INDEPENDENT TRANSVERSAL TOTAL DOMINATION VERSUS TOTAL DOMINATION IN TREES

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Abstract

A subset of vertices in a graph G is a total dominating set if every vertex in G is adjacent to at least one vertex in this subset. The total domination number of G is the minimum cardinality of any total dominating set in G and is denoted by $\gamma_t(G)$. A total dominating set of G having nonempty intersection with all the independent sets of maximum cardinality in G is an independent transversal total dominating set. The minimum cardinality of any independent transversal total dominating set is denoted by $\gamma_{tt}(G)$. Based on the fact that for any tree T, $\gamma_t(T) \leq \gamma_{tt}(T) \leq \gamma_t(T) + 1$, in this work we give several relationships between $\gamma_{tt}(T)$ and $\gamma_t(T)$ for trees T which are leading to classify the trees which are satisfying the equality in these bounds.

Keywords: independent transversal total domination number, total domination number, independence number, trees.

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1. Introduction

Researches concerning domination and/or independence in graphs are very frequently presented by several researchers for some decades. One common kind of investigation deals with studies on domination and independence properties of trees, and among these investigations common results concern finding characterizations (constructive, algorithmic or even theoretical) of the trees satisfying a determined property or achieving a given value of one invariant. As some recent examples, we cite for instances [1, 5, 7], although there is a long list of them throughout the literature. In the present article, we continue the research in this issue by finding several relationships between the independent transversal total domination number and the total domination number of trees.

Throughout this work we consider $G = (V, E)$ as a simple graph of order n and size m. That is, graphs that are finite, undirected, and simple. Given a vertex v of G, $N_G(v)$ represents the *open neighborhood* of v, i.e., the set of all neighbors of v in G and the *degree* of v is $d(v) = |N_G(v)|$. The *minimum* and *maximum degrees* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For any two vertices u and v, the *distance* $d(u, v)$ between u and v is the minimum number of edges on a path between u and v. Given a set of vertices S of G, we use $G - S$ to denote the graph obtained from G by removing all the vertices of S and the edges incident with them. If $S = \{v\}$ for some vertex v, then we simply write $G - v$. Also, the subgraph of G induced by $D \subset V$ will be denoted by $G[D]$.

A set $D \subset V(G)$ is a *total dominating set* of G if every vertex in $V(G)$ is adjacent to at least one vertex in D. The *total domination number* of G is the minimum cardinality of any total dominating set of G and is denoted by $\gamma_t(G)$. A $\gamma_t(G)$-set is a total dominating set of cardinality $\gamma_t(G)$. For more information on total domination see the recent book [6] or the latest survey [4].

We now consider a kind of partial total domination version in a graph G. Consider $A \subset V(G)$ is a given set of vertices of G. We say that $D_A \subset V(G)$ is a *total dominating set with respect to A in G* (from now on, an *A-total dominating set in G* for short), if every vertex in A is adjacent to at least one vertex in D_A (note that there is no inclusion relation between A and D_A, i.e., not necessarily $D_A \subset A$ nor $A \cap D_A \neq \emptyset$ and so on). The *total domination number with respect to A in G* (from now on, *A-total domination number* in G for short), is the minimum cardinality of any A-total dominating set in G, and is denoted by $\gamma_t(A)$. A $\gamma_t(A)$-set is an A-total dominating set in G of cardinality $\gamma_t(A)$. Clearly, if $A = V(G)$, then the A-total domination number becomes the standard total domination number.

In other words, definitions above can be roughly understood as follows. A $\gamma_t(A)$-set D_A is a set of minimum cardinality in G that dominates every vertex of A. Every vertex from $A \cap D_A$ must have a neighbor in D_A, but a vertex from

$(V(G) \setminus A) \cap D_A$ does not need to have a neighbor in D_A. Moreover, one does not need to dominate vertices from $V(G) \setminus (A \cup D_A)$ with D_A, although it could happen.

A set S of vertices is *independent* if S induces an edgeless graph. An independent set of maximum cardinality is a *maximum independent set* of G. The *independence number* of G is the cardinality of a maximum independent set of G and is denoted by $\beta(G)$. An independent set of cardinality $\beta(G)$ is called a $\beta(G)$-*set*.

