Universal lines in graphs

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Abstract

In a metric space $M=(X,d)$, a line induced by two distinct points $x,x'\in X$, denoted by $\mathcal{L}_M\{x,x'\}$, is the set of points given by

$$\mathcal{L}_M\{x,x'\}=\{z\in X:\ d(x,x')=d(x,z)+d(z,x') \text{ or } d(x,x')=|d(x,z)-d(z,x')|\}.$$

A line $\mathcal{L}_M\{x,x'\}$ is universal whenever $\mathcal{L}_M\{x,x'\}=X$.

Chen and Chvátal [Disc. Appl. Math. 156 (2008), 2101-2108.] conjectured that in any finite metric space $M=(X,d)$ either there is a universal line, or there are at least $|X|$ different (nonuniversal) lines. A particular problem derived from this conjecture consists of investigating the properties of M that determine the existence of a universal line, and the problem remains interesting even if we can check that M has at least $|X|$ different lines. Since the vertex set of any connected graph, equipped with the shortest path distance, is a metric space, the problem automatically becomes of interest in graph theory. In this paper, we address the problem of characterizing graphs that have universal lines. We consider several scenarios in which the study can be approached by analysing the existence of such lines in primary subgraphs. We first discuss the wide class of separable graphs, and then describe some particular cases, including those of block graphs, rooted product graphs and corona graphs. We also discuss important classes of nonseparable graphs, including Cartesian product graphs, join graphs and lexicographic product graphs.

Keywords: Lines in graphs, universal lines, distance in graph, product graphs, metric spaces.

Mathematics Subject Classification: 30L99, 05C12, 05C76

1 Introduction

In a metric space $M=(X,d_M)$ a line induced by two distinct points $x,x'\in X$ is defined by

$$\mathcal{L}_M\{x,x'\}=\{z\in X:\ d_M(x,x')=d_M(x,z)+d_M(z,x') \text{ or } d_M(x,x')=|d_M(x,z)-d_M(z,x')|\}.$$

A line $\mathcal{L}_M\{x,x'\}$ is universal whenever $\mathcal{L}_M\{x,x'\} = X$.

For instance, if $M = \mathbb{R}^n$ is the n-dimensional Euclidean space, then for any pair of points x,x', the line $\mathcal{L}_{\mathbb{R}^n}\{x,x'\}$, equipped with its usual metric inherited from $\mathbb{R}^n$, is isometric to the one-dimensional Euclidean space. Hence, the n-dimensional Euclidean space has a universal line if and only if $n = 1$. Now, if a metric space M is equipped with the discrete metric[1], then $\mathcal{L}_M\{x,x'\} = \{x,x'\}$ for every pair of points $x,x' \in X$.

There are metric spaces with metrics that may differ from the discrete metric yet generate the same topology. Such spaces are called *discrete metric spaces* [16], i.e., a metric space M is called a discrete metric space if and only if all its subsets are open (and therefore closed) in M. For instance, the set $\mathbb{N}$, with its usual metric inherited from the one-dimensional Euclidean space $\mathbb{R}$, is a discrete metric space; every finite metric space is a discrete metric space; and the vertex set of any connected graph, equipped with the shortest path distance, is a discrete metric space.

In this context, the following conjecture was stated by Chen and Chvátal [6] for the case finite metric spaces, where $\ell(M)$ denotes the number of distinct lines in M.

Conjecture 1.1. [6] *Any finite metric space $M = (X,d_M)$ with at least two points and $\ell(M) < |X|$ has a universal line.*

Conjecture 1.1 is an attempt to generalize a classic theorem of Euclidean geometry which asserts that any noncollinear set of n points in the plane determines at least n distinct lines. The problem remains open and, in particular, it was shown in [1] that any finite metric space on n points ($n \geq 2$) with no universal line has at least $(\frac{1}{\sqrt{2}} - o(1))\sqrt{n}$ distinct lines. Recently, Chvátal [7] described the main progress related to the conjecture, and pointed out twenty-nine related open problems plus three additional conjectures.

A problem derived from Conjecture 1.1 consists of investigating the properties of metric spaces having a universal line, and the problem remains interesting even if we can check that $\ell(M) \geq |X|$. In this paper, we deal with related problems for the particular case of graphs.

To ease the presentation we will refer to the line induced by two distinct vertices $u,v \in V(G)$ of a graph G as $\mathcal{L}_G\{u,v\}$. For instance, $\mathcal{L}_{K_n}\{u,v\} = \{u,v\}$ for every pair of distinct vertices u,v of a complete graph K_n. Hence, a complete graph of order n has a universal line if and only if $n = 2$. In contrast, it is easy to check that for any pair of adjacent vertices $u,v \in V(T)$ of a tree T, the line $\mathcal{L}_T\{u,v\}$ is universal. This implies that any nontrivial tree has a universal line. Another interesting example is the cycle C_4, where $\mathcal{L}_{C_4}\{u,v\}$ is universal for every $u,v \in V(C_4)$.

Given a graph G, let $\mu(G)$ be the number of pair of vertices of G inducing a universal line. In this paper we face the following problems.

Problem 1.2 (Existence)**.** *Determine necessary and sufficient conditions for a graph to have a universal line, i.e., graphs with $\mu(G) \geq 1$.*

Problem 1.3 (Uniqueness)**.** *Determine necessary and sufficient conditions for a graph to have exactly one pair of vertices inducing a universal line, i.e., graphs with $\mu(G) = 1$.*

The problem of characterizing all graphs where all lines are universal was solved in [13].

[1] The discrete metric is given by $d_M(x,y) = 0$ if $x = y$ and $d_M(x,y) = 1$ otherwise.

