Correlation Regimes in International Equity and Bond Returns

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Abstract

Measuring comovements across international financial markets is important for policy purposes and portfolio management. We develop a new approach to analyse such comovements in relation to key state variables, such as equity market volatility and short-term interest rates. These state variables can identify regimes in comovements through a fast, tractable threshold model. The advantage over existing methods is that our model can be easily estimated and does not have a serious bias, even when applied to large asset portfolios. Out-of-sample portfolio evaluation shows that our method outperforms the standard dynamic conditional correlation approach, especially during the recent global financial crisis when financial market comovements experienced substantial regime shifts. These comovement regimes-shifts are associated with higher equity market volatility and lower interest rates. Overall, we contribute to the empirical literature by shedding new light on the fundamental drivers of comovements across international financial markets which can help guide analysis of systemic risk, asset allocation and hedging.

Keywords: International Equity and Bond Comovements, Threshold, State Variables, Stock Market Volatility, Interest Rates.

JEL Classifications: C14; C58; G10.

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1. Introduction

International financial markets are highly interconnected, and financial crises and economic downturns can easily spread across economies. Therefore, measuring the comovements across international financial markets is a major concern for policy-markers. The academic literature on international comovements is abundant --see, for example, Ang and Bekaert (2002), Longin and Solnik, 1995, 2001, Ramchand and Susmel, 1998, Bekaert et 2009, and Christoffersen et al., 2012, BenSaïda et al., 2018, Si et al., 2019, among others. One of the most popular models to capture the time-varying structure of financial market comovements is the Dynamic Conditional Correlation (DCC). The DCC assumes that comovoments evolve linearly according to a simple GARCH-type structure (Engle, 2002). It is simple to estimate the DCC parameters with *correlation targeting,* which consists in substituting the unconditional correlation by the sample counterpart (see Engle, 2009, Ch. 11)[2]. Yet, little is known about the main determinants of the comovements.

A different way of thinking about time-varying comovements to uncover their determinants is to define different *states of the world* or *regimes*, and allow for the dynamic behaviour of international return comovements to depend on the regime that occurs at any given point in time. A key issue here is the modelling of shifts between the regimes. While Pelletier (2006) examines latent regime-shifts in comovements, Silvennoinen and Teräsvirta (2015) explicitly identify the regimes using observable *state variables* in their Smooth Transition Conditional Correlation (STCC) setup[3]. However, because of the smooth transition nature of the regime, correlation targeting is not possible. Consequently, feasible estimation can be an issue even when applied to small portfolios.

The current paper adapts insights from the STCC modelling to propose a fast, tractable Threshold Conditional Correlation (TCC) approach to overcome the STCC's problem. Our method can be useful to academics and practitioners thanks to its advantage over existing methods in that it can be easily estimated even when applied to large portfolios. The TCC is similar in spirit to the STCC but it allows for a discrete regime-

[2] There are several refinements of the DCC model such as the consistent DCC (cDCC) by Aielli (2013), the generalized DCC model (Cappiello et al., 2006), the threshold varying conditional correlation model (Kwan et al., 2009), the semi-parametric and non-parametric correlations models (Hafner et al., 2006, Long et al., 2011, and Aslanidis and Casas, 2013), among others. More recently, Adams et al (2017) find strong evidence of structural breaks in the DCC.

[3] Note Berben and Jansen (2005) propose a special case of a STCC model. In particular, their model is bivariate with a time trend as the switching variable, while the framework in Silvennoinen and Teräsvirta (2009) is multivariate and their switching variable can be deterministic or stochastic.

switch, which makes TCC more computationally attractive than STCC. Unlikely the STCC, correlation targeting is possible in the TCC, through a quasi maximum likelihood estimator (QMLE) derived from the grid search-based sample correlation matrix of the data. In a 2-regime framework, this would *eliminate* $N(N$-1) parameters from the optimization procedure. Moreover, our Monte Carlo simulations show that our methodolody generally does not have any serious bias even when applied to large portfolios[4].

Using this threshold method, we provide new empirical evidence on the determinants of time-varying comovements across international equities and government bonds. Our findings make two novel contributions to the literature. First, we show that high equity market volatility and low short-term interest rates are the key drivers of equity and bond comovements, respectively. Second, based on out-of-sample portfolio evaluation, our method outperforms the popular Dynamic Conditional Correlation model, especially during the recent global financial crisis, a period characterized by significant shifts in the level of long-run asset comovements due to higher volatility and lower interest rates. Overall, our findings contribute to the empirical literature on time-varying comovements across international financial markets by shedding new light on the fundamental drivers of these comovements. Our results offer interesting insights into investment and diversification that can be applied to analyse systemic risk, asset allocation and hedging. Thus, they can be helpful to policy makers for market design and setting financial regulations.

The paper is organised as follows. Section 2 presents the threshold conditional correlation model and discusses estimation issues and the testing of constant correlations. Section 3 reports the main in-sample results while out-of-sample analysis is discussed in Section 4. Finally, Section 5 concludes.

2. Methodology

2.1 Model specification

Consider the following N-dimensional vector process of asset returns (r_t):

$$r_t = \mu + u_t \qquad t = 1, \dots, T \tag{1}$$

[4]Results are reported in the Appendix A.2.

where μ denotes the vector of mean returns. The return innovations u_t are assumed to be a vector martingale difference (MDS) process such that:

$$E(u_t|\Im_{t-1}) = 0 \quad (2)$$

$$Var(u_t|\Im_{t-1}) = H_t \quad (3)$$

where $\Im_{t-1}$ is the σ–field generated by all the available information up through time $t-1$, $H_t \equiv [\sigma_{ij,t}]$ is the positive-definite time-varying conditional covariance matrix of u_t. Each of the univariate innovation processes follows:

$$u_{i,t} = h_{i,t}^{1/2}\eta_{i,t} \text{ for } i = 1, \dots, N \quad (4)$$

where the terms $\eta_{i,t}$ form a sequence of serially uncorrelated but cross-sectionally correlated random variables with mean zero and variance one, for each of the asset returns $i = 1, \dots, N$. Each $h_{i,t}$ is the asset-specific volatility term that can be modelled as GARCH or realized variance/volatility (RV) or any other univariate volatility process[5].

Rather than modelling the off-diagonal elements of H_t directly, we use the following decomposition (e.g., Bollerslev, 1990 and Engle, 2002, among others):

$$H_t = D_t R_t D_t \quad (5)$$

which allows the focus to be placed on the conditional correlation matrix $R_t \equiv [\rho_{ij,t}]$, while $D_t \equiv diag(h_{1,t}^{1/2}, \dots, h_{N,t}^{1/2})$ is the diagonal standard deviation (volatility) matrix. For the positive definiteness of H_t it is sufficient to require the correlation matrix R_t to be positive definite at each point in time. It follows that the innovation term in Eq. (1) can also be written as:

$$u_t = H_t^{1/2}\varepsilon_t = D_t R_t^{1/2}\varepsilon_t, \quad \varepsilon_t \sim iid(0, I_N) \quad (6)$$

We propose the following model for the conditional correlations:

$$R_t = R_1 I[s_{t-1} < \gamma] + R_2(1 - I[s_{t-1} < \gamma]) \quad (7)$$

where R_1 and R_2 are (constant) positive definite correlation matrices, and $I[s_{t-1} < \gamma]$ is an indicator function with s_{t-1} being the state variable and γ the threshold value. Therefore, R_1 and R_2 represent the correlation regimes with correlations being in regime R_1 when the state variable s_{t-1} is below γ, while being in regime R_2 when the state variable s_{t-1} is above γ.

[5] In the application, we use the realized variance based on the sum of squared daily returns over the week.

2.2 Grid search-QMLE method

An appealing feature of a model with threshold conditional correlations relates to the simplified estimation procedure. Let the parameters of interest be the mean vector μ parameters, the volatility matrix D_t parameters (denoted by ϕ), the parameters of the correlation matrices R_1 and R_2 (denoted by ψ) and the threshold value γ[6]. Estimation of the TCC model can be carried out by quasi maximum likelihood estimation (QMLE). Given a value of γ, the Gaussian log-likelihood is given by:

$$\begin{aligned}\ln L(\theta|\gamma) &= -\tfrac{1}{2}\textstyle\sum_t\big(N\ln(2\pi)+\ln(|H_t|)+(r_t-\mu)'H_t^{-1}(r_t-\mu)\big)\\ &= -\tfrac{1}{2}\textstyle\sum_t\big(N\ln(2\pi)+\ln(|D_tR_tD_t|)+(r_t-\mu)'D_t^{-1}R_t^{-1}D_t^{-1}(r_t-\mu)\big)\\ &= -\tfrac{1}{2}\textstyle\sum_t\Big(N\ln(2\pi)+\sum_i\ln h_{i,t}+\ln(|R_t|)+(r_t-\mu)'D_t^{-1}R_t^{-1}D_t^{-1}(r_t-\mu)\Big)\\ &= -\tfrac{1}{2}\textstyle\sum_t\big(N\ln(2\pi)+\sum_i\ln h_{i,t}+\ln(|R_t|)+\eta_t{}'R_t^{-1}\eta_t\big) \qquad (8)\end{aligned}$$

where $\theta\equiv(\mu',\phi',\psi')'$ collects the mean, volatility and correlations parameters and $\eta_t=D_t^{-1}(r_t-\mu)$ represents the vector of standardized return innovations with unit variance. Considering the correlation structure in Eq. (7) reduces enormously the computational complexity. By direct substitution:

$$\begin{aligned}ln\,L(\theta|\gamma) = -\tfrac{N}{2}T\,ln(2\pi)-\tfrac{1}{2}\textstyle\sum_t\sum_i ln\,h_{i,t}-\tfrac{1}{2}\sum_t I[s_{t-1}<\gamma](ln(|R_1|)+\eta_t{}'R_1^{-1}\eta_t)\\ -\tfrac{1}{2}\textstyle\sum_t(1-I[s_{t-1}<\gamma])(ln(|R_2|)+\eta_t{}'R_2^{-1}\eta_t) \qquad (9)\end{aligned}$$