A total dominating set of G which intersects every independent set of maximum cardinality in G is called an *independent transversal total dominating set*. The minimum cardinality of an independent transversal total dominating set is called the *independent transversal total domination number* of G and is denoted by $\gamma_{tt}(G)$. An independent transversal total dominating set of cardinality $\gamma_{tt}(G)$ is a $\gamma_{tt}(G)$-*set*, see [2].

Let T be a tree. A *leaf* of T is a vertex of degree one. A *support vertex* of T is a vertex of degree at least two adjacent to a leaf, and a *semi-support vertex* is a vertex adjacent to a support vertex which is not a leaf or a support vertex. By an *isolated support vertex* of T we mean an isolated vertex of the subgraph induced by the support vertices of T. The set of leaves is denoted by $L(T)$, the set of support vertices is denoted by $S(T)$ and the set of semi-support vertices is denoted by $SS(T)$. Moreover, $S^*(T)$ is the set of isolated support vertices of T, $L^*(T)$ represents the set of leaves adjacent to vertices of $S^*(T)$ and the set $SS^*(T)$ contains all semi-support vertices adjacent to vertices of $S^*(T)$ which are not adjacent to any vertex of $S(T) \setminus S^*(T)$.

The independent transversal total domination number of a graph G has recently been introduced in [2], where several of its combinatorial and computational properties were presented. Among them, interesting bounds in terms of $\gamma_t(G)$ and $\delta(G)$ were proved for $\gamma_{tt}(G)$, namely, for any graph G, $\gamma_t(G) \leq \gamma_{tt}(G) \leq \gamma_t(G) + \delta(G)$. A realization result concerning such parameter was also proved. That is, for every positive integers a, b, c with $a \leq b \leq a + c$, there exists a graph G of minimum degree $\delta(G) = c$ such that $\gamma_t(G) = a$ and $\gamma_{tt}(G) = b$. An interesting particular case of the previous bound stands up for the class of tree graphs. Clearly, since $\delta(T) = 1$ for any tree T, this bound leads to the following sandwich-style result already presented in [2].

Observation 1 [2]**.** *If T is a tree on at least two vertices, then*

$$\gamma_t(T) \leq \gamma_{tt}(T) \leq \gamma_t(T) + 1.$$

One can immediately think into classifying the trees into two types, according to which value for the independence transversal total domination number they achieve. In this sense, an open problem concerning characterizing the trees T for

which either $\gamma_{tt}(T) = \gamma_t(T)$ or $\gamma_{tt}(T) = \gamma_t(T)+1$ is satisfied was already presented in [2]. It is therefore our goal to present conditions which are characteristic for these two classes of trees and which bring more insight to problems that occurs dealing with it.

In concordance with this objective, in this paper we assume that $|S(T)| \geq 2$ since the case $S(T) = 0$ (T is a P_2 and $\gamma_{tt}(T) = \gamma_t(T) = 2$) and $S(T) = 1$ (T is a star S_n and $\gamma_{tt}(T) = \gamma_t(T) = 2$) are straightforward.

2. Relating $\gamma_t(T)$ and $\gamma_{tt}(T)$ for a Tree T

In order to easily proceed with our exposition from now on we say that a tree T belongs to the family $\mathcal{T}_0$, if $\gamma_{tt}(T) = \gamma_t(T)$ or T is in the family $\mathcal{T}_1$, if $\gamma_{tt}(T) = \gamma_t(T)+1$. In Figure 1 we show two examples of trees T_6 and T_7 where $T_6 \in \mathcal{T}_1$ and $T_7 \in \mathcal{T}_0$. For T_6 note that the set $D = \{s_1, s_2\}$ is a unique $\gamma_t(T_6)$-set and there is a unique $\beta(T_6)$-set formed by the set $H = \{h_1, h_2, h_3, h_4\}$. Since $D \cap H = \emptyset$, $S = D \cup \{h_1\}$ is a $\gamma_{tt}(T_6)$-set and $\gamma_{tt}(T_6) = \gamma_t(T_6) + 1$. On the other hand, for T_7 we note that the set $D = \{s_1, s_2, ss\}$ is a unique $\gamma_t(T_7)$-set and there is a unique $\beta(T_7)$-set formed by the $H = \{h_1, h_2, h_3, h_4, ss\}$. Since $D \cap H = \{ss\}$, D is a $\gamma_{tt}(T_7)$-set, which leads to $\gamma_{tt}(T_7) = \gamma_t(T_7)$.
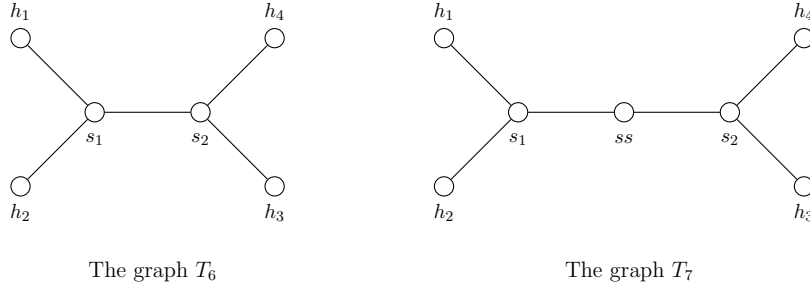