Theorem 1.4 (Totality, [13]). *If G is a connected nontrivial graph, then $\mu(G) = \binom{|V(G)|}{2}$ if and only if $G \cong P_n$ or $G \cong C_4$.*

In this paper, we address Problems 1.2 and 1.3 by analysing the existence of universal lines in primary subgraphs. In particular, in Section 3 we discuss the wide class of separable graphs. We first discuss the general case, and then describe some particular cases, including those of block graphs, bridgeless graphs where all cycles are odd, rooted product graphs and corona graphs. The remaining sections concern the case of nonseparable graphs. Sections 4, 5, and 6 are devoted to Cartesian product graphs, join graphs and lexicographic product graphs, respectively.

2 Notation, terminology and examples

Throughout the paper, we will use the notation $G \cong H$ if G and H are isomorphic graphs, and $u \sim v$ if u and v are adjacent vertices. Given a graph G, the open neighbourhood of a vertex $v \in V(G)$ will be denoted by $N_G(v)$. As usual, if $N_G(v) = V(G) \setminus \{v\}$, then we will say that v is a universal vertex of G, while if $N_G(v) = \varnothing$, then we will say that v is an isolated vertex of G. The degree of v will be denoted by $\deg_G(v) = |N_G(v)|$, while the minimum degree of G will be denoted by $\delta(G)$, i.e. $\delta(G) = \min\{\deg_G(v) : v \in V(G)\}$.

Given two distinct vertices $u, v \in V(G)$ of a connected graph G, we write $[uv]$ for the *geodesic closure* of $\{u, v\}$, i.e.,

$$[uv] = \{w \in V(G) : d_G(u,v) = d_G(u,w) + d_G(w,v)\}.$$

Hence, $w \in \mathcal{L}_G\{u,v\}$ if and only if $w \in [uv]$ or $u \in [wv]$ or $v \in [uw]$.

In general, the *geodesic closure* of a non-singleton set $S \subseteq V(G)$ is defined to be

$$[S] = \bigcup_{u,v \in S} [uv].$$

A subset $S \subseteq V(G)$ is a *geodetic set* of G if $[S] = V(G)$. The study of geodetic sets in graphs was introduced about 30 years ago by Harary and his coworkers [4, 5, 11]. The *geodetic number* of G, denoted by $\mathrm{g}(G)$, is defined as the minimum cardinality among all geodetic sets of G.

The following straightforward remark describes the case of universal lines induced by geodetic sets.

Remark 2.1. *Let G be a connected graph. If $\mathrm{g}(G) = 2$, then G has a universal line.*

The number of geodetic sets of cardinality two will be denoted by $\mu_{\mathrm{g}}(G)$. An example of graph with $\mu_{\mathrm{g}}(G) = 1$ is shown in Figure 1.

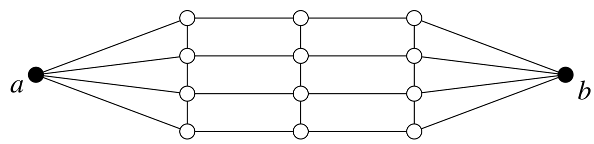

Figure 1: A graph G with $\mathrm{g}(G) = 2$ and $\mu_{\mathrm{g}}(G) = 1$. In this case, $\mathcal{L}_G\{a,b\}$ is a universal line induced by the geodetic set $\{a,b\}$.

An edge e of a connected graph G is a bridge if $G - e$ is not connected. The following remark is straightforward.

Remark 2.2. *If $\{u,v\}$ is a bridge of a connected graph G, then $\mathcal{L}_G\{u,v\}$ is universal.*

The remark above describes a particular case of universal lines induced by adjacent vertices. The number of pairs of adjacent vertices $g, g' \in V(G)$ such that $\mathcal{L}_G\{g,g'\}$ is universal, will be denoted by $\mu_a(G)$. For instance, in the case of the standard cube Q_3, we have that $\mu_{\mathrm{g}}(Q_3) = 4$, $\mu_a(G) = 12$ and $\mu(G) = \mu_{\mathrm{g}}(Q_3) + \mu_a(Q_3) = 16$. Figure 2, shows a 2-connected graph G where $\mathcal{L}_G\{x,y\}$ is a universal line induced by two adjacent vertices. In this case, $\mu_a(G) = 2$, $\mu_{\mathrm{g}}(G) = 0$ and $\mu(G) = \mu_a(G) + \mu_{\mathrm{g}}(G) = 2$.

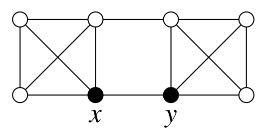

Figure 2: $\mathcal{L}_G\{x,y\}$ is a universal line induced by two adjacent vertices.

A wide family of graphs where $\mu_a(G)$ equals the size of G is the class of connected bipartite graphs. As observed by Beaudou et al. [3], the following result solves Conjecture 1.1 for the case of bipartite graphs.

Theorem 2.3. [3] *If G is a connected bipartite graph of order at least two, then $\mu(G) \geq 1$.*

We would emphasize that Conjecture 1.1 was also proved in [3] for connected chordal graphs.

Theorem 2.4. [3] *If G is a connected chordal graph of order at least two, then either $\ell(G) \geq |V(G)|$ or $\mu(G) \geq 1$.*

A graph G is distance-hereditary if for any connected induced subgraph H of G and for any pair of vertices $x, y \in V(H)$, $d_H(x,y) = d_G(x,y)$. The following lemma was stated in [2].

Lemma 2.5. [2] *If $x, y \in V(G)$ are adjacent vertices of a connected distance-hereditary graph, then either they belong to a triangle or $\mathcal{L}_G\{x,y\}$ is universal.*

This Lemma was used by Aboulker and Kapadia [2] to prove Conjecture 1.1 for the case of distance-hereditary graphs.

Theorem 2.6. [2] *If G is a connected distance-hereditary graph of order at least two, then either $\ell(G) \geq |V(G)|$ or $\mu(G) \geq 1$.*

There are universal lines that are not induced, neither by geodetic sets nor by adjacent vertices. Figure 3 shows a graph G such that $\mathrm{g}(G) > 2$ where no pair of adjacent vertices induces a universal line, but there are two vertices $a, b \in V(G)$ such that $d_G(a,b) \geq 2$ and $\mathcal{L}_G\{a,b\}$ is universal. In this case, $\mu(G) = 1$, while $\mu_{\mathrm{g}}(G) = \mu_a(G) = 0$.

