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Bounds on the number of numerical semigroups of a given genus

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ABSTRACT

Lower and upper bounds are given for the number n_g of numerical semigroups of genus g. The lower bound is the first known lower bound while the upper bound significantly improves the only known bound given by the Catalan numbers. In a previous work the sequence n_g is conjectured to behave asymptotically as the Fibonacci numbers. The lower bound proved in this work is related to the Fibonacci numbers and so the result seems to be in the direction to prove the conjecture. The method used is based on an accurate analysis of the tree of numerical semigroups and of the number of descendants of the descendants of each node depending on the number of descendants of the node itself.

1. Introduction

Let $\mathbb{N}_0$ denote the set of all non-negative integers. A *numerical semigroup* is a subset Λ of $\mathbb{N}_0$ containing 0, closed under summation and with finite complement in $\mathbb{N}_0$. The elements in the complement $\mathbb{N}_0 \setminus \Lambda$ are called the *gaps* of the numerical semigroup and $|\mathbb{N}_0 \setminus \Lambda|$ is its *genus*. The largest gap is the *Frobenius number* of Λ and it is at most two times the genus minus one. If it equals this bound then the numerical semigroup is said to be symmetric.

Some results have been proved related to the number of numerical semigroups of a given Frobenius number [1] and the number of symmetric semigroups of a given Frobenius number (and thus, the number of symmetric semigroups of a given genus) [2,3]. In this work we address the problem of counting the number of numerical semigroups of a given genus.

We denote by n_g the number of numerical semigroups of genus g. It is easy to check that $n_0 = 1$ and $n_1 = 1$. The values up to n_{16} were computed by Nivaldo Medeiros and Shizuo Kakutani, and the values up to n_{50} can be found in [4]. It is proved in [5] that any numerical semigroup can be represented by a unique Dyck path of order given by its genus and thus $n_g \leqslant C_g$ where C_g denotes the Catalan number, $C_g = \frac{1}{g+1}\binom{2g}{g}$. It is conjectured in [4] that the sequence (n_g) asymptotically behaves like the Fibonacci sequence. More precisely, $n_g \geqslant n_{g-1} + n_{g-2}$, for $g \geqslant 2$; $\lim_{g\to\infty} \frac{n_{g-1}+n_{g-2}}{n_g} = 1$; $\lim_{g\to\infty} \frac{n_g}{n_{g-1}} = \phi$, where ϕ is the golden ratio.

Let F_i denote the ith Fibonacci number starting by $F_0 = 0, F_1 = 1$. We prove that

$$2F_g \leqslant n_g \leqslant 1 + 3 \cdot 2^{g-3}.$$

The method used is based on an accurate analysis of the tree of numerical semigroups and of the number of descendants of the descendants of each node depending on the number of descendants of the node itself. From this analysis auxiliary generating trees are constructed and the bounds are deduced. The generating trees we obtain are similar to those in [6–8].

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$$
\begin{aligned}
A_2 &= \{1,3\}\\
A_3 &= \{0,0,2,4\}\\
A_4 &= \{0,0,1,1,3,5\}\\
A_5 &= \{0,0,0,0,1,1,2,2,4,6\}\\
A_6 &= \{0,0,0,0,0,0,1,1,1,1,2,2,3,3,5,7\}\\
A_7 &= \{0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,2,2,2,2,3,3,4,4,6,8\}
\end{aligned}
$$

Fig. 1. First multisets A_g as in Lemma 1.

2. Some results on combinatorics

Lemma 1. *The multisets A_g defined recursively by $A_2 = \{1, 3\}$,*

$$A_g = \{g+1\} \cup \left(\bigcup_{m \in A_{g-1}} \{0, 1, \dots, m-1\} \right) \setminus \{g-2\}$$

for $g > 2$ (see Fig. 1) satisfy, if $g \geqslant 2$,

$$A_g = \left(\overbrace{\{0,0,\dots,0\}}^{2F_{g-2}} \cup \overbrace{\{1,1,\dots,1\}}^{2F_{g-3}} \cup \overbrace{\{2,2,\dots,2\}}^{2F_{g-4}} \cup \cdots \cup \overbrace{\{g-4,g-4\}}^{2F_2} \cup \overbrace{\{g-3,g-3\}}^{2F_1} \right) \cup \{g-1, g+1\}$$

and

$$|A_g| = 2F_g.$$

Proof. Both results can be proved by induction and are a consequence from the fact that, for $i \geqslant 2$, $F_i = 1 + \sum_{j=1}^{i-2} F_j$. This in turn can be proved by induction. Indeed, it is obvious for $i = 2$. If $i > 2$, by the induction hypothesis $F_{i-1} = 1 + \sum_{j=1}^{i-3} F_j$ and hence $F_i = F_{i-1} + F_{i-2} = 1 + \sum_{j=1}^{i-2} F_j$. □

