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Interfaces with Other Disciplines

Sequential bankruptcy problems

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ABSTRACT

In this paper, we analyze sequential bankruptcy problems, which generalize bankruptcy problems. They cover the problems of sharing water in a transboundary river and of allocating expedition rewards in projects. We propose the upwards mechanism for generalizing rules for bankruptcy problems to rules for sequential bankruptcy problems. Further, we characterize the upwards constrained equal awards, the upwards constrained equal losses, and the upwards proportional rules on the basis of upwards composition and upwards path independence.

1. Introduction

Bankruptcy problems are one of the simplest, yet most interesting, economic problems. In a bankruptcy problem, there is a group of agents with rightful claims over a scarce estate. The question is how to "fairly" share the estate among the agents. There is an extensive literature on bankruptcy problems from a game theoretical perspective (cf. Aumann and Maschler, 1985; Curiel, Maschler, and Tijs, 1987; O'Neill, 1982, as seminal papers) as well as from an axiomatic perspective (cf. Dagan, 1996; Dietzenbacher, 2018; Gallice, 2019; Herrero & Villar, 2001; Moreno-Ternero & Vidal-Puga, 2020; Moulin, 1987; Peters, Schröder, & Vermeulen, 2019; Tsay & Yeh, 2019). For a survey on bankruptcy literature we refer to Thomson (2003, 2015).

Bankruptcy problems have been applied in a variety of economic situations like in sharing the emission of CO_2 (Gutiérrez, Llorca, Sánchez-Soriano, & Mosquera, 2018 and Duro, Giménez-Gómez, & Vilella, 2020), passepartout problems (Estévez-Fernández, Borm, & Hamers, 2012), museum pass problems (Bergantiños & Moreno-Ternero, 2015, 2016; Casas-Mendez, Fragnelli, & Garcia-Jurado, 2011, 2014), sharing the revenues from broadcasting sport events (Bergantiños & Moreno-Ternero, 2019) and in two-sided matching problems (Estévez-Fernández, Borm, & Lazarova, 2016). Recently, Ansink and Weikard (2012) analyses the problem of sharing water in a transboundary river in the framework of bankruptcy problems where the agents are linearly ordered (see Madani, Zarezadeh, & Morid, 2014, and Mianabadi, Mostert, Zarghami, & van de Giesen, 2014, for instance). This linear order is given by the position in the river and the natural flow of its water. Moreover, at each location along the river, there is exactly one agent and the total water inflow at each location restricts the maximum amount of water an agent can get. Allowing more than one agent at each location enriches the problem since allows the representation of its different uses, e.g. human consumption, crop irrigation, electricity, industry, etc. By allowing to differentiate amongst water uses, it is possible to take priorities of claimants during drought times (Moulin, 2000; Thomson, 2003).

Estévez-Fernández, Borm, and Hamers (2007), and Bergantiños and Lorenzo (2019); Estévez-Fernández (2012), and Bordley, Keisler, and Logan (2019) in a more general setting, analyze allocation of delay costs and expedition rewards in projects within a game theoretical framework. In many projects, specially when developing high technology, there is a race against the clock and high incentives are given in order to finish the project before the planned time. For a project to be expedited, the activities need to coordinate and cooperate for this aim. Estévez-Fernández et al. (2007) and Estévez-Fernández (2012) model the allocation of revenues from expedited projects using a bankruptcy approach where the agents have a linear order based on the minimum slack of all the "paths" in which they are involved. Besides, the slack of the paths also restrict the maximum amount the activities in a path can obtain from the total reward.

In this paper, we introduce sequential bankruptcy problems, which form a theoretical framework that supports the works of Ansink and Weikard (2012), Estévez-Fernández et al. (2007), and Estévez-Fernández (2012), and Bergantiños and Lorenzo (2019). In a sequential bankruptcy problem, there is an ordered partition of

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the set of agents, $\langle N_1, \ldots, N_r \rangle$, and r estates, $E_1, \ldots, E_r$, such that the members in N_1 claim on E_1, the members in N_2 claim on E_1 and E_2, and so forth. Moreover, each estate E_l is not high enough to satisfy the total claim of the members in $N_l, \ldots, N_r$. Relating it back to the work of Ansink and Weikard (2012), the problem of sharing water in a transboundary river can be modeled as a sequential bankruptcy problem where $|N_l| = 1$ for each $l \in \{1, \ldots, r\}$. Regarding the work of Estévez-Fernández (2012), the allocation of revenues from expedited projects can be modeled as a sequential bankruptcy problem where: (i) there are as many elements in the partition of agents as groups of paths with same slack that are involved in the project expedition and (ii) the total estate over which a group can claim is the contribution of the group to the total reward of the expedition.

We introduce the upwards mechanism for generalizing rules for bankruptcy problems to rules for sequential bankruptcy problems. This name is inspired in the literature of sharing water in a river. Given a bankruptcy rule f, the upwards mechanism generates a sequential bankruptcy rule by starting to share the estate "downstream" and, subsequently, moving upwards while updating the claims at each stage of the procedure. We focus on three of the most used rules for bankruptcy problems: the proportional, the constrained equal awards, and the constrained equal losses rules. Inspired in Dagan (1996) and Herrero and Villar (2001), we characterize the upwards constrained equal awards rule by means of upwards composition, the upwards constrained equal losses rule by means of upwards path independency, and the upwards proportional rule is characterize using upwards self-duality and upwards composition, as well as upwards self-duality and upwards path independency.

The remainder of the paper is organized as follows. Section 2 surveys bankruptcy problems and introduces sequential bankruptcy problems. In Section 3, we introduce the upwards mechanism and provide characterizations for the upwards constrained equal awards, the upwards constrained equal losses, and the upwards proportional rules. We conclude with Section 4.

2. Preliminaries

2.1. Bankruptcy problems

In this section, we give a brief survey of existing concepts in the literature of bankruptcy problems and introduce sequential bankruptcy problems.

