH, there is no $O(pol(n+m)2^{o(n)})$ algorithm solving* 1-LocDom *on general graphs of order n and size m.*

Proof. Membership in NP is easy to see. We start our reduction from a Vertex Cover instance (G, k), where $G = (V, E)$ is a graph of order n and size m, and k is an integer. Notice that we could assume that G is even 3-regular and planar. Next, we replace each edge by an edge gadget as indicated in Fig. 3, yielding a new graph G' of order $n + 3m$ and size $7m$. We make the following observations:

- If C is a vertex cover of G, then we can take C and additionally select one vertex y from each edge gadget to obtain a dominating set $D \supseteq C$ with $|D| = |C| + m$, as indicated in Fig. 3.
- With this selection of D, each vertex belonging to one edge gadget is neighbor to one vertex in D that itself is only adjacent to vertices within this edge gadget. Hence, (*) vertices within different edge gadgets can be easily differentiated in the sense that for any two such vertices u, v, $(N(u) \cap D) \Delta (N(v) \cap D) \neq \emptyset$. Moreover, the two vertices x, z added by the construction and not belonging to D can be differentiated, as exactly one of them is neighbor to

some vertex from $D \cap V$. Finally, as vertices from V always belong to three edge gadgets, they are differentiated by observation (*). This means that D actually forms a 1-locating dominating set.

- If D is some 1-locating dominating set for G', first observe that since the vertices named x, y in Fig. 3 obey $N[x] = N[y]$, at least one of them must belong to D. If both of them belong to D, then we can create another 1-locating dominating set by removing x from D and putting in v instead; this leads to the situation depicted in Fig. 3. Now, if $|D| \leq m + k$, then $|D \cap V| \leq k$, and from each edge gadget at least one vertex belongs to $D \cap V$. Hence, $D \cap V$ forms a vertex cover in the original graph G.

As the construction clearly preserves planarity and each vertex in G' is of degree at most six, the claim follows. □

By the proof of Theorem 37, we can now conclude:

Corollary 41. *Assuming ETH, there is no $O(pol(n + m)2^{o(n)})$ algorithm solving* AdjDim *on graphs of order n and size m.*

As explained in Remark 36, Theorem 1 can be used to deduce furthermore:

Corollary 42. *Assuming ETH, there is no $O(pol(n + m)2^{o(n)})$ algorithm solving* Dim *on graphs of order n and size m.*

From Remark 23 and Lemma 39, we can conclude, as membership in NP is easy to see:

Theorem 43. LocAdjDim *is NP-complete. Moreover, assuming ETH, there is no $O(pol(n + m)2^{o(n)})$ algorithm solving* LocAdjDim *on graphs of order n and size m.*

We provide an alternative proof of the previous theorem in the Appendix of this paper. That proof is a direct reduction from 3-SAT and is, in fact, very similar to the textbook proof for the NP-hardness of Vertex Cover. This also proves that LocAdjDim is NP-complete when restricted to planar instances.

As explained in Remark 36, we can (now) use Theorem 17 together with Theorem 43 to conclude the following hitherto unknown complexity result. (Membership in NP is again easy to see.)

Theorem 44. LocDim *is NP-complete. Moreover, assuming ETH, there is no $O(pol(n + m)2^{o(n)})$ algorithm solving* LocDim *on graphs of order n and size m.*

Notice that the reduction explained in Remark 36 does not help find any hardness results on planar graphs. Hence, we leave it as an open question whether or not LocDim is NP-hard also on planar graph instances.

Furthermore, let us point to the fact that the twin equivalence classes are quite easy to compute. Therefore, the formula shown in Theorem 28 allows us to conclude that singleton twin equivalence classes are essential for the NP-hardness results that we obtained in this section.

We conclude this section by pointing out into another direction, namely from AdjDim (and from LocAdjDim) to Test Cover.

Mainly motivated from computational biology, De Bontridder et al. discussed the Test Cover problem [3]. Further variants were introduced in [2]. This problem can be stated as follows: Given a set of substances S and a set T of tests $t : S \to \{0, 1\}$, is it possible to find a test cover $C \subseteq T$ containing at most k tests that discriminate all substances, i.e.,

$$\forall s_1, s_2 \in S, s_1 \neq s_2 \exists t \in C : t(s_1) \neq t(s_2).$$

Clearly, k is here the natural parameter (in the sense of parameterized complexity), as it is an upper bound on the size of some adjacency generator in the case of AdjDim.