Following Engle (2002) and Engle and Sheppard (2001), we divide the estimation procedure of θ into two steps: the mean and volatility estimation first and then the correlation estimation. In particular, conditional on the mean and volatility estimates[7], $\tilde{\mu},\tilde{\phi}$, the conditional log-likelihood is given (except for constants) by:

$$\begin{aligned}\ln L\big(\psi|\tilde{\mu},\tilde{\phi},\gamma\big) &= -\frac{1}{2}\sum\nolimits_t(\ln(|R_t|)+\tilde{\eta}_t{}'R_t^{-1}\tilde{\eta}_t)\\ &= \tfrac{1}{2}\textstyle\sum_t I[s_{t-1}<\gamma](\ln(|R_1|)+\tilde{\eta}_t'R_1^{-1}\tilde{\eta}_t)\\ &\quad -\tfrac{1}{2}\textstyle\sum_t(1-I[s_{t-1}<\gamma])(\ln(|R_2|)+\tilde{\eta}_t'R_2^{-1}\tilde{\eta}_t) \qquad (10)\end{aligned}$$

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[6] In the case of realized volatility measures, there are only *N* parameters, one for each of the unconditional variance of the series, i.e. $D_t=D$.

[7] These estimations are consistent under mild conditions from each of the series.

Note that holding γ fixed, *correlation targeting* is still possible cutting down, in the way of Bollerslev (1990), the computacional cost of the maximization iterative methods. The sample covariance matrices of the estimated standardized innovations computing the quasi-maximum likelihood correlation estimator of the conditional log-likelihood Eq. (10) are given:

$$\begin{aligned}\tilde{R}_1(\gamma) &\equiv \frac{1}{\sum_t I[s_{t-1}<\gamma]}\sum_t \tilde{\eta}_t\tilde{\eta}_t' I[s_{t-1}<\gamma] \\ \tilde{R}_2(\gamma) &\equiv \frac{1}{\sum_t (1-I[s_{t-1}<\gamma])}\sum_t \tilde{\eta}_t\tilde{\eta}_t'(1-I[s_{t-1}<\gamma])\end{aligned} \quad (11)$$

where $\left(\tilde{R}_1(\gamma), \tilde{R}_2(\gamma)\right)$ are the sample covariance matrices of $\tilde{\eta}_t = \tilde{D}_t^{-1}(r_t - \tilde{\mu})$ in each regime, which are almost surely (a.s.) positive definite by construction.

Notice in other regime-switching correlation models such as the STCC, as the evaluation of the likelihood function requires one $N \times N$ matrix inversion for each time period, maximization of $\ln L\left(\psi\middle|\tilde{\mu}, \tilde{\phi}, \gamma\right)$ by iterative methods can be quite costly even for small cross-sections. As an example, in contrast to the STCC, in a two-regime TCC with N=13, the N(N-1)=156 correlation estimates are actually *eliminated* from the optimization procedure. Put differently, the 156 correlations are estimated by their corresponding sample moments, and not through intensive iterative optimization methods as in the case of the other regime-switching models such as the STCC. Thus the TCC estimation procedure involves no computation cost.

Finally, estimation of γ is given by:

$$\tilde{\gamma} \equiv \underset{\gamma\in[\gamma_L,\gamma_U]}{\arg\max} \ln L\left(\tilde{R}_1(\gamma), \tilde{R}_2(\gamma)\middle|\tilde{\mu}, \tilde{\phi}\right) \quad (12)$$

where $\ln L\left(\tilde{R}_1(\gamma), \tilde{R}_2(\gamma)\middle|\tilde{\mu}, \tilde{\phi}\right) = -\frac{1}{2}\sum_t\left(\ln\left(\left|\tilde{R}_t\right|\right) + \tilde{\eta}_t{}'\tilde{R}_t^{-1}\tilde{\eta}_t\right)$ with $\tilde{R}_t = \tilde{R}_1 I(s_{t-1} < \gamma) + \tilde{R}_2[1 - I(s_{t-1} < \gamma)]$.

That is, $\ln L\left(\tilde{R}_1(\gamma), \tilde{R}_2(\gamma)\middle|\tilde{\mu}, \tilde{\phi}\right)$ is the value achieved in the maximization of Eq. (10).

The maximization problem in Eq. (12) can be solved by a grid search. A detailed explanation of the implementation of the grid search is given in the next section[8].

2.3 Computational issues and bias

In practice, we implement the above grid-search maximization algorithm as follows[9]. We sort the distinct values of the observations on the threshold variable s_{t-1}. We eliminate the smallest and largest 15%[10] observations and search over the remaining $T*0.70$ observations of s_{t-1}. In our empirical application, we search over 979*0.7 = 685 observations. For the bootstrap methods discussed in the Appendix, a full grid search involves 685*300 = 205500 test evaluations (due to 300 bootstrap replications).

It is now well known in the literature that multivariate volatility models often suffer from serious bias in high dimensions (Pakel et al., 2020). To address this issue we study the finite-sample properties of the threshold parameter of the TCC by Monte Carlo simulations. The results are presented in the Appendix A2 and in Table A2. We find the TCC generally does not suffer from a serious bias even when the number of series becomes very large (*N*=100).

2.4 Threshold test

When considering the TCC model it is important to determine whether the change in correlation is statistically significant. The relevant null hypothesis of no threshold effect (or constancy) is $H_0: R_1 = R_2$. To that purpose and for exposition purposes, we redefine Eq. (7) as follows:

$$R_t = R + \Lambda I(s_{t-1} < \gamma) \qquad (13)$$

where $R \equiv R_2$ and $\Lambda \equiv R_1 - R_2$. With this notation, the null of constancy is tested as $H_0^{\Lambda}: \Lambda = 0$. We perform an extension of the Lagrange Multiplier test developed by Tse (2000). More specifically, for a given γ, we denote $s(\gamma)$ as the *N*-element score vector, $s(\gamma) = \partial \ln L(\theta|\gamma)/\partial\theta$, where θ is the parameter vector under the alternative hypothesis. Then, the partial derivatives with respect to λ_{ij} parameters (with $\Lambda \equiv [\lambda_{ij}]$) are:

[8] If γ were known, following Engle (2002), consistency of the two-step procedure could be proved based on Newey and McFadden (1994) Theorem 6.1. In practice, γ is unknow and, therefore, one should consider additional conditions in the spirit of Hansen (2000).

[9] For more details on the grid search method and theoretical results in threshold models, we refer to see the seminal paper by Tong and Lim (1980) and the excellent contributions by Hansen (1996, 2000), among others.

[10] Other trimming parameter choices such as 5% deliver similar results.

$$\frac{\partial \ln L(\theta|\gamma)}{\partial \lambda_{ij}} = \frac{1}{2} I(s_{t-1} < \gamma)\left(\eta_{i,t}^* \eta_{j,t}^* - (\rho + \lambda)^{ij}\right) \quad (14)$$

where $\eta_t^* = \left(\eta_{1,t}^*, \dots, \eta_{N,t}^*\right)' = R_t^{-1} \eta_t$ and $[R + \Lambda]^{-1} \equiv \left\{(\rho + \lambda)^{ij}\right\}$[11]. Note under null, $R_1 = R_2$ ($\Lambda = 0$), so that $\eta_t^* = \left(\eta_{1,t}^*, \dots, \eta_{N,t}^*\right)' = R^{-1} \eta_t$ and $(\rho + \lambda)^{ij} = \rho^{ij}$. In this case, η_t^* are just the standardized returns calculated as in Bolleslev (1990).

Further, we denote $S(\gamma)$ as the $T \times N$ matrix the rows of which are the partial derivatives $\partial \ln L(\theta|\gamma)/\partial \theta'$, for $t = 1, \dots, T$. The *LM* statistic for testing the null H_0^Λ, denoted as $LMC(\gamma)$, is calculated by using the sum of the cross products of the first derivates of $\ln L_t$:

$$LMC(\gamma) = \hat{s}(\gamma)' \left(\hat{S}(\gamma)' \hat{S}(\gamma)\right)^{-1} \hat{s}(\gamma) \quad (15)$$

where the hats denote evaluation at $\hat{\theta}$ under the null hypothesis. If γ was known, under the usual regularity conditions, the distribution of $LMC(\gamma)$ would be asymptotically χ^2 distributed.

In the present paper, however, we are interested in cases where γ is unknown. In such cases, one has to contruct a test statistic that does not take γ as given. This illustrates the problem of an unidentified nuisance parameter under the null hypothesis, that is, the parameter γ is only identified under the alternative (the so-called Davies problem, Davies 1987, 2002). In consequence, *LM*-type statistics constructed with γ treated as a parameter do not possess their standard large sample asymptotic distribution.

Here we adopt a common method proposed to address this issue and consider the *Sup LMC* and the *Ave LMC* tests statistics (based on the analyses of Andrews and Ploberger, 1994, and Hansen, 1996). The *Sup LMC* is simply the maximum of the individual *LMC* statistics:

$$Sup\, LMC = \max_{\gamma_L \le \gamma \le \gamma_U} \left(LMC(\gamma)\right) \quad (16)$$

The *Ave LMC* is the simple average of the individual *LMC* statistics:

$$Ave\, LMC = \frac{1}{k} \sum_{\gamma=\gamma_L}^{\gamma_U} LMC(\gamma) \quad (17)$$

[11] In our application, we use a nonparametric measure of variance (realized variance). If estimated by a parametric GARCH, for details regarding the derivatives with respect to the variance parameters, we refer to Tse (2000).

where k is the number of grid points. Note that as the function $LMC(\gamma)$ is non-differentiable in γ, we perform a grid search- LMC evaluation over $[\gamma_L, \gamma_U]$ (see Section 2.3 for details).

