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NORMAL FORMS FOR SINGULARITIES OF ONE DIMENSIONAL HOLOMORPHIC VECTOR FIELDS

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Abstract. We study the normal form of the ordinary differential equation $\dot{z} = f(z)$, $z \in \mathbb{C}$, in a neighbourhood of a point $p \in \mathbb{C}$, where f is a one-dimensional holomorphic function in a punctured neighbourhood of p. Our results include all cases except when p is an essential singularity. We treat all the other situations, namely when p is a regular point, a pole or a zero of order n. Our approach is based on a formula that uses the flow associated with the differential equation to search for the change of variables that gives the normal form.

1. Introduction and Main Result

This note provides a proof for the following result:

Theorem 1.1. *Let $f(z)$ be a one-dimensional holomorphic function in a punctured neighbourhood $\mathcal{U} \subset \mathbb{C}$ of a point p. Consider the ordinary differential equation*

$$\frac{dz}{dt} = f(z), \quad z \in \mathcal{U}, \quad t \in \mathbb{C}. \tag{1.1}$$

Then, in a neighbourhood of p, this equation is conformally conjugate, in a neighbourhood of the origin, to

(a) $\dot{z} = 1$, *if* $f(p) \neq 0$,
(b) $\dot{z} = f'(p)z$, *if p is a zero of f of order 1 (i.e. $f'(p) \neq 0$),*
(c) $\dot{z} = z^n/(1 + cz^{n-1})$, *where $c = \mathrm{Res}(1/f, p)$, if p is a zero of f of order $n > 1$,*
(d) $\dot{z} = 1/z^n$, *if p is a pole of order n.*

Statements (a), (b) and (c) are well-known, see for instance [1]. To our knowledge statement (d), as it is presented here, is new. Just topological conjugacy between equation (1.1) and $\dot{z} = 1/z^n$ has been proved in [4, 5, 6, 8]. In [3] a similar result is also given. Our approach is based in the one dimensional version of a nice formula, as far as we know, firstly introduced in the unpublished thesis of Pazzi [7] and recently used in [2, 9, 10]. This formula allows to look for the conjugacy between the original differential equation (1.1) and its normal form by using the flow of

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the primer. In Lemma 2.1, for the sake of completeness, we prove it in the one dimensional framework. Notice that the case of f having an essential singularity at p is not covered by the above theorem. In a forthcoming paper we will consider this situation.

Also we observe that by using Theorem 1.1 it is easy to describe the phase portrait, for $t \in \mathbb{R}$, of meromorphic equations in a small punctured neighbourhood of their singularities and poles. Furthermore, except for the simple zeroes of f, it can be shown that the index at the point fully characterizes the dynamics near the singularity.

2. Proof of main Theorem

Without loss of generality, we assume that the point p in Theorem 1.1 is moved to the origin. The solution of (1.1) passing through $z \in \mathcal{U}$ at $t = 0$ is denoted by $\varphi_f(t, z)$.

Let f and g be analytic functions in a punctured neighbourhood of 0. It is said that the differential equations $\dot{z} = f(z)$ and $\dot{z} = g(z)$ are *conformally conjugated near the origin* if there exists a conformal map $\Phi : \mathcal{V} \to \mathcal{W}$, $\mathcal{V}$ and $\mathcal{W}$ being two open neighbourhoods of 0, such that $\Phi(0) = 0$ and

$$\Phi(\varphi_f(t, z)) = \varphi_g(t, \Phi(z)), \quad \text{for all} \quad z \in \mathcal{V} \setminus \{0\}, \tag{2.1}$$

and all t for which the above expressions are well defined and the corresponding points are in $\mathcal{V}$ and $\mathcal{W}$.

Lemma 2.1 ([2, 10])**.** *Let $z, w \in \mathbb{C}$ be given implicitly by the equation $z = \varphi_f(h(w), w)$. Then if h is a holomorphic function, the expression of* (1.1) *in the w-variable is*

$$\dot{w} = \frac{f(w)}{1 + f(w)h'(w)}\,.$$

Proof. By taking derivative with respect to t of $z = \varphi_f(h(w), w)$, we obtain

$$\dot{z} = \big(h'(w)\frac{\partial \varphi_f}{\partial t}(h(w), w) + \frac{\partial \varphi_f}{\partial z}(h(w), w)\big)\dot{w}\ .$$