In an earlier version of this paper, we wanted to build a direct bridge (reduction) from AdjDim to Test Cover, preserving the parameter, but this attempt was flawed. However, the problems are similar enough to allow using some of the typical arguments for Test Cover also for AdjDim. Without introducing the notions from parameterized complexity formally, we nonetheless use them in the following, referring the interested reader to the most recent monograph [13].

Theorem 45. AdjDim *(with the natural parameter) is in FPT.*

Proof. Consider an instance $(G = (V, E), k)$ of AdjDim.

Assume D is an adjacency basis with at most k vertices. Any two vertices not in D must have different adjacency vectors with respect to D. As there are at most 2^k such adjacency vectors, $|V \setminus D| \leq 2^k$, or we face a NO instance. Hence, we can report NO if $|V| > k + 2^k$. Otherwise, $|V| \leq k + 2^k$, so that membership in FPT is shown by this kernel result. □

With nearly the same proof we can show:

Corollary 46. 1-LocDom *(with the natural parameter) is in FPT.*

From the approximation point of view, let us mention (in-) approximability results as obtained in [36,46]. In particular, inapproximability of 1-LocDom readily transfers to inapproximability of AdjDim and this in turn leads to inapproximability results for Dim as in Remark 36; also see [24]. Similar inapproximability results have been established for Test Cover in [3].

It remains unclear if a similar argument holds for LocAdjDim. The problem is that the number 2^k provides no upper bound on the number of vertices that do not go into the local adjacency generator of size at most k. For instance, consider a wheel $W = (V, E)$ where every second spoke is missing, i.e., $V = \{c, 1, 2, \ldots, 2n\}$, $\langle\{1, 2, \ldots, 2n\}\rangle \cong C_{2n}$, and the remaining edges are $\{c, 2i\}$ with $i = 1, \ldots, n$. Then, $\{c\}$ is a local adjacency generator. Any two adjacent vertices can be distinguished by c.

6. Conclusions

We have studied four dimension parameters in graphs. In particular, establishing concise formulas for corona product graphs allowed to deduce NP-hardness results (and similar hardness claims) for all these graph parameters, based on known results, in particular on Vertex Cover and on Dominating Set problems. We hope that the idea of using such types of non-trivial (combinatorial) formulas for computational hardness proofs can be also applied in other situations.

For instance, observe that reductions based on formulas as derived in Theorem 1 clearly preserve the natural parameter of these problems, which makes this approach suitable for Parameterized Complexity. However, let us notice here that Dim is unlikely to be fixed-parameter tractable under the natural parameterization (i.e., an upper bound on the metric dimension) even for subcubic graph instances; see [24]. We have pointed to the differences with the parameterized complexity of AdjDim above. Let us only mention one interesting open question. Does AdjDim admit a polynomial-size kernel, or does it rather behave like Test Cover?

From a computational point of view, let us mention (in-) approximability results as obtained in [36,46]. In particular, inapproximability of 1-LocDom readily transfers to inapproximability of AdjDim and this in turn leads to inapproximability results for Dim as in Remark 36; also see [24].

All these computational hardness results, as well as the various different applications that led to the introduction of these graph dimension parameters, also open up the quest for moderately exponential-time algorithms, i.e., algorithms that should find an optimum solution for any of our dimension problems in time $O(\text{poly}(n + m)c^n)$ on graphs of size m and order n for some $c < 2$, or also to finding polynomial-time algorithms for special graph classes. In this context, we mention results on trees, series–parallel and distance-regular graphs [11,21,26].

In view of the original motivation for introducing these graph parameters, it would be interesting to study their complexity on geometric graphs. Notice that the definition of a metric generator is not exclusively referring to (finite) graphs, which might lead us even back to the common roots of graph theory and topology.