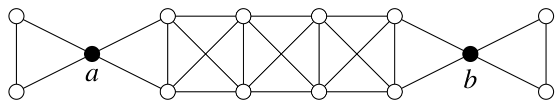

Figure 3: $\mathcal{L}_G\{a,b\}$ is universal.

The graph in Figure 4 does not have universal line.

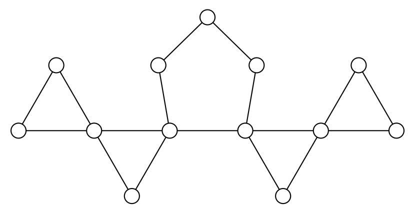

Figure 4: A graph G with $\mu(G) = 0$.

For the remainder of the paper, definitions will be introduced whenever a concept is needed. In particular, this is the case for concepts, notation and terminology to be used only once or in an specific section.

3 Separable graphs

Let $\mathcal{H} = \{G_1, \ldots, G_k\}$ be a family of pairwise disjoint (nontrivial) connected graphs. Consider a connected graph G constructed from $\mathcal{H}$ in the following way. First, we select one vertex of G_1, one vertex of G_2, and identify these two vertices. Afterwards, continue this procedure inductively. That is, if r graphs $G_1, \ldots, G_r$ have been used in the construction, where $r \in \{2, \ldots, k-1\}$, then select one vertex in the already constructed graph (this vertex may be one of the already selected vertices) and one vertex of G_{r+1}, and then identify these two vertices. We say that G is obtained by *point-attaching* from $\mathcal{H}$ and that the graphs in $\mathcal{H}$ are the *primary subgraphs* of G. Furthermore, the vertices of G obtained by identifying two vertices of different primary subgraphs are the *attachment vertices* of G. Obviously, the attachment vertices are cut vertices of G. We denote by $A(G)$ the set of attachment vertices of G and by $A(G_i)$ the set of attachment vertices of G belonging to $V(G_i)$. Figure 5 illustrates a sketch of a graph obtained in this manner.

The construction described above was introduced by Deutsch and Klavžar in [8], where they used it to compute the Hosoya polynomials of graphs. After that, this construction has been used by several authors. For instance, it was used in [9] to study the terminal Hosoya polynomial of composite graphs, in [15] to compute the local metric dimension of graphs, and in [14] to compute the metric dimension.

Observe that any graph, constructed by point-attaching from a family of connected graphs, has a tree-like structure, where the primary subgraphs are its building stones.

We would highlight the following remark, which is ease to see from the inductive construction described above.

Remark 3.1. *Let G be a graph constructed by point-attaching from a family $\mathcal{H}$ of connected graphs. Then the following statements hold.*

(i) $d_G(x,y) = d_{G_i}(x,y)$ *for every* $G_i \in \mathcal{H}$ *and* $x,y \in V(G_i)$.

(ii) *If* $G_i, G_j \in \mathcal{H}$ *are two different graphs, then* $|V(G_i) \cap V(G_j)| = |A(G_i) \cap A(G_j)| \leq 1$.

(iii) *Let* $G_i \in \mathcal{H}$ *and* $G_j \in \mathcal{H} \setminus \{G_i\}$. *For any* $u \in V(G_j)$, *there exists* $a \in A(G_i)$ *such that* $d_G(u,a) = \min\{d_G(u,w) : w \in V(G_i)\}$.

A connected nontrivial graph is said to be *separable* if it can be disconnected by removing one vertex, ie., if it has a cut vertex. Now, a *nonseparable graph* is a nontrivial connected graph containing no cut vertices.

Remark 3.2. *Every separable graph is obtained by point-attaching from a family of nonseparable graphs.*

The class of separable graphs contains several well-known families of graphs. For instance, this is the case of cactus graphs, block graphs, chains of graphs, circuits of graphs, corona product graphs, rooted product graphs, bouquets of graphs, etc. Some of these families of graphs are described below.

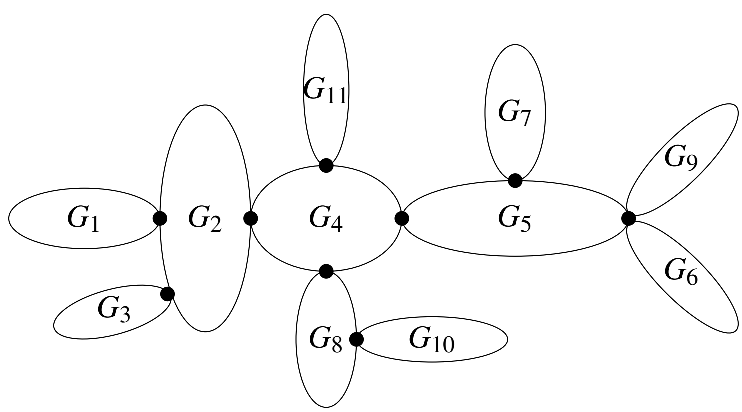

Figure 5: Sketch of a graph G constructed by point-attaching from a family $\mathcal{H} = \{G_1, \ldots, G_{11}\}$ of primary subgraphs.

The following result shows that the problem of determining the existence of universal lines in separable graphs carries over to the problem of terminating the existence of universal lines in their primary subgraphs.

Theorem 3.3 (Existence)**.** *A separable graph G has a universal line if and only if there exists a primary subgraph G_i of G and $x,y \in V(G_i)$ such that $\mathcal{L}_{G_i}\{x,y\}$ is universal and* $A(G_i) \cap ([xy] \setminus \{x,y\}) = \varnothing$.

Proof. Let G be a graph obtained by point-attaching from a family of connected nontrivial graphs $\mathcal{H} = \{G_1, \ldots, G_k\}$.