Thus

$$\dot{z} = \big(h'(w)f(z) + \frac{\partial \varphi_f}{\partial z}(h(w), w)\big)\dot{w}\ . \tag{2.2}$$

To compute $\partial\varphi_f(t, z)/\partial z$ we note that it is the solution of the Cauchy problem given by the variational equation $\dot{u} = f'(\varphi_f(t, z))u$ with initial condition $u(0) = 1$. Observe that the function $f(\varphi_f(t, z))/f(z)$ is the solution of this Cauchy problem. Hence by the uniqueness of solutions we have

$$\frac{\partial \varphi_f}{\partial z}(t, z) = \frac{f(\varphi_f(t, z))}{f(z)}.$$

Evaluating the above expression at $(t, z) = (h(w), w)$ we obtain

$$\frac{\partial \varphi_f}{\partial z}(h(w), w) = \frac{f(z)}{f(w)}.$$

Consequently (2.2) becomes

$$f(z) = \big(h'(w)f(z) + \frac{f(z)}{f(w)}\big)\dot{w},$$

as desired. $\square$

This lemma points out that a suitable choice for the function h may reduce equation (1.1) to its normal form, as it is stated in Theorem 1.1. Before proving the theorem we show, in two steps, a useful condition for two differential equations to be conformally conjugated.

Lemma 2.2. *The equation $z = \varphi_f(t, w)$, associated to the flow of* (1.1) *is, in a punctured neighbourhood of the origin, implicitly given by the equation*

$$\exp\Big(\int_w^z \frac{1}{cf(s)}\,ds\Big) = \exp\big(\frac{t}{c}\big), \quad \text{when } c := \operatorname{Res}(1/f, 0) \neq 0, \tag{2.3}$$

or by

$$\int_w^z \frac{1}{f(s)}\,ds = t, \quad \text{when } \operatorname{Res}(1/f, 0) = 0. \tag{2.4}$$

Observe that the case $\operatorname{Res}(1/f, 0) = 0$ *also includes the situation in which $1/f$ is holomorphic at $z = 0$.*

Proof. To show the first statement, where $c \neq 0$, we claim that, on one hand, both sides of equation (2.3) are well defined in a punctured neighbourhood of the origin, and, on the other hand, the solution, $z = \eta(t, w)$ (such that $\eta(0, w) = w$), of the equation (2.3) is indeed the flow associated to equation (1.1). To see the claim we integrate $\dot z = f(z)$ in the "formal" way to obtain

$$\int_w^z \frac{1}{f(s)}\,ds = t. \tag{2.5}$$

The left-hand side of the above equation is not well defined since its value depends, in general, of the path from w to z in $\mathbb{C}$. In other words, the integral is only defined in a neighbourhood of $z = 0$ slit along the negative real line. However, from the Residue Theorem, we know that such difference is either zero, if the two paths do not cross the negative real line, or it is $\pm 2\pi i$ times the residue of $1/f$ at $z = 0$, if they enclose $z = 0$. Thus, dividing by $c \neq 0$ and taking exponential, we uniquely determine the value of the left hand side of equation (2.3) and, moreover, it has meaning in a whole punctured neighbourhood of $z = 0$ (including the negative real line).

To complete the proof of the claim, we need to show that the solution, $z = \eta(t, w)$ (such that $\eta(0, w) = w \neq 0$), of the equation (2.3) is the flow associated to equation (1.1). To do so we use the Implicit Function Theorem. If we denote by $F(z, w, t) = 0$ the equation (2.3) we note that

$$\frac{\partial}{\partial z}F(z, w, t) = \exp\Big(\int_w^z \frac{1}{cf(s)}\,ds\Big)\frac{1}{cf(z)} \neq 0.$$

Hence we know of the existence of $z = \eta(t, w)$ with $\eta(0, w) = w$. Now, if we compute $\frac{\partial}{\partial t}\eta(t, w)$, we easily get

$$\exp\Big(\int_w^{\eta(t,w)} \frac{1}{cf(s)}\,ds\Big)\frac{1}{cf(\eta(t, w))}\frac{\partial}{\partial t}\eta(t, w) = \frac{1}{c}\exp(\frac{t}{c}),$$

or, equivalently, $\frac{\partial}{\partial t}\eta(t, w) = f(\eta(t, w))$, as desired.