First, we present notation that will be used throughout the article. For $x \in \mathbb{R}$, $x_+ = \max\{0, x\}$. Let N be a finite set. We denote by $\mathbf{0} \in \mathbb{R}^N$ the vector with all zeros. For $x \in \mathbb{R}^N$ and $S \subseteq N$, $x_S \in \mathbb{R}^S$ denotes the projection of x in $\mathbb{R}^S$ and $x(S) = \sum_{i \in S} x_i$. Let $r \in \mathbb{N}$, $r \le |N|$, and let $\langle N_1, \ldots, N_r \rangle$ be a partition of N. For $l_1, \ldots, l_s \in \{1, \ldots, r\}$ and $x \in \mathbb{R}^{|N|}$, we denote $x^{l_1, \ldots, l_s} \in \mathbb{R}^{|N|}$ with

$$x_l^{l_1, \ldots, l_s} = \begin{cases} x_l & \text{if } l \in \{l_1, \ldots, l_s\}, \\ 0 & \text{otherwise.} \end{cases}$$

With minor abuse of language, for $y_1 \in \mathbb{R}^{N_1}, \ldots, y_{N_r} \in \mathbb{R}^{N_r}$, we denote $y^{l_1, \ldots, l_s} \in \mathbb{R}^N$ with

$$y_{N_l}^{l_1, \ldots, l_s} = \begin{cases} y_{N_l} & \text{if } l \in \{l_1, \ldots, l_s\}, \\ \mathbf{0}_{N_l} & \text{otherwise.} \end{cases}$$

In a bankruptcy problem, a finite group of agents have a rightful claim over a scarce estate. Formally, a **bankruptcy problem** is described by a tuple (N, E, c) where N is the set of agents, $E \in \mathbb{R}_+$ is the estate, and $c \in \mathbb{R}_+^N$ is the vector of claims, with c_i the claim of $i \in N$ on E, that satisfies $c(N) \ge E$. Let $\mathcal{B}^N$ denote the set of bankruptcy problems with set of agents N. For notational easiness, a bankruptcy problem is denoted $(E, c) \in \mathcal{B}^N$.

For $(E, c) \in \mathcal{B}^N$, the **aggregate loss** of a bankruptcy problem is the difference between the total claim and the estate, that is, $L(E, c) = c(N) - E$. If no confusion is to be expected, we write L instead of $L(E, c)$. For a claimant $i \in N$, the **minimal right** of i is the amount of estate available, if any, once all other claimants have received their full claim. Another, less used, interpretation of the minimal right of i is as follows. Pessimistically assuming that agent i is going to pay the aggregate loss, his minimal right is the part of the claim that is left, if any, after subtracting the aggregate loss:

$$m_i(E, c) = (E - c(N \setminus \{i\}))_+ = (c_i - L)_+.$$

If no confusion is to be expected, we write m_i instead of $m_i(E, c)$. It is well established that $\sum_{i \in N} m_i \le E$ (cf. Curiel et al., 1987; O'Neill, 1982).

A **bankruptcy rule** is a function f that assigns to each bankruptcy problem $(E, c) \in \mathbb{B}^N$ a vector $f(E, c) \in \mathbb{R}^N$ satisfying

$$0 \le f(E, c) \le c \text{ and } \sum_{i \in N} f_i(E, c) = E.$$

The three more relevant bankruptcy rules in the literature are the proportional rule, the constrained equal awards rule, and the constrained equal losses rule. The **proportional rule**, Prop, allocates the estate among the agents proportionally to their claims: For every $(E, c) \in \mathcal{B}^N$, $\text{Prop}(E, c) = \frac{E}{c(N)} c$. The **constrained equal awards rule**, CEA, allocates the estate as equal as possible among the agents, considering that they do not get more than their claims: For every $(E, c) \in \mathcal{B}^N$, $\text{CEA}(E, c) = (\min\{c_i, \alpha\})_{i \in N}$ with $\alpha \in \mathbb{R}$ such that $\sum_{i \in N} \min\{c_i, \alpha\} = E$. The **constrained equal losses rule**, CEL, allocates the losses as equal as possible among the agents, considering that they do not get a negative amount: For every $(E, c) \in \mathcal{B}^N$, $\text{CEL}(E, c) = ((c_i - \beta)_+)_{i \in N}$ with $\beta \in \mathbb{R}$ such that $\sum_{i \in N} (c_i - \beta)_+ = E$. For a survey on more bankruptcy rules and their properties, we refer to Thomson (2003, 2015).

2.2. Sequential bankruptcy problems

Before defining sequential bankruptcy problems, we give two introductory examples. Ambec and Sprumont (2002) initiated a mainstream of literature on sharing the water of an international river among the agents located along the river. Ansink and Weikard (2012) further analyses this problem in the framework of bankruptcy problems where the agents are linearly ordered. They propose a mechanism that transforms bankruptcy rules into rules for sharing water in a river. One feature of the problem in Ansink and Weikard (2012) is that in each river location, there is exactly one agent. We now consider that at each location, there may be more than one agent. For instance, in one specific location, one agent might represent the water needed for human consumption, another might represent the water needed for crop irrigation, and another one might represent the water needed for industry. By allowing more than one agent at each river location, we can also take priorities of claimants into account: under water shortage, human consumption may have priority over water needed for industry. For more on bankruptcy problems with priorities, see Moulin (2000) and Thomson (2003).

Example 2.1. Consider a river with three locations, 1,2,3, along the river. We may assume that location 1 is the location that is most upstream, location 2 is the location that is halfway, and location 3 is the most downstream one. In location 1 there is a total inflow of $E_1 = 6$ units and there are two claimants, A and B, with claims $c_A = 2$ and $c_B = 3$; in location 2 there is a total inflow of $E_2 = 5$ units and there are three claimants, C, D, and E, with claims $c_C = 1$, $c_D = 2$, and $c_E = 3$; and in location 3 there is a total inflow of $E_3 = 3$ units and there is one claimant, F, with claim $c_F = 5$.

Here, the agents at location 1 have rights over the inflow of 6 units; the agents at location 2 have rights over the total inflow at

Table 1
Planned and realization times of the activities and the paths in Example 2.2.

Activity	Planned time	Realization time	Expedition
A	5	2	3
B	7	3	4
C	10	7	3
D	4	3	1
E	2	1	1
F	3	2	1
Path			
A-B	**12**	5	
C	10	**7**	
D-E-F	9	6	

locations 1 and 2 of $6+5=11$ units; and the agent at location 3 has rights over the total inflow of the river: $6+5+3=14$ units.

In this specific situation, agents A and B have an inflow that totally satisfies their claims and let 1 unit of water go downstream. The agents C, D, and E have a total of $5+1$ units of water inflow, in order to satisfy their demands. Agent F has a claim of 5 and an insufficient inflow of 3 units to satisfy his claim. Therefore, agents A, B, C, D and E may not be able to fulfill all their claims.

Estévez-Fernández et al. (2007), and Estévez-Fernández (2012) in a more general setting, analyze allocation of delay costs and expedition rewards in projects within a game theoretical framework. A project consists of a set of activities that are interconnected and need to be carried out during a period of time in order to achieve a particular aim. Examples of projects are the construction of a building, the organization of a congress, or the development of the hyperloop. A planned project specifies its activities, their interconnections, and the planned time to carry out each activity. This allows us to have a planned duration of the project. In many projects, specially when developing high technology, there is a race against the clock and high incentives are given in order to finish the project before the planned time. For a project to be expedited, the activities need to coordinate and cooperate for this aim as the following example illustrates.

