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Total Domination in Rooted Product Graphs

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Abstract: During the last few decades, domination theory has been one of the most active areas of research within graph theory. Currently, there are more than 4400 published papers on domination and related parameters. In the case of total domination, there are over 580 published papers, and 50 of them concern the case of product graphs. However, none of these papers discusses the case of rooted product graphs. Precisely, the present paper covers this gap in the theory. Our goal is to provide closed formulas for the total domination number of rooted product graphs. In particular, we show that there are four possible expressions for the total domination number of a rooted product graph, and we characterize the graphs reaching these expressions.

Keywords: total domination; domination; rooted product graph

Let G be a graph. The open neighborhood of a vertex $v \in V(G)$ is defined to be $N(v) = \{u \in V(G) : u \text{ is adjacent to } v\}$. A set $S \subseteq V(G)$ is a dominating set of G if $N(v) \cap S \neq \varnothing$ for every vertex $v \in V(G) \setminus S$. Let $\mathcal{D}(G)$ be the set of dominating sets of G. The domination number of G is defined to be,

$$\gamma(G) = \min\{|S| : S \in \mathcal{D}(G)\}.$$

A set $S \subseteq V(G)$ is a total dominating set, TDS, of a graph G without isolated vertices if every vertex $v \in V(G)$ is adjacent to at least one vertex in S. Let $\mathcal{D}_t(G)$ be the set of total dominating sets of G.

The total domination number of G is defined to be,

$$\gamma_t(G) = \min\{|S| : S \in \mathcal{D}_t(G)\}.$$

By definition, $\mathcal{D}_t(G) \subseteq \mathcal{D}(G)$, so that $\gamma(G) \leq \gamma_t(G)$.

We define a $\gamma_t(G)$-set as a set $S \in \mathcal{D}_t(G)$ with $|S| = \gamma_t(G)$. The same agreement will be assumed for optimal parameters associated with other characteristic sets defined in the paper. For instance, a $\gamma(G)$-set will be a set $S \in \mathcal{D}(G)$ with $|S| = \gamma(G)$.

The theory of domination in graphs has been extensively studied. For instance, there are more than 4400 papers already published on domination and related parameters. In particular, we cite the following books [1,2]. In the case of total domination, there are over 580 published papers and one book [3]. Among these papers on total domination in graphs, there are over 50 which concern the case of product graphs. Surprisingly, none of these papers discusses the case of rooted product graphs. The present paper covers that gap in the theory.

In order to present our results, we need to introduce some additional notation and terminology. The closed neighborhood of $v \in V(G)$ is defined to be $N[v] = N(v) \cup \{v\}$. A vertex $v \in V(G)$ is universal if $N[v] = V(G)$, while it is a leaf if $|N(v)| = 1$. The set of leaves of G will be denoted by $\mathcal{L}(G)$. A support vertex is a vertex v with $N(v) \cap \mathcal{L}(G) \neq \varnothing$. The set of support vertices of G will be denoted by $\mathcal{S}(G)$. If v is a vertex of a graph G, then the vertex-deletion subgraph $G - \{v\}$ is the subgraph of G induced by $V(G) \setminus \{v\}$. By analogy, we define the subgraph $G - S$ for an arbitrary subset $S \subseteq V(G)$.

The concept of rooted product graph was introduced in 1978 by Godsil and McKay [4]. Given a graph G of order $\mathrm{n}(G)$ and a graph H with root vertex v, the rooted product graph $G \circ_v H$ is defined as the graph obtained from G and H by taking one copy of G and $\mathrm{n}(G)$ copies of H and identifying the i^{th} vertex of G with the root vertex v in the i^{th} copy of H for every $i \in \{1, 2, \ldots, \mathrm{n}(G)\}$. If H or G is a trivial graph, then $G \circ_v H$ is equal to G or H, respectively. In this sense, hereafter we will only consider graphs G and H with no isolated vertex.

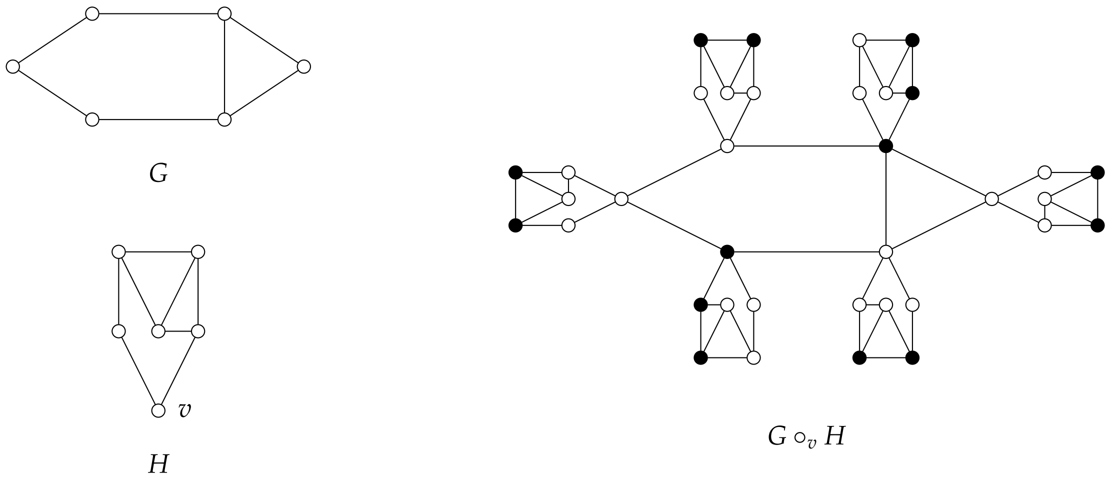

Figure 1. The set of black-coloured vertices forms a $\gamma_t(G \circ_v H)$-set.

Figure 1 shows an example of a rooted product graph. In this case, the set of black-coloured vertices forms a TDS of $G \circ_v H$ and $\gamma_t(G \circ_v H) = 14 = \gamma(G) + \mathrm{n}(G)(\gamma_t(H) - 1)$.

For every $x \in V(G)$, $H_x \cong H$ will denote the copy of H in $G \circ_v H$ containing x. The restriction of any set $S \subseteq V(G \circ_v H)$ to $V(H_x)$ will be denoted by S_x, and the restriction to $V(H_x - \{x\})$ will be denoted by S_x^-; i.e., $S_x = S \cap V(H_x)$ and $S_x^- = S_x \setminus \{x\}$. In some cases, we will need to define $S \subseteq V(G \circ_v H)$ from the sets $S_x \subseteq V(H_x)$ as $S = \cup_{x \in V(G)} S_x$.