As in other threshold models, the asymptotic distribution of the threshold effect statistics generally depends upon moments of the sample. For this reason, we propose a bootstrap implementation of the threshold test[12].

3. Empirical results

3.1 Data and preliminary statistics

We estimate the correlations of a portfolio containing equities and bonds as in Cappiello et al (2006) and Ehrmann, Fratzscher and Rigobon (2011). In particular, we include the stock market indices of Australia (ASX 200), Britain (FTSE 100), France (CAC 40), Germany (DAX 30), Hong Kong (HANG SENG), Japan (NIKKEI 225) and the US (S&P 500). We also consider the 10-year constant maturity bonds of Britain (UK 10Y), France (FR 10Y), Germany (GER 10Y), Italy (IT 10Y), Japan (JP 10Y) and the US (US 10Y). All data is obtained from DataStream. The sample period starts on June 1992 and ends on March 2011, which yields 980 weekly observations. This long time span features several episodes of financial market turbulence (e.g., Asian crisis, LTCM, Russian crisis, terrorist attacks in September 2001 and the recent financial crisis) as well as tranquil periods.

Descriptive statistics of the data are presented in Table 1. During this period, bonds provide lower standard deviations, but also surprisingly, higher returns compared to equities. As expected, the standardized returns (standardized by the realized standard deviation) are less skewed and less fat-tailed than the raw returns.

3.2 Modelling cycle

As mentioned before, we divide the estimation procedure into two separate estimations: the mean and variance estimation first and then the correlation estimation. In particular, for the conditional variances we calculate the realized volatility using daily returns. We test whether the realized variances sufficiently capture the dominant volatility dynamics

[12] A detailed description of the procedure and a simulation study is reported in the Appendix A.1.

in the data. For this, we apply the Ljung-Box statistic for testing for autocorrelation up to $m = \sqrt{T}$ lags in the squared standardized returns. The results (last column of Table 1) show that the null hypothesis of no serial correlation is not rejected. Hence, we conclude that the realized variances capture the volatilities dynamics in the data quite adequately.[13]

As for the correlations, we first consider the two financial markets/blocks (equities and bonds), separately. For each market, we estimated bivariate and higher-dimensional TCC models. The bivariate results just gave us some rough idea on how "homogeneous" are the blocks in terms of the chosen state variable for the conditional correlations. The bivariate TCC results are very much in line with the market-wide findings, and are not reported for space considerations.

We make use of the following candidate state variables (1-week lag) derived directly from our data (for each market/block, separately):

- Realized variances of each block return series, $rv_{i,t}$.
- Principal components of realized variances.
- Standardized return series, $r_{i,t}/\sqrt{rv_{i,t}}$.
- Absolute value of standardized return series.
- Principal components of standardized return series.
- Absolute value of principal components of standardized return series.
- The CBOE (Chicago Board of Options Exchange) VIX volatility index. The VIX is considered a leading measure of the equity market's near term volatility, and is a proxy for risk aversion and aggregate uncertainty. In practice, we calculated the realized mean of the VIX index over the week using daily data.
- The US short term interest rate (3-month money market rate) as a proxy for business cycle conditions in the US and elsewhere.

In total, we employ between 43 and 50 candidate state variables for each market/block.

3.3 Main in-sample results

The results for the two markets are reported in Tables 2-3. For each market we estimated the TCC models by using each candidate state variable. In the tables we report the correlation matrices obtained by using the best-fit state variable. For equities the best fit is obtained by using the equity market VIX volatility index (see Tables 2a-b). The model

[13] Nevertheless, we also estimated a standard GARCH(1,1) process and our results remained qualitatively the same.

gives a threshold estimate of $\hat{\gamma} = 21.334$, and therefore it is possible to speak about high versus low volatility regimes (see Figure 1). During high volatility periods (in the late 1990s, early 2000s and after 2007-08), correlations are higher than during calm periods. That is, the investors' uncertainty is shown by an increase in the correlations. This result is in line with other studies in the literature, for example Ang and Bekaert (2002), Longin and Solnik (1995, 2001), Ramchand and Susmel (1998), among others.

However, for bonds the best fit is obtained by using the US short rate as state variable (results are reported in Tables 3a-b). The threshold estimate is 2.85, which implies that there are two regimes: a low interest rate regime (when $\hat{\gamma} < 2.85$) and a high interest rate regime (when $\hat{\gamma} \geq 2.85$) (see Figure 2). During the low interest rate regime (from early to mid 2000s and after 2007-08) correlations are stronger as opposed to periods of high interest rates. The low interest rate regime reflects periods during which business cycle conditions are weak, so higher bond correlations are expected for high-quality bonds (a result which corroborates with Diebold and Yilmaz, 2015).

We also compare our TCC estimates to those obtained from a STCC model. To keep the comparison feasible we run bivariate TCC and STCC for equities using the equity market VIX volatility and for bonds using the US short rate as the state variables. Figures 3-4 plot the log-likelihood values obtained from the two models (note STCC has one extra parameter). As seen, the fit of the models is almost identical. Actually, the slope parameter in the STCC model is generally estimated large implying abrupt shifts between the regimes. This can be explained by the fact that in financial markets information is readily available so regime shifts in correlations tend to be abrupt.

The next step of our analysis is to test for the significance of the threshold effect by applying the *Sup* LMC and *Ave* LMC tests in Eq. (16-17). The test was applied for the two markets separately using the *best* state variables from the aforementioned analysis. The p-values were calculated using the model-based bootstrap experiment with B=300 replications (described in Section A.1 in the Appendix). The results, reported in Table 4, show that all four p-values of the test are effectively zero. Thus, we conclude that there is strong support for the proposed threshold specification for both markets when the *best* state variables are used.

One important issue is whether the two financial markets can be unified in a single threshold framework. In particular, given the different state variables (and thresholds) for equities and bonds, we ask the following questions. Can we estimate a two-threshold TCC specification for these two financial markets in a single step? Furthermore, can we test and allow, where appropriate, different parts in the correlation matrix to have their own switching mechanism? If so, how do we guarantee that the resulting correlation matrix is positive definite?

In Section A.3 in the Appendix we address all the aforementioned questions theoretically. In practice, we test for a restricted four-regime TCC where the correlations of equities have two regimes that depend only on equity market volatility VIX, and similarly the correlations of bonds have two regimes that depend only on the US short rate as state variable. As with the cross-correlations between equities and bonds, we consider four regimes associated with both state variables. Under the alternative hypothesis, the model is an unrestricted four-regime TCC with all correlations depending on the two state variables. Our test results, in Table A.4, show that the null hypothesis of a restricted four-regime TCC is strongly rejected. Thus, we proceed to estimate the unrestricted four-regime TCC, which is also our preferred model in the out-of-sample portfolio evaluation exercise presented in Section 4[14].

As a diagnostic test, we further study the possibility that our four-regime TCC model captures all the regime-switching dynamics of the return comovements. To this end, we test for structural breaks in the estimated stardardized return errors $\hat{\varepsilon}_t = \hat{R}_t^{-1/2}\hat{\eta}_t$ (obtained from the four-regime TCC) using the recently developed correlation change-point detection algorithm proposed by Wied, Krämer, and Dehling (2012) at a biraviate setting. Our results show that the stardardized return errors do not generally suffer from significant structural breaks. For example, only 5 out of 78 correlations pairs show some evidence of a structural break. Ideally, we would have liked to carry out a multivariate structural break test for all correlation pairs jointly to determine the overall significance level of the test, but this is not available in the Wied, Krämer, and Dehling (2012) setting. Therefore, we conclude that the four-regime TCC captures the dominant comovement dynamics in the data.

4. Out-of sample analysis

4.1 Asset allocation test

[14] For space considerations, the estimation results of this model are also reported in the Appendix in Table A3.

In this section, we investigate the implications of time-varying correlations on asset allocation. The literature has mainly centred on evaluating the statistical performance of correlation models rather than their economic significance. In contrast, we focus on the latter. In particular, we examine the performance of the TCC with four regimes against the dynamic conditional correlation (DCC) model using the test proposed by Engle and Colacito (2006)[15].