So far, we only focussed on proving computational hardness results for the four graph dimension notions that we studied in this paper. This is usually only the beginning of an algorithmic research line that deals with the following Research Questions:

- Describe the boundary between polynomial-time solvability and NP-hardness in terms of graph classes. We already explicitly mentioned several results on computing the four graph dimension parameters when restricted to planar graphs.
- Investigate the approximability of the graph parameters, viewed as minimization problems.
- Study aspects of parameterized complexity for these graph parameters, further to what we said above.

Let us conclude with indicating other possible future research directions. Given some metric D on the vertex set of a (connected) graph $G = (V, E)$ and some vertex set $S \subseteq V$, one can define the following relation $\sim_{D,S}$ on V:

$$u \sim_{D,S} v \iff \forall x \in S : D(x, u) = D(x, v).$$

Clearly, for any D, S, $\sim_{D,S}$ is an equivalence relation on V. Moreover, S is a metric generator (with respect to the metric D) if and only if all equivalence classes of $\sim_{D,S}$ are singleton sets. We can then introduce the *D-dimension* as the size of the smallest metric generator with respect to the metric D on G.

So far, we focussed on the metrics d_G and $d_{G,2}$. One might also study other metrics, like $d_{G,k}(x, y) = \min\{d_G(x, y), k\}$ for $k > 2$. This way, also other notions of metric bases can be investigated, as well as the according graph dimension parameters. First studies might focus on combinatorial aspects.

From the above point of view, we can call a set S a *local metric generator* (with respect to the metric D on G) if each equivalence class of $\sim_{D,S}$ forms an independent set. We can then introduce the *local D-dimension* as the size of the smallest local metric generator with respect to the metric D on G. So, we propose to study computational hardness questions for the problems related to these graph parameters.

Also, 1-locating dominating sets have been studied (actually, independently introduced) in connection with coding theory [34]. Recall that these sets are basically adjacency bases. Therefore, it might be interesting to try to apply some of the information-theoretic arguments on variants of metric dimension, as well. Conversely, the notion of locality used in this paper connects to the idea of correcting only 1-bit errors in codes. These interconnections deserve further studies.

In view of the many different motivations, also the study of computational aspects of other variants of dimension parameters could be of interest. We only mention here the notions of resolving dominating sets [4] and independent resolving sets [10]. Recall again that identifying codes are nothing else than dominating adjacency generators. Also, little is known about the robustness notions and other variants discussed in connection with wireless networks for identifying codes in [16–18,27,37,41], let alone discussing combinations of robustness with other dimension parameters.

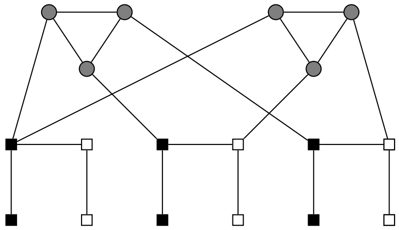

Fig. 4. A small example illustrating the overall structure of the reduction for the clauses $(x \vee y \vee z)$ and $(x \vee \neg y \vee \neg z)$. There are three variables in the formula, x, y, z, in that order. To each of them, an induced P_4 belongs whose vertices are colored black and white. The three black middle vertices of these paths are the positive literal vertices, while the three white middle vertices of these paths are the negative literal vertices. The two triangle-shaped parts of the graph are gadgets for clauses. Here, they represent $(x \vee y \vee z)$ (gray, on the left) and $(x \vee \neg y \vee \neg z)$ (gray, on the right).

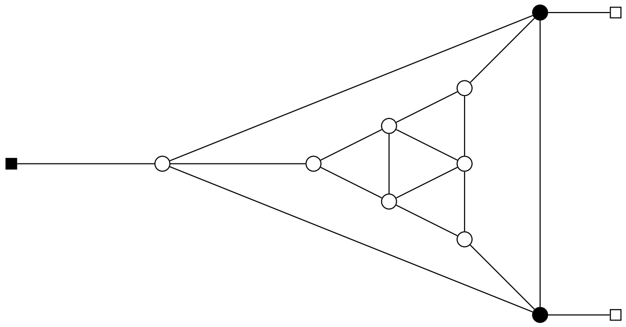

Fig. 5. The clause gadget illustration. The square-shaped vertices do not belong to the gadget, but they are the three literal vertices in variable gadgets that correspond to the three literals in the clause.