Since $V(G \circ_v H) = \cup_{x \in V(G)} V(H_x)$, we have that for every set $S \subseteq V(G \circ_v H)$,

$$|S| = \sum_{x \in V(G)} |S_x| = \sum_{x \in V(G)} |S_x^-| + |S \cap V(G)|. \tag{1}$$

A basic problem in the study of product graphs consists of finding closed formulas or sharp bounds for specific invariants of the product of two graphs and expressing these in terms of parameters of the graphs involved in the product. In this sense, for recent results on rooted product graphs, we cite the following works [5–19]. As we can expect, the products of graphs are not alien to applications in other fields. In particular, in [5] the authors show that several important classes of chemical graphs can be expressed as rooted product graphs, and as described in [20], there exist a number of molecular graphs of high-tech interest that can be generated using the rooted product of graphs.

1. Closed Formulas for the Total Domination Number

The following three lemmas will be the main tools to deduce our results.

Lemma 1. *Given a graph H with no isolated vertex and any $v \in V(H) \setminus \mathcal{S}(H)$, the following statements hold.*

(i) $\gamma_t(H - \{v\}) \geq \gamma_t(H) - 1$.
(ii) *If $\gamma_t(H - \{v\}) = \gamma_t(H) - 1$, then the following statements hold.*

   (a) *$N(v) \cap S = \varnothing$ for every $\gamma_t(H - \{v\})$-set S.*

(b) *There exists a $\gamma_t(H)$-set S such that $v \notin S$.*

(iii) *If $\gamma_t(H - \{v\}) > \gamma_t(H)$, then $v \in S$ for every $\gamma_t(H)$-set S.*

Proof. Let $v \in V(H) \setminus \mathcal{S}(H)$ and S a $\gamma_t(H - \{v\})$-set. For every $u \in N(v)$ we have that $S \cup \{u\}$ is a TDS of H, which implies that $\gamma_t(H) \leq |S \cup \{u\}| \leq \gamma_t(H - \{v\}) + 1$. Therefore, (i) follows.

Now, in order to prove (ii), we assume that $|S| = \gamma_t(H) - 1$. If there exists a vertex $y \in N(v) \cap S$, then S is also a TDS of H, which is a contradiction. Therefore, $N(v) \cap S = \varnothing$ and so (a) follows. In addition, for any $y \in N(v)$, the set $S \cup \{y\}$ is a $\gamma_t(H)$-set not containing v. Therefore, (b) also follows.

Finally, we proceed to prove (iii). If there exists a $\gamma_t(H)$-set D such that $v \notin D$, then D is also a TDS of $H - \{v\}$, and so $\gamma_t(H - \{v\}) \leq |D| = \gamma_t(H)$. Therefore, we conclude that if $\gamma_t(H - \{v\}) > \gamma_t(H)$, then $v \in D$ for every $\gamma_t(H)$-set D, which completes the proof. □

Lemma 2. *Let H be a graph and $v \in V(H)$. If v is not a universal vertex and $H - N[v]$ does not have isolated vertices, then*

$$\gamma_t(H - N[v]) \geq \gamma_t(H) - 2.$$

Furthermore, if $\gamma_t(H - \{v\}) = \gamma_t(H) - 1$, then

$$\gamma_t(H) - 2 \leq \gamma_t(H - N[v]) \leq \gamma_t(H) - 1.$$

Proof. Assume that v is not a universal vertex and $H - N[v]$ does not have isolated vertices. Let S be a $\gamma_t(H - N[v])$-set and $u \in N(v)$. Since $S \cup \{u, v\}$ is a TDS of H, we have that $\gamma_t(H) \leq |S \cup \{u, v\}| = \gamma_t(H - N[v]) + 2$, as required.

Now, assume $\gamma_t(H - \{v\}) = \gamma_t(H) - 1$. In this case, by Lemma 1 (ii) we have that $N(v) \cap D = \varnothing$ for every $\gamma_t(H - \{v\})$-set D, which implies that D is a TDS of $H - N[v]$, and so $\gamma_t(H - N[v]) \leq |D| = \gamma_t(H - \{v\}) = \gamma_t(H) - 1$. Therefore, the result follows. □

Lemma 3. *Given a $\gamma_t(G \circ_v H)$-set S and a vertex $x \in V(G)$, the following statements hold.*

(i) $|S_x| \geq \gamma_t(H) - 1$.

(ii) *If $|S_x| = \gamma_t(H) - 1$, then $N(x) \cap S_x = \varnothing$.*

Proof. Let $x \in V(G)$. Notice that every vertex in $V(H_x) \setminus \{x\}$ is adjacent to some vertex in S_x. For any $y \in N(x) \cap V(H_x)$, the set $S_x \cup \{y\}$ is a TDS of H_x, and so $\gamma_t(H) = \gamma_t(H_x) \leq |S_x \cup \{y\}| = |S_x| + 1$. Therefore, (i) follows.

Finally, assume that $|S_x| = \gamma_t(H) - 1$. If there exists a vertex $y \in N(x) \cap S_x$, then S_x is a TDS of H_x, which is a contradiction. Therefore, $N(x) \cap S_x = \varnothing$, and so (ii) follows. □

Given a $\gamma_t(G \circ_v H)$-set S, we define the following subsets of $V(G)$ associated with S.

$$\mathcal{A}_S = \{x \in V(G) : |S_x| \geq \gamma_t(H)\} \text{ and } \mathcal{B}_S = \{x \in V(G) : |S_x| = \gamma_t(H) - 1\}.$$

These sets will play an important role in the inference results. By Lemma 3, $V(G) = \mathcal{A}_S \cup \mathcal{B}_S$. In particular, if $\mathcal{A}_S = \varnothing$, then $\gamma_t(G \circ_v H) = \mathrm{n}(G)(\gamma_t(H) - 1)$, and as we will show in the proof of Theorem 2, if $\mathcal{B}_S = \varnothing$, then $\gamma_t(G \circ_v H) = \mathrm{n}(G)\gamma_t(H)$. As we can expect, these are the extreme values of $\gamma_t(G \circ_v H)$.