Suppose that we have two different time series of the covariance matrices $\{H_t^j\}_{j=1}^2$ and a set of hypothesized vectors of expected returns $\{\mu^k\}_{k=1}^N, k = 1, \dots, N$, (divided by the required excess return, μ_0), where $j = 1,2$ corresponds to the benchmark (DCC) and alternative (TCC) models. For each period we calculate a set of optimal portfolio weights, $w_t^{j,k}$ based on a covariance matrix and on an expected return. The portfolio´s return is given by:

$$\pi_t^{j,k} = \left(w_t^{j,k}\right)'(r_t - \bar{r}) \tag{18}$$

where $w_t^{j,k} = \frac{\left(H_t^j\right)^{-1}(\mu^k)}{(\mu^k)'\left(H_t^j\right)^{-1}(\mu^k)}$ and $\bar{r}$ denotes (sample) mean returns. The *Engle-Colacito* test statistic is calculated as the difference between realized portfolio variance obtained from the DCC and the TCC model:

$$u_t^k = \left(\pi_t^{1,k}\right)^2 - \left(\pi_t^{2,k}\right)^2 \tag{19}$$

Under the null hypothesis, the expected value of u_t^k is zero for all k implying that the threshold model does not reduce portfolio's variance compared to the DCC. To improve the sampling properties of the test, we also perform the weighted version of the test where we divide u_t^k by its standard deviation:

$$v_t^k = u_t^k\left[2\left(u_t^{k\prime}(H_t^1)^{-1}u_t^k\right)\left(u_t^{k\prime}(H_t^2)^{-1}u_t^k\right)\right]^{1/2} \tag{20}$$

We estimate jointly:

$$\begin{aligned} u_t^1 &= \beta + \epsilon_{u,t}^1 \\ u_t^2 &= \beta + \epsilon_{u,t}^2 \\ &\cdots \\ u_t^N &= \beta + \epsilon_{u,t}^N \end{aligned} \tag{21a}$$

or the weighted version of the test

[15] Note that for both TCC and DCC we use the same realized volatility measure, so that any differences are due to the different correlation structure.

$$
\begin{aligned}
v_t^1 &= \beta + \epsilon_{v,t}^1 \\
v_t^2 &= \beta + \epsilon_{v,t}^2 \\
&\cdots \\
v_t^N &= \beta + \epsilon_{v,t}^N
\end{aligned} \tag{21b}
$$

Thus, the null hypothesis of equal variances is $H_0 \colon \beta = 0$. In a multivariate setting, one problem in running Eq. (21a-b) is the choice of the appropriate vector of expected returns, which may lead to an unbearable number of possible combinations. We follow Engle and Colacito (2006) and focus on hedging portfolios. More specifically, we select vectors of expected returns for which one entry is equal to one, while everything else is zero. That is, one asset is hedged against all the others.

To apply the test, we perform an out-of-sample evaluation using a rolling window estimation scheme. More specifically, we produce one-step-ahead forecasts at time *m*=579 using data indexed 1,…,m and are compared to $R_{m+1}(R_{579+1})$. The estimation window is then rolled forward one step and the second set of forecasts are obtained using observation 2,…,*m*+1 and are compared to $R_{m+2}(R_{579+2})$. This procedure is thus iterated and the last forecasts are obtained using observations *T*-*m*,…,*T*-1 and are compared to $R_T(R_{979})$.This yields a sequence of *n*=*T*-*m* (979-579=400) out-of-sample correlation forecasts obtained from the DCC and from the unrestricted four-regime TCC.

Initially, forecasts are calculated for the period after July 17, 2003 (400 forecasts) and the corresponding portfolio evaluation results are reported in Table 5. Furthermore, to investigate any effect of the global financial crisis, we also test the models for the subperiod before mid-2007, July 17, 2003 to May 10, 2007, (200 forecasts) and for the subperiod after mid-2007, May 17, 2007 to March 10, 2011, (200 forecasts). The later subsample covers a period with significant shifts in the level of long-run correlations (see, for example, Figures 1-2) and, therefore, is expected to be relatively more informative about which of the models might deliver the best forecasts.

Table 5 holds the results. For the total forecast period the test is highly significant in favour of the TCC model. Note, however, that for the period before the global financial crisis the DCC is the preferred specification based on both versions of the test. More interestingly, the picture changes in the period after 2007 and the TCC clearly dominates the DCC. This finding may be explained by the fact that as correlations became more volatile after 2007 a TCC model allowing for significant shifts in the level of long-run correlations could capture those shifts better than a DCC that mainly allows for slowly changing correlations (due to persistence). Furthermore, this finding corroborates with the results in Aslanidis and Casas (2013) that show in Monte Carlo simulations the DCC is best for slowly changing correlations compared to semi-parametric and non-parametric correlation methods. Overall, we conclude the threshold method with four regimes can give rise to a significant reduction in the portfolio's variance compared to the DCC benchmark particularly during the recent turbulent crisis.

4.2. Density forecasting

The interest in density forecasts is expanding rapidly in the finance literature. In this section, we evaluate the performance of the models based on their ability to deliver better density forecasts. To that end, denote the competing one-step-ahead predictive density obtained from the TCC and DCC models by $\hat{f}_{m,t}^{TCC}(r_{t+1})$ and $\hat{f}_{m,t}^{DCC}(r_{t+1})$, respectively (we consider the same forecast period as before). The subscripts indicate that the time-*t* forecasts are measurable functions of the m most recent observations (using the rolling window scheme described above)[16]. We restrict attention to the logarithmic scoring rule (see, Amisano and Giacomini, 2007, and Diks et al., 2014, among others):

$$S^l\left(\hat{f}_{t,m}; r_{t+1}\right) = \log \hat{f}_{t,m}(r_{t+1}) \tag{22}$$

Based on the $n \equiv T - m$ (n=400) forecast observations, the sample average of the log score differences is given by:

$$\bar{d}_{m,n} = n^{-1}\left(\sum_{t=m}^{T-1} \log \hat{f}_{m,t}^{TCC}(r_{t+1}) - \sum_{t=m}^{T-1} \log \hat{f}_{m,t}^{DCC}(r_{t+1})\right) \tag{23}$$

We test the null hypothesis of equal density forecast accuracy (that is, the DCC model is as accurate as the TCC model) $H_0: E\left(\bar{d}_{m,n}\right) = 0$ using a Diebold-Mariano type statistic:

$$\frac{\bar{d}_{m,n}}{\sqrt{\hat{\sigma}_{m,n}^2 / n}} \tag{24}$$

where $\hat{\sigma}_{m,n}^2$ is a heteroskedastic and autocorrelation-consistent (HAC) variance estimator of $\sigma_{m,n}^2 = Var\left(\sqrt{n}\bar{d}_{m,n}\right)$. Under the null hypothesis, as $n \to \infty$ with m fixed the test statistic in Eq (24) is asymptotically standard normal distributed (Giacomini and White, 2006). A significantly positive test statistic implies that the TCC model is more accurate than the DCC model in terms of density forecasting.

[16] In practice, we keep the estimation window fixed over time.

The results are shown in Table 6. The main picture is similar to the previously shown Engle-Colacito test results. Over the entire period, the four-regime TCC delivers more accurate density forecasts than the DCC at the 5% significance level. However, for the period before the global financial crisis the log score difference is negative in favour of the DCC though not statistically significant. Nevertheless, in the turbulent after 2007 period the TCC improves and beats the DCC forecast at the 5% level.

5. Conclusions

Examining the comovements across international equity and bond markets is a main concern for policy makers and academics. In this article, we propose a flexible methodology to identify international comovement regimes. The advantage of this new approach over existing methods is that it is easy to estimate and does not have a serious bias, even when it is applied to large portfolios. Empirically, we find strong evidence that high equity market volatility and low short-term interest rates are the key drivers of equity and bond comovements, respectively. Moreover, out-of-sample evaluation shows our approach outperforms the popular dynamic conditional correlation, especially during the recent global financial crisis, a period characterized by significant shifts in the long-run level of comovements due to higher volatility and lower interest rates. Overall, our results offer interesting insights into investment and diversification that can be applied to analyse systemic risk, asset allocation and hedging.

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Table 1: Summary statistics

Abbr.	Mean	St.dev	Skewness	Kurtosis	Standardized Skewness	Standardized Kurtosis	LB- Squared Standardized
Australia Equities - ASX 200	0.103	2.067	-0.581	5.713	-0.156	2.035	0.856
UK Equities - FTSE 100	0.079	2.336	-0.569	5.873	-0.097	2.104	0.275
Hong Kong Equities – HANG SENG	0.139	3.694	-0.565	6.917	-0.123	1.855	0.189
France Equities – CAC 40	0.070	2.848	-0.346	5.266	-0.130	2.100	0.766
US Equities - S&P 500	0.116	2.416	-1.139	14.98	-0.148	2.114	0.550
Germany Equities - DAX 30	0.140	3.018	-0.575	6.256	-0.150	2.104	0.157
Japan Equities - NIKKEI 225	-0.055	3.043	-0.285	5.567	-0.039	2.084	0.214
Germany bonds - GER 10Y	0.139	1.583	0.372	5.009	-0.081	2.138	0.409
France bonds - FR 10Y	0.147	1.568	0.288	4.927	-0.095	2.141	0.703
Italy bonds - IT 10Y	0.151	1.813	-0.237	7.050	-0.100	2.058	0.129
Japan bonds - JP 10Y	0.128	1.666	0.639	7.201	-0.033	2.144	0.268
UK bonds - UK 10Y	0.130	1.532	-0.486	5.875	-0.104	2.126	0.574
US bonds – US 10Y	0.116	1.017	0.031	4.633	-0.096	2.028	0.499

Notes: The standardized skewness and kurtosis are the skewness and kurtosis of the returns standardized by the realized standard deviation. The Ljung-Box statistic (*p*-value) tests autocorrelation up to 31 lags in the squares of standardized returns. Source is DataStream.