Example 2.2. Consider a project with six activities which interconnections are given in Fig. 2.

The planned time of the activities are $p(\mathrm{A})=5$, $p(\mathrm{B})=7$, $p(\mathrm{C})=10$, $p(\mathrm{D})=4$, $p(\mathrm{E})=2$ and $p(\mathrm{F})=3$. Hence, the planned time of the project is

$$\max\{p(\mathrm{A})+p(\mathrm{B}), p(\mathrm{C}), p(\mathrm{D})+p(\mathrm{E})+p(\mathrm{F})\}$$
$$=\max\{5+7, 10, 4+2+3\}=12.$$

The manager wants the project to be expedited. For this, he is willing to pay a reward of € 1000 per unit of expedition. To expedite the whole project, activities in path A-B need to be expedited. If only activities in A-B are expedited and C, D, E, F act according to plan, the project can have a maximum expedition of 2 units of time. If C coordinates with A and B to expedite the project, while D, E, and F act according to plan, they could increase the total expedition by at most one extra unit of time. To further expedite the project, A, B and C also need the cooperation of activities in the path D-E-F. Let the realization times of the project be those given on Table 1.

In this situation, A claims € 3000, B claims € 4000, C claims € 3000, and D, E, and F claim € 1000 each. Here, A and B are needed to bring the total expedition of the project ($12-7=5$), and have rights over the total reward of € 5000. Activity C contributes to the expedition of $10-7=3$ units of time and has rights over € 3000 of the total reward. Activities D, E, and F contribute to the expedition of $9-7=2$ units of time and have rights over € 2000 of the total reward.

We now introduce sequential bankruptcy problems. Let N be a finite set of claimants and let $\langle N_1,\dots,N_r\rangle$ be an ordered partition of N. A **sequential bankruptcy problem** is a tuple $(\langle N_1,\dots,N_r\rangle, (E_1,\dots,E_r), c)$ where $E_1,\dots,E_r\in\mathbb{R}_+$ are the estates to be shared among the claimants such that the members of N_1 claim over E_1, the members of N_2 claim over E_1+E_2, and, in general, the members of N_l claim over $\sum_{\lambda=1}^{l}E_\lambda$ for $l\in\{1,\dots,r\}$; and $c\in\mathbb{R}_+^N$ is the vector of claims satisfying $c(N)\geq\sum_{\lambda=1}^{r}E_\lambda$. Similar to Ansink and Weikard (2012), we impose the following assumption to guarantee meaningfulness of sequential bankruptcy problems.

Assumption:

$$\sum_{\lambda=l}^{r}c(N_\lambda)\geq\sum_{\lambda=l}^{r}E_\lambda \quad \text{for all } l\in\{1,\dots,r\}.$$

Let $\mathcal{B}^{N_1,\dots,N_r}$ denote the set of bankruptcy problems with set of agents $N=\bigcup_{\lambda=1}^{r}N_\lambda$ and ordered partition $\langle N_1,\dots,N_r\rangle$. For notational easiness, a sequential bankruptcy problem is denoted $(\underline{E},c)\in\mathcal{B}^{N_1,\dots,N_r}$, where $\underline{E}=(E_1,\dots,E_r)$.

For $(\underline{E},c)\in\mathcal{B}^{N_1,\dots,N_r}$, the **aggregate loss** for the members of N_k is defined by $L_k(\underline{E},c)=c(N_k)-E_k$. Unlike in bankruptcy problems, $L_k(\underline{E},c)$ may be negative. If no confusion is to be expected, we write L_k instead of $L_k(\underline{E},c)$. Our assumption can now be restated as

$$DL_l=\sum_{\lambda=l}^{r}L_\lambda\geq 0 \quad \text{for all } l\in\{1,\dots,r\}$$

where DL_l represents the downwards aggregate loss for the members of $N_l,\dots,N_r$.

Example 2.3. The problem of sharing water in a river in Example 2.1 can be seen as a sequential bankruptcy problem where $N=\{\mathrm{A,B,C,D,E,F}\}$, $N_1=\{\mathrm{A,B}\}$, $N_2=\{\mathrm{C,D,E}\}$, and $N_3=\{\mathrm{F}\}$; $E_1=6$, $E_2=5$, and $E_3=3$; and $c=(2,3,1,2,3,5)$.

The expedition project problem in Example 2.2 can be seen as a sequential bankruptcy problem where $N=\{\mathrm{A,B,C,D,E,F}\}$, $N_1=\{\mathrm{D,E,F}\}$, $N_2=\{\mathrm{C}\}$, and $N_3=\{\mathrm{A,B}\}$; $E_1=2000$, $E_2=1000$, and $E_3=2000$; and $c=(3000,4000,3000,1000,1000,1000)$.

A **sequential bankruptcy rule** or **rule** is a function f that assigns to each sequential bankruptcy problem $(\underline{E},c)\in\mathcal{B}^{N_1,\dots,N_r}$ a vector $f(\underline{E},c)\in\mathbb{R}^N$ satisfying

$$0\leq f(\underline{E},c)\leq c, \tag{1}$$

$$\sum_{\lambda=1}^{l}\sum_{i\in N_\lambda}f_i(\underline{E},c)\leq\sum_{\lambda=1}^{l}E_\lambda \text{ for each } l=1,\dots,r-1, \quad \text{and} \tag{2}$$

$$\sum_{\lambda=1}^{r}\sum_{i\in N_\lambda}f_i(\underline{E},c)=\sum_{\lambda=1}^{r}E_\lambda. \tag{3}$$

For $k\in\{1,\dots,r\}$ and $E_1=\ldots=E_k=0$, conditions (1) and (2) imply $x_{N_l}=\mathbf{0}_{N_l}$ for each $l\in\{1,\dots,k\}$.

2.3. Properties

One way to discern among several division rules is by looking at their properties. We now pay attention to generalizing basic properties for bankruptcy rules to properties for sequential bankruptcy rules. We first consider invariance under claims truncation (see Curiel et al., 1987). It refers to the upper bound for claims. It says that those claims that are over the estate should not be rewarded. Hence the allocation should not depend on that part of the claim that is greater than the maximum available quantity: For each $(E,c)\in\mathcal{B}^N$, $f(E,c)=f(E,c^E)$ where $c_i^E=\min\{c_i,E\}$ for all $i\in N$.