Theorem 1. *For any graphs G and H with no isolated vertex and any $v \in V(H)$,*

$$\mathrm{n}(G)(\gamma_t(H) - 1) \leq \gamma_t(G \circ_v H) \leq \mathrm{n}(G)\gamma_t(H).$$

Furthermore, if $\gamma_t(H - \{v\}) = \gamma_t(H) - 1$, then

$$\gamma_t(G \circ_v H) \leq \gamma_t(G) + \mathrm{n}(G)(\gamma_t(H) - 1).$$

Proof. The lower bound follows from Lemma 3, as for any $\gamma_t(G \circ_v H)$-set S,

$$\gamma_t(G \circ_v H) = |S| = \sum_{x \in V(G)} |S_x| \geq \mathrm{n}(G)(\gamma_t(H) - 1).$$

Now, we proceed to prove the upper bound. Let $D \subseteq V(G \circ_v H)$ such that D_x is a $\gamma_t(H_x)$-set for every $x \in V(G)$. It is readily seen that D is a TDS of $G \circ_v H$. Hence,

$$\gamma_t(G \circ_v H) \leq |D| = \sum_{x \in V(G)} |D_x| = \sum_{x \in V(G)} \gamma_t(H_x) = \mathrm{n}(G)\gamma_t(H).$$

From now on, assume $\gamma_t(H - \{v\}) = \gamma_t(H) - 1$. Notice that, by assumption, $H - \{v\}$ does not have isolated vertices.

Let $W \subseteq V(G \circ_v H)$ such that $W_x^- = W_x \setminus \{x\}$ is a $\gamma_t(H_x - \{x\})$-set for every $x \in V(G)$ and $W \cap V(G)$ is a $\gamma_t(G)$-set. Clearly, W is a TDS of $G \circ_v H$, which implies that

$$\gamma_t(G \circ_v H) \leq |W \cap V(G)| + \sum_{x \in V(G)} |W_x^-| = \gamma_t(G) + \sum_{x \in V(G)} \gamma_t(H_x - \{x\}) = \gamma_t(G) + \mathrm{n}(G)(\gamma_t(H) - 1).$$

Therefore, the result follows. □

The following lemma is another important tool for determining all possible values of $\gamma_t(G \circ_v H)$.

Lemma 4. *Given a $\gamma_t(G \circ_v H)$-set S with $\mathcal{B}_S \neq \varnothing$, the following statements hold.*

(i) *If $\mathcal{B}_S \cap S \neq \emptyset$, then $\gamma_t(G \circ_v H) = \mathrm{n}(G)(\gamma_t(H) - 1)$.*
(ii) *If $\mathcal{B}_S \cap S = \varnothing$, then $\gamma_t(H - \{v\}) = \gamma_t(H) - 1$, and as a consequence,*

$$\gamma(G) + \mathrm{n}(G)(\gamma_t(H) - 1) \leq \gamma_t(G \circ_v H) \leq \gamma_t(G) + \mathrm{n}(G)(\gamma_t(H) - 1).$$

Proof. First, we proceed to prove (i). Given a fixed $x' \in \mathcal{B}_S \cap S$, let $D \subseteq V(G \circ_v H)$ such that for every $x \in V(G)$ the set D_x is induced by $S_{x'}$. Obviously, D is a TDS of $G \circ_v H$. Hence, $\gamma_t(G \circ_v H) \leq |D| = \sum_{x \in V(G)} |D_x| = \mathrm{n}(G)|S_{x'}| = \mathrm{n}(G)(\gamma_t(H) - 1)$. Therefore, Theorem 1 leads to $\gamma_t(G \circ_v H) = \mathrm{n}(G)(\gamma_t(H) - 1)$.

In order to prove (ii), assume that $\mathcal{B}_S \cap S = \varnothing$, and let $x \in \mathcal{B}_S$. By Lemma 3 we have that $N[x] \cap S_x = \varnothing$. So, $x \notin \mathcal{S}(H_x)$ and S_x is a TDS of $H_x - \{x\}$. Hence, $\gamma_t(H - \{v\}) = \gamma_t(H_x - \{x\}) \leq |S_x| = \gamma_t(H) - 1$, and so Lemma 1 leads to $\gamma_t(H - \{v\}) = \gamma_t(H) - 1$. Therefore, by Theorem 1 we have that $\gamma_t(G \circ_v H) \leq \gamma_t(G) + \mathrm{n}(G)(\gamma_t(H) - 1)$.

Moreover, since $N[x] \cap S_x = \varnothing$ for every $x \in \mathcal{B}_S$, we have that $\mathcal{A}_S$ is a dominating set of G. Hence,

$$\begin{aligned} \gamma_t(G \circ_v H) &= \sum_{x \in \mathcal{A}_S} |S_x| + \sum_{x \in \mathcal{B}_S} |S_x| \\ &\geq |\mathcal{A}_S|\gamma_t(H) + |\mathcal{B}_S|(\gamma_t(H) - 1) \\ &\geq |\mathcal{A}_S| + \mathrm{n}(G)(\gamma_t(H) - 1) \\ &\geq \gamma(G) + \mathrm{n}(G)(\gamma_t(H) - 1). \end{aligned}$$

Therefore, the result follows. □

Next we give one of the main results of this section, which states the four possible values of $\gamma_t(G \circ_v H)$.

Theorem 2. *Let G and H be two graphs with no isolated vertex. For any $v \in V(H)$,*

$$\gamma_t(G \circ_v H) \in \{\mathrm{n}(G)(\gamma_t(H) - 1), \gamma(G) + \mathrm{n}(G)(\gamma_t(H) - 1), \gamma_t(G) + \mathrm{n}(G)(\gamma_t(H) - 1), \mathrm{n}(G)\gamma_t(H)\}.$$

Proof. Let S be a $\gamma_t(G \circ_v H)$-set and consider the subsets $\mathcal{A}_S, \mathcal{B}_S \subseteq V(G)$ associated with S. We distinguish the following cases.

Case 1. $\mathcal{B}_S = \emptyset$. In this case, for any $x \in V(G)$ we have that $|S_x| \geq \gamma_t(H)$, and as a consequence, $\gamma_t(G \circ_v H) = \sum_{x \in V(G)} |S_x| \geq \mathrm{n}(G)\gamma_t(H)$. Thus, Theorem 1 leads to the equality $\gamma_t(G \circ_v H) = \mathrm{n}(G)\gamma_t(H)$.