Assume that there exists a primary subgraph $G_i \in \mathcal{H}$ having a universal line $\mathcal{L}_{G_i}\{x,y\}$ such that $A(G_i) \cap ([xy] \setminus \{x,y\}) = \varnothing$. Let $G_j \in \mathcal{H} \setminus \{G_i\}$ and $u \in V(G_j)$. Let $v \in A(G_i)$ such that $d_G(u,v) = \min\{d_G(u,w) : w \in V(G_i)\}$. Since $x \in [vy]$ or $y \in [vx]$, we conclude that $x \in [uy]$ or $y \in [ux]$, which implies that $\mathcal{L}_G\{x,y\}$ is universal.

Conversely, let $\mathcal{L}_G\{x,y\}$ be a universal line. If there exists $G_i \in \mathcal{H}$ such that $x,y \in V(G_i)$, then the set $A(G_i) \cap ([xy] \setminus \{x,y\})$ has to be empty, otherwise for any $a \in A(G_i) \cap ([xy] \setminus \{x,y\})$ there exists $G_j \in \mathcal{H} \setminus \{G_i\}$ such that $a \in V(G_j)$ and $(V(G_j) \setminus \{a\}) \cap \mathcal{L}_G\{x,y\} = \varnothing$, which is a contradiction.

From now on, assume that $\{x,y\} \not\subseteq V(G_i)$ for every $G_i \in \mathcal{H}$. First, consider the case in which $x \in V(G_i)$, $y \in V(G_j)$, $i \neq j$ and there exists $a \in A(G_i) \cap A(G_j)$. Obviously, $x \neq a$ and it is readily seen that $\mathcal{L}_{G_i}\{x,a\}$ is universal. Thus, a reasoning analogous to the one described above allows us to conclude that $A(G_i) \cap ([xa] \setminus \{x,a\}) = \varnothing$.

Now, assume that $x \in V(G_i)$, $y \in V(G_j)$, $i \neq j$ and $A(G_i) \cap A(G_j) = \varnothing$. In this case, there exists $G_l \in \mathcal{H}$ such that every vertex of G_l lies on a shortest path from x to y. Let $a,b \in A(G_l)$ such that $d_G(x,a) = \min\{d_G(x,w) : w \in V(G_l)\}$ and $d_G(y,b) = \min\{d_G(y,w) : w \in V(G_l)\}$. Obviously, $\mathcal{L}_{G_l}\{a,b\}$ is universal and, as above, we can check that $A(G_l) \cap ([ab] \setminus \{a,b\}) = \varnothing$. Therefore, the result follows. □

In the following subsections we give applications of this result to specific families of separable graphs.

3.1 The case of block graphs

A block of a graph G is a maximal nonseparable subgraph of G. That is, a block of G is a nonseparable subgraph of G that is not a proper subgraph of any nonseparable subgraph of G. A connected graph G is a *block graph* if every block of G is a complete graph. Since every block graph is obtained by point-attaching from a family of complete graphs, Theorem 3.3 leads to the following result.

Corollary 3.4 (Existence and uniqueness)**.** *If G is a block graph, then the following statements hold.*

(i) $\mu(G) > 0$ *if and only if K_2 is a block of G.*

(ii) $\mu(G) = 1$ *if and only if exactly one block of G is isomorphic to K_2.*

3.2 The case of bridgeless graphs where all cycles are odd

A graph is said to be *bridgeless* or *isthmus-free* if it contains no bridges. Figure 4 shows a connected bridgeless graph.

Theorem 3.5. *If G is a connected bridgeless graph where all cycles are odd, then $\mu(G) = 0$.*

Proof. Let G be a connected bridgeless graph where all cycles are odd, and let $x,y \in V(G)$. First, consider the particular case $G \cong C_n$, where n is odd. Let $V(C_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ where consecutive vertices are adjacent. Without loss of generality, we can assume that $x = v_0, v_1, \ldots, v_l = y$ is a shortest path from x to y in C_n. If l is odd, then for $j = \frac{n+l}{2}$ we have that $d_{C_n}(x, v_j) = d_{C_n}(y, v_j) = \frac{n-l}{2}$, and so $v_j \notin \mathcal{L}_{C_n}\{x,y\}$. Now, if l is even, then for $i = \frac{n+l+1}{2}$ we have that $d_{C_n}(x, v_i) = \frac{n-l-1}{2}$ and $d_{C_n}(y, v_i) = \frac{n-l+1}{2}$, which implies that $v_i \notin \mathcal{L}_{C_n}\{x,y\}$. Therefore, no line of C_n is universal.

For the proof of the general case we apply Theorem 3.3 considering the fact that every connected bridgeless graph is obtained by point-attaching from a family of odd cycles. To show this, we only need to prove that all cycles of G are edge disjoint. Suppose, to the contrary, that there are two cycles C_k and C_t of G sharing some edges. Let $V(C_k) = \{u_1, \ldots, u_k\}$ and $V(C_t) = \{v_1, \ldots, v_t\}$ such that consecutive vertices are adjacent. We can assume, without loss of generality, that the common vertices of these cycles form a path $u_1 = v_1, \ldots, u_r = v_r$, where $r \geq 2$. Hence, the cycle C, whose vertex set is $V(C) = \{v_r = u_r, u_{r+1}, \ldots, u_k, u_1 = v_1, v_t, v_{t-1}, \ldots, v_{r+1}\}$, has even length $|V(C)| = k + t - 2r + 2$, as k and t are odd, which is a contradiction.

Notice that from two cycles that share two or more disjoint paths, we can always identify two cycles that share exactly one path, and then we can use the previous procedure to reach a contradiction. Therefore, the result follows. □

3.3 The case of rooted product graphs

Let G be a graph and H a graph with root vertex v. The rooted product graph $G \circ_v H$ is defined as the graph obtained from one copy of G and $|V(G)|$ copies of H, identifying the i^{th} vertex of G with vertex v in the i^{th} copy of H for each $i \in \{1, \ldots, |V(G)|\}$. For every $x \in V(G)$, the copy of H in $G \circ_v H$ containing x, which is isomorphic to H, will be denoted by H_x. Since $G \circ_v H$ can be obtained by point-attaching from the family $\{G\} \cup \{H_x : x \in V(G)\}$, Theorem 3.3 leads to the following result.