Appendix

Theorem 47. LocAdjDim *is NP-complete, even when restricted to planar instances.*

Proof. Membership in NP is easy to see.

For the hardness part, we propose a reduction from 3-SAT that is similar to the standard textbook reduction for proving NP-hardness of Vertex Cover, confer, e.g., [20]. An illustration of the construction is shown in Fig. 4.

For each Boolean variable, we introduce four vertices that form a P_4. This path will be induced in the final graph, and connections to other graph parts would be only possible via the two middle vertices. The combinatorial claim is that finally exactly one of the two middle vertices should be in any local adjacency basis of the graph instance that we construct. In the following, these two middle vertices are called literal vertices, as whether or not they belong to the local adjacency generator determines whether or not the literal is set to true. Due to this intention, we will call one of the two middle vertices positive literal vertex and the other one negative literal vertex.

We introduce one clause gadget per clause. This is a graph of order nine depicted in Fig. 5. We claim that we need at least two vertices from each of these clause gadgets in any local adjacency basis. Two are only sufficient if some of the literal vertex neighbors from variable gadgets are in the local adjacency basis. This can be seen in Fig. 5 by considering the vertices colored black. Also, we need at least two vertices in any local adjacency basis that are from the "innermost" six vertices in each gadget. We assume that the three outermost vertices of each gadget are numbered like 1, 2, 3.

The overall structure of the graph $G = (V, E)$ belonging to some formula F given by some set X of n variables and some set of m 3-element clauses C is as follows:

- Introduce an induced P_4, called $p(x)$ for each variable $x \in X$.
- Introduce a subgraph $g(c)$ of order nine for each clause $c \in C$.
- Assume that there is some order $<$ on X, which transfers to the set $X(c)$ of variables occurring in clause c. Hence, we can refer to the ith vertex in $X(c)$.
- An edge interconnects the positive literal vertex of $p(x)$ with the outermost vertex o of $g(c)$ if and only if the literal x occurs in c, x is the ith variable in $X(c)$ and o is the vertex number i.
- An edge interconnects the negative literal vertex of $p(x)$ with the outermost vertex o of $g(c)$ if and only if the literal $\bar{x}$ occurs in c, x is the ith variable in $X(c)$ and o is the vertex number i.
- There are no further edges in the graph G.

Hence, $|V| = 4n + 9m$, $|E| = 3n + 18m$.

The overall claim is that there is a local adjacency basis of size at most (and also exactly) $2m+n$ if and only if the given 3-SAT formula F was satisfiable. Our previous reasoning already explained that the local adjacency dimension of the union of the variable and clause gadget graphs is at least $2m+n$. Furthermore, we claim that for any local adjacency basis that does not contain any vertex of some path $p(x)$, there exists another local adjacency basis that contains a middle vertex from $p(x)$. Having a local adjacency basis A for G with two vertices from each clause gadget and one middle vertex from each variable gadget, we obtain a satisfying assignment of the formula F by setting x to true if and only if the positive literal of $p(x)$ belongs to A. This way, x is set to false if and only if the negative literal of $p(x)$ belongs to A. Conversely, any satisfying assignment of F yields a local adjacency basis for G by first putting the positive literal of $p(x)$ into the basis if x is set to true by the assignment and by then putting the negative literal of $p(x)$ into the basis if x is set to false. Moreover, as the assignment was assumed to be satisfying, each clause c is satisfied, so that at least one of the literal vertices neighboring some outermost vertices of $g(c)$ was put into the basis. Now, two innermost vertices from $g(c)$ could be put into the basis (as shown in Fig. 5) to finally produce a local adjacency basis of size $n+2m$ as required.

To show NP-hardness for planar instances, we recall Lichtenstein's construction for Planar Vertex Cover; see [38]: We only have to introduce cycles of length $4m$ instead of paths P_4 in our reduction; this enables "individual" vertices in the local adjacency generator that correspond to occurrences of literals in the clauses. Also, the cycle structure of the variables (in Lichtenstein's framework) can be implemented by appropriate interconnections of the variable cycle gadgets. As $\dim_{A,l}(C_{4m}) = m$, this sketch should suffice to show that LocAdjDim is NP-complete, even when restricted to planar instances. □

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