Lemma 2. *The multisets B_g defined recursively by $B_2 = \{1, 3\}$,*

$$B_g = \{0, g+1\} \cup \left(\bigcup_{m \in B_{i-1}} \{1, 2, \dots, m\} \right) \setminus \{g, g-2\}$$

for $g > 2$ (see Fig. 2) satisfy, if $g > 2$,

$$B_g = \{0\} \cup \left(\overbrace{\{1,1,\dots,1\}}^{3\cdot 2^{g-4}} \cup \overbrace{\{2,2,\dots,2\}}^{3\cdot 2^{g-5}} \cup \cdots \cup \overbrace{\{g-3,g-3,g-3\}}^{3\cdot 2^{0}} \right) \cup \{g-2, g-1, g+1\}$$

and

$$|B_g| = 1 + 3 \cdot 2^{g-3}.$$

Proof. Both results can be proved by induction and are a consequence from the fact that, for $i \geqslant 0$, $2^i = 1 + \sum_{j=0}^{i-1} 2^j$. This in turn can be proved by induction. □

3. Taking out generators from a semigroup

Every numerical semigroup can be generated by a finite set of elements and a minimal set of generators is unique (see for instance [2]). Given a numerical semigroup Λ of genus g and Frobenius number f, $\Lambda \cup \{f\}$ is a numerical semigroup and its genus is $g-1$. So, any numerical semigroup of genus g can be obtained from a numerical semigroup of genus $g-1$ by removing one element larger than its Frobenius number. It is easy to check that when removing such an element from a numerical semigroup, the set obtained is a numerical semigroup if and only if the removed element belongs to the set of minimal generators. This gives a recursive procedure to obtain all numerical semigroups of genus g from all numerical

$$B_2 = \{1, 3\}$$
$$B_3 = \{0, 1, 2, 4\}$$
$$B_4 = \{0, 1, 1, 1, 2, 3, 5\}$$
$$B_5 = \{0, 1, 1, 1, 1, 1, 1, 2, 2, 2, 3, 4, 6\}$$
$$B_6 = \{0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 3, 3, 3, 4, 5, 7\}$$
$$B_7 = \{0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,\\ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 4, 4, 4, 5, 6, 8\}$$

Fig. 2. First multisets B_g as in Lemma 2.

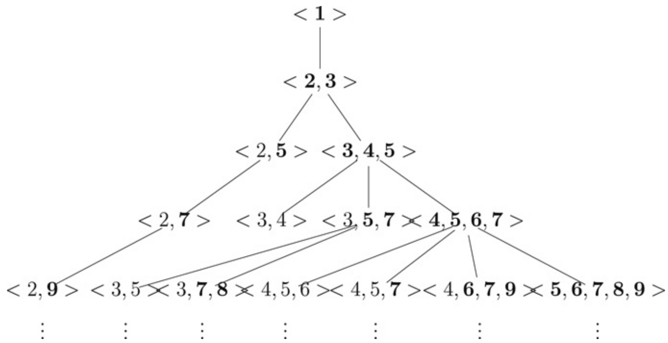

Fig. 3. Recursive construction of numerical semigroups of genus g from numerical semigroups of genus $g-1$. Generators larger than the conductor are written in bold face.

semigroups of genus $g-1$ by taking out, one by one, each generator that is larger than the Frobenius number for each numerical semigroup.

We can think of a tree whose root corresponds to the numerical semigroup $\mathbb{N}_0$, each numerical semigroup of genus g is a node at distance g from the root, and the children of a numerical semigroup are the numerical semigroups obtained when removing one by one each of its minimal generators which are larger than its Frobenius number. This construction was already considered in [9,3,10]. We depicted this tree in Fig. 3. We wrote $\langle \lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_n} \rangle$ to denote the numerical semigroup generated by $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_n}$. We used boldface letters for the minimal generators that are larger than the Frobenius number.

We say that a numerical semigroup is *ordinary* if it is equal to $\{0\} \cup \{i \in \mathbb{N}_0 : i \geqslant c\}$ for some non-negative integer c. In the next lemma we prove that if a node corresponding to a non-ordinary numerical semigroup in the semigroup tree has k descendants, then its descendants have at least $0, \ldots, k-1$ and at most $1, \ldots, k$ descendants, respectively.