Corollary 3.6 (Existence)**.** *Let G and H be two connected graph of order at least two, and let $v \in V(H)$. Then $\mu(G \circ_v H) \geq 1$ if and only if $\mu_a(G) \geq 1$ or there exist two vertices $h, h' \in V(H)$ such that $\mathcal{L}_H\{h,h'\}$ is universal and $v \notin [hh'] \setminus \{h,h'\}$.*

Theorem 3.7 (Uniqueness)**.** *Let G and H be two connected graph of order at least two, and let $v \in V(H)$. Then $\mu(G \circ_v H) = 1$ if and only if the following conditions hold.*

(i) $\mu_a(G) = 1$.

(ii) *If there exists a universal line $\mathcal{L}_H\{h,h'\}$, then $v \in [hh'] \setminus \{h,h'\}$.*

Proof. For any $x \in V(G)$ and $h \in V(H)$, let h_x be the vertex of H_x corresponding to h.

Assume $\mu(G \circ_v H) = 1$. By Corollary 3.6, $\mu_a(G) \geq 1$ or there exist two vertices $h, h' \in V(H)$ such that $\mathcal{L}_H\{h,h'\}$ is universal and $v \notin [hh'] \setminus \{h,h'\}$. Now, if $\mu_a(G) = k \geq 1$, then there are k pairs $x, y \in V(G)$ of adjacent vertices such that $\mathcal{L}_G\{x,y\}$ is universal, and $\mathcal{L}_{G \circ_v H}\{x,y\}$ is also universal. Therefore, (i) follows. On the other side, if there exists a universal line

$\mathcal{L}_H\{h,h'\}$ such that $v \notin [hh'] \setminus \{h,h'\}$, then for any $x \in V(G)$, the line $\mathcal{L}_{G\circ_v H}\{h_x,h'_x\}$ is universal, which is a contradiction, as in such a case, $\mu(G\circ_v H) \geq |V(G)|$. This implies that $\mu(H) = 0$ or $v \in [hh'] \setminus \{h,h'\}$ for every universal line $\mathcal{L}_H\{h,h'\}$. Therefore, (ii) follows.

Conversely, assume that (i) and (ii) hold. Let $x,y \in V(G\circ_v H)$ such that $\mathcal{L}_{G\circ_v H}\{x,y\}$ is universal. We differentiate the following cases.

Case 1. $x,y \in V(G)$. Since $V(G) \subseteq \mathcal{L}_{G\circ_v H}\{x,y\}$, we conclude that $\mathcal{L}_G\{x,y\}$ is universal. Now, if $x \not\sim y$, then for every vertex $g \in V(G)$ lying between x and y, we have that $V(H_g) \not\subseteq \mathcal{L}_{G\circ_v H}\{x,y\}$, which is a contradiction. Hence, $x \sim y$ and, since $\mu_a(G) = 1$, they form the only pair of adjacent vertices of G that induce a universal line in $G\circ_v H$.

Case 2. $x,y \in V(H_g) \setminus \{g\}$ for some vertex $g \in V(G)$. In this case, $\mathcal{L}_{H_g}\{x,y\}$ is universal and, by assumption, $g \in [xy] \setminus \{x,y\}$. Thus, for any $z \in V(G\circ_v H) \setminus (V(H_g) \cup \{g\})$, we have that $z \notin \mathcal{L}_{G\circ_v H}\{x,y\}$, which is a contradiction.

Case 3. There exists $g \in V(G)$ such that $x \in V(H_g) \setminus \{g\}$ and $y \notin V(H_g) \setminus \{g\}$. As above, $\mathcal{L}_{H_g}\{x,g\}$ is universal, which is a contradiction, as $g \notin [xg] \setminus \{x,g\}$.

Therefore, according to the three cases above, we conclude that $\mu(G\circ_v H) = 1$. □

3.4 The case of corona product graphs

Given two graphs G and H, the corona product $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G and $|V(G)|$ copies of H, and making the i^{th} vertex of G adjacent to every vertex of the i^{th} copy of H for every $i \in \{1,\dots,|V(G)|\}$.

The particular case $K_1 \odot H$ is known as the join graph $K_1 + H$. To prove our results, we need to state the following straightforward lemma.

Lemma 3.8. *Let H be a graph, $h,h' \in V(H)$ and $V(K_1) = \{v\}$. Then the following statements hold.*

(i) *The line $\mathcal{L}_{K_1+H}\{h,v\}$ is universal if and only if $\deg_H(h) = 0$.*

(ii) *If $\mathcal{L}_{K_1+H}\{h,h'\}$ is a universal line, then $v \in [hh']$.*

Theorem 3.9 (Existence)**.** *Let G be a connected graph of order at least two and H a graph. Then $\mu(G\odot H) \geq 1$ if and only if $\mu_a(G) \geq 1$ or $\delta(H) = 0$.*

Proof. Notice that $G\odot H \cong G\circ_v (K_1+H)$, where v is the vertex of K_1. Hence, by Corollary 3.6, $\mu(G\odot H) \geq 1$ if and only if $\mu_a(G) \geq 1$ or there exist two vertices $h,h' \in V(K_1+H)$ such that $\mathcal{L}_{K_1+H}\{h,h'\}$ is universal and $v \notin [hh'] \setminus \{h,h'\}$. Therefore, from Lemma 3.8 we deduce that $\mu(G\odot H) \geq 1$ if and only if $\mu_a(G) \geq 1$ or $\delta(H) = 0$. □

Theorem 3.10 (Uniqueness)**.** *Let G be a connected graph of order at least two and H a graph. Then $\mu(G\odot H) = 1$ if and only if $\mu_a(G) = 1$ and $\delta(H) \geq 1$.*

Proof. Since $G\odot H \cong G\circ_v (K_1+H)$, where v is the vertex of K_1, the result is a direct consequence of Theorem 3.7 and Lemma 3.8. □

4 The case of Cartesian product graphs

Given two graphs G and H, the *Cartesian product* $G\Box H$ is the graph with vertex set $V(G\Box H) = V(G)\times V(H)$, where two vertices $(g,h),(g',h')$ are adjacent in $G\Box H$ if and only if, either $g = g'$ and $hh' \in E(H)$ or $h = h'$ and $gg' \in E(G)$. The distance between two vertices (g,h) and (g',h') is given by

$$d_{G\Box H}((g,h),(g',h')) = d_G(g,g') + d_H(h,h').$$

Since $G\Box H \cong H\Box G$, explanations of symmetrical cases will be omitted.