Figure 1. A tree T_6 from a family $\mathcal{T}_1$ and a tree T_7 from a family $\mathcal{T}_0$.

Next we present some primary results (some of them are already known), which will be useful later.

Lemma 2 [3]. *For any connected graph G with diameter at least three, there exists a $\gamma_t(G)$-set that contains no leaves of G.*

Observation 3. *If v is a support vertex of a tree T, then v is in every $\gamma_t(T)$-set.*

The following lemma is obvious and we state it without a proof.

Lemma 4. *If v is a leaf of a tree T, then every $\beta(T)$-set contains either v or the support adjacent to v.*

Corollary 5. *If T is a tree of order $n \geq 3$, then there exists a $\beta(T)$-set containing all the leaves of T.*

In contrast with Lemma 2, Observation 3 and Lemma 4 imply that for every tree T from $\mathcal{T}_1$, every $\gamma_t(T)$-set must be without leaves. Otherwise, there exists a $\gamma_t(T)$-set that contains some leaves and its support vertices, and by Lemma 4, we have that $T \in \mathcal{T}_0$. Notice that the above condition is trivially fulfilled whenever $S^*(T) = \emptyset$, and one can expect more problems if $S^*(T) \neq \emptyset$. In concordance with this, we next make a separation of the study in the cases where $S^*(T) = \emptyset$ or $S^*(T) \neq \emptyset$.

2.1.   The tree T is without isolated support vertices

In this subsection we deal with trees for which $S^*(T) = \emptyset$. This means that every support vertex has a neighbor among the support vertices, and in view of Observation 3, we need to consider vertices in $V(T) \setminus (L(T) \cup S(T))$. The following theorem describes these relationships.

Theorem 6. *Let T be a tree of order at least four with $S^*(T) = \emptyset$, let $F = T - (L(T) \cup S(T))$ and let $A = V(F) \setminus SS(T)$. The tree T belongs to $\mathcal{T}_1$ if and only if for every $\gamma_t(A)$-set D_A in F there exists a $\beta(F)$-set B such that $B \cap D_A = \emptyset$.*

Proof. Let T be a tree of order at least four with $S^*(T) = \emptyset$. By Observation 3, every support vertex is in every $\gamma_t(T)$-set. Since $S^*(T) = \emptyset$, every support vertex has a support vertex as a neighbor and every support vertex is totally dominated by every $\gamma_t(T)$-set. Let $F = T - (L(T) \cup S(T))$ and $A = V(F) \setminus SS(T)$.

Suppose first that for every $\gamma_t(A)$-set D_A in F there exists a $\beta(F)$-set B such that $B \cap D_A = \emptyset$. Let D_A be any $\gamma_t(A)$-set in F. We will show that $D = D_A \cup S(T)$ is a $\gamma_t(T)$-set. As already mentioned, every support vertex is dominated by another support vertex and hence by D. Clearly, every leaf is adjacent to its support vertex and therefore, also dominated by D. Also, every vertex from A is adjacent to a vertex from D_A by the definition. Finally, every vertex from $SS(T)$ is also adjacent to a support vertex which is in D. Altogether, D is a total dominating set of T. On the other hand, all vertices from $S(T)$ must be in any $\gamma_t(T)$-set and all vertices that have no neighbor in $S(T)$ are in A. By the definition of a $\gamma_t(A)$-set in F, there exists no set of smaller cardinality that is an A-total dominating set in F. Hence, D is a $\gamma_t(T)$-set. Let now $Q = B \cup L(T)$. Clearly, Q is an independent set, as $B \subset V(F)$ and there are no edges between leaves of T and vertices of F. Moreover, Q is a $\beta(T)$-set because B is a $\beta(F)$-set and by Corollary 5. Now, $B \cap D_A = \emptyset$ implies that $Q \cap D = \emptyset$ and therefore, $\gamma_{tt}(T) > \gamma_t(T)$. By Observation 1 we have $\gamma_{tt}(T) = \gamma_t(T) + 1$ and $T \in \mathcal{T}_1$.