The latter statement of the lemma, *i.e.* when $\operatorname{Res}(1/f, 0) = 0$, follows from the fact that, under this hypothesis, the left-hand side of equation (2.5) is well defined in a punctured neighbourhood of the origin. $\square$

Proposition 2.3. *Let $f(z)$ and $g(z)$ be holomorphic functions in a punctured neighbourhood of the origin and set $c = \operatorname{Res}(1/f, 0)$. Assume that the function*

$$H(z) := \frac{f(z) - g(z)}{f(z)g(z)} = \frac{1}{g(z)} - \frac{1}{f(z)} \tag{2.6}$$

is a holomorphic map at $z = 0$, and define $h(z) = \int_0^z H(s)\, ds$. Then the transformation $z = \varphi_f(h(w), w)$, which is not necessarily bijective, transforms the equation $\dot z = f(z)$ into the equation $\dot w = g(w)$. Moreover, in a punctured neighbourhood of the origin, the transformation can be implicitly written as

$$\exp\Big(\int_w^z \frac{1}{cg(s)}\, ds\Big) = \exp\Big(\frac{1}{c}\int_0^z \big(\frac{1}{g(s)} - \frac{1}{f(s)}\big)\, ds\Big), \tag{2.7}$$

when $c \neq 0$ or as

$$\int_w^z \frac{1}{g(s)}\, ds = \int_0^z \big(\frac{1}{g(s)} - \frac{1}{f(s)}\big)\, ds, \tag{2.8}$$

when $c = 0$. Moreover, this later equation simplifies to

$$\int_0^w \frac{1}{g(s)}\, ds = \int_0^z \frac{1}{f(s)}\, ds, \tag{2.9}$$

if $1/g(z)$ and $1/f(z)$ are both holomorphic at the origin.

Remark 2.4. Note that by using the above proposition, but interchanging the roles of f and g, it can be shown that the local inverse of the map $z = \varphi_f(h(w), w)$ is given by $w = \varphi_g(-h(z), z)$.

Corollary 2.5. *Under the hypotheses of Proposition 2.3, let $w = W(z)$ be a solution, in a punctured neighbourhood of the origin, of (2.7) if $c \neq 0$ or of (2.8) if $c = 0$, satisfying that $W(z)$ tends to 0 when z tends to 0. If this solution can be extended to a conformal map in a full neighbourhood of the origin, then the ordinary differential equations*

$$\frac{dz}{dt} = f(z), \quad \text{and} \quad \frac{dz}{dt} = g(z),$$

are conformally conjugated near the origin.

Proof of Proposition 2.3. We define $w = W(z)$ as a solution of $z = \varphi_f(h(w), w)$, satisfying $W(0) = 0$, where h is an unknown function. By Lemma 2.1, the suitable condition over h which implies that the expression of $\dot z = f(z)$ in the variable w is $\dot w = g(w)$, reads as

$$h'(w) = H(w) = \frac{f(w) - g(w)}{f(w)g(w)} = \frac{1}{g(w)} - \frac{1}{f(w)}.$$

By hypothesis, the right hand side of the above equation is holomorphic. Hence choosing $h(w) = \int_0^w H(s)\, ds$, the transformation given by $z = \varphi_f(\int_0^w H(s)\, ds, w)$, transforms $\dot z = f(z)$ into $\dot w = g(w)$. When $\operatorname{Res}(1/f, 0) = c \neq 0$, we apply equation (2.3) in Lemma 2.2 to write equation $z = \varphi_f(t, w)$ as

$$\exp\Big(\int_w^z \frac{1}{cf(s)}\, ds\Big) = \exp(\frac{t}{c}).$$

Hence equation $z = \varphi_f(h(w), w)$ writes as

$$\exp\Big(\int_w^z \frac{1}{cf(s)}\, ds\Big) = \exp\big(\frac{h(w)}{c}\big) = \exp\Big(\frac{1}{c}\int_0^w \big(\frac{1}{g(s)} - \frac{1}{f(s)}\big)\, ds\Big).$$

which is equivalent to (2.7) by using that

$$\int_0^w \big(\frac{1}{g(s)}-\frac{1}{f(s)}\big)\,ds=\int_0^z \big(\frac{1}{g(s)}-\frac{1}{f(s)}\big)\,ds+\int_z^w \big(\frac{1}{g(s)}-\frac{1}{f(s)}\big)\,ds.$$

When $c=0$ we can perform similar computations but using (2.4) instead of (2.3) to arrive to the desired result. Finally if both, $1/f(z)$ and $1/g(z)$ are holomorphic at zero, easy manipulations transform equation (2.8) into (2.9). $\square$

It only remains to prove the main theorem.