Lemma 3. *Let Λ be a non-ordinary numerical semigroup. Suppose that $\{\lambda_{i_1} < \lambda_{i_2} < \cdots < \lambda_{i_k}\}$ are the minimal generators of Λ which are larger than the Frobenius number. Then the number of minimal generators of the numerical semigroup $\Lambda \setminus \{\lambda_{i_j}\}$ which are larger than its Frobenius number is*

- *at least $k-j$,*
- *at most $k-j+1$.*

Proof. It is obvious that the number of minimal generators which are larger than the Frobenius number is at least $k-j$ because all elements in $\Lambda \setminus \{\lambda_{i_j}\}$ which are minimal generators in Λ are also minimal generators in $\Lambda \setminus \{\lambda_{i_j}\}$ and the new Frobenius number is λ_{i_j}.

The elements in $\Lambda \setminus \{\lambda_{i_j}\}$ which are not minimal generators in Λ and become minimal generators in $\Lambda \setminus \{\lambda_{i_j}\}$ must be of the form $\lambda_{i_j} + \lambda_r$ for some $\lambda_r \in \Lambda$. Let λ_1 be the smallest non-zero element of Λ. If $\lambda_r > \lambda_1$ then $\lambda_{i_j} + \lambda_r - \lambda_1 > \lambda_{i_j}$ and hence $\lambda_{i_j} + \lambda_r = \lambda_1 + \lambda_s$ for some $\lambda_s \in \Lambda \setminus \{\lambda_{i_j}\}$, and $\lambda_{i_j} + \lambda_r$ is not a minimal generator of $\Lambda \setminus \{\lambda_{i_j}\}$. So, the only element that is not a minimal generator of Λ and that may be a minimal generator of $\Lambda \setminus \{\lambda_{i_j}\}$ is $\lambda_{i_j} + \lambda_1$. $\square$

Lemma 4. *The ordinary semigroup $\Lambda = \{0, g+1, g+2, \ldots\}$ has minimal set of generators $\{g+1, g+2, \ldots, 2g+1\}$ and*

(1) *$\Lambda \setminus \{g+1\}$ has $g+2$ minimal generators larger than its Frobenius number,*
(2) *$\Lambda \setminus \{g+2\}$ has g minimal generators larger than its Frobenius number,*
(3) *$\Lambda \setminus \{g+r\}$, with $r > 2$, has $g-r+1$ minimal generators larger than its Frobenius number.*

Proof. The first item is obvious.

By the same argument as in the proof of Lemma 3, the only element that is not a minimal generator of Λ and that may be a minimal generator of $\Lambda \setminus \{\lambda_{i_j}\}$ is $\lambda_{i_j} + \lambda_1$. It is easy to prove that if $r = 2$ then $\lambda_{i_j} + \lambda_1$ is a minimal generator while if $r > 2$, it is not. $\square$

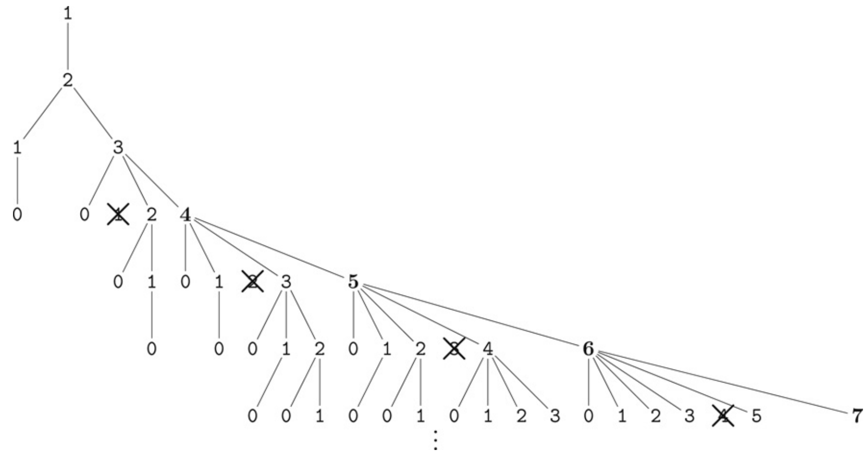

Fig. 4. Tree A. It is a subtree of the tree of numerical semigroups.