Let now $T \in \mathcal{T}_1$, which implies that $\gamma_{tt}(T) = \gamma_t(T) + 1$. By Observation 1 and the definition of class $\mathcal{T}_1$, for every $\gamma_t(T)$-set D there exists a $\beta(T)$-set Q such

that $D \cap Q = \emptyset$. As all support vertices are in D by Observation 3, all leaves must be in Q by Lemma 4 and maximality of Q. We claim that $D_A = D \cap V(F)$ and $B = Q \cap V(F)$ are the desired $\gamma_t(A)$-set and $\beta(F)$-set, respectively. Clearly, $B \cap D_A = \emptyset$ follows immediately from $D \cap Q = \emptyset$. Also, D_A is an A-total dominating set in F, since D is a total dominating set of T, and there are no edges between vertices of A and $V(T) \setminus V(F)$. If D_A is not a $\gamma_t(A)$-set in F, then there exists a set D'_A which is a $\gamma_t(A)$-set in F with $|D'_A| < |D_A|$. This yields a contradiction with D being a $\gamma_t(T)$-set, since $D'_A \cup S(T)$ is a total dominating set of T of cardinality less than $|D|$. Hence, D_A is a $\gamma_t(A)$-set in F. Similarly, if B is not a $\beta(F)$-set, then there exists an independent set B' of $V(F)$ such that $|B| < |B'|$. Again this yields a contradiction with B being a $\beta(F)$-set, since $L(T) \cup B'$ is an independent set of larger cardinality than Q. This final contradiction ends the proof. ∎

We remark that the A-domination features used in Theorem 6 concern the subgraph F of the tree T, namely D_A is an A-total dominating set in F, but no relationship with vertices in $V(T) \setminus V(F)$ exists. We next observe some particular case of the theorem above.

Corollary 7. *Let T be a tree of order at least four with $S^*(T) = \emptyset$. If $V(T) = L(T) \cup S(T) \cup SS(T)$, then T belongs to $\mathcal{T}_1$.*

Proof. Since $V(T) = L(T) \cup S(T) \cup SS(T)$, the set A from Theorem 6 is an empty set. Hence the condition of Theorem 6 is trivially fulfilled and $T \in \mathcal{T}_1$. ∎

Notice that a special case of the corollary above occurs also when $SS(T) = \emptyset$, which will be discussed in Theorem 12, while particularizing some cases of the main results of this work. By Observation 1, a tree T is either in $\mathcal{T}_1$ or in $\mathcal{T}_0$. The next theorem follows directly from negation of condition of Theorem 6.

Theorem 8. *Let T be a tree of order at least four with $S^*(T) = \emptyset$, let $F = T - (L(T) \cup S(T))$ and let $A = V(F) \setminus SS(T)$. The tree T belongs to $\mathcal{T}_0$ if and only if there exists a $\gamma_t(A)$-set D_A in F such that for every $\beta(F)$-set B it follows $B \cap D_A \neq \emptyset$.*

2.2. The tree T contains isolated support vertices

In this part we consider trees with $S^*(T) \neq \emptyset$. An important difference can occur in this case. Namely, it can happen that there exists a $\gamma_t(T)$-set which contains a leaf. Such a case immediately implies that $T \in \mathcal{T}_0$ by Lemma 4. Therefore the following terminology is natural. A tree T is a *good tree*, if there exists a $\gamma_t(T)$-set which contains at least one leaf, and T is a *bad tree* otherwise, that is, every $\gamma_t(T)$-set is without leaves. The next theorem and its proof are similar to

Theorem 6 and its proof, but they contain some important differences, mainly because the definitions of F and A are different from that ones of Theorem 6.