Case 2. $\mathcal{B}_S \neq \emptyset$. If $\mathcal{B}_S \cap S \neq \emptyset$, then from Lemma 4 (i) we have that $\gamma_t(G \circ_v H) = \mathrm{n}(G)(\gamma_t(H) - 1)$.

From now on we assume that $\mathcal{B}_S \cap S = \emptyset$. Hence, Lemma 4 (ii) leads to

$$\gamma(G) + \mathrm{n}(G)(\gamma_t(H) - 1) \leq \gamma_t(G \circ_v H) \leq \gamma_t(G) + \mathrm{n}(G)(\gamma_t(H) - 1).$$

We only need to prove that $\gamma_t(G \circ_v H)$ only can take the extreme values. To this end, we shall need to introduce the following notation. Let $\mathcal{A}'_S = \{x \in \mathcal{A}_S : |S_x| = \gamma_t(H)\}$ and $\mathcal{A}''_S = \mathcal{A}_S \setminus \mathcal{A}'_S$.

Subcase 2.1. There exists $x' \in \mathcal{A}'_S$ such that $S_{x'}$ is a $\gamma_t(H_{x'})$-set containing x'. From a fixed vertex $y \in \mathcal{B}_S$ and any $\gamma(G)$-set D, we can construct a set $W \subseteq V(G \circ_v H)$ as follows. If $x \in D$, then W_x is induced by $S_{x'}$, while if $x \in V(G) \setminus D$, then W_x is induced by S_y. Notice that W is a TDS of $G \circ_v H$, which implies that $\gamma_t(G \circ_v H) \leq |W| = \gamma(G) + \mathrm{n}(G)(\gamma_t(H) - 1)$. Therefore, $\gamma_t(G \circ_v H) = \gamma(G) + \mathrm{n}(G)(\gamma_t(H) - 1)$.

Subcase 2.2. $\mathcal{A}'_S = \emptyset$ or for any $x \in \mathcal{A}'_S$, either S_x is not a $\gamma_t(H_x)$-set or $x \notin S_x$. If $\mathcal{A}'_S \neq \emptyset$, then every vertex $x \in \mathcal{A}'_S$ satisfies one of the following conditions.

(a) S_x is a $\gamma_t(H_x)$-set such that $x \notin S_x$.
(b) S_x is not a TDS of H_x and $x \in S_x$.

Notice that we do not consider the case where S_x is not a TDS of H_x and $x \notin S_x$, as in this case we can replace S with the $\gamma_t(G \circ_v H)$-set $(S \setminus S_x) \cup S'_x$ for some $\gamma_t(H_x)$-set S'_x. In such a case, if $x \in S'_x$, then we proceed as in Subcase 2.1, while if $x \notin S'_x$, then x satisfies (a).

Let us construct a TDS X of G as follows.

- $\mathcal{A}_S \subseteq X$.
- For any $x \in \mathcal{A}'_S$ which satisfies condition (a) and $N(x) \cap S \cap V(G) = \emptyset$, we choose one vertex $y \in N(x) \cap V(G)$ and set $y \in X$.
- For any $x \in \mathcal{A}''_S$ with $N(x) \cap S \cap V(G) = \emptyset$, we choose one vertex $y \in N(x) \cap V(G)$ and set $y \in X$.

We proceed to show that X is a TDS of G. If $x \in V(G) \setminus X$, then either $x \in \mathcal{B}_S$ or $x \in \mathcal{A}'_S \setminus S$. If $x \in \mathcal{B}_S$, then $N(x) \cap S \cap \mathcal{A}_S \neq \emptyset$, which implies that $N(x) \cap X \neq \emptyset$. Obviously, if $x \in \mathcal{A}'_S \setminus S$, then $N(x) \cap X \neq \emptyset$, by definition of X. Now, let $x \in X$. If $x \in \mathcal{A}''_S \cup (\mathcal{A}'_S \setminus S)$, then $N(x) \cap X \neq \emptyset$ by definition. If $x \in \mathcal{A}'_S \cap S$, then x satisfies condition (b). This implies that $N(x) \cap S_x = \emptyset$. Hence, there exists a vertex $y \in N(x) \cap V(G) \cap S \subseteq X$, as desired.

Therefore, X is a TDS of G, which implies that $\gamma_t(G) \leq |X| \leq 2|\mathcal{A}''_S| + |\mathcal{A}'_S|$. Thus,

$$\begin{aligned}
\gamma_t(G \circ_v H) &\geq \sum_{x \in \mathcal{A}''_S} |S_x| + \sum_{x \in \mathcal{A}'_S} |S_x| + \sum_{x \in \mathcal{B}_S} |S_x| \\
&\geq |\mathcal{A}''_S|(\gamma_t(H) + 1) + |\mathcal{A}'_S|\gamma_t(H) + |\mathcal{B}_S|(\gamma_t(H) - 1) \\
&\geq (2|\mathcal{A}''_S| + |\mathcal{A}'_S|) + \mathrm{n}(G)(\gamma_t(H) - 1) \\
&\geq \gamma_t(G) + \mathrm{n}(G)(\gamma_t(H) - 1),
\end{aligned}$$

which completes the proof. □

Later on, we will characterize the graphs that reach each of the previous expressions. However, we have to admit that when applying some of these characterizations we will need to calculate the total domination number of $H - \{v\}$ or $H - N[v]$ which may not be easy. Before giving the above mentioned characterizations, we shall show a simple example in which we can observe that these expressions of $\gamma_t(G \circ_v H)$ are realizable.

Example 1. *Let G be a graph with no isolated vertex. If H is one of the graphs shown in Figure 2, then the resulting values of $\gamma_t(G \circ_v H)$ for some specific roots are described below.*

- $\gamma_t(G \circ_{v'} H_2) = 3\,\mathrm{n}(G) = \mathrm{n}(G)(\gamma_t(H_2) - 1)$.
- $\gamma_t(G \circ_v H_2) = \gamma(G) + 3\,\mathrm{n}(G) = \gamma(G) + \mathrm{n}(G)(\gamma_t(H_2) - 1)$.
- $\gamma_t(G \circ_v H_1) = \gamma_t(G) + 2\,\mathrm{n}(G) = \gamma_t(G) + \mathrm{n}(G)(\gamma_t(H_1) - 1)$.
- $\gamma_t(G \circ_{v'} H_1) = \gamma_t(G \circ_{v''} H_1) = 3\,\mathrm{n}(G) = \mathrm{n}(G)\gamma_t(H_1)$.