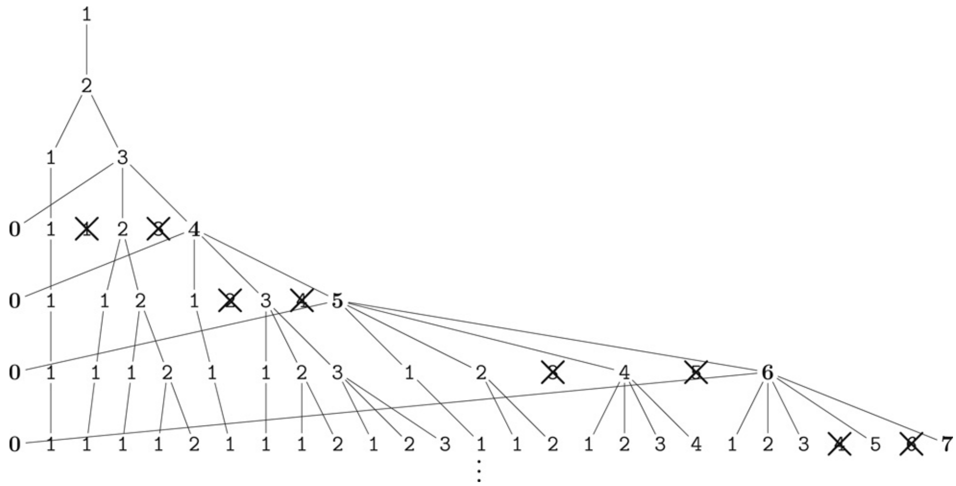

Fig. 5. Tree B. It is a *supertree* of the tree of numerical semigroups.

Theorem 5. *The number n_g of numerical semigroups of genus g satisfies $2F_g \leqslant n_g$ for all $g \geqslant 2$ and $2F_g \leqslant n_g \leqslant 1 + 3 \cdot 2^{g-3}$ for all $g \geqslant 3$.*

Proof. Consider the tree A represented in Fig. 4. It is recursively defined as follows: Its root is labeled as 1 and it has a single descendant which is labeled as 2. This descendant in turn has two descendants labeled as 1 and 3. At each level g, the number of descendants of a node is equal to its label. From level $g = 3$ on, if the label of a node is k then the labels of its descendants are $0, \ldots, k-1$ except for the node with label $k = g+1$, whose descendants have labels $0, \ldots, k-3, k-1, k+1$. It turns out that, for $g \geqslant 2$, the labels of nodes at distance g from the root are exactly those in A_g, where A_g is defined as in Lemma 1.

On the other hand consider the tree B represented in Fig. 5. It is recursively defined as follows: Its root is labeled as 1 and it has a single descendant which is labeled as 2. This descendant in turn has two descendants labeled as 1 and 3. At each level g, the number of descendants of a node is equal to its label. From level $g = 3$ on, if the label of a node is k then the labels of its descendants are $1, \ldots, k$ except for the node with label $k = g+1$, whose descendants have labels $0, \ldots, k-3, k-1, k+1$. In this case, for $g \geqslant 2$, the labels of nodes at distance g from the root are exactly those in B_g, where B_g is defined as in Lemma 2.

It is now easy to check that, by Lemmas 3 and 4, the semigroup tree (Fig. 3) contains A as a subtree and is contained in B. Thus, $|A_g| \leqslant n_g \leqslant |B_g|$ for $g \geqslant 2$. By Lemmas 1 and 2 it follows that $2F_g \leqslant n_g$ for all $g \geqslant 2$ and $2F_g \leqslant n_g \leqslant 1 + 3 \cdot 2^{g-3}$ for all $g \geqslant 3$. $\square$

In Table 1 one can compare for g up to 30 the actual values of n_g with the bounds given in Theorem 5 and also with the bound given by the Catalan numbers proved in [5]. The values n_g are from [4].

Table 1
Values of $2F_g$, n_g, $1+3\cdot 2^{g-3}$, and C_g for g up to 30.

g	$2F_g$	n_g	$1+3\cdot 2^{g-3}$	C_g
0		1		1
1		1		1
2	2	2		2
3	4	4	4	5
4	6	7	7	14
5	10	12	13	42
6	16	23	25	132
7	26	39	49	429
8	42	67	97	1430
9	68	118	193	4862
10	110	204	385	16796
11	178	343	769	58786
12	288	592	1537	208012
13	466	1001	3073	742900
14	754	1693	6145	2674440
15	1220	2857	12289	9694845
16	1974	4806	24577	35357670
17	3194	8045	49153	129644790
18	5168	13467	98305	477638700
19	8362	22464	196609	1767263190
20	13530	37396	393217	6564120420
21	21892	62194	786433	24466267020
22	35422	103246	1572865	91482563640
23	57314	170963	3145729	343059613650
24	92736	282828	6291457	1289904147324
25	150050	467224	12582913	4861946401452
26	242786	770832	25165825	18367353072152
27	392836	1270267	50331649	69533550916004
28	635622	2091030	100663297	263747951750360
29	1028458	3437839	201326593	1002242216651368
30	1664080	5646773	402653185	3814986502092304

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