Theorem 9. *Let T be a tree of order at least four with $S^*(T) \neq \emptyset$, let $F = T - ((L(T) \setminus L^*(T)) \cup (S(T) \setminus S^*(T)))$ and let $A = V(F) \setminus (SS(T) \setminus SS^*(T))$. The tree T belongs to $\mathcal{T}_1$ if and only if T is a bad tree and for every a $\gamma_t(A)$-set D_A in F there exists a $\beta(F)$-set B such that $B \cap D_A = \emptyset$.*

Proof. Let T be a tree of order at least four with $S^*(T) \neq \emptyset$. By Observation 3, every support vertex is in every $\gamma_t(T)$-set. Let $F = T - ((L(T) \setminus L^*(T)) \cup (S(T) \setminus S^*(T)))$ and let $A = V(F) \setminus (SS(T) \setminus SS^*(T))$.

Suppose first that T is a bad tree and for every $\gamma_t(A)$-set D_A in F there exists a $\beta(F)$-set B such that $D_A \cap B = \emptyset$. Let D_A be any $\gamma_t(A)$-set in F. We will show that $D = D_A \cup S(T)$ is a $\gamma_t(T)$-set. As already mentioned, every support vertex which is not in $S^*(T)$ is dominated by another support vertex, and hence, by D. Clearly, every leaf is adjacent to its support vertex, and therefore, also dominated by D. Also, every vertex from A is adjacent to a vertex from D_A by the definition (recall that also vertices from $S^*(T)$ are in A). Finally, every vertex from $SS(T)$ is also adjacent to a support vertex which is in D. Altogether, D is a total dominating set of T. On the other hand, all the vertices from $S(T)$ must be in any $\gamma_t(T)$-set and all vertices that have no neighbor in $S(T) \setminus S^*(T)$ are in A. By the definition of the $\gamma_t(A)$-set in F, there exists no set of smaller cardinality than D_A, which is an A-total dominating set in F. Because there is no edge between vertices of A and vertices of $S(T) \setminus S^*(T)$, they have no influence on any $\gamma_t(A)$-set in F. Hence, D is a $\gamma_t(T)$-set. Moreover, $D \cap L^*(T) = \emptyset$, since T is a bad tree. Let now $S = B \cup (L(T) \setminus L^*(T))$ (notice that by the maximality of B, and because $B \cap D_A = \emptyset$, all leaves from $L^*(T)$ must be in B). Clearly, S is an independent set as $B \subset V(F)$, and there are no edges between leaves in $L(T) \setminus L^*(T)$ of T and vertices of F. Moreover, S is a $\beta(T)$-set because B is a $\beta(F)$-set and by Corollary 5. Now, $B \cap D_A = \emptyset$ implies that $S \cap D = \emptyset$, and therefore, $\gamma_{tt}(T) > \gamma_t(T)$. By Observation 1 we have $\gamma_{tt}(T) = \gamma_t(T) + 1$ and $T \in \mathcal{T}_1$.

Let now $T \in \mathcal{T}_1$, which implies that $\gamma_{tt}(T) = \gamma_t(T) + 1$. By Observation 1 and by the definition, for every $\gamma_t(T)$-set D there exists a $\beta(T)$-set S such that $D \cap S = \emptyset$. By Lemma 4, D contains no leaves and T is therefore a bad tree. As all support vertices are in D by Observation 3, it follows that all leaves must be in S, by Lemma 4 and maximality of S. We claim that $D_A = D \cap V(F)$ and $B = S \cap V(F)$ are the desired $\gamma_t(A)$-set in F and $\beta(F)$-set, respectively. Clearly, $B \cap D_A = \emptyset$ follows immediately from $D \cap S = \emptyset$. Also, D_A is an A-total dominating set in F, since D is a total dominating set of T and there are no edges between vertices of A and $S(T) \setminus S^*(T)$. If D_A is not a $\gamma_t(A)$-set in F, then there exists a set D'_A which is a $\gamma_t(A)$-set in F with $|D'_A| < |D_A|$. This

yields a contradiction with D being a $\gamma_t(T)$-set, since the set $D'_A \cup S(T)$ is a total dominating set of T of cardinality less than D. Hence, D_A is a $\gamma_t(A)$-set in F. Similarly, if B is not a $\beta(F)$-set, then there exists an independent set B' of $V(F)$ such that $|B| < |B'|$. Again this yields a contradiction with B being a $\beta(F)$-set, since $(L(T) \setminus L^*(T)) \cup B'$ is an independent set of larger cardinality than S. This final contradiction completes the proof. ∎

Since the result above is mainly based on bad trees, it would be desirable to describe their properties in order to give more insight into their structure. This is next presented.