For these cases, it is not difficult to construct a $\gamma_t(G \circ_v H)$-set. For instance, a $\gamma_t(G \circ_v H_2)$-set S can be formed as follows. Given a fixed $\gamma(G)$-set X, we take S in such a way that the set S_x is induced by $\{a, b, v', v\}$ for every $x \in X$, and induced by $\{a, b, c\}$ for every $x \in V(G) \setminus X$.

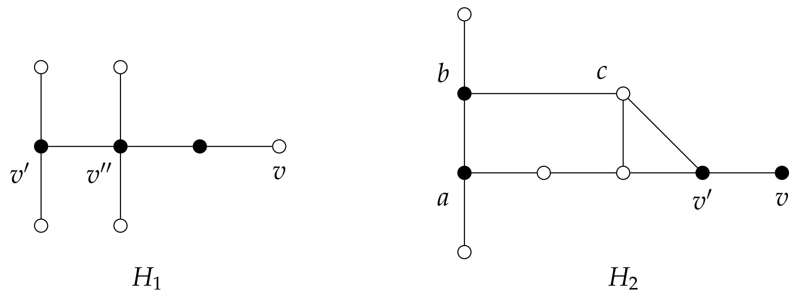

Figure 2. The set of black-coloured vertices forms a $\gamma_t(H_i)$-set for $i \in \{1, 2\}$. The set $\{v', v''\}$ forms a $\gamma_t(H_1 - \{v\})$-set, while $\{a, b, c\}$ forms a $\gamma_t(H_2 - \{v\})$-set.

As we have observed in Lemma 2, if $v \in V(H)$ is not a universal vertex and $H - N[v]$ does not have isolated vertices, then $\gamma_t(H - N[v]) \geq \gamma_t(H) - 2$. Next we show that the extreme case $\gamma_t(H - N[v]) = \gamma_t(H) - 2$ characterizes the graphs with $\gamma_t(G \circ_v H) = \mathrm{n}(G)(\gamma_t(H) - 1)$.

Theorem 3. *Given two graphs G and H with no isolated vertex and $v \in V(H)$, the following statements are equivalent.*

(i) $\gamma_t(G \circ_v H) = \mathrm{n}(G)(\gamma_t(H) - 1)$.
(ii) *v is a universal vertex of H or $\gamma_t(H - N[v]) = \gamma_t(H) - 2$.*

Proof. First, assume that (i) holds. Let S be a $\gamma_t(G \circ_v H)$-set. If v is a universal vertex of H, then we are done. Assume that $v \in V(H)$ is not a universal vertex. In this case, Lemma 3 leads to $\mathcal{B}_S = V(G)$ and $N(x) \cap S_x = \varnothing$ for every $x \in \mathcal{B}_S$. Thus, $\mathcal{B}_S \cap S$ is a dominating set of G and for any $x \in \mathcal{B}_S \cap S$ we have that $H_x - N[x]$ does not have isolated vertices and $S_x \setminus \{x\}$ is a TDS of $H_x - N[x]$, which implies that $\gamma_t(H - N[v]) = \gamma_t(H_x - N[x]) \leq |S_x \setminus \{x\}| = \gamma_t(H) - 2$. Hence, Lemma 2 leads to $\gamma_t(H - N[v]) = \gamma_t(H) - 2$. Therefore, (ii) follows.

Conversely, assume that (ii) holds. If v is a universal vertex of H, then $V(G)$ is a TDS of $G \circ_v H$, which implies that $\gamma_t(G \circ_v H) \leq |V(G)| = \mathrm{n}(G) = \mathrm{n}(G)(\gamma_t(H) - 1)$. Thus, by Theorem 1 we conclude that $\gamma_t(G \circ_v H) = \mathrm{n}(G)(\gamma_t(H) - 1)$.

From now on, we assume that v is not a universal vertex. For any $x \in V(G)$, let D'_x be a $\gamma_t(H_x - N[x])$-set and $D_x = D'_x \cup \{x\}$. Observe that $D = \cup_{x \in V(G)} D_x$ is a TDS of $G \circ_v H$, which implies that $\gamma_t(G \circ_v H) \leq |D| = \mathrm{n}(G)(\gamma_t(H - N[v]) + 1) = \mathrm{n}(G)(\gamma_t(H) - 1)$. By Theorem 1 we conclude that $\gamma_t(G \circ_v H) = \mathrm{n}(G)(\gamma_t(H) - 1)$, which completes the proof. □

Lemma 5. *Let G and H be two graphs with no isolated vertex and $v \in V(H) \setminus \mathcal{S}(H)$. If $\gamma_t(H - \{v\}) \geq \gamma_t(H)$, then*

$$\gamma_t(G \circ_v H) \in \{\mathrm{n}(G)\gamma_t(H), \mathrm{n}(G)(\gamma_t(H) - 1)\}.$$

Proof. By Theorem 1 we have that $\gamma_t(G \circ_v H) \leq \mathrm{n}(G)\gamma_t(H)$. Let S be a $\gamma_t(G \circ_v H)$-set. If $|S| = \mathrm{n}(G)\gamma_t(H)$, then we are done. Suppose that $|S| < \mathrm{n}(G)\gamma_t(H)$. Hence, there exists $x \in V(G)$ such that $|S_x| < \gamma_t(H)$, which implies that $x \in \mathcal{B}_S$ by Lemma 3. Since $\gamma_t(H - \{v\}) \geq \gamma_t(H)$, Lemma 4 (ii) leads to $x \in S$, and by Lemma 4 (i) we deduce that $\gamma_t(G \circ_v H) = \mathrm{n}(G)(\gamma_t(H) - 1)$. □

Lemma 6. *Let G and H be two graphs with no isolated vertex and $v \in V(H)$. If v belongs to every $\gamma_t(H)$-set, then*

$$\gamma_t(G \circ_v H) \in \{\mathrm{n}(G)\gamma_t(H), \mathrm{n}(G)(\gamma_t(H) - 1)\}.$$

Proof. We first consider the case where $v \in V(H) \setminus \mathcal{S}(H)$. By Lemma 1 we deduce that $\gamma_t(H - \{v\}) \geq \gamma_t(H)$, and so Lemma 5 leads to the result. Now, assume that $v \in \mathcal{S}(H)$ and let S be a $\gamma_t(G \circ H)$-set. If $\gamma_t(G \circ_v H) = \mathrm{n}(G)\gamma_t(H)$, then we are done. Thus, we assume that $\gamma_t(G \circ_v H) < \mathrm{n}(G)\gamma_t(H)$. In such a case, there exists $x \in \mathcal{B}_S$, and since $x \in \mathcal{S}(H_x)$, it follows that $x \in \mathcal{S}(G \circ H)$. Therefore, $x \in S$, and by Lemma 4 (i) we deduce that $\gamma_t(G \circ_v H) = \mathrm{n}(G)(\gamma_t(H) - 1)$, which completes the proof. □

We are now ready to characterize the graphs with $\gamma_t(G \circ_v H) = \gamma(G) + \mathrm{n}(G)(\gamma_t(H) - 1)$.

