The k-adjacency dimension of graphs

Alejandro Estrada-Moreno ⋆ ⋆⋆

Department of Computer Engineering and Mathematics, Universitat Rovira i Virgili
Tarragona, Spain
alejandro.estrada@urv.cat

1 Introduction

A generator of a metric space (X, d) is a set $S \subset X$ of points in the space with the property that every point of X is uniquely determined by the distances from the elements of S. Given a simple and connected graph $G = (V, E)$, we consider the function $d_G : V \times V \to \mathbb{N} \cup \{0\}$, where $d_G(x, y)$ is the length of a shortest path between u and v and $\mathbb{N}$ is the set of positive integers. Then (V, d_G) is a metric space. A vertex $v \in V$ is said to *distinguish* two vertices x and y if $d_G(v, x) \neq d_G(v, y)$. A set $S \subset V$ is said to be a *metric generator* for G if any pair of vertices of G is distinguished by some element of S. A minimum cardinality metric generator is called a *metric basis*, and its cardinality the *metric dimension* of G, denoted by $\dim(G)$.

Uniquely determining the localization of an intruder in a network was the problem that motivated Slater in [7] to use the notion of metric dimension of a graph, where the metric generators were called *locating sets*. The concept of metric dimension of a graph was also introduced by Harary and Melter in [5], where metric generators were called *resolving sets*.

The concept of adjacency generator[3] was introduced by Jannesari and Omoomi in [6] as a tool to study the metric dimension of lexicographic product graphs. This concept has been studied further by Fernau and Rodríguez-Velázquez in [3,4] where they showed that the (local) metric dimension of the corona product of a graph of order n and some non-trivial graph H equals n times the (local) adjacency metric dimension of H. As a consequence of this strong relation they showed that the problem of computing the adjacency metric dimension is NP-hard. A set $S \subset V$ of vertices in a graph $G = (V, E)$ is said to be an *adjacency generator* for G if for every two vertices $x, y \in V - S$ there exists $s \in S$ such that s is adjacent to exactly one of x and y. A minimum cardinality adjacency generator is called an *adjacency basis* of G, and

⋆ PhD advisors: I. G. Yero (UCA), J. A. Rodríguez-Velázquez (URV)
⋆⋆ Other collaborators: Y. Ramírez Cruz (URV)
[3] Adjacency generators were called adjacency resolving sets in [6].

its cardinality the *adjacency dimension* of G, denoted by $\mathrm{adim}(G)$. Since any adjacency basis is a metric generator, $\dim(G) \leq \mathrm{adim}(G)$. Besides, for any connected graph G of diameter at most two, $\mathrm{adim}(G) = \dim(G)$. As pointed out in [3,4], any adjacency generator of a graph $G = (V, E)$ is also a metric generator in a suitably chosen metric space. Given the distance function $d_{G,2} : V \times V \to \mathbb{N} \cup \{0\}$, where

$$d_{G,2}(x, y) = \min\{d_G(x, y), 2\}.$$

Note that the metric dimension of $(V, d_{G,2})$ is equal to $\mathrm{adim}(G)$.

We introduced the concept of k-adjacency generator in [1,2]. We say that a set $S \subseteq V(G)$ is a *k-adjacency generator* for G if for every two vertices $x, y \in V(G)$, there exist at least k vertices $w_1, w_2, ..., w_k \in S$ such that

$$d_{G,2}(x, w_i) \neq d_{G,2}(y, w_i), \text{ for every } i \in \{1, ..., k\},$$

A minimum k-adjacency generator is called a *k-adjacency basis* of G and its cardinality, the *k-adjacency dimension* of G, is denoted by $\mathrm{adim}_k(G)$.

The general objective of our work is to study the problem of finding the k-adjacency dimension of a graph.

2 Some results on the k-adjacency dimension of graphs

In this section we present some results that allow us to compute the largest integer k' such that there exists a k'-adjacency basis, as well as, the k-adjacency dimension of several families of graphs. We also give some tight bounds on the k-adjacency dimension of a graph.

A graph G is said to be a *k-adjacency dimensional graph* if k is the largest integer such that there exists a k-adjacency basis. Given a connected graph G and two different vertices $x, y \in V(G)$, we denote by $\mathcal{C}_G(x, y)$ the set of vertices that distinguish the pair x, y with regard to the metric (1), *i.e.*, $\mathcal{C}_G(x, y) = \{z \in V(G) : \ d_{G,2}(z, y) \neq d_{G,2}(z, y)\}$. We define the following global parameter

$$\mathcal{C}(G) = \min_{x,y \in V(G)} \{|\mathcal{C}_G(x, y)|\}.$$

Theorem 1. [1] *A graph G is k-adjacency dimensional if and only if $k = \mathcal{C}(G)$. Moreover, $\mathcal{C}(G)$ can be computed in $O(n^3)$ time.*

The problem of computing the value k for which a given graph is k-adjacency dimensional is polynomial as we showed in Theorem 1. It was shown in [3,4] that the problem of finding the adjacency dimension of a graph is NP-complete. The NP-completeness of the problem of finding the k-adjacency dimension of a graph has not studied for $k > 1$. However, it is interesting to study this invariant for particular classes of graphs, specially for value of k greater than one.

We now present some results that allow us to compute the k-adjacency dimension of several families of graphs. We also give some tight bounds on the k-adjacency dimension of a graph.