Table 2a: 2-regime TCC correlations for equities
Regime 1: $VIX_{t-1} < 21.334$ (low volatility)

	ASX 200	FTSE 100	HANG SENG	CAC 40	S&P 500	DAX 30	NIKKEI 225
ASX 200	1						
FTSE 100	0.408	1					
HANG SENG	0.363	0.399	1				
CAC 40	0.358	0.694	0.372	1			
S&P 500	0.346	0.516	0.364	0.508	1		
DAX 30	0.409	0.643	0.400	0.717	0.516	1	
NIKKEI 225	0.364	0.317	0.264	0.316	0.322	0.293	1

Table 2b: 2-regime TCC correlations for equities
Regime 2: $VIX_{t-1} \geq 21.334$ (high volatility)

	ASX 200	FTSE 100	HANG SENG	CAC 40	S&P 500	DAX 30	NIKKEI 225
ASX 200	1						
FTSE 100	0.581	1					
HANG SENG	0.625	0.596	1				
CAC 40	0.603	0.860	0.597	1			
S&P 500	0.617	0.713	0.588	0.773	1		
DAX 30	0.609	0.771	0.619	0.870	0.729	1	
NIKKEI 225	0.573	0.474	0.577	0.516	0.454	0.503	1

Table 3a: 2-regime TCC correlations for bonds

Regime 1: $US\ short - rate_{t-1} < 2.85$ (low interest rate)

	GER 10Y	FR 10Y	IT 10Y	JP 10Y	UK 10Y	US 10Y
GER 10Y	1					
FR 10Y	0.985	1				
IT 10Y	0.935	0.958	1			
JP 10Y	0.447	0.418	0.365	1		
UK 10Y	0.738	0.718	0.691	0.346	1	
US 10Y	0.404	0.361	0.289	0.407	0.391	1

Table 3b: 2-regime TCC correlations for bonds

Regime 2: $US\ short - rate_{t-1} \geq 2.85$ (high interest rate)

	GER 10Y	FR 10Y	IT 10Y	JP 10Y	UK 10Y	US 10Y
GER 10Y	1					
FR 10Y	0.911	1				
IT 10Y	0.741	0.763	1			
JP 10Y	0.354	0.326	0.211	1		
UK 10Y	0.581	0.596	0.533	0.208	1	
US 10Y	0.287	0.288	0.290	0.125	0.407	1

Table 4: Constancy tests

	$Sup\ LMC$	$Ave\ LMC$
Equities	202.1 (0.000)***	129.8 (0.000)***
Bonds	287.4 (0.000)***	146.1 (0.000)***

Notes: $Sup\ LMC$ and $Ave\ LMC$ statistics, see Eq. (A.4-A.5). Bootstrapped *p*-values in parentheses, where the number of bootstrap replications is *B*=300. The tests are performed using the following state variables: VIX_{t-1} for equities and $US\ short - rate_{t-1}$ for bonds.

***/**/* indicate significance at the 1%/5%/10% level.

Table 5: Asset allocation test (out-of-sample)

Equities and bonds Jointly	*Engle-Colacito* test (unweighted)	*Engle-Colacito* test (weighted)
17/07/03-10/03/11 (400 for)	-0.090***	-0.213***
17/07/03-10/05/07 (200 for)	0.143***	0.094***
17/05/07-10/03/11 (200 for)	-0.326***	-0.529***

Notes: Engle-Colacito (2006) estimates of coefficients for testing that the TCC produces a smaller variance than the DCC. Negative values are evidence in favour of the TCC.
***/**/* indicate significance at the 1%/5%/10% level.

Table 6: One-step-ahead density forecasts

Equities and bonds Jointly	Average score difference and tests of equal predictive accuracy
17/07/03-10/03/11 (400 for)	3.541**
17/07/03-10/05/07 (200 for)	-0.212
17/05/07-10/03/11 (200 for)	7.294**

Notes: The table presents the average score difference for the logarithmic scoring rule in Eq. (18). A significantly positive test statistic implies that the TCC is superior to the DCC.
***/**/* indicate significance at the 1%/5%/10% level.

Figure 1: VIX volatility state variable (equities)

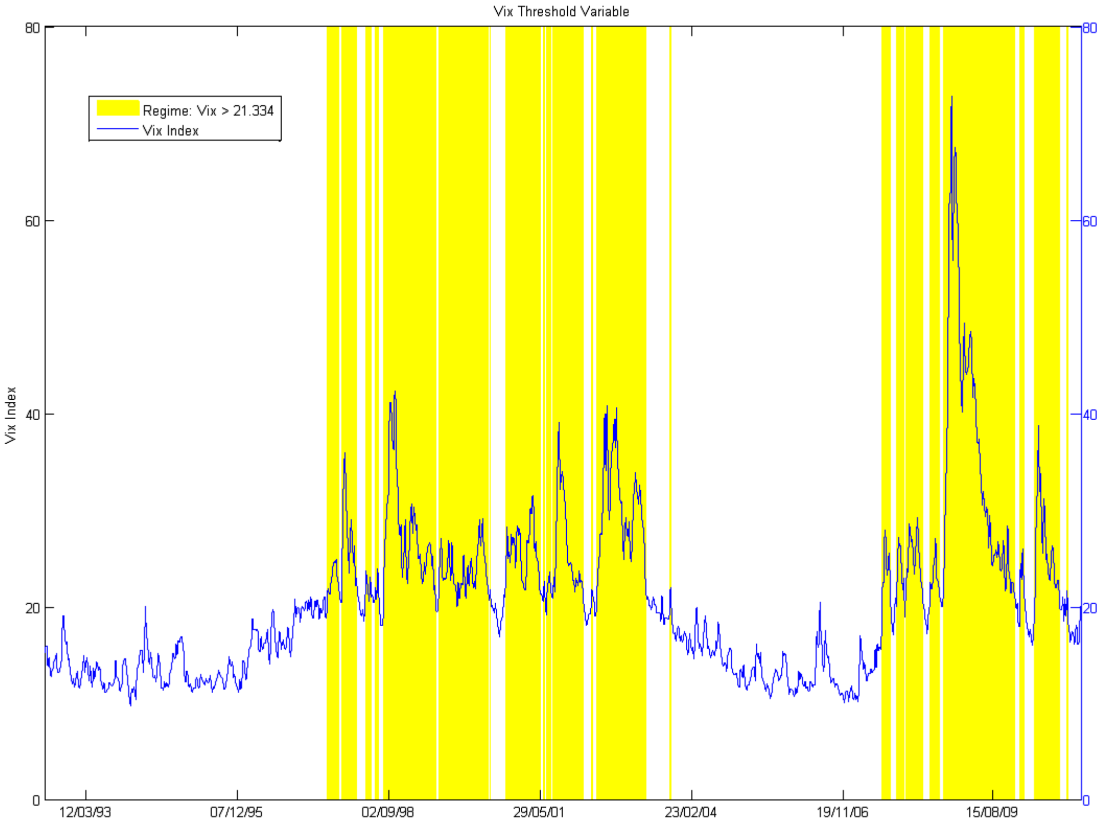


Figure 2: US short rate state (bonds)

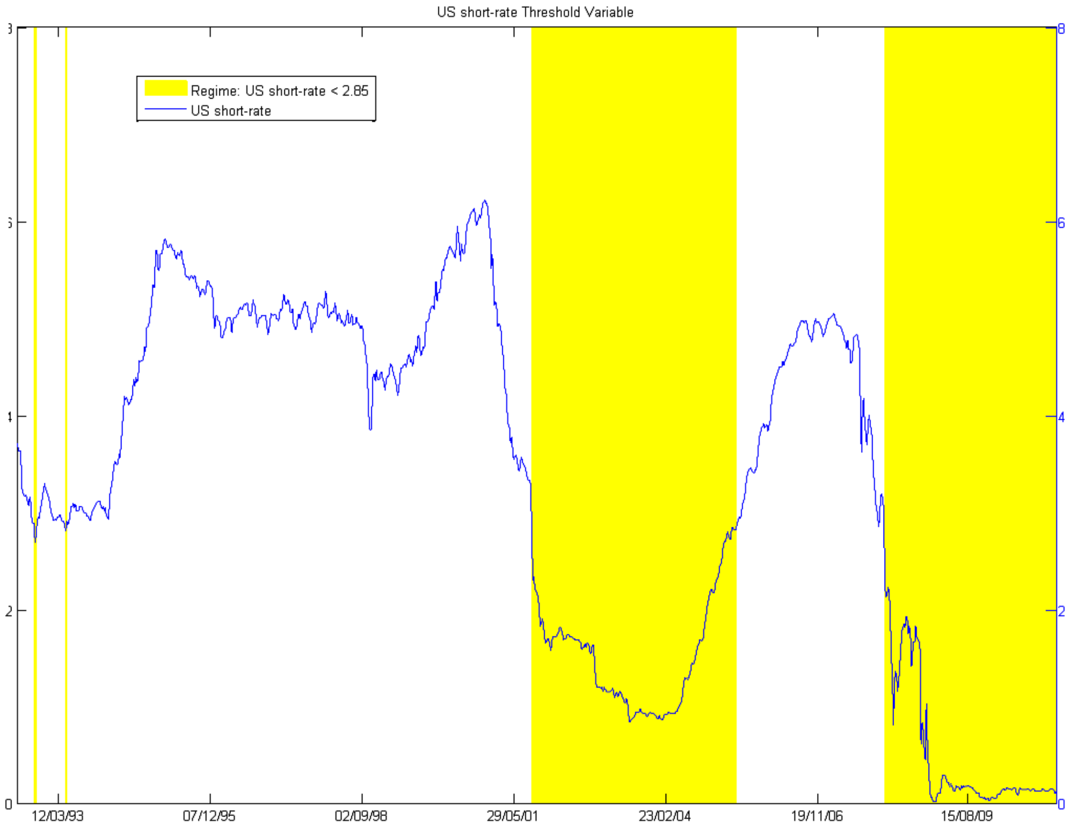


Figure 3: Bivariate TCC vs STCC: Log-likelihood values (equities)