Theorem 4.1 (Existence and uniqueness)**.** *Let G and H be two connected nontrivial graphs. Then $\mu(G\Box H) > 0$ if and only if one of the following conditions holds.*

(i) $\mu_a(G) \geq 1$ *or* $\mu_a(H) \geq 1$.

(ii) $\mathrm{g}(G) = 2$ *and* $\mathrm{g}(H) = 2$.

Furthermore, $\mu(G\Box H) \neq 1$.

Proof. First, assume that (i) holds. Without loss of generality, we can assume that there exist two vertices $g,g' \in V(G)$ such that $g \sim g'$ and $\mathcal{L}_G\{g,g'\}$ is universal. Notice that for any $x \in V(G)$, we have that $g \in [xg']$ or $g' \in [xg]$. Let $h \in V(H)$ and $(x,y) \in V(G\Box H)$. If $g \in [xg']$, then

$$\begin{aligned} d_{G\Box H}((x,y)(g',h)) &= d_G(x,g') + d_H(y,h) \\ &= d_G(x,g) + d_G(g,g') + d_H(y,h) \\ &= d_{G\Box H}((x,y)(g,h)) + d_{G\Box H}((g,h)(g',h)), \end{aligned}$$

while if $g' \in [xg]$, then

$$\begin{aligned} d_{G\Box H}((x,y)(g,h)) &= d_G(x,g) + d_H(y,h) \\ &= d_G(x,g') + d_G(g',g) + d_H(y,h) \\ &= d_{G\Box H}((x,y)(g',h)) + d_{G\Box H}((g,h)(g',h)). \end{aligned}$$

Therefore, $\mathcal{L}_{G\Box H}\{(g,h),(g',h)\}$ is universal. Notice that, in this case, $\mu(G\Box H) \geq \mu_a(G)|V(H)|$.

Now, assume that (ii) holds. Let $\{g,g'\}$ be a geodetic set of G and $\{h,h'\}$ a geodetic set of H. In this case, for any $(x,y) \in V(G\Box H)$,

$$\begin{aligned} d_{G\Box H}((g,h)(g',h')) =& d_G(g,g') + d_H(h,h') \\ =& d_G(g,x) + d_G(x,g') + d_H(h,y) + d_H(y,h') \\ =& d_{G\Box H}((g,h)(x,y)) + d_{G\Box H}((x,y)(g',h')). \end{aligned}$$

Therefore, $\mathcal{L}_{G\Box H}\{(g,h),(g',h')\}$ is universal. Notice that, in this case, $\mu(G\Box H) \geq 2$ as $\mathcal{L}_{G\Box H}\{(g,h'),(g',h)\}$ is also universal.

Conversely, assume that there exist two distinct vertices $(g,h),(g',h') \in V(G\Box H)$ such that $\mathcal{L}_{G\Box H}\{(g,h),(g',h')\}$ is universal. We differentiate the following two cases.

Case 1. $h = h'$. If $d_g(g,g') \geq 2$, then for any $h'' \in N_H(h)$ and $g'' \in [gg'] \cap N_G(g)$ we have $d_{G\Box H}((g',h),(g,h)) = d_G(g',g'') + 1 = d_G(g',g'') + d_H(h,h'') = d_{G\Box H}((g',h),(g'',h''))$, and so $(g'',h'') \notin \mathcal{L}_{G\Box H}\{(g,h),(g',h')\}$, which is a contradiction. Hence, $g \sim g'$. Now, observe that if we take K_1 as the trivial graph with $V(K_1) = \{h\}$, then $G\Box K_1 \cong G$ and, since $\mathcal{L}_{G\Box H}\{(g,h),(g',h)\}$ is universal, $\mathcal{L}_{G\Box K_1}\{(g,h),(g',h)\}$ is also universal. This implies that $\mathcal{L}_G\{g,g'\}$ is universal. Therefore, (i) follows.

Case 2. $g \neq g'$ and $h \neq h'$. If $H \cong K_2$, then (i) follows. Assume $H \ncong K_2$, and let $v \in V(H) \setminus \{h,h'\}$. We consider first the case in which $h \sim h'$. Observe that $(g',v) \notin [(g',h')(g,h)]$ and $(g,h) \notin [(g',v)(g',h')]$. Hence, $(g',h') \in [(g',v)(g,h)]$, and so $h' \in [vh]$, which implies that $\mathcal{L}_H\{h,h'\}$ is universal. Therefore, (i) follows. The same conclusion is reached if $g \sim g'$.

From now on, assume $d_H(h,h') \geq 2$ and $d_G(g,g') \geq 2$. Let $y \in N_H(h) \cap [hh']$. Suppose that $\{g,g'\}$ is not a geodetic set of G. In such a case, there exist two adjacent vertices $x, g'' \in V(G)$ such that $g'' \in [gg']$ and $x \notin [gg']$.