Proposition 10. *Let T be a tree with $S^*(T) \neq \emptyset$. The tree T is a bad tree if and only if for every vertex $v \in S^*(T)$ and for every $u \in N(v) \cap SS(T) \cap D$, there exists $x \in N(u) \setminus \{v\}$ with $N(x) \cap D = \{u\}$, for every $\gamma_t(T)$-set D.*

Proof. Let T be a bad tree, let D be a $\gamma_t(T)$-set and let $v \in S^*(T)$. Since T is a bad tree, leaves adjacent to v are not in D and so, v is totally dominated by at least one of its semi-support vertices, say u (that is $u \in D$). If all vertices at distance two from u, which are not neighbors of v (note that there are neighbors of v which are at distance two from u, for instance other semi-support vertices and all leaves adjacent to v) are in D, then $(D \setminus \{u\}) \cup \{y\}$ is a $\gamma_t(T)$-set for some leaf $y \in N(v)$. This is a contradiction with T being a bad tree. So, u totally dominates at least one vertex different from v, say x, which is not totally dominated by any other vertex from D (notice that x may be in D or not). Thus, one direction of the proof is done.

On the other hand, if, namely, there exists $v \in S^*(T)$ such that there is $u \in N(v) \cap SS(T) \cap D$ for which every $x \in N(u) \setminus \{v\}$ satisfies that $N(x) \cap D \neq \{u\}$ for some $\gamma_t(T)$-set D, then $D' = D \setminus \{u\} \cup \{y\}$ is also a $\gamma_t(T)$-set for any leaf $y \in N(v)$. Thus, T is a good tree, which completes the proof. ∎

As in the previous subsection, we can obtain a characterization of a tree in the class $\mathcal{T}_0$ by negating the condition of Theorem 9.

Theorem 11. *Let T be a tree of order at least four with $S^*(T) \neq \emptyset$, let $F = T - ((L(T) \setminus L^*(T)) \cup (S(T) \setminus S^*(T)))$ and let $A = V(F) \setminus (SS(T) \setminus SS^*(T))$. The tree T belongs to $\mathcal{T}_0$ if and only if either T is a good tree, or T is a bad tree and there exists a $\gamma_t(A)$-set D_A in F such that $B \cap D_A \neq \emptyset$ for every $\beta(F)$-set B.*

2.3. Particularizing some situations

The two main results of the previous subsections are frequently difficult to deal with while trying to classify a tree to be in $\mathcal{T}_0$ or in $\mathcal{T}_1$. In concordance with this, in this subsection, we give some more useful necessary conditions for a tree to be either in $\mathcal{T}_0$ or in $\mathcal{T}_1$.

Theorem 12. *Let T be any tree of order n. If $|S(T)| + |L(T)| = n$, then $T \in \mathcal{T}_1$.*

Proof. Clearly, $|S(T)| + |L(T)| = n$ implies that every vertex of T is either a support or a leaf. Also it implies that $|L(T)| \geq |S(T)|$ because every support is adjacent to at least one leaf. Therefore, $L(T)$ is a $\beta(T)$-set. On the other hand, notice that $S(T)$ is the unique $\gamma_t(T)$-set and does not intersect every $\beta(T)$-set, which means $\gamma_{tt}(T) \geq \gamma_t(T) + 1$. By Observation 1, $\gamma_{tt}(T) = \gamma_t(T) + 1$ and $T \in \mathcal{T}_1$. ∎

Theorem 13. *Let T be a tree of order n such that $|S(T)| + |L(T)| = n - 1$.*

(a) *If $|S^*(T)| > 0$, then $T \in \mathcal{T}_0$.*

(b) *If $|S^*(T)| = 0$, then $T \in \mathcal{T}_1$.*

Proof. Let B be any $\beta(T)$-set. If $|S(T)| + |L(T)| = n - 1$, then there exists a unique $s' \in SS(T)$. Clearly $s' \in B$.

(a) Let $v \in S^*(T)$. Since v is an isolated support, it cannot be adjacent to any support. Moreover, since T is not a star ($|S^*(T)| > 0$), v must be adjacent to s'. Now, in order to totally dominate v and its adjacent leaves, $S(T) \cup \{s'\}$ can be chosen as a $\gamma_t(T)$-set, according to Lemma 2. Moreover, $S(T) \cup \{s'\}$ intersects B, and since B is arbitrary, $S(T) \cup \{s'\}$ is an independent transversal total dominating set. So $\gamma_{tt}(T) \leq \gamma_t(T)$ and by Observation 1, the result follows.