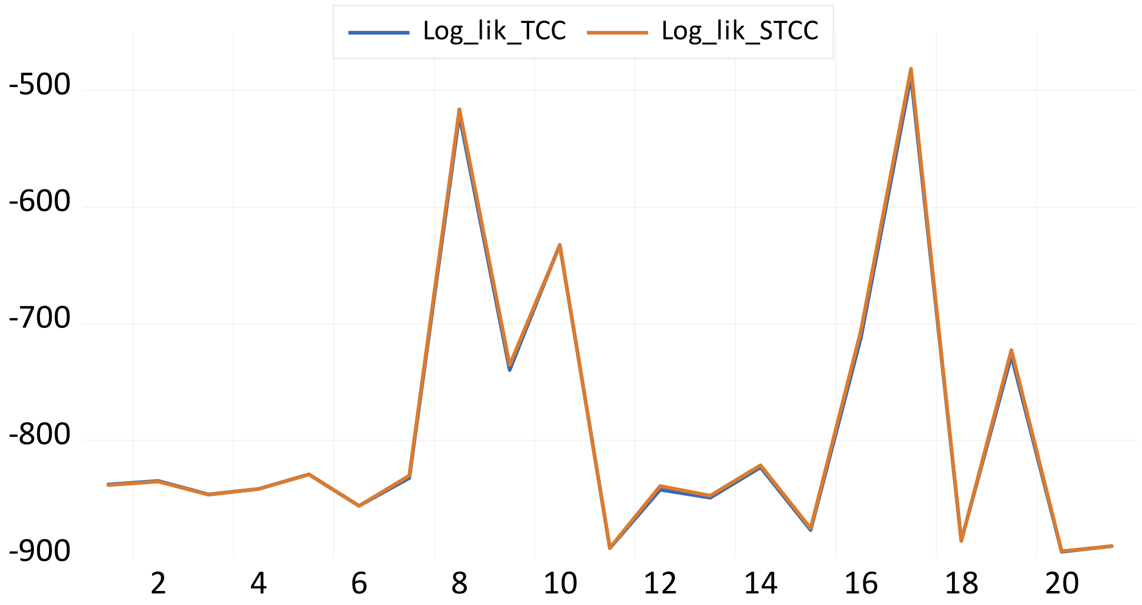
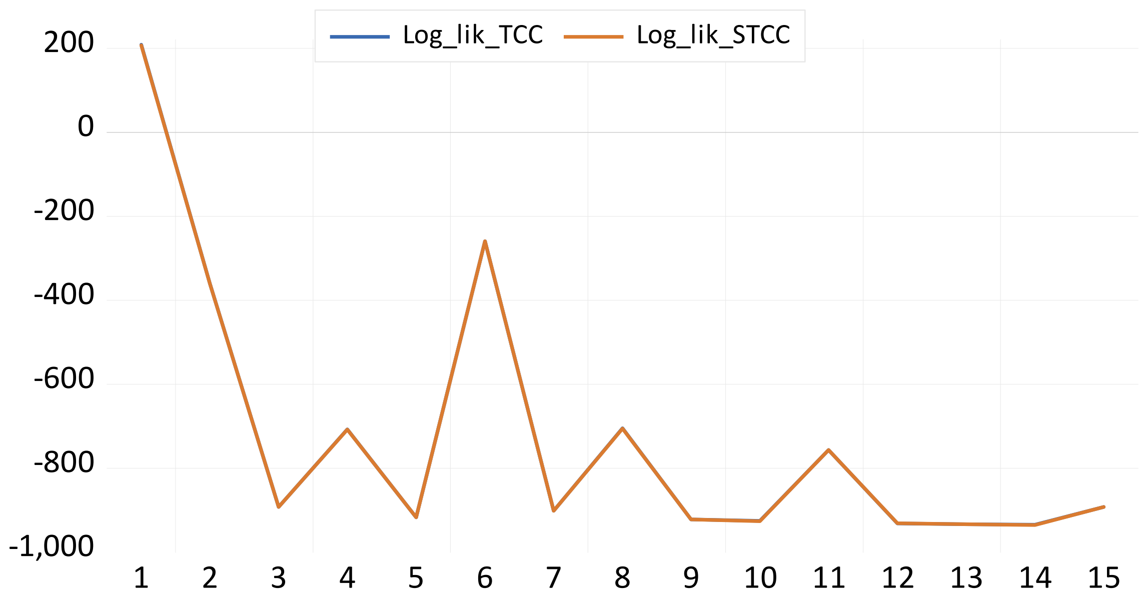


Figure 4: Bivariate TCC vs STCC: Log-likelihood values (bonds)



Appendix

A.1 *Bootstrap implementation of threshold test and simulation results*

As in other threshold models, the asymptotic distribution of the threshold effect statistics generally depends upon moments of the sample. For this reason, we propose a bootstrap implementation of the threshold test performed in the following computer algorithm format:

Algorithm (*Model-based Bootstrap procedure*)

1. $l = 1$
2. Generate $\left\{\varepsilon_t^{(l)}\right\}_{t=1}^{T}$ by resampling from $\{\hat{\varepsilon}_t\}_{t=1}^{T}$, with $\hat{\varepsilon}_t = \hat{R}_t^{-1/2}\hat{\eta}_t$ (estimated standardized errors under the alternative hypothesis, threshold conditional correlation).
3. Calculate $\left\{r_t^{(l)}\right\}_{t=1}^{T}$ from $r_t^{(l)} = \hat{\mu} + \hat{D}_t\hat{R}^{1/2}\varepsilon_t^{(l)}$ (model under the null hypothesis, constant conditional correlation).
4. Compute *Sup* $LMC^{(l)}$ and *Ave* $LMC^{(l)}$ for the bootstrap sample series $\left\{r_t^{(l)}\right\}_{t=1}^{T}$.
5. $l = l + 1$. Go to step 2 while $l \leq B$.
6. Estimate the *p*-value, from the bootstrap approximation

$$p_{sub} = \frac{1}{B}\sum_{l=1}^{B} I\left[Sup\ LMC^{(l)} \geq Sup\ LMC\right] \quad \text{(A.1)}$$

$$p_{ave} = \frac{1}{B}\sum_{l=1}^{B} I\left[Ave\ LMC^{(l)} \geq Ave\ LMC\right] \quad \text{(A.2)}$$

Note that in step 2, we resample $\{\hat{\varepsilon}_t\}_{t=1}^{T}$ from the vector $\hat{\varepsilon}_t$ simultaneously from all its elements. As far as the realized volatility is concerned, we treat this series as given holding its values fixed in repeated bootstrap samples.

We study the finite-sample properties of the bootstrap implementation of the threshold effect test by Monte Carlo simulations. We simulate a bivariate volatility model with normal errors. The parameter values for the data generated processes (DGPs) are as follows:

$$r_{1,t} = h_{1,t}^{1/2}\eta_{1,t}, \quad h_{1,t} = 0.4rv_{1,t}, \quad rv_{1,t} = 1 + \left(0.15\eta_{1,t-1}^2 + 0.8\right)rv_{1,t-1} \quad \text{(A.3)}$$

$$r_{2,t} = h_{2,t}^{1/2}\eta_{2,t}, \quad h_{2,t} = 0.2rv_{2,t}, \quad rv_{2,t} = 1 + \left(0.2\eta_{2,t-1}^2 + 0.7\right)rv_{2,t-1} \quad \text{(A.4)}$$

$$\begin{pmatrix}\eta_{1,t}\\ \eta_{2,t}\end{pmatrix} \sim N\left[\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}\right] \quad \text{(A.5)}$$

For the correlation parameter, we tried four different values $\rho = \{0.8, 0.3, -0.3, -0.8\}$. To calculate the test statistic in Eq. (15) we need to compute the partial derivatives given in Eq. (14). For this we use as a state variable the realized volatility $rv_{1,t}$ or the first principal component of the two series. The sample size is *T*=500, the number of Monte Carlo simulations is *M*=300, and the number of bootstrap replications is *B*=300.

Table A.1 holds the rejection frequencies at significance levels of 1%, 5% and 10%. As seen, the results show the actual size of the test is close to the nominal size. Thus, we conclude the test does not generally seem to suffer from any size distortion.

A.2 *High dimensional problems*

Multivariate volatility models can suffer from serious bias when the data consists of a large number of assets (Pakel et al., 2020). In light of this result, an important question is whether the proposed TCC model is free from such bias. To address this issue, we study the finite-sample properties of the threshold parameter γ when the number of series *N* becomes large. Following Pakel et al. (2020), we present a Monte Carlo exercise where we examine the mean bias and root mean square error (RMSE) of the threshold estimate for $N = \{3, 10, 50, 100\}$. The series of returns are simulated according to a two-regime TCC with $\mu = 0$. Further, for the state variable $s_t = \phi s_{t-1} + e_t$ we tried two different processes $\phi = \{0.5, 0.9\}$ with $e_t \sim N(0,1)$. For the threshold parameter, we choose γ such that for each replication we guarantee that $Prob(s_t < \gamma) = 0.5$, so both regimes are equally likely to occur. The sample size is *T*=2000 while the number of Monte Carlo simulations is *M*=2500. The correlation matrices (R_1, R_2) were generated from a single-factor model, so $R_j^{l_1 l_2} = \pi_j^{l_1} \pi_j^{l_2}$ for $l_1 \neq l_2$ and 1 if $l_1 = l_2$ with $l_1, l_2 = 1, \dots, N$ (*j*=1,2 indicates the number of regimes). The vectors π_j^l are distributed according to independent beta distributions.

We consider two different scenarios for the correlations.

Scenario 1: The average (regime-depedent) correlations in the cross section are $E\left(R_1^{l_1 l_2}\right) = 0.25$ and $E\left(R_2^{l_1 l_2}\right) = 0.5$ and the standard deviation is equal to 0.1 in both cases, $\pi_1^l \sim Beta(5.9907, 5.9907)$ and $\pi_2^l \sim Beta(14.083, 5.8332)$.

Scenario 2: The average (regime-depedent) correlations in the cross section are $E\left(R_1^{l_1 l_2}\right) = 0.25$ and $\left(R_2^{l_1 l_2}\right) = 0.75$ and the standard deviation is equal to 0.1 in both cases, $\pi_1^l \sim Beta(5.9907{,}5.9907)$ and $\pi_2^l \sim Beta(14.273{,}2.208)$.