Theorem 4. *Let G and H be two graphs with no isolated vertex and $v \in V(H)$. The following statements are equivalent.*

(i) $\gamma_t(G \circ_v H) = \gamma(G) + \mathrm{n}(G)(\gamma_t(H) - 1)$.
(ii) $\gamma_t(H - N[v]) = \gamma_t(H - \{v\}) = \gamma_t(H) - 1$, *and in addition,* $\gamma_t(G) = \gamma(G)$ *or there exists a* $\gamma_t(H)$*-set* D *such that* $v \in D$.

Proof. First, assume that (i) holds. Since $1 \leq \gamma(G) < \mathrm{n}(G)$, by Lemma 6, $v \notin \mathcal{S}(H)$, so that from Lemma 5 we deduce that $\gamma_t(H - \{v\}) \leq \gamma_t(H) - 1$ and Lemma 1 leads to $\gamma_t(H - \{v\}) = \gamma_t(H) - 1$. Hence, by Lemma 2 it follows that $\gamma_t(H - N[v]) \in \{\gamma_t(H) - 2, \gamma_t(H) - 1\}$ and by Theorem 3 we obtain that $\gamma_t(H - N[v]) = \gamma_t(H) - 1$.

Now, let S be a $\gamma_t(G \circ_v H)$-set. Since $1 \leq \gamma(G) < \mathrm{n}(G)$, Lemma 3 leads to $\mathcal{A}_S \neq \varnothing$ and $\mathcal{B}_S \neq \varnothing$. Additionally, by Lemma 4 we deduce that $\mathcal{B}_S \cap S = \varnothing$, and by Lemma 3 we have that $N(x) \cap S_x = \varnothing$ for every $x \in \mathcal{B}_S$. Hence, $\mathcal{A}_S$ is a dominating set of G and $\mathcal{A}_S \cap S \neq \varnothing$. Thus, $\gamma_t(G \circ_v H) \geq |\mathcal{A}_S| + \mathrm{n}(G)(\gamma_t(H) - 1) \geq \gamma(G) + \mathrm{n}(G)(\gamma_t(H) - 1) = \gamma_t(G \circ_v H)$, which implies that $\mathcal{A}_S$ is a $\gamma(G)$-set and for every $x \in \mathcal{A}_S \cap S$ we have that $|S_x| = \gamma_t(H)$. Therefore, there exists $x \in \mathcal{A}_S \cap S$ such that S_x is a $\gamma_t(H_x)$-set or $\mathcal{A}_S$ is a $\gamma_t(G)$-set, which implies that (ii) holds.

Conversely, assume that (ii) holds. As above, let S be a $\gamma_t(G \circ_v H)$-set. Since $\gamma_t(H - \{v\}) = \gamma_t(H) - 1$, by Theorem 1, $\gamma_t(G \circ_v H) \leq \gamma_t(G) + \mathrm{n}(G)(\gamma_t(H) - 1)$.

Suppose that $\mathcal{B}_S = \varnothing$. In such a case, $\gamma_t(G \circ_v H) = \mathrm{n}(G)\gamma_t(H)$, which implies that $\gamma(G) < \gamma_t(G) = \mathrm{n}(G)$, and so $G \cong \cup K_2$. Let $A \cup B = V(G)$ be the bipartition of the vertex set of G, i.e., every edge has one endpoint in A and the other one in B. Thus, for every $x \in V(G)$ we define a subset $Y_x \subseteq V(H_x)$ as follows. If $x \in A$, then Y_x is a $\gamma_t(H_x)$-set which contains x, while if $x \in B$, then Y_x is a $\gamma_t(H_x - \{x\})$-set. Hence, $Y = \cup_{x \in V(G)} Y_x$ is a TDS of $G \circ_v H$ and so $\gamma_t(G \circ_v H) \leq |Y| = \mathrm{n}(G)\gamma_t(H) - \frac{\mathrm{n}(G)}{2} < \mathrm{n}(G)\gamma_t(H)$, which is a contradiction. From now on we assume that $\mathcal{B}_S \neq \varnothing$.

If there exists a vertex $x \in \mathcal{B}_S \cap S$, then by Lemma 3 we have that $N(x) \cap S_x = \varnothing$, which implies that $S_x \setminus \{x\}$ is a TDS of $H_x - N[x]$. Hence, $\gamma_t(H - N[v]) = \gamma_t(H_x - N[x]) \leq |S_x \setminus \{x\}| = \gamma_t(H) - 2$, which is a contradiction with the assumption $\gamma_t(H - N[v]) = \gamma_t(H) - 1$. Therefore, $\mathcal{B}_S \cap S = \varnothing$, and by Lemma 4 we deduce that $\gamma_t(G \circ_v H) \geq \gamma(G) + \mathrm{n}(G)(\gamma_t(H) - 1)$.