Proof of Theorem 1.1. We deal first with statements (a) and (d), *i.e.* for f having either a regular point ($n=0$) or a pole of order n at the origin. Hence $n\in\mathbb{N}\cup\{0\}$. In these cases $f(z)=1/(z^nG(z))$ with $G(0)=g_0\neq 0$. We choose $g(z)=1/(g_0z^n)$. Notice that $H(z)=z^n(g_0-G(z))$ is a holomorphic map at $z=0$ and we are under the hypotheses of Proposition 2.3. Furthermore, since $1/f(z)$ and $1/g(z)$ are also holomorphic, again by Proposition 2.3, the transformation $z=\varphi_f(h(w),w)$, writes as (2.9), which is

$$\int_0^w g_0s^n\,ds=\int_0^z s^nG(s)\,ds.$$

Thus

$$w=W(z):=z\sqrt[n+1]{\frac{(n+1)\int_0^z s^nG(s)\,ds}{g_0z^{n+1}}},$$

is a conformal change of variables (notice that $W(0)=0$ and $W'(0)\neq 0$) between $\dot z=1/(z^nG(z))$ and $\dot w=1/(g_0w^n)$. A new change of variables $w\to\alpha w$, for a convenient $\alpha\in\mathbb{C}$, finishes the proof of cases (a) and (d).

We consider now statement (b). Write $f(z)=z/G(z)$, with $G(0)=g_0\neq 0$ and $g(z)=z/g_0$. Thus $H(z)$ is a holomorphic map at $z=0$ and we are again under the hypotheses of Proposition 2.3. Arguing like in the precedent case, but using equation (2.7) instead of (2.9), we get that $z=\varphi_f(h(w),w)$ writes as

$$\exp\Big(\int_w^z \frac{1}{s}\,ds\Big)=\exp\Big(\frac{1}{g_0}\int_0^z \frac{g_0-G(s)}{s}\,ds\Big).$$

It is easy to see that the function

$$w=W(z):=z\exp\Big(\int_0^z \frac{G(s)-g_0}{g_0s}\,ds\Big),$$

is a solution of the above equation. Furthermore it is a conformal map and satisfies $W(0)=0, W'(0)\neq 0$. Hence by Corollary 2.5 it is a conformal conjugacy between $\dot z=z/G(z)$ and $\dot w=w/g_0$, as stated.

Finally we deal with statement (c). In this case $f(z)=z^n/G(z)$ where $n\geq 2$ is an integer number, and $G(0)=g_0\neq 0$. As usually, if $G(z)=g_0+g_mz^m+O(m+1)$, by considering a conformal change of variables of the form $z=z_1(1+az_1^m)$ we get $\dot z_1=z_1^n/(g_0+[(m+1-n)ag_0+g_m]z_1^m+O(m+1))$. Hence, if $1\leq m\leq n-2$, we can choose the parameter a in such a way that $\dot z_1=z_1^n/(g_0+O(m+1))$, and so, it is not restrictive to assume that $f(z)=z^n/(g_0+g_{n-1}z^{n-1}+\tilde G(z))$, $\tilde G(z)/z^n$ being analytic at the origin. We consider $g(z)=z^n/(g_0+g_{n-1}z^{n-1})$, and we proceed as in the previous cases. It is convenient, however, to deal with the cases $\mathrm{Res}(1/f,0)=g_{n-1}\neq 0$ and $g_{n-1}=0$ separately. We start with the case $g_{n-1}\neq 0$.