Theorem 2 (Monotony of the k-adjacency dimension). [1] *Let G be a k-adjacency dimensional graph and let k_1, k_2 be two integers. If $1 \leq k_1 < k_2 \leq k$, then* $\operatorname{adim}_{k_1}(G) < \operatorname{adim}_{k_2}(G)$.

Corollary 1. [1] *Let G be a k-adjacency dimensional graph of order n.*
(i) (i) (i)
1. *For any $r \in \{2, ..., k\}$,* $\operatorname{adim}_r(G) \geq \operatorname{adim}_{r-1}(G) + 1$.
2. *For any $r \in \{1, ..., k\}$,* $\operatorname{adim}_r(G) \geq \operatorname{adim}(G) + (r - 1)$.
3. *For any $r \in \{1, ..., k-1\}$,* $\operatorname{adim}_r(G) < n$.

Theorem 3. [1] *Let G be a k-adjacency dimensional graph of order $n \geq 2$. Then* $\operatorname{adim}_k(G) = n$ *if and only if* $\mathcal{C}_k(G) = V(G)$.

Since $\mathcal{C}_G(x, y) = \mathcal{C}_{\overline{G}}(x, y)$ for all $x, y \in V(G)$, we deduce the following result, which was previously observed for $k = 1$ by Jannesari and Omoomi in [6].

Observation 1 [1] *For any nontrivial graph G and $k \in \{1, 2, \ldots, \mathcal{C}(G)\}$,*

$$\operatorname{adim}_k(G) = \operatorname{adim}_k(\overline{G}).$$

Proposition 1. [1] *If G is a graph of order $n \geq 2$, then* $\operatorname{adim}_k(G) = k$ *if and only if $k \in \{1, 2\}$ and $G \in \{P_2, P_3, \overline{P}_2, \overline{P}_3\}$*

Theorem 4. [1] *For any graph G of order $n \geq 7$ and $k \in \{1, \ldots, \mathcal{C}(G)\}$,* $\operatorname{adim}_k(G) \geq k + 2$.

Observation 2 [1] *A graph G of order greater than or equal to four satisfies* $\operatorname{adim}_3(G) = 4$ *if and only if $G \in \{P_4, C_5\}$.*

Observation 3 [1] *A graph G of order $n \geq 5$ satisfies that* $\operatorname{adim}_4(G) = 5$ *if and only if $G \cong C_5$.*

The join $G+H$ of two vertex-disjoint graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph with vertex set $V(G + H) = V_1 \cup V_2$ and edge set $E(G + H) = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$.

Theorem 5. [1] *For any nontrivial graph H, the following assertions are equivalent:*
(i) (i) (i)
1. *There exists a k-adjacency basis A of H such that $|A - N_H(y)| \geq k$, for all $y \in V(H)$.*
2. $\operatorname{adim}_k(K_1 + H) = \operatorname{adim}_k(H)$.

Corollary 2. [1] *For any graph H of diameter $D(H) \geq 6$ and $k \in \{1, \ldots, \mathcal{C}(K_1 + H)\}$, $\mathrm{adim}_k(K_1 + H) = \mathrm{adim}_k(H)$.*

Corollary 3. [1] *Let H be a graph of girth $\mathtt{g}(H) \geq 5$ and minimum degree $\delta(H) \geq 3$. Then for any $k \in \{1, \ldots, \mathcal{C}(K_1+H)\}$, $\mathrm{adim}_k(K_1+H) = \mathrm{adim}_k(H)$.*

Theorem 6. [1] *Let G and H be two nontrivial graphs. Then the following assertions are equivalent:*

(i)

1. *There exists a k-adjacency basis A_G of G and a k-adjacency basis A_H of H such that $|(A_G - N_G(x)) \cup (A_H - N_H(y))| \geq k$, for all $x \in V(G)$ and $y \in V(H)$.*
2. $\mathrm{adim}_k(G + H) = \mathrm{adim}_k(G) + \mathrm{adim}_k(H)$.

Acknowledgement. Martí-Franquès Research grants Programme of Universitat Rovira i Virgili.

References

[1] A. Estrada-Moreno, Y. Ramírez-Cruz, J. A. Rodríguez-Velázquez, On the adjacency dimension of graphs, arXiv:1501.04647 [math.CO].

[2] A. Estrada-Moreno, I. G. Yero, J. A. Rodríguez-Velázquez, The k-metric dimension of the lexicographic product of graphs, Submitted.

[3] H. Fernau, J. A. Rodríguez-Velázquez, On the (adjacency) metric dimension of corona and strong product graphs and their local variants: combinatorial and computational results, arXiv:1309.2275 [math.CO].

[4] H. Fernau, J. A. Rodríguez-Velázquez, Notions of metric dimension of corona products: combinatorial and computational results, in: Computer science—theory and applications, vol. 8476 of Lecture Notes in Comput. Sci., Springer, Cham, 2014, pp. 153–166.

[5] F. Harary, R. A. Melter, On the metric dimension of a graph, Ars Combinatoria 2 (1976) 191–195.

[6] M. Jannesari, B. Omoomi, The metric dimension of the lexicographic product of graphs, Discrete Mathematics 312 (22) (2012) 3349–3356.

[7] P. J. Slater, Leaves of trees, Congressus Numerantium 14 (1975) 549–559.