It is still necessary to prove that $\gamma_t(G \circ_v H) \leq \gamma(G) + \mathrm{n}(G)(\gamma_t(H) - 1)$. If $\gamma(G) = \gamma_t(G)$, then we are done. Assume $\gamma(G) < \gamma_t(G)$. Now we take a $\gamma(G)$-set X and for every $x \in V(G)$ we define a set $Z_x \subseteq V(H_x)$ as follows. If $x \in X$, then Z_x is a $\gamma_t(H_x)$-set such that $x \in Z_x$, while if $x \in V(G) \setminus X$, then Z_x is a $\gamma_t(H_x - \{x\})$-set. Notice that $Z = \cup_{x \in V(G)} Z_x$ is a TDS of $G \circ_v H$. Therefore, $\gamma_t(G \circ_v H) \leq |Z| = \gamma(G) + \mathrm{n}(G)(\gamma_t(H) - 1)$, as required. □

Next we proceed to characterize the graphs with $\gamma_t(G \circ_v H) = \gamma_t(G) + \mathrm{n}(G)(\gamma_t(H) - 1)$. Notice that it is excluded the case $G \cong \cup K_2$. In such a case, $\gamma_t(G) = \mathrm{n}(G)$, and so $\gamma_t(G) + \mathrm{n}(G)(\gamma_t(H) - 1) = \mathrm{n}(G)\gamma_t(H)$, which implies that the characterization of this particular case can be derived by elimination from Theorems 3 and 4. Analogously, the case $\gamma(G) = \gamma_t(G)$ is excluded, as it was discusses in Theorem 4.

Theorem 5. *Let $G \ncong \cup K_2$ and H be two graphs with no isolated vertex such that $\gamma(G) < \gamma_t(G)$, and let $v \in V(H)$. The following statements are equivalent.*

(i) $\gamma_t(G \circ_v H) = \gamma_t(G) + \mathrm{n}(G)(\gamma_t(H) - 1)$.

(ii) $\gamma_t(H - \{v\}) = \gamma_t(H) - 1$ *and* $v \notin D$ *for every* $\gamma_t(H)$*-set* D.

Proof. First, assume that (i) holds. Since, $G \ncong \cup K_2$, we have that $\gamma_t(G) < \mathrm{n}(G)$. Thus, by Lemma 6, $v \notin \mathcal{S}(H)$ and then by Lemma 5 we deduce that $\gamma_t(H - \{v\}) \leq \gamma_t(H) - 1$ and Lemma 1 leads to $\gamma_t(H - \{v\}) = \gamma_t(H) - 1$.

Suppose that there exists a $\gamma_t(H)$-set containing v. Let X be a $\gamma(G)$-set. For every $x \in V(G)$ we define a set $Z_x \subseteq V(H_x)$ as follows. If $x \in X$, then Z_x is a $\gamma_t(H_x)$-set such that $x \in Z_x$, while if $x \in V(G) \setminus X$, then Z_x is a $\gamma_t(H_x - \{x\})$-set. Notice that $Z = \cup_{x \in V(G)} Z_x$ is a TDS of $G \circ_v H$. Therefore, $\gamma_t(G \circ_v H) \leq |Z| = \gamma(G) + \mathrm{n}(G)(\gamma_t(H) - 1)$, which is a contradiction, as $\gamma_t(G) > \gamma(G)$. Therefore, $v \notin D$ for every $\gamma_t(H)$-set D, which implies that (ii) follows.

Conversely, assume that (ii) holds. Since $\gamma_t(H - \{v\}) = \gamma_t(H) - 1$, by Theorem 1 we have that $\gamma_t(G \circ_v H) \leq \gamma_t(G) + \mathrm{n}(G)(\gamma_t(H) - 1)$. Let S be a $\gamma_t(G \circ_v H)$-set. If $\mathcal{B}_S = \varnothing$, then $\gamma_t(G \circ_v H) = \mathrm{n}(G)\gamma_t(H)$, and so $\gamma_t(G) = \mathrm{n}(G)$, which is a contradiction, as $G \ncong \cup K_2$. Hence, from now on we assume that $\mathcal{B}_S \neq \varnothing$.

If there exists a vertex $x \in \mathcal{B}_S \cap S$, then for any vertex $y \in N(v) \cap V(H_x)$, the set $S_x \cup \{y\}$ is a $\gamma_t(H_x)$-set, which is a contradiction. Thus, $\mathcal{B}_S \cap S = \varnothing$, and so by Lemma 3, $\mathcal{A}_S$ is a dominating set of G. Moreover, by Lemma 4 and Theorem 2 we deduce that either $\gamma_t(G \circ_v H) = \gamma(G) + n(G)(\gamma_t(H) - 1)$ or $\gamma_t(G \circ_v H) = \gamma_t(G) + n(G)(\gamma_t(H) - 1)$. Now, let $\mathcal{A}_S = A^- \cup A^+$ where $x \in A^-$ if $x \in \mathcal{A}_S$ and $N(x) \cap \mathcal{A}_S = \varnothing$. Let $B \subseteq \mathcal{B}_S$ such that $|B| \leq |A^-|$ and $N(x) \cap B \neq \varnothing$ for every $x \in A^-$. Obviously, $B \cup A^+$ is a total dominating set of G, and so $\gamma_t(G) + n(G)(\gamma_t(H) - 1) \leq |B \cup A^+| + n(G)(\gamma_t(H) - 1) \leq |\mathcal{A}_S| + n(G)(\gamma_t(H) - 1) \leq \gamma_t(G \circ_v H)$. Therefore, the result follows. □

From Theorem 2 we learned that there are four possible expressions for $\gamma_t(G \circ_v H)$. In the case of the first three expressions, the graphs (and the root) reaching the equality were characterized in Theorems 3–5. In the case of the expression $\gamma_t(G \circ_v H) = \mathrm{n}(G)\gamma_t(H)$, the corresponding characterization can be derived by elimination from the previous results, although it must be recognized that the formulation of such a characterization is somewhat cumbersome. To conclude this section, we will just give a couple of examples where this expression is obtained.

The following result shows an example where $\gamma_t(G \circ_v H) = \mathrm{n}(G)\gamma_t(H)$, which covers the cases in which v is a neighbor of a support vertex, excluding the case where v is the only leaf adjacent to its support.

Proposition 1. *Let G and H be two graphs with no isolated vertex and $v \in V(H)$. If there exists $u \in N(v)$ such that $N(u) \cap (\mathcal{L}(H) \setminus \{v\}) \neq \varnothing$, then*

$$\gamma_t(G \circ_v H) = \mathrm{n}(G)\gamma_t(H).$$

Proof. Assume first that $v \notin \mathcal{S}(H)$. Let D be a $\gamma_t(H - \{v\})$-set. Since $u \in \mathcal{S}(H - \{v\})$, we have that $u \in D$. Hence, D is a TDS of H, and so $\gamma_t(H - \{v\}) = |D| \geq \gamma_t(H)$. Therefore, Lemma 5 leads to $\gamma_t(G \circ_v H) = \mathrm{n}(G)\gamma_t(H)$ or $\gamma_t(G \circ_v H) = \mathrm{n}(G)(\gamma_t(H) - 1)$. Now, suppose that $\gamma_t(G \circ_v H) = \mathrm{n}(G)(\gamma_t(H) - 1)$. Let S be a $\gamma_t(G \circ_v H)$-set. By Lemma 3, $\mathcal{B}_S = V(G)$ and $N(x) \cap S_x = \varnothing$

for every $x \in \mathcal{B}_S$, which is a contradiction, as $N(x) \cap \mathcal{S}(H_x) \neq \emptyset$ and $\mathcal{S}(H_x) \subseteq S_x$. Therefore, $\gamma_t(G \circ_v H) = \mathrm{n}(G)\gamma_t(H)$.