By using (2.7) we get that $z = \varphi_f(h(w), w)$ writes as

$$\exp\Big(\int_w^z \frac{g_0 + g_{n-1}s^{n-1}}{g_{n-1}s^n}\,ds\Big) = \exp\Big(\int_0^z \frac{-\tilde{G}(s)}{g_{n-1}s^n}\,ds\Big). \tag{2.10}$$

Although the above equation is well defined in a punctured neighbourhood $\mathcal{N}$ of the origin, in order to take logarithms (principal determination), we consider it restricted to $\mathcal{N}$ slit along the negative real line. Thus, it writes as

$$\frac{g_0}{1-n}\Big[\frac{1}{z^{n-1}} - \frac{1}{w^{n-1}}\Big] + g_{n-1}\log\frac{z}{w} + \int_0^z \frac{\tilde{G}(s)}{s^n}\,ds = 0, \tag{2.11}$$

We note that if $g_{n-1} = 0$, using (2.8) instead of (2.7), we would have arrived also to (2.11). So, in what follows, we proceed with the two cases together. By multiplying the above equation by z^{n-1} and introducing the new variable $u = z/w$ we get the equation

$$F(u,z) = \frac{g_0}{1-n}\big[1 - u^{n-1}\big] + g_{n-1}z^{n-1}\log u + z^{n-1}\int_0^z \frac{\tilde{G}(s)}{s^n}\,ds = 0.$$

Note that $F(1,0) = 0$ and $\partial F(1,0)/\partial u \neq 0$. Therefore, by the Implicit Function Theorem, near $(u,z) = (1,0)$, there exists a unique analytic function $u = U(z)$ such that $F(U(z), z) \equiv 0$ and $U(0) = 1$. Hence $w = W(z) := z/U(z)$ is a conformal solution of (2.11) on $\mathcal{N}$ slit along the negative real line, which is indeed a solution of (2.10) on $\mathcal{N}$. Furthermore, since it is also conformal at the origin and $W(0) = 0$, by using Corollary 2.5 we know that it gives a conformal change of variables between the equations $\dot{z} = z^n/(g_0 + g_{n-1}z^{n-1} + \tilde{G}(z))$ and $\dot{w} = w^n/(g_0 + g_{n-1}w^{n-1})$.

By introducing a final change of the form $w \to \alpha w$ we can impose that $g_0 = 1$ and the expression of the normal form given in the theorem follows. It is easy to check that $\mathrm{Res}(1/f, 0) = g_{n-1}$ is an invariant for conformal conjugation, see Remark 2.6 or [1]. Hence c is $\mathrm{Res}(1/f, 0)$ and the proof of statement (c) in Theorem 1.1 follows. □

Remark 2.6. As we have already mentioned, in [1] it is proved that under the hypotheses of Proposition 2.3 a necessary condition for equations $\dot{z} = f(z)$ and $\dot{z} = g(z)$ to be conformally conjugated near the origin is that $\mathrm{Res}(1/f, 0) = \mathrm{Res}(1/g, 0)$. Notice that the condition (2.6) in Proposition 2.3, implies, among other things, that $\mathrm{Res}(1/f, 0) = \mathrm{Res}(1/g, 0)$.

Remark 2.7. Observe that in case that both $\dot{z} = f(z)$ and $\dot{z} = g(z)$ have a pole at the origin (not necessarily of the same order), the map given by (2.9) transforms one equation into the other. However, this transformation is not bijective (so, it is not a change of variables) unless both poles have the same order.

Remark 2.8. In case (c) of Theorem 1.1, it is not difficult to find other simple models for the normal form. For instance, the one given in the statement can be replaced by $\dot{z} = z^n - c^{1/(n-1)}z^{n+1}$ or by $\dot{z} = z^n - cz^{2n-1}$.

Remark 2.9. The main results given in this paper remain true for real analytic vector fields. Only some constants have to be added in the real normal forms.

Concluding Remarks. If we are not interested in a constructive proof of items (a), (b) or (d) of Theorem 1.1 we can check the following assertions, which give a straightforward proof (modulus a final rescalling of the variable w in cases (a) and (d)):

(i) For $n \in \mathbb{N} \cup \{0\}$ the conformal change of variables
$$w = z \sqrt[n+1]{\frac{(n+1)\int_0^z s^n G(s)\,ds}{g_0 z^{n+1}}},$$
transforms $\dot{z} = 1/(z^n G(z))$ with $G(0) = g_0 \neq 0$ into $\dot{w} = 1/(g_0 w^n)$.

(ii) The conformal change of variables
$$w = z \exp\Big(\int_0^z \frac{G(s) - g_0}{g_0 s}\,ds\Big),$$
transforms $\dot{z} = z/G(z)$ with $G(0) = g_0 \neq 0$ into $\dot{w} = w/g_0$.

As it has been shown in the previous section, the change of variables which proves item (c) is not explicit.

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