Table A.2 summarizes the mean bias and root mean square error (RMSE) values. As seen, in an equally likely regimes scenario, the TCC model does not suffer from a serious bias even when the number of series becomes very large (*N*=100). Moreover, when the state variable is a persistent process, $\phi = 0.9$, although the bias is higher (as expected), importantly, it does not increase with the dimension of the cross section. Note also that the performance of the model is quite similar across the two correlation scenarios. These results are very encouraging and show that the TCC can handle well even highly dimensional problems.

A.3 *Estimating and testing for large TCC models*

In this section we analyse a two-threshold TCC specification for the shares and bonds jointly. Based on previous analysis, consider two groups of assets such as equities and bonds with conditional correlation matrix given by:

$$Var\begin{pmatrix}\eta_t^s \\ \eta_t^b\end{pmatrix} = \begin{pmatrix} R_t^s & R_t^{sb} \\ R_t^{bs} & R_t^b \end{pmatrix} = R_t \qquad \text{(A.6)}$$

where η_t^s is a $n_1 \times 1$ vector of standardized equity return innovations and η_t^b is a $n_2 \times 1$ vector of standardized bond return innovations. The upper block of the diagonal of R_t denoted by R_t^s is the correlation matrix of the equities, while the lower block of the diagonal of R_t denoted by R_t^b is the correlation matrix of the bonds. Then, the off-diagonal blocks R_t^{sb} (and $R_t^{bs} = \left(R_t^{sb}\right)'$) are the cross-correlations between equities and bonds.[17]

A four-regime (two-threshold) TCC model for equities and bonds can be written as follows:

$$R_t = R_1 I\left(s_{t-1}^s < \gamma_s \wedge s_{t-1}^b < \gamma_b\right) + R_2 I\left(s_{t-1}^s < \gamma_s \wedge s_{t-1}^b \geq \gamma_b\right) + R_3 I\left(s_{t-1}^s \geq \gamma_s \wedge s_{t-1}^b < \gamma_b\right) + R_4 I\left(s_{t-1}^s \geq \gamma_s \wedge s_{t-1}^b \geq \gamma_b\right) \qquad \text{(A.7)}$$

[17] Note that this part of the correlation matrix is neither a correlation matrix with ones on the diagonal nor necessarily a square matrix.

where $R_i = \begin{pmatrix} R_i^s & R_i^{sb} \\ R_i^{bs} & R_i^b \end{pmatrix}$, i =1,…,4 are the regime-dependent correlation matrices. In this setting, it is of interest to test whether s_{t-1}^b implies a statistically significant change in the correlations only for the bonds and/or s_{t-1}^s produces a statistically significant change in the correlations only for the equities. Regarding the cross-correlations between equities and bonds, without loss of generality, we allow for both variables to act as state variables. In particular, the relevant null hypothesis to test is given by:[18]

$$H_0: R_1^s = R_2^s \wedge R_3^s = R_4^s \wedge R_1^b = R_3^b \wedge R_2^b = R_4^b \tag{A.8}$$

Under H_0, the restricted two-threshold TCC implies the following regimes:

$$R_t = R_1 = \begin{bmatrix} R_{low}^s & R_1^{sb} \\ R_1^{bs} & R_{low}^b \end{bmatrix}, \text{ when } I\left(s_{t-1}^s < \gamma_s \wedge s_{t-1}^b < \gamma_b\right) = 1 \tag{A.9}$$

$$R_t = R_2 = \begin{bmatrix} R_{low}^s & R_2^{sb} \\ R_2^{bs} & R_{high}^b \end{bmatrix}, \text{ when } I\left(s_{t-1}^s < \gamma_s \wedge s_{t-1}^b \geq \gamma_b\right) = 1, \tag{A.10}$$

$$R_t = R_3 = \begin{bmatrix} R_{high}^s & R_3^{sb} \\ R_3^{bs} & R_{low}^b \end{bmatrix}, \text{ when } I\left(s_{t-1}^s \geq \gamma_s \wedge s_{t-1}^b < \gamma_b\right) = 1, \tag{A.11}$$

$$R_t = R_4 = \begin{bmatrix} R_{high}^s & R_4^{sb} \\ R_4^{bs} & R_{high}^b \end{bmatrix}, \text{ when } I\left(s_{t-1}^s \geq \gamma_s \wedge s_{t-1}^b \geq \gamma_b\right) = 1, \tag{A.12}$$

where $R_1^s = R_2^s \equiv R_{low}^s$, $R_3^s = R_4^s \equiv R_{high}^s$, $R_1^b = R_3^b \equiv R_{low}^b$ and $R_2^b = R_4^b \equiv R_{high}^b$.

In this case, the correlations of equities have two regimes that depend only on s_{t-1}^s, and similarly the correlations of bonds have two regimes that depend only on s_{t-1}^b. On the other hand, the cross-correlations between equities and bonds have four regimes depending on both s_{t-1}^s and s_{t-1}^b. Interestingly, note the null hypothesis does not impose any reduction in the number of thresholds, so both threshold parameters are identified. In practice, we apply a classical Wald-type test taking $\hat{\gamma}_s$ and $\hat{\gamma}_b$ as known, given the super-consistency of both estimators[19].

Under the alternative hypothesis, the model is an unrestricted four-regime TCC with all correlations switching at the same time *t*. In this case, estimation can be carried

[18] Notice that the null can be decomposed in two null hypotheses:
$H_0: R_1^s = R_2^s \wedge R_3^s = R_4^s$ and $H_0: R_1^b = R_3^b \wedge R_2^b = R_4^b$. Under the former null the correlations of equities are governed only by s_{t-1}^s while under the latter, the correlations of bonds are driven only by s_{t-1}^b.

[19] In single equation models $\hat{\gamma}$ converges to γ at a rate T (for more details on the super-consistency of the threshold estimator, see the seminar paper by Chan, 1990). Although not explicitly shown, under the appropriate conditions, this result is assumed to generalize to TCC models.

out by extending the estimation method described in Section 2.2 to a two-dimensional grid search Quasi Maximum Likelihood (ML) procedure. On the other hand, under the null hypothesis the model is a restricted two-threshold TCC model that allows for different parts in the correlation matrix to be governed by different state variables. In this setting, one important issue is to guarantee that the resulting correlation matrix is positive definite. To achive this we propose a two-step estimation method. In the first step we use an unrestricted four-regime TCC model to obtain a consistent estimator of R_t in Eq. (A.7):

$$\hat{R}_t = \begin{pmatrix} \hat{R}_t^s & \hat{R}_t^{sb} \\ \hat{R}_t^{bs} & \hat{R}_t^b \end{pmatrix} \tag{A.13}$$

Then we calculate:

$$\hat{R}_t^* = \begin{bmatrix} \hat{R}_t^{s^{-1/2}} & 0 \\ 0 & \hat{R}_t^{b^{-1/2}} \end{bmatrix} \hat{R}_t \begin{bmatrix} \hat{R}_t^{s^{-1/2}} & 0 \\ 0 & \hat{R}_t^{b^{-1/2}} \end{bmatrix} = \begin{bmatrix} I & \tilde{R}_t^{sb^*} \\ \tilde{R}_t^{bs^*} & I \end{bmatrix} \tag{A.14}$$

which is positive definite given that $\hat{R}_t$ is positive definite.

In the second step we use two-regime TCC models for equities and bonds, separately (as in the previous section) and obtain consistent estimates for R_t^s and R_t^b, denoted by $\tilde{R}_t^s$ and $\tilde{R}_t^b$. Finally, the restricted conditional correlation matrix estimator is given by:

$$\tilde{R}_t = \begin{bmatrix} \tilde{R}_t^{s^{1/2}} & 0 \\ 0 & \tilde{R}_t^{b^{1/2}} \end{bmatrix} \hat{R}_t^* \begin{bmatrix} \tilde{R}_t^{s^{1/2}} & 0 \\ 0 & \tilde{R}_t^{b^{1/2}} \end{bmatrix} \tag{A.15}$$

Again, the positive definiteness of $\hat{R}_t^*$ guarantees the positive definiteness of the restricted conditional correlation matrix estimator $\tilde{R}_t$.

A grid search procedure is followed in the estimation of the four-regime TCC model with two state variables. In this case, we carry out a two-dimensional grid that in total requires 685*685 = 469225 function evaluations. In practice, however, we only search for values that require each of the four regimes to include at least 15% of the sample, so the grid search is less numerically intense.

Table A.1: Finite-sample properties of threshold effect test

State variable		Realized volatility			Principal component		
		1%	5%	10%	1%	5%	10%
$\rho = 0.8$	*Sup LMC*	0.026	0.053	0.100	0.016	0.080	0.136
	Ave LMC	0.023	0.053	0.103	0.026	0.093	0.130
$\rho = 0.3$	*Sup LMC*	0.010	0.046	0.100	0.013	0.070	0.120
	Ave LMC	0.006	0.050	0.096	0.016	0.066	0.126
$\rho = -0.3$	*Sup LMC*	0.010	0.056	0.086	0.026	0.066	0.130
	Ave LMC	0.013	0.060	0.106	0.020	0.056	0.136
$\rho = -0.8$	*Sup LMC*	0.000	0.056	0.116	0.010	0.056	0.106
	Ave LMC	0.013	0.043	0.096	0.010	0.073	0.116

Notes: Actual size of the threshold effect test (rejection frequencies). The sample size is *T*=500, the number of Monte Carlo simulations is *M*=300, and the number of bootstrap replications is *B*=300.