Notice that $d_G(g,x) = d_G(g,g'') + 1$, $d_G(g',x) = d_G(g',g'') + 1$ and $d_G(g,g') = d_G(g,g'') + d_G(g'',g')$. Hence,

$$d_{G\Box H}((g,h),(x,y)) = d_G(g,x) + 1 = d_G(g,g'') + 2$$

and

$$\begin{aligned} d_{G\Box H}((g',h'),(x,y)) &= d_G(g',x) + d_H(h',y) \\ &= d_G(g',g'') + 1 + d_H(h',y) \\ &= d_G(g',g'') + d_H(h',h). \end{aligned}$$

This implies that $(x,y) \notin [(g,h)(g',h')]$, as

$$\begin{aligned} d_{G\Box H}((g,h),(x,y)) + d_{G\Box H}((x,y),(g',h')) &= d_G(g,g'') + 2 + d_G(g',g'') + d_H(h',h) \\ &= d_{G\Box H}((g,h),(g',h')) + 2 \\ &\neq d_{G\Box H}((g,h),(g',h')). \end{aligned}$$

Hence, $(g,h) \in [(x,y)(g',h')]$ or $(g',h') \in [(x,y)(g,h)]$, which implies that

$$\begin{aligned} d_G(g,g'') + d_G(g'',g') + d_H(h,h') &= d_G(g,g') + d_H(h,h') \\ &= d_{G\Box H}((g,h),(g',h')) \\ &= |d_{G\Box H}((x,y),(g,h)) - d_{G\Box H}((x,y),(g',h'))| \\ &= |d_G(g,g'') + 2 - d_G(g'',g') - d_H(h,h')|. \end{aligned}$$

This implies that $d_G(g,g'') = -1$, which is impossible, or $d_G(g'',g') + d_H(h,h') = 1$, which is a contradiction again, as $d_H(h,h') \geq 2$. Thus, $\{g,g'\}$ is a geodetic set of G, and by analogously we deduce that $\{h,h'\}$ is a geodetic set of H. Therefore, (ii) follows.

Finally, to check that $\mu(G\Box H) \neq 1$ we only need to observe that we have shown that if (i) holds, then $\mu_a(G) \geq 1$ and $\mu(G\Box H) \geq \mu_a(G)|V(H)|$ or $\mu_a(H) \geq 1$ and $\mu(G\Box H) \geq \mu_a(H)|V(G)|$. Furthermore, we have shown that (ii) leads to $\mu(G\Box H) \geq 2$. Therefore, the result follows. □

5 The case of join graphs

The join $G+H$ of two disjoint graphs G and H is defined as the graph obtained by taking one copy of G and one copy of H and joining by an edge each vertex of G with each vertex of H. Since $G+H \cong H+G$, explanations of symmetrical cases will be omitted.

In order to state the next result, we need to introduce the following family $\mathcal{G}$ of graphs such that $G \in \mathcal{G}$ if and only if $G \cong N_2$ or there exists a graph G' such that $G \cong N_2 + G'$. Notice that $\mathcal{G} \setminus \{N_2\}$ is the class of graphs of diameter two with $\mathrm{g}(G) = 2$.

Theorem 5.1 (Existence)**.** *Let G and H be two graphs. Then $\mu(G+H) > 0$ if and only if one of the following conditions holds.*

(i) *$G \in \mathcal{G}$ or $H \in \mathcal{G}$.*

(ii) *$\delta(G) = 0$ and $\delta(H) = 0$.*

Proof. First, assume $G \in \mathcal{G}$. Let $V(N_2) = \{x,y\}$ in such a way that either $G \cong N_2$ or there exists a graph G' such that $G \cong N_2 + G'$. Since $\{x,y\}$ is an independent set and $N_{G+H}(x) = N_{G+H}(y) = V(G+H) \setminus \{x,y\}$, we conclude that $\mathcal{L}_{G+H}\{x,y\}$ is universal.

Now, if there exist two vertices $g \in V(G)$ and $h \in V(H)$ such that $\deg_G(g) = \deg_H(h) = 0$, then it is also very easy to check that $\mathcal{L}_{G+H}\{g,h\}$ is universal.

Conversely, assume that here exist two vertices $u,v \in V(G+H)$ such that $\mathcal{L}_{G+H}\{u,v\}$ is universal. We differentiate two cases.

Case 1. $u,v \in V(G)$. First, observe that if $u \sim v$, then $V(H) \not\subseteq \mathcal{L}_{G+H}\{u,v\}$, which implies that $u \not\sim v$. Now, if there exists $w \in V(G) \setminus \{u,v\}$, then $\{u,v\}$ has to be a subset of $N_G(w)$, to ensure that $w \in \mathcal{L}_{G+H}\{u,v\}$. Therefore, $G \in \mathcal{G}$, i.e., (i) follows.

Case 2. $u \in V(G)$ and $v \in V(H)$. If there exists $u' \in V(G) \setminus \{u\}$, then $u' \notin N_G(u)$, to ensure that $u' \in \mathcal{L}_{G+H}\{u,v\}$. Thus, $\deg_G(u) = \delta(G) = 0$. By analogy we see that $\delta(H) = 0$, and so (ii) follows. □

In order to state the next result, we need to introduce some additional notation. Let $\mathcal{G}_u$ be the family of graphs such that $G \in \mathcal{G}_u$ if and only if there exist exactly two vertices $x,y \in V(G)$ such that $N_G(x) = N_G(y) = V(G) \setminus \{x,y\}$. Now, let $\mathcal{O}$ be the family of graphs such that $G \in \mathcal{O}$ if and only if $\delta(G) = 0$. Obviously, $\mathcal{G} \cap \mathcal{O} = \{N_2\}$ and $\mathcal{G}_u \subseteq \mathcal{G}$.

Theorem 5.2 (Uniqueness)**.** *Let G and H be two graphs. Then $\mu(G+H) = 1$ if and only if exactly one of the following conditions holds.*

(i) *$|\{G,H\} \cap \mathcal{O}| \leq 1$, $G \notin \mathcal{G} \cup \{K_1\}$ and $H \in \mathcal{G}_u$.*

(ii) *$G \cong K_1$ and $H \in \mathcal{G}_u \setminus \{N_2\}$.*

(iii) *G has exactly one isolated vertex and H has exactly one isolated vertex.*

Proof. Assume $\mu(G+H) \geq 1$. Let $\mathcal{L}_{G+H}\{x,y\}$ be a universal line. By Theorem 5.1, we can distinguish the following cases.