When considering sequential bankruptcy problems, the question we first need to address is what is our reference point for truncation. It is clear that claimants can never get more than the "total estate" over which they are claiming. A rule f satisfies **invariance under claims truncation** if for each $(\underline{E}, c) \in \mathcal{B}^{N_1,\ldots,N_r}$,

$$f(\underline{E}, c) = f(\underline{E}, c^{\underline{E}})$$

where $c_i^{\underline{E}} = \min\{c_i, \sum_{\lambda=1}^{k} E_\lambda\}$ for all $i \in N_k$ and all $k \in \{1,\ldots,r\}$.

We can relax invariance under claims truncation by considering the property for those sequential bankruptcy problems with only one positive estate. A rule f satisfies **weak invariance under claims truncation** if for each $k \in \{1,\ldots,r\}$ and each $E_k > 0$ with $(\underline{E}^k, c) \in \mathcal{B}^{N_1,\ldots,N_r}$,

$$f(\underline{E}^k, c) = f(\underline{E}^k, c^{E_k})$$

where $c_i^{E_k} = \min\{c_i, E_k\}$ for all $i \in \bigcup_{\lambda=k}^{r} N_\lambda$ and $c_i^{E_k} = 0$ for all $i \in \bigcup_{\lambda=1}^{k-1} N_\lambda$. It readily follows that invariance under claims truncation implies weak invariance under claims truncation.

Next, we consider equal treatment of equals (cf. Thomson, 2003). This property says that the same claims should be rewarded with the same allocation: For each $(E, c) \in \mathcal{B}^N$ and $i, j \in N$ with $c_i = c_j$, $f_i(E, c) = f_j(E, c)$.

First, we need to decide when two claimants are considered equal. For a start, they need to have equal claims. Besides, they also need to hold their claim over the same amount of "total estate". A rule f satisfies **equal treatment of equals** if for each $(\underline{E}, c) \in \mathcal{B}^{N_1,\ldots,N_r}$, each $k, l \in \{1,\ldots,r\}$ with $\sum_{\lambda=1}^{k} E_\lambda = \sum_{\lambda=1}^{l} E_\lambda$, and each $i \in N_k$ and $j \in N_l$ with $c_i = c_j$,

$$f_i(\underline{E}, c) = f_j(\underline{E}, c).$$

Therefore, if $i \in N_k$ and $j \in N_l$, with $k < l$, are equals, then, $c_i = c_j$ and $E_{k+1} = \ldots = E_l = 0$.

To conclude, we generalize minimal rights first (cf. Thomson, 2003). It establishes a minimal level of allocation for each agent, since it works as a lower bound. Firstly, for each claimant we identify an amount considered to be a minimum right (the difference between the estate and the remaining claims if it is non-negative, and zero otherwise). Then, each claimant receives its minimal right, and the bankruptcy problem is revised to distribute the remaining estate by applying the rule: For each $(E, c) \in \mathcal{B}^N$, $f(E, c) = m(E, c) + f(E - \sum_{j \in N} m_j(E, c), c - m(E, c))$. In sequential bankruptcy problems, the minimal right of claimant $i \in N_k$ is the part of the estate left, if any, after all other claimants have been fully compensated. For this, we first need to define the **additional estate** that is available to N_k from the upstream claimants. Let $(\underline{E}, c) \in \mathcal{B}^{N_1,\ldots,N_r}$ and let $AE_0 = 0$. For $l = 1$,

$$AE_1 = (AE_0 + E_1 - c(N_1))_+ = (AE_0 - L_1)_+$$

is the part of the estate E_1 that the members of N_1 are not claiming and is the additional estate for N_2. Then, the members of N_2 can guarantee a claim over $E_2 + AE_1$. For $l \in \{2,\ldots,r-1\}$,

$$AE_l = (AE_{l-1} + E_l - c(N_l))_+ = (AE_{l-1} - L_l)_+$$

is the additional estate for N_{l+1} and the members of N_{l+1} can guarantee a claim over $E_{l+1} + AE_l$. By the assumption on sequential bankruptcy problems, $AE_l - \sum_{\lambda=l+1}^{r} L_\lambda \leq 0$ for $l \in \{1,\ldots,r-1\}$.

The minimal right of $i \in N_k$, $k \in \{1,\ldots,r\}$, is defined by

$$m_i(\underline{E}, c) = \left(AE_{k-1} + E_k - c(N_k \setminus \{i\}) + \sum_{\lambda=k+1}^{r} (E_\lambda - c(N_\lambda)) \right)_+ = \left(AE_{k-1} + c_i - \sum_{\lambda=k}^{r} L_\lambda \right)_+ .$$

We denote $m(\underline{E}, c) = (m_i(\underline{E}, c))_{i \in N}$. If no confusion is to be expected, we write m_i instead of $m_i(\underline{E}, c)$ and m instead of $m(\underline{E}, c)$.

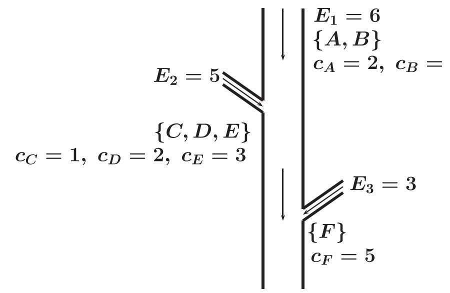

Fig. 1. Sharing the water in a river in Example 2.1.

A rule f satisfies **strong minimal rights first** if for each $(\underline{E}, c) \in \mathcal{B}^{N_1,\ldots,N_r}$,

$$f(\underline{E}, c) = m + f\left(\bar{E}_1 - \sum_{i \in N_1} m_i, \ldots, \bar{E}_r - \sum_{i \in N_r} m_i, c - m \right)$$

with $\bar{E}_l = \min\{c(N_l), AE_{l-1} + E_l\}$ for $l \in \{1,\ldots,r\}$.

Easily, $m(\bar{E}_1,\ldots,\bar{E}_r, c) = m(\underline{E}, c)$. Therefore, minimal rights first is very restrictive since it implies $f(\underline{E}, c) = f(\bar{E}_1,\ldots,\bar{E}_r, c)$ for every $(\underline{E}, c) \in \mathcal{B}^{N_1,\ldots,N_r}$ with $m(\underline{E}, c) = \mathbf{0}$. We consider, for that reason, the following alternative. A rule f satisfies **minimal rights first** if for each $k \in \{1,\ldots,r\}$ and each $E_k > 0$ with $(\underline{E}^k, c) \in \mathcal{B}^{N_1,\ldots,N_r}$,

$$f(\underline{E}^k, c) = m + f\left(0,\ldots,0, E_k - \sum_{\lambda=k}^{r} \sum_{i \in N_\lambda} m_i, 0,\ldots,0, c - m \right).$$

The following result provides an expression of the minimal right of the claimants based on their claim and the aggregate loss on the different sets of claimants. The proof follows straightforwardly from the definition of minimal right and is, therefore, omitted.