Table A2: Finite-sample properties of threshold estimate γ

	Average correlation in cross section $\left(E\left(R_1^{l_1 l_2}\right) = 0.25, E\left(R_2^{l_1 l_2}\right) = 0.5\right)$			
	$s_t = 0.5 s_{t-1} + e_t$		$s_t = 0.9 s_{t-1} + e_t$	
N	Bias	RMSE	Bias	RMSE
3	0.001	0.165	-0.006	0.298
10	-0.006	0.042	-0.013	0.090
50	-0.003	0.018	-0.009	0.040
100	-0.002	0.025	-0.010	0.086
	Average correlation in cross section $\left(E\left(R_1^{l_1 l_2}\right) = 0.25, E\left(R_2^{l_1 l_2}\right) = 0.75\right)$			
	$s_t = 0.5 s_{t-1} + e_t$		$s_t = 0.9 s_{t-1} + e_t$	
N	Bias	RMSE	Bias	RMSE
3	-0.002	0.017	-0.007	0.031
10	-0.002	0.005	-0.005	0.009
50	-0.002	0.003	-0.004	0.007
100	-0.002	0.003	-0.004	0.008

Notes: Monte Carlo simulations for the TCC model. The sample size is *T*=2000 and the number of Monte Carlo simulations is *M*=2500. The standard deviation is 0.1 in both scenarios and in both regimes.

Table A3a: 4-regime TCC correlations for equities and bonds
Regime 1: $VIX_{t-1} < 21.378 \wedge US\ short - rate_{t-1} < 2.850$

	ASX 200	FTSE 100	HANG SENG	CAC 40	S&P 500	DAX 30	NIKKEI 225	GER 10Y	FR 10Y	IT 10Y	JP 10Y	UK 10Y	US 10Y
ASX 200	1												
FTSE 100	0.578	1											
HANG SENG	0.302	0.419	1										
CAC 40	0.609	0.862	0.439	1									
S&P 500	0.515	0.709	0.451	0.755	1								
DAX 30	0.529	0.790	0.430	0.877	0.737	1							
NIKKEI 225	0.415	0.584	0.443	0.598	0.599	0.567	1						
GER 10Y	0.059	-0.089	0.061	-0.155	-0.004	-0.221	-0.145	1					
FR 10Y	0.060	-0.093	0.071	-0.154	-0.004	-0.224	-0.144	0.997	1				
IT 10Y	0.065	-0.079	0.093	-0.126	0.006	-0.194	-0.114	0.964	0.972	1			
JP 10Y	-0.016	-0.160	0.066	-0.201	-0.040	-0.224	-0.296	0.503	0.498	0.462	1		
UK 10Y	0.035	-0.155	0.110	-0.130	-0.034	-0.177	-0.172	0.805	0.804	0.776	0.445	1	
US 10Y	-0.086	-0.243	-0.131	-0.257	-0.228	-0.299	-0.237	0.386	0.364	0.293	0.381	0.455	1

Table A3b: 4-regime TCC correlations for equities and bonds
Regime 2: $VIX_{t-1} < 21.378 \wedge US\ short - rate_{t-1} \geq 2.850$

	ASX 200	FTSE 100	HANG SENG	CAC 40	S&P 500	DAX 30	NIKKEI 225	GER 10Y	FR 10Y	IT 10Y	JP 10Y	UK 10Y	US 10Y
ASX 200	1												
FTSE 100	0.361	1											
HANG SENG	0.403	0.404	1										
CAC 40	0.279	0.635	0.357	1									
S&P 500	0.301	0.457	0.344	0.428	1								
DAX 30	0.382	0.595	0.403	0.661	0.445	1							
NIKKEI 225	0.369	0.240	0.215	0.231	0.240	0.207	1						
GER 10Y	0.064	-0.066	0.097	-0.118	0.051	-0.066	-0.079	1					
FR 10Y	0.014	-0.039	0.089	-0.018	0.065	-0.063	-0.113	0.898	1				
IT 10Y	0.123	0.046	0.180	0.071	0.110	0.052	-0.006	0.641	0.684	1			
JP 10Y	-0.012	-0.159	-0.009	-0.215	-0.043	-0.182	-0.239	0.402	0.380	0.210	1		
UK 10Y	0.084	0.021	0.152	0.032	0.127	0.012	0.001	0.554	0.571	0.487	0.230	1	
US 10Y	0.029	0.035	0.082	0.046	0.265	0.060	-0.001	0.282	0.291	0.295	0.139	0.403	1

Table A3c: 4-regime TCC correlations for equities and bonds
Regime 3: $VIX_{t-1} \geq 21.378 \wedge US\ short - rate_{t-1} < 2.850$

	ASX 200	FTSE 100	HANG SENG	CAC 40	S&P 500	DAX 30	NIKKEI 225	GER 10Y	FR 10Y	IT 10Y	JP 10Y	UK 10Y	US 10Y
ASX 200	1												
FTSE 100	0.680	1											
HANG SENG	0.716	0.640	1										
CAC 40	0.715	0.918	0.665	1									
S&P 500	0.696	0.769	0.673	0.815	1								
DAX 30	0.723	0.867	0.698	0.928	0.820	1							
NIKKEI 225	0.681	0.556	0.649	0.573	0.532	0.597	1						
GER 10Y	-0.100	-0.097	-0.072	-0.143	-0.109	-0.137	-0.038	1					
FR 10Y	-0.040	-0.028	0.035	-0.072	-0.037	-0.071	0.022	0.974	1				
IT 10Y	0.028	0.045	0.075	0.006	0.025	-0.008	0.087	0.906	0.943	1			
JP 10Y	-0.260	-0.230	-0.219	-0.242	-0.178	-0.271	-0.331	0.397	0.347	0.274	1		
UK 10Y	-0.058	-0.152	-0.037	-0.107	-0.092	-0.119	-0.101	0.670	0.635	0.606	0.258	1	
US 10Y	-0.323	-0.375	-0.349	-0.430	-0.417	-0.425	-0.308	0.443	0.378	0.297	0.443	0.350	1

Table A3d: 4-regime TCC correlations for equities and bonds
Regime 4: $VIX_{t-1} \geq 21.378 \wedge US\ short - rate_{t-1} \geq 2.850$

	ASX 200	FTSE 100	HANG SENG	CAC 40	S&P 500	DAX 30	NIKKEI 225	GER 10Y	FR 10Y	IT 10Y	JP 10Y	UK 10Y	US 10Y
ASX 200	1												
FTSE 100	0.490	1											
HANG SENG	0.539	0.561	1										
CAC 40	0.499	0.810	0.546	1									
S&P 500	0.547	0.669	0.521	0.741	1								
DAX 30	0.508	0.686	0.559	0.816	0.652	1							
NIKKEI 225	0.462	0.387	0.508	0.457	0.381	0.412	1						
GER 10Y	-0.032	-0.148	-0.056	-0.176	-0.059	-0.152	-0.048	1					
FR 10Y	-0.013	-0.140	-0.050	-0.151	-0.011	-0.138	-0.032	0.946	1				
IT 10Y	0.000	-0.113	-0.058	-0.144	-0.023	-0.118	-0.006	0.965	0.944	1			
JP 10Y	-0.070	-0.104	0.037	-0.138	-0.143	-0.100	-0.093	0.276	0.234	0.234	1		
UK 10Y	0.009	-0.083	0.048	-0.007	0.069	0.005	-0.032	0.658	0.672	0.660	0.175	1	
US 10Y	-0.096	-0.127	-0.176	-0.133	-0.081	-0.146	-0.167	0.295	0.283	0.274	0.101	0.409	1

Table A4: Testing for large TCC (equities and bonds jointly)

	Wald statistic (p-value)
$H_0: R_1^s = R_2^s \wedge R_3^s = R_4^s$	259.5 (0.000)***
$H_0: R_1^b = R_3^b \wedge R_2^b = R_4^b$	224.5 (0.000)***
$H_0: R_1^s = R_2^s \wedge R_3^s = R_4^s \wedge R_1^b = R_3^b \wedge R_2^b = R_4^b$	503.5 (0.000)***

Notes: Tests a restricted two-threshold TCC vs. an unrestricted two-threshold TCC. The first null hypothesis tests for the significance of $US\ short - rate_{t-1}$ as a state variable (additional to VIX_{t-1}) for the correlation matrix of the equities. The number of restrictions is 42. The second null tests for the significance of VIX_{t-1} as a state variable (additional to $US\ short - rate_{t-1}$) for the correlation matrix of the bonds. The number of restrictions is 30. The third null hypothesis tests jointly for the aforementioned two hypotheses. ***/**/* indicate significance at the 1%/5%/10% level.

Table A5: Data Overview

Symbol	Name
ASX 200	S&P/ASX 200 - PRICE INDEX (Australia)
FTSE 100	FTSE 100 - PRICE INDEX (United Kingdom)
HANG SENG	HANG SENG - PRICE INDEX (Hong Kong)
CAC 40	CAC 40 - PRICE INDEX (France)
S&P 500	S&P 500 COMPOSITE - PRICE INDEX (United States)
DAX 30	DAX 30 PERFORMANCE - PRICE INDEX (Germany)
NIKKEI 225	NIKKEI 225 STOCK AVERAGE – PRICE INDEX (Japan)
GER 10Y	BD BENCHMARK 10 YEAR DS GOVT. INDEX - TOT RETURN IND (~U$) (Germany)
FR 10Y	FR BENCHMARK 10 YEAR DS GOVT. INDEX - TOT RETURN IND (~U$)(France)
IT 10Y	IT BENCHMARK 10 YEAR DS GOVT. INDEX - TOT RETURN IND (~U$) (Italy)
JP 10Y	JP BENCHMARK 10 YEAR DS GOVT. INDEX - TOT RETURN IND (~U$) (Japan)
UK 10Y	UK BENCHMARK 10 YEAR DS GOVT. INDEX - TOT RETURN IND (~U$) (United Kingdom)
US 10Y	US BENCHMARK 10 YEAR DS GOVT. INDEX - TOT RETURN IND (~U$) (United States)