(b) $|S^*(T)| = 0$ implies that every support is adjacent to at least one other support. So, $S(T)$ is a total dominating set, which is also the unique $\gamma_t(T)$-set. Thus, $S(T)$ is a $\gamma_t(T)$-set and may not intersect B (in particular it does not intersect the $\beta(T)$-set mentioned in Corollary 5). This means that at least an extra vertex is needed in any $\gamma_t(T)$-set to be a $\gamma_{tt}(T)$-set. So, $\gamma_{tt}(T) \geq \gamma_t(T) + 1$. By Observation 1, $\gamma_{tt}(T) \leq \gamma_t(T) + 1$, which completes the proof. ∎

From now on, we center our attention on those trees for which $|S(T)| + |L(T)| < n - 1$. Herein we denote by $P(u, v)$ the set of vertices of the shortest path between u and v and by $B_{u,v} = P(u, v) \cap B$, where B is a maximum independent set. Moreover, given a $\gamma_t(T)$-set D that contains no leaves, we set $SS_D(T) = SS(T) \cap D$.

Proposition 14. *Let T be a tree such that $|S(T)| + |L(T)| < n - 1$ and $S^*(T) = S(T)$. If D is a $\gamma_t(T)$-set that contains no leaves, then for any $\beta(T)$-set B containing all leaves of T, $SS_D(T) \cap B \neq \emptyset$ holds.*

Proof. Let h_1 and h_2 be two diametrical leaves of T. Assume the subgraph induced by $P(h_1, h_2)$ is $h_1 s_1 v_1 \cdots v_r s_2 h_2$, where $h_1, h_2 \in B$, $s_1, s_2 \in S(T)$, and $v_1, v_r \in SS(T)$ since $S^*(T) = S(T)$. Note that v_1 and v_r exist because $S^*(T) = S(T)$, $|S(T)| + |L(T)| < n - 1$ and h_1 and h_2 are diametrical.

Now, suppose that $SS_D(T) \cap B = \emptyset$ for some $\beta(T)$-set B containing all the leaves of T. Thus $v_1, v_r \notin B$. Let B'_{h_1,h_2} be a maximum independent set containing h_1 and h_2 in the subgraph induced by $P(h_1, h_2)$. It is not difficult to observe that $B'_{h_1,h_2} \cap \{v_1, v_r\} \neq \emptyset$, and that $|B'_{h_1,h_2}| > |B_{h_1,h_2}|$, since $v_1, v_r \notin B$. Consider now $A_{h_1,h_2} = \{b_1, \ldots, b_k\}$ as the set of vertices of degree at least three in T belonging to B'_{h_1,h_2}, but not to B_{h_1,h_2}. If $A_{h_1,h_2} = \emptyset$, then $(B \setminus B_{h_1,h_2}) \cup B'_{h_1,h_2}$ is an independent set in T of cardinality larger than $|B|$, a contradiction. So $A_{h_1,h_2} \neq \emptyset$.

Let $b_i \in A_{h_1,h_2}$ and let h_ℓ be a leaf of T such that $P(b_i, h_\ell) \cap P(h_1, h_2) = \{b_i\}$. Among all such leaves we choose for h_ℓ the one that is closest to b_i. Moreover, assume $B_{b_i,h_\ell} = \{b_i, b_{\ell_1}, \ldots, b_{\ell_q}, h_\ell\}$. Notice that, since $h_\ell \in B$ and $SS_D(T) \cap B = \emptyset$, it follows $b_{\ell_q} \notin \{s_\ell, ss_\ell\}$, where $s_\ell, ss_\ell \in P(b_i, h_\ell)$ are the support and semi-support vertices, respectively, with h_ℓ adjacent to s_ℓ. Proceeding as above, for a set B'_{b_i,h_ℓ} we observe that $B'_{b_i,h_\ell} \cap \{s_\ell, ss_\ell\} \neq \emptyset$. Thus, $|B'_{b_i,h_\ell}| \geq |B_{b_i,h_\ell}|$.