Proposition 2.1. *Let* $(\underline{E}, c) \in \mathcal{B}^{N_1,\ldots,N_r}$.

(i) $AE_{k-1} - \sum_{\mu=k}^{r} L_\lambda = \max_{1 \leq \lambda \leq k} \left\{ -\sum_{\mu=\lambda}^{r} L_\mu \right\} \leq 0$ *for every* $k \in \{2,\ldots,r\}$.

(ii) *For* $i \in N_k$, $m_i(\underline{E}, c) = \max_{1 \leq \lambda \leq k} \left\{ \left(c_i - \sum_{\mu=\lambda}^{r} L_\mu \right)_+ \right\}$.

3. Upwards mechanism for sequential bankruptcy problems

In this section, we introduce the upwards mechanism for sequential bankruptcy problems. It generalizes rules for bankruptcy problems to rules for sequential bankruptcy problems. The name of upwards mechanism is inspired by the connection of sequential bankruptcy problems and problems of sharing water in a river: The mechanism starts allocating the estates from downstream and moves upwards by updating the claims at each stage. We start this section by reconsidering Example 2.1.

Example 3.1. Reconsider the problem of sharing water in a river in Example 2.1 (see Fig. 1). As pointed out in Example 2.3, it can be interpreted as a sequential bankruptcy problem with $N = \{A, B, C, D, E, F\}$, $N_1 = \{A, B\}$, $N_2 = \{C, D, E\}$, and $N_3 = \{F\}$; $E_1 = 6$, $E_2 = 5$, and $E_3 = 3$; and $c = (2, 3, 1, 2, 3, 5)$.

To allocate the available water among the claimants, we will use the constrained equal awards rule. First, recall that the water inflow E_3 can only be shared among the claimants in N_3 as water only flows downstream. We can apply the constrained equal awards rule to $(E_3, (\mathbf{0}_{N_1}, \mathbf{0}_{N_2}, c_{N_3})) \in \mathcal{B}^N$:

$$x^3 = \mathrm{CEA}(3, (0, 0, 0, 0, 0, 5)) = (0, 0, 0, 0, 0, 3).$$

Second, the water inflow E_2 can only be shared among the claimants in N_2 and N_3 as water only flows downstream. Since in

Activity	Predecessors
A	
B	A
C	
D	
E	D
F	D, E

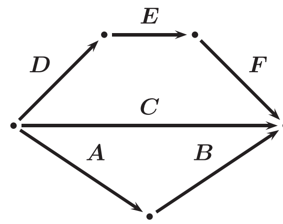

Fig. 2. Activities interconnections and project representation in Example 2.2.

Table 2
Upwards mechanism applied to the constrained equal awards, constrained equal losses, and proportional rules in Example 3.1.

Rule	A	B	C	D	E	F
CEA^{Up}	$1\frac{5}{9}$	$1\frac{5}{9}$	1	2	$2\frac{8}{9}$	5
CEL^{Up}	$1\frac{2}{3}$	$2\frac{2}{3}$	$\frac{2}{3}$	$1\frac{2}{3}$	$2\frac{2}{3}$	$4\frac{2}{3}$
Prop^{Up}	$1\frac{1}{2}$	$2\frac{1}{4}$	$\frac{29}{32}$	$1\frac{13}{16}$	$2\frac{23}{32}$	$4\frac{13}{16}$

our first step agent F has already got 3 units of water, we need to update his claim to $5-3=2$. We can now apply the constrained equal awards rule to $(5,(0,0,1,2,3,2))\in\mathcal{B}^N$:

$$x^2=\text{CEA}(5,(0,0,1,2,3,2))=\left(0,0,1,1\frac{1}{3},1\frac{1}{3},1\frac{1}{3}\right).$$

Third and last, the water inflow E_1 can be shared among all the claimants. Since in our second step agents C, D, E and F have already got water, we need to update their claims: for C we have $1-1=0$, for D we have $2-1\frac{1}{3}=\frac{2}{3}$, for E we have $3-1\frac{1}{3}=1\frac{2}{3}$, and for F we have $2-1\frac{1}{3}=\frac{2}{3}$. We can now apply the constrained equal awards rule to $(6,(2,3,0,\frac{2}{3},1\frac{2}{3},\frac{2}{3}))\in\mathcal{B}^N$:

$$x^1=\text{CEA}\left(6,\left(2,3,0,\frac{2}{3},1\frac{2}{3},\frac{2}{3}\right)\right)=\left(1\frac{5}{9},1\frac{5}{9},0,\frac{2}{3},1\frac{5}{9},\frac{2}{3}\right).$$

The upwards mechanism applied to the constrained equal awards rule leads to the allocation:

$$x^3+x^2+x^1=\left(1\frac{5}{9},1\frac{5}{9},1,2,2\frac{8}{9},5\right).$$

In Table 2, we give the allocations obtained by applying the upwards mechanism to the constrained equal awards, constrained equal losses, and proportional rules.

Given a bankruptcy rule f, the **upwards mechanism** generates a rule for sequential bankruptcy problems, f^{Up}, that assigns to each $(\underline{E},c)\in\mathcal{B}^{N_1,\dots,N_r}$, a vector $f^{\text{Up}}(\underline{E},c)\in\mathbb{R}^N$ defined by

$$f^{\text{Up}}(\underline{E},c)=\sum_{k=1}^{r}x^k$$

where $x^r=f(E_r,c^r)$ and, for $k=r-1,\dots,1$, x^k is recursively defined by

$$x^k=f\left(E_k,c^{k,\dots,r}-\sum_{\lambda=k+1}^{r}x^\lambda\right).$$

Recall that $c^{k,\dots,r}=(\mathbf{0}_{N_1},\dots,\mathbf{0}_{N_{k-1}},c_{N_k},\dots,c_{N_r})$.

Our assumption on sequential bankruptcy problems and the boundedness constraints of bankruptcy rules ensure that all the problems above are bankruptcy problems.

Following the idea behind the upwards mechanism, a rule f is an upwards sequential bankruptcy rule or upwards rule if for every $(\underline{E},c)\in\mathcal{B}^{N_1,\dots,N_r}$, $x^{r+1}=\mathbf{0}$ and x^k recursively defined by

$$x^k=f\left(\underline{E}^k,c^{k,\dots,r}-\sum_{\lambda=k+1}^{r+1}x^\lambda\right)$$

for $k=r,r-1,\dots,2,1$, imply

$$f(\underline{E},c)=\sum_{\lambda=1}^{r}x^\lambda.$$

Recall that $\underline{E}^k=(0,\dots,0,E_k,0,\dots,0)$. Thus, if f is an upwards rule, it is the same to directly apply f to $(\underline{E},c)$, or to apply the rule to $(\underline{E}^r,c^r)$ and, after updating the claims, work all the way up to $(\underline{E}^1,d^1)$, where d^1 represents the updated claims. Clearly, the upwards mechanism generates upwards rules.