Now, if $v \in \mathcal{S}(H)$, then $u, v \in \mathcal{S}(G \circ_v H)$. Hence, for every $\gamma_t(G \circ_v H)$-set S and every vertex $x \in V(G)$, we have that S_x is a TDS of H_x. Thus, $\mathcal{B}_S = \emptyset$, which implies that $\gamma_t(G \circ_v H) = \mathrm{n}(G)\gamma_t(H)$, as required. □

We next consider another example where $\gamma_t(G \circ_v H) = \mathrm{n}(G)\gamma_t(H)$.

Proposition 2. *Let G and H be two graphs with no isolated vertex and $v \in V(H) \setminus \mathcal{S}(H)$. If $\gamma_t(H - \{v\}) \geq \gamma_t(H)$ and v does not belong to any $\gamma_t(H)$-set, then*

$$\gamma_t(G \circ_v H) = \mathrm{n}(G)\gamma_t(H).$$

Proof. If $\gamma_t(H - \{v\}) \geq \gamma_t(H)$, then by Lemma 5 we have that $\gamma_t(G \circ_v H) = \mathrm{n}(G)\gamma_t(H)$ or $\gamma_t(G \circ_v H) = \mathrm{n}(G)(\gamma_t(H) - 1)$. Now, assume that v does not belong to any $\gamma_t(H)$-set. If $\gamma_t(G \circ_v H) = \mathrm{n}(G)(\gamma_t(H) - 1)$, then $\mathcal{B}_S = V(G)$. Hence, by Lemma 4 (ii) there exists $x \in \mathcal{B}_S \cap S$, which is a contradiction as from any $x' \in N(x) \cap V(H_x)$ the set $S_x \cup \{x'\}$ is a $\gamma_t(H_x)$-set containing x. Therefore, $\gamma_t(G \circ_v H) = \mathrm{n}(G)\gamma_t(H)$. □

2. An Observation on the Domination Number

It was shown in [15] that there are two possibilities for the domination number of a rooted product graph. Since the graphs reaching these expressions have not been characterized, we consider that it is appropriate to derive a result in this direction. Specifically, we will provide a characterization in Theorem 7.

Theorem 6. [15] *For any nontrivial graphs G and H and any $v \in V(H)$,*

$$\gamma(G \circ_v H) \in \{\mathrm{n}(G)\gamma(H), \gamma(G) + \mathrm{n}(G)(\gamma(H) - 1)\}.$$

In order to derive our result, we need to introduce the following two lemmas.

Lemma 7. [21] *Let H be a graph. For any vertex $v \in V(H)$,*

$$\gamma(H - \{v\}) \geq \gamma(H) - 1.$$

Lemma 8. *For any $\gamma(G \circ_v H)$-set D and any vertex $x \in V(G)$,*

$$|D_x| \geq \gamma(H) - 1.$$

Furthermore, if $|D_x| = \gamma(H) - 1$, then $N[x] \cap D_x = \emptyset$.

Proof. Let $x \in V(G)$. Notice that every vertex in $V(H_x) \setminus \{x\}$ is adjacent to some vertex in D_x. Since $D_x \cup \{x\}$ is a dominating set of H_x, we have that $\gamma(H) = \gamma(H_x) \leq |D_x \cup \{x\}| \leq |D_x| + 1$, as required.

Now, assume that $|D_x| = \gamma(H) - 1$. If $N[x] \cap D_x \neq \emptyset$, then D_x is a dominating set of H_x, which is a contradiction as $|D_x| = \gamma(H_x) - 1$. Therefore, the result follows. □

Theorem 7. *For any pair of nontrivial graphs G and H, and any $v \in V(H)$,*

$$\gamma(G \circ_v H) = \begin{cases} \gamma(G) + \mathrm{n}(G)(\gamma(H) - 1) & \text{if } \gamma(H - \{v\}) = \gamma(H) - 1, \\ \mathrm{n}(G)\gamma(H) & \text{otherwise.} \end{cases}$$

Proof. By Theorem 6 we only need to prove that $\gamma(G \circ_v H) = \gamma(G) + \mathrm{n}(G)(\gamma(H) - 1)$ if and only if $\gamma(H - \{v\}) = \gamma(H) - 1$.

We first assume $\gamma(H - \{v\}) = \gamma(H) - 1$. Let $D \subseteq V(G \circ_v H)$ such that $D_x^- = D_x \setminus \{x\}$ is a $\gamma(H_x - \{x\})$-set for every $x \in V(G)$, and $D \cap V(G)$ is a $\gamma(G)$-set. It is readily seen that D is a dominating set of $G \circ_v H$, which implies that $\gamma(G \circ_v H) \leq |D| = \gamma(G) + \sum_{x \in V(G)} |D_x^-| = \gamma(G) + \mathrm{n}(G)(\gamma(H) - 1)$, and by Theorem 6 we conclude that the equality holds.

Conversely, assume $\gamma(G \circ_v H) = \gamma(G) + \mathrm{n}(G)(\gamma(H) - 1)$. Let S be a $\gamma(G \circ_v H)$-set. Since $|S| < \mathrm{n}(G)\gamma(H)$, there exists $x \in V(G)$ such that $|S_x| < \gamma(H)$. Hence, by Lemma 8, $|S_x| = \gamma(H) - 1$ and $N[x] \cap S_x = \varnothing$. This implies that S_x is a dominating set of $H_x - \{x\}$, and so $\gamma(H - \{v\}) = \gamma(H_x - \{x\}) \leq |S_x| = \gamma(H) - 1$. By Lemma 7 we conclude that $\gamma(H - \{v\}) = \gamma(H) - 1$, which completes the proof. $\square$

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