We continue an analogous procedure and consider the vertices belonging to a set A_{b_i,h_ℓ} (in the case that $A_{b_i,h_\ell} \neq \emptyset$). We again construct a set $P(z_i, h_q)$ in the manner, where $z_i \in P(b_i, h_\ell)$, and repeat the process. Clearly, this procedure always finish at a finite number of steps, since the graph is finite.

Such process of construction gives at the end an independent set of cardinality larger than $|B|$ which is not possible. Therefore, it must happen that $SS_D(T) \cap B \neq \emptyset$ for every $\beta(T)$-set B containing all leaves of T. ∎

With the result above in mind we give another sufficient condition for classifying a tree to $\mathcal{T}_0$.

Theorem 15. *Let T be a tree of order n. If $|S(T)| + |L(T)| < n - 1$ and $S^*(T) = S(T)$, then $T \in \mathcal{T}_0$.*

Proof. Let D be a $\gamma_t(T)$-set that contains no leaves. Clearly, every support vertex of T is in D, and for every support vertex s there is a semi-support vertex ss adjacent to s, which is in D, since $S^*(T) = S(T)$. We consider now any $\beta(G)$-set B. If B does not contain all the leaves of T, then B contains at least one support vertex. So, $D \cap B \neq \emptyset$. Now, if B contains all leaves of T, then by Proposition 14, B satisfies $SS_D(T) \cap B \neq \emptyset$. So, $D \cap B \neq \emptyset$ and as a consequence, D is an independent transversal total dominating set. Therefore, $\gamma_{tt}(T) \leq \gamma_t(G)$ and by Observation 1 the equality follows, which leads to $T \in \mathcal{T}_0$. ∎

A particular case of the result above can be given as follows for a specific class of trees.

Corollary 16. *Let T be a tree different from a star. If the distance between any two leaves is even, then $T \in \mathcal{T}_0$.*

Proof. Let the distance between any two leaves of T be even. If two different support vertices are adjacent, then two leaves attached to them are at distance three, which is not possible. Therfore, $S^*(T) = S(T)$. Moreover, since T is not a star, it must happen $|S(T)| + |L(T)| < n-1$. Hence, Proposition 14 and Theorem 15 lead to $T \in \mathcal{T}_0$. ∎

We end our exposition with a simple result for the class of trees T such that $|S(T)| + |L(T)| < n - 1$ and $|S^*(T)| < |S(T)|$. Clearly, in such situation, there are adjacent support vertices. Let $S_a(T) = S(T) \setminus S^*(T)$, i.e., $S_a(T)$ is the set of non-isolated support vertices, let $L_a(T)$ be the set of leaves adjacent to a vertex in $S_a(T)$, and let $SS_a(T)$ be the set of semi-support vertices adjacent to a vertex in $S_a(T)$. By using these sets, the next result is an immediate consequence of Corollary 7.

Corollary 17. *Let T be a tree of order n with $|S(T)| + |L(T)| < n - 1$. If $V(T) = L_a(T) \cup S_a(T) \cup SS_a(T)$, then $T \in \mathcal{T}_1$.*

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REFERENCES

[1] C. Brause, M.A. Henning and M. Krzywkowski, *A characterization of trees with equal 2-domination and 2-independence numbers*, Discrete Math. Theor. Comput. Sci. **19** (2017) #1.
doi:10.23638/DMTCS-19-1-1

[2] A. Cabrera Martínez, J.M. Sigarreta Almira and I.G. Yero, *On the independence transversal total domination number of graphs*, Discrete Appl. Math. **219** (2017) 65–73.
doi:10.1016/j.dam.2016.10.033

[3] M. Chellali and T.W. Haynes, *A note on the total domination of a tree*, J. Combin. Math. Combin. Comput. **58** (2006) 189–193.

[4] M.A. Henning, *A survey of selected recent results on total domination in graphs*, Discrete Math. **309** (2009) 32–63.
doi:10.1016/j.disc.2007.12.044

[5] M.A. Henning and S.A. Marcon, *A constructive characterization of trees with equal total domination and disjunctive domination numbers*, Quaest. Math. **39** (2016) 531–543.
doi:10.2989/16073606.2015.1096860

[6] M.A. Henning and A. Yeo, Total Domination in Graphs (Springer, New York, 2013).
doi:10.1007/978-1-4614-6525-6

[7] Z. Li and J. Xu, *A characterization of trees with equal independent domination and secure domination numbers*, Inform. Process. Lett. **119** (2017) 14–18.
doi:10.1016/j.ipl.2016.11.004