The following result states that the properties of invariance under claims truncation, equal treatment of equals, and minimal rights first in the class of bankruptcy problems are inherited in the class of sequential bankruptcy problems by the upwards mechanism. The proof is postponed to the supplementary material.

Proposition 3.1. *Let f be a bankruptcy rule.*

(i) *If f satisfies invariance under claims truncation, then, f^{Up} satisfies invariance under claims truncation.*
(ii) *If f satisfies invariance under claims truncation, then, f^{Up} satisfies weak invariance under claims truncation.*
(iii) *If f satisfies equal treatment of equals, then, f^{Up} satisfies equal treatment of equals.*
(iv) *If f satisfies minimal rights first, then, f^{Up} satisfies minimal rights first.*

Next, we turn our attention to the properties of duality (cf. Aumann & Maschler, 1985), composition (cf. Young, 1988), and path independence (cf. Moulin, 1987).[1] Two bankruptcy rules f and f^{D} are dual if f shares rewards in the same way as f^{D} allocates losses and vice versa: For each $(E,c)\in\mathcal{B}^N$, $f(E,c)=c-f^{\text{D}}(L,c)$. A rule f is self-dual if $f^{\text{D}}=f$.

The difficulty of generalizing duality to sequential bankruptcy problems lies in how to define the losses at a given estate E_k, since this estate is shared by agents in $N_k,\dots,N_r$. We are going to use an upwards approach, although other generalizations may also be possible. Given $(\underline{E},c)\in\mathcal{B}^{N_1,\dots,N_r}$, recall that $DL_k=\sum_{\lambda=l}^{r}L_\lambda$ represents the downwards aggregate loss for the members of $N_k,\dots,N_r$ and $DL_k\geq 0$ for all $k\in\{1,\dots,r\}$ by assumption. We denote $\underline{DL}=(DL_1,\dots,DL_r)$. It follows $(\underline{DL}^k,c)\in\mathcal{B}^{N_1,\dots,N_r}$ for all $k\in\{1,\dots,r\}$. The sequential bankruptcy problem $(\underline{DL}^k,c)$ represents the problem of sharing the sum of the downwards aggregate loss in groups $N_k,\dots,N_r$ among their members. Two rules f and f^{UpD} are upwards dual if f shares rewards in the same way as f^{UpD} allocates downward losses in an upwards manner. The rule f^{UpD} is the **upwards dual rule** of f if for each $(\underline{E},c)\in\mathcal{B}^{N_1,\dots,N_r}$,

$$f^{\text{UpD}}(\underline{E},c)=c-f\left(\underline{DL}^1,c-(c^{2,\dots,r}-x^2)\right)$$

where $x^r=f\left(\underline{DL}^r,c^r\right)$ and for $k=r-1,\dots,1$,

$$x^k=f\left(\underline{DL}^k,c^{k,\dots,r}-(c^{k+1,\dots,r}-x^{k+1})\right).$$

[1] We follow here the nomenclature in Herrero and Villar (2001).

The rule f is **upwards self-dual** if $f = f^{\mathrm{UpD}}$.

Remark 3.2. Let f^{UpD} be the upwards dual rule of f and let $(\underline{E}, c) \in \mathcal{B}^{N_1,\dots,N_r}$ with $E_1 = \ldots = E_{k-1} = 0$ and $E_k > 0$ for some $k \in \{2, \dots, r\}$. Using that $f^{\mathrm{UpD}}(\underline{E}, c) = \sum_{l=1}^{r} (c^{l,\dots,r} - x^l - (c^{l+1,\dots,r} - x^{l+1}))$, with $c^{r+1} = x^{r+1} = \mathbf{0}$, it readily follows that

$$f^{\mathrm{UpD}}(\underline{E}, c) = c^{k,\dots,r} - x^k.$$

The following result establishes the one-to-one relation between an upwards rule and its dual. Besides, it shows that given two dual bankruptcy rules, the corresponding upwards rules obtained with the upwards mechanism are also upwards dual. The proof is postponed to the supplementary material.

Lemma 3.3.

(i) *Let f be an upwards rule and let f^{UpD} be its upwards dual rule. Then, f^{UpD} is also an upwards rule and f is the upwards dual rule of f^{UpD}.*

(ii) *Let f and g be two dual bankruptcy rules. Then, f^{Up} and g^{Up} are upwards dual.*

A bankruptcy rule f satisfies composition if given two bankruptcy problems with same claimants and same claims, but where one of the estates is higher than the other, it is the same to directly apply f to the problem with the higher estate than to first apply f to the problem with the lowest estate and second apply f to the estates difference after updating the claims: For each $(E, c), (E', c) \in \mathcal{B}^N$ with $E \geq E'$ and $x = f(E', c)$, $f(E, c) = f(E', c) + f(E - E', c - x)$.

Ansink and Weikard (2015) have translated composition of bankruptcy problems to rules for the river claims problem, which can be analyzed in the context of sequential bankruptcy problems where each element of the partition has exactly one claimant. They propose three composition properties: (i) river composition, (ii) composition downstream, and (iii) composition upstream. Unfortunately, the updated estate vector combined with the updated claims in the definition of river composition need not lead to a sequential bankruptcy problem. Besides, their definition of composition upstream is equivalent to the concept of upwards rule.

The main challenge when generalizing composition to sequential bankruptcy problems is making sure that the updated estate vector and updated claims remain in the class of sequential bankruptcy problems. For this, we continue to apply an upwards methodology. A rule f satisfies upwards composition if given two sequential bankruptcy problems $(\underline{E}, c)$ and $(\underline{E}', c)$ that only differ in estate k with $E'_k \leq E_k$, then, it is the same to apply the rule to $(\underline{E}, c)$ than to apply first the rule to $(\underline{E}'^{k,\dots,r}, c)$ and second to the updated estate vector and claims. A rule f satisfies **upwards composition** if for each $(\underline{E}, c) \in \mathcal{B}^{N_1,\dots,N_r}$, each $k \in \{1, \dots, r\}$, and each $E'_k \in \mathbb{R}_+$ with $E'_k \leq E_k$, $x^k = f(\underline{E}'^k + \underline{E}^{k+1,\dots,r}, c^{k,\dots,r})$ implies

$$f(\underline{E}, c) = f(\underline{E}^{1,\dots,k} - \underline{E}'^k, c - x^k) + f(\underline{E}'^k + \underline{E}^{k+1,\dots,r}, c^{k,\dots,r}).$$

Lemma 3.4. *Any rule satisfying upwards composition is an upwards rule.*

Proof. Let f satisfy upwards composition and let $(\underline{E}, c) \in \mathcal{B}^{N_1,\dots,N_r}$. Let $E'_r = E_r$ and $x^r = f(\underline{E}^r, c^r)$. By upwards composition,

$$f(\underline{E}, c) = f(\underline{E} - \underline{E}^r, c - x^r) + f(\underline{E}^r, c^r).$$

Let $E'_k = E_k$ and $x^k = f\left(\underline{E}^k, c^k - \sum_{\lambda=k+1}^{r} x^\lambda\right)$ for $k = r-1, \dots, 1$. Re-iteratively applying upwards composition, we get

$$f(\underline{E}, c) = \sum_{\lambda=1}^{r} x^l.$$

□

The following result states that composition in the class of bankruptcy problems is inherited in the class of sequential bankruptcy problems by the upwards mechanism. The proof is postponed to the supplementary material.

Proposition 3.5. *Let f be a bankruptcy rule satisfying composition, then, f^{Up} satisfies upwards composition.*

Theorem 3.6. *The upwards constrained equal awards rule is the only rule satisfying equal treatment of equals, weak invariance under claims truncation, and upwards composition.*

Proof. It is well established that the constrained equal awards rule for bankruptcy problems satisfies equal treatment of equals, invariance under claims truncation, and composition. By Propositions 3.1 and 3.5, the upwards constrained equal awards rule satisfies equal treatment of equals, weak invariance under claims truncation, and upwards composition as well. We now prove uniqueness. Let f be a rule satisfying equal treatment of equals, weak invariance under claims truncation, and upwards composition. By Lemma 3.4, f is an upwards rule. Next, we show $f = \mathrm{CEA}^{\mathrm{Up}}$. Let $(\underline{E}, c) \in \mathcal{B}^{N_1,\dots,N_r}$. By upwards composition, $x^r = f(\underline{E}^r, c^r)$ implies

$$f(\underline{E}, c) = f(\underline{E} - \underline{E}^r, c - x^r) + f(\underline{E}^r, c^r).$$

Following the same lines as in Dagan (1996), it is readily seen that equal treatment of equals, weak invariance under claims truncation, and upwards composition imply $f(\underline{E}^r, c^r) = \mathrm{CEA}^{\mathrm{Up}}(\underline{E}^r, c^r)$ and

$$f(\underline{E}, c) = f(\underline{E} - \underline{E}^r, c - x^r) + \mathrm{CEA}^{\mathrm{Up}}(\underline{E}^r, c^r)$$

with $x^r = \mathrm{CEA}^{\mathrm{Up}}(\underline{E}^r, c)$. Recursively, following the same lines as above, for $k \in \{1, \dots, r-1\}$, $x^r = f(\underline{E}^r, c^r) = \mathrm{CEA}^{\mathrm{Up}}(\underline{E}^r, c^r)$ and $x^l = f\left(\underline{E}^l, c^{l,\dots,r} - \sum_{\lambda=l+1}^{r} x^\lambda\right) = \mathrm{CEA}^{\mathrm{Up}}\left(\underline{E}^l, c^{l,\dots,r} - \sum_{\lambda=l+1}^{r} x^\lambda\right)$ for $l \in \{k, \dots, r-1\}$, imply

$$\begin{aligned} f(\underline{E}, c) &= f\left(\underline{E}^{1,\dots,k} - \underline{E}^k, c - \sum_{\mu=\lambda+1}^{r} x^\mu\right) + \sum_{\lambda=k}^{r} f\left(\underline{E}^\lambda, c^{\lambda,\dots,r} - \sum_{\mu=\lambda+1}^{r} x^\mu\right) \\ &= f\left(\underline{E}^{1,\dots,k} - \underline{E}^k, c - \sum_{\mu=\lambda+1}^{r} x^\mu\right) + \sum_{\lambda=k}^{r} \mathrm{CEA}^{\mathrm{Up}}\left(\underline{E}^\lambda, c^{\lambda,\dots,r} - \sum_{\mu=\lambda+1}^{r} x^\mu\right) \end{aligned}$$

Therefore, for $k = 1$,

$$f(\underline{E}, c) = \sum_{\lambda=1}^{r} \mathrm{CEA}^{\mathrm{Up}}\left(\underline{E}^\lambda, c^{\lambda,\dots,r} - \sum_{\mu=\lambda+1}^{r} x^\mu\right) = \mathrm{CEA}^{\mathrm{Up}}(\underline{E}, c).$$

□

As an immediate consequence of Proposition 3.1 and Theorem 3.6, we have

Corollary 3.7. *The upwards constrained equal awards rule is the only rule satisfying equal treatment of equals, invariance under claims truncation, and upwards composition.*

A bankruptcy rule f satisfies path independence if given two bankruptcy problems with same claimants and same claims, but where one of the estates is higher than the other, it is the same to directly apply f to the problem with the lowest estate than to first apply f to the problem with the highest estate and second use this allocation as the new vector of claims to share the lowest estate: For each $(E, c), (E', c) \in \mathcal{B}^N$ with $E \geq E'$, $f(E', c) = f(E', f(E, c))$.

Ansink and Weikard (2015) have defined path independence of bankruptcy problems for rules to the river claims problem. Recall that river claims problem can be seen as sequential bankruptcy problems where all elements in the partition are singletons. They propose river path independence where, if the water level at some points in the river is less than the anticipated, it is the same to allocate the water of the new situation using the original vector of claims, or using the initially established allocation. The boundedness conditions 1 and 2 allow for a similar definition of path independence in the context of sequential bankruptcy problems: A rule f satisfies **path independence** if for each $(\underline{E}, c), (\underline{E}', c) \in \mathcal{B}^{N_1,\dots,N_r}$, with $E'_k \leq E_k$ for each $k \in \{1, \dots, r\}$,

$$f(\underline{E}', c) = f(\underline{E}', f(\underline{E}, c)).$$

In this paper, we are putting forward upwards mechanisms to translate bankruptcy rules to the more general setting of sequential bankruptcy problems. Therefore, we also propose a new concept for path independence which is based on the upwards idea. A rule f satisfies **upwards path independence** if for each $(\underline{E}, c) \in \mathcal{B}^{N_1,\dots,N_r}$, each $k \in \{1, \dots, r\}$, and each $E'_k \in \mathbb{R}_+$ with $E'_k \leq E_k$, $\underline{E}' = \underline{E} - \underline{E}^k + \underline{E}'^k$, $d^k = f(\underline{E}^{k,\dots,r}, c^{k,\dots,r})$, $x^{k+1} = f(\underline{E}'^{k+1,\dots,r}, c^{k+1,\dots,r})$ and $x'^k = f(\underline{E}'^k, d^k - x^{k+1})$, imply

$$f(\underline{E}', c) = f(\underline{E}'^{1,\dots,k-1}, c - x'^k - x^{k+1}) + f(\underline{E}'^k, d^k - x^{k+1}) + f(\underline{E}'^{k+1,\dots,r}, c^{k+1,\dots,r}).$$

Following the same lines as Lemma 3.4, one can see that path independence implies being an upwards rule. The proof is, therefore, omitted.

Lemma 3.8. *Any rule satisfying upwards path independence is an upwards rule.*

It is readily seen that upwards path independence is equivalent to path independence together with being an upwards rule.

Inspired by Herrero and Villar (2001), two properties $\mathcal{P}$ and $\mathcal{P}^{\mathrm{UpD}}$ are **upwards dual** when a rule f satisfies $\mathcal{P}$ if, and only if, its upwards dual rule f^{UpD} satisfies $\mathcal{P}^{\mathrm{UpD}}$. A property $\mathcal{P}$ is **upwards self-dual** when a rule f satisfies $\mathcal{P}$ if, and only if, its upwards dual rule f^{UpD} satisfies $\mathcal{P}$ as well.

The proof of the following result follows the same lines as the proof of Theorem 0 in Herrero and Villar (2001). It is, therefore, omitted.

Theorem 3.9 (Herrero and Villar, 2001)**.** *Let a rule f be characterized by a set of independent properties $\mathcal{P}_1, \dots, \mathcal{P}_s$. Let $\mathcal{P}_1^{\mathrm{UpD}}, \dots, \mathcal{P}_s^{\mathrm{UpD}}$ be the dual properties of $\mathcal{P}_1, \dots, \mathcal{P}_s$, respectively. Then, the upwards dual rule f^{UpD} is characterized by $\mathcal{P}_1^{\mathrm{UpD}}, \dots, \mathcal{P}_s^{\mathrm{UpD}}$ and these properties are also independent.*

The following result states that upwards composition and upwards path independence are upwards dual properties, and minimal rights first and weak invariance under claims truncation are upwards dual properties as well. The proof of these results follow the same lines as their counterparts in the framework of bankruptcy problems and is, therefore, omitted.

Lemma 3.10.

(i) *Upwards composition and upwards path independence are upwards dual.*
(ii) *Minimal rights first and weak invariance under claims truncation are upwards dual.*
(iii) *Equal treatment of equals is upwards self-dual.*

The following result is a direct consequence of Lemma 3.3 (ii), Theorems 3.6 and 3.9, and Lemma 3.10.

Theorem 3.11. *The upwards constrained equal losses rule is the only rule satisfying equal treatment of equals, minimal rights first, and upwards path independence.*

Corollary 3.12. *The upwards constrained equal losses rule is the only upwards rule satisfying equal treatment of equals, minimal rights first, and path independence.*

To conclude this section, we characterize the upwards proportional rule. For this, we need to introduce one last property. Let f be a rule. Let $c \in \mathbb{R}^N$. For $k \in \{1, \dots, r\}$, let $p_f^{k,c} : [0, \sum_{l=k}^r c(N_l)] \to \mathbb{R}^N$ be defined by $p_f^{k,c}(E_k) = f(\underline{E}^k, c)$. We say that f is **weak continuous** if for each $c \in \mathbb{R}^N$ and each $k \in \{1, \dots, r\}$, $p_f^{k,c}$ is continuous.

As remarked in Herrero and Villar (2001), Young's characterization of the proportional rule uses continuity. However, only continuity with respect to the estate is needed. In our case, we only need the requirement of weak continuity, which follows from both upwards composition and upwards path independence. The proof of the following result follows the same lines as the proof of Theorem 3.6 and is, therefore, omitted.

Theorem 3.13.

(i) *The upwards proportional rule is the only rule satisfying upwards composition and upwards self-duality.*
(ii) *The upwards proportional rule is the only rule satisfying upwards path independence and upwards self-duality.*

4. Concluding remarks

In this paper, we have introduced sequential bankruptcy problems as a generalization of bankruptcy problems, which includes river problems and allocation of expedition rewards in projects. We have put forward the upwards method to translate rules for bankruptcy problems to rules for sequential bankruptcy problems. We have translated basic properties of bankruptcy rules to the framework of sequential bankruptcy problems. For properties involving a change of estate, we have used the upwards philosophy to adapt them to the new setting. Further, we have characterized the upwards versions of three well-known bankruptcy rules (constrained equal awards, constrained equal losses, and proportional rules) on the basis of upwards composition and upwards path independence.

Ansink and Weikard (2012) and Ansink and Weikard (2015) put forward sequential sharing rules for river problems. They start sharing the “most upstream estate” between the agent claiming only on that estate (agent 1) and a fictitious agent representing the “downstream agents”, which claim is the aggregated loss of all downstream agents. Agent 1 leaves with his share and the allocation given to the fictitious player is added to the second “most upstream estate”. Second, they share the updated second “most upstream estate” between the remaining agent claiming only on that estate and a fictitious agent representing his “downstream agents”, which claim is the aggregated loss of all his downstream agents. Following the same idea, all estates are updated and consecutively shared until arriving to the “most downstream” agent that gets his own updated estate. This technique can easily be generalized to sequential bankruptcy problems, which we call the “*downstream mechanism*” to distinguish both settings and to be consistent with the upwards mechanism.

Inspired by the two-step procedure for bankruptcy problems with a priori unions in Borm, Carpente, Casas-Méndez, and Hendrickx (2005), we propose the *two-steps mechanism* for generalizing rules for bankruptcy problems to rules for sequential bankruptcy problems. In a “first step”, a chosen bankruptcy rule is applied to the bankruptcy problem with r agents where the estate is the sum of all estates and the claim of agent k is the total claim of the members of N_k truncated with respect to the sum of the estates $E_k, \dots, E_r$, $k \in \{1, \dots, r\}$; in a “second step”, the same rule is used to divide the allocation of each group $N_1, \dots, N_r$ among their members.

It is easily seen, although technically convoluted, that the properties of invariance under claims truncation and minimal rights first are inherited when applying the downwards and the two-steps mechanisms. However, equal treatment of equals is not inherited in the class of sequential bankruptcy problems by these two mechanisms. The same applies to composition and path independence and their generalizations in the form of upwards composition and upwards path independence since the downwards and two-steps mechanisms do not generate upwards rules. Future research should further study characterizations of rules generated by the downstream and the two-steps mechanisms.

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Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.ejor.2020.10.038.

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