Case 1. $|\{G,H\}\cap\mathcal{O}|\leq 1$ and $\{G,H\}\cap\mathcal{G}\neq\emptyset$. Since $\deg(G)>0$ or $\deg(H)>0$, every pair of adjacent vertices in $G+H$ form a triangle, and so x and y are not adjacent in $G+H$, which implies that $N_{G+H}(x)=N_{G+H}(y)=V(G+H)\setminus\{x,y\}$. Now, if $G,H\in\mathcal{G}$, then $\mu(G+H)\geq 2$, and so we can assume, without loss of generality, that $G\notin\mathcal{G}$. In such a case, $H\in\mathcal{G}$ and $x,y\in V(H)$. Observe that, if $G\cong K_1$, then $H\not\cong N_2$, and so $\mu(G+H)=1$ if and only if $G+H\not\cong K_1+N_2\cong P_3$ and also x and y form the only pair of vertices of H with $N_H(x)=N_H(y)=V(H)\setminus\{x,y\}$. Therefore, $\mu(G+H)=1$ if and only if either (i) holds or (ii) holds.

Case 2. $G,H\in\mathcal{O}$. Notice that $\mathcal{G}\cap\mathcal{O}=\{N_2\}$. For any vertex $g\in V(G)$ and $h\in V(H)$ such that $\deg_G(g)=\deg_H(h)=0$, the line $\mathcal{L}_{G+H}\{g,h\}$ is universal. Therefore, $\mu(G+H)=1$ if and only if (iii) hods. □

6 The case of lexicographic product graphs

Let G and H be two graphs. The *lexicographic product* of G and H is the graph $G\circ H$ whose vertex set is $V(G\circ H)=V(G)\times V(H)$, where two vertices (g,h) and (g',h') are adjacent if and only if $gg'\in E(G)$ or $g=g'$ and $hh'\in E(H)$. Notice that for any $g\in V(G)$ the subgraph of $G\circ H$ induced by $\{g\}\times V(H)$ is isomorphic to H. For simplicity, we will denote this subgraph by H_g. For a basic introduction to the lexicographic product of two graphs we suggest the books [10, 12].

The following claim, which states the distance formula in the lexicographic product of two graphs, is one of our main tools.

Remark 6.1. [10] *For any connected graph G of order $\mathrm{n}(G)\geq 2$ and any graph H, the following statements hold.*

(i) $d_{G\circ H}((g,h),(g',h'))=d_G(g,g')$ *for* $g\neq g'$.

(ii) $d_{G\circ H}((g,h),(g,h'))=\min\{2,d_H(h,h')\}$.

As in Section 5, we shall use the notation $\mathcal{G}$ for the family of graphs such that $G\in\mathcal{G}$ if and only if $G\cong N_2$ or there exists a graph G' such that $G\cong N_2+G'$.

Theorem 6.2 (Existence)**.** *Let G be a connected graph and H a nontrivial graph. Then $\mu(G\circ H)>0$ if and only if at least one of the following conditions holds.*

(i) *G has a universal vertex and $H\in\mathcal{G}$.*

(ii) $\mu_a(G)\geq 1$ *and* $\delta(H)=0$.

Proof. Let $g,g'\in V(G)$ and $h,h'\in V(H)$. We proceed to show that $\mathcal{L}_{G\circ H}\{(g,h),(g',h')\}$ is universal if and only if one of the following conditions holds.

(a) $g=g'$ is a universal vertex, $h\neq h'$ and $N_H(h)=N_H(h')=V(H)\setminus\{h,h'\}$.

(b) $\mathcal{L}_G\{g,g'\}$ is universal, $d_G(g,g')=1$ and $\deg_H(h)=\deg_H(h')=0$.

By simple inspection we can see that if the pair $(g,h),(g',h') \in V(G) \times V(H)$ satisfies (a) or (b), then $\mathcal{L}_{G\circ H}\{(g,h),(g',h')\}$ is universal.

Conversely, assume that $\mathcal{L}_{G\circ H}\{(g,h),(g',h')\}$ is universal. We differentiate two cases.

Case 1. $g = g'$. If $d_H(h,h') = 1$, then $N_G(g) \times V(H) \not\subseteq \mathcal{L}_{G\circ H}\{(g,h),(g,h')\}$, which is a contradiction and, as a result, $h \not\sim h'$. If there exists $h'' \in V(H) \setminus (N_H(h') \cup \{h,h'\})$, then $d_{G\circ H}((g,h''),(g,h')) = 2 = d_{G\circ H}((g,h),(g,h'))$, and so $(g,h'') \notin \mathcal{L}_{G\circ H}\{(g,h),(g,h')\}$, which is a contradiction. Hence, $N_H(h') = V(H) \setminus \{h,h'\}$ and, analogously, $N_H(h) = V(H) \setminus \{h,h'\}$. Finally, if there exists $g'' \in V(G)$ such that $d_G(g,g'') \geq 2$, then

$$d_{G\circ H}((g,h),(g,h')) = 2 \leq d_G(g,g'') = d_{G\circ H}((g,h),(g'',h)) = d_{G\circ H}((g,h'),(g'',h)).$$

Thus, $(g'',h) \notin \mathcal{L}_{G\circ H}\{(g,h),(g,h')\}$, which is a contradiction, and so g is a universal vertex of G. Therefore, (a) follows.

Case 2. $g \neq g'$. Note that h and h' are not necessarily different. Suppose that there exits $z \in V(G) \setminus \mathcal{L}_G\{g,g'\}$. In such a case,

$$\begin{aligned} d_{G\circ H}((g,h),(g',h')) &= d_G(g,g') \\ &\neq d_G(g,z) + d_G(z,g') \\ &= d_{G\circ H}((g,h),(z,h)) + d_{G\circ H}((z,h),(g',h')) \end{aligned}$$

and, analogously, $d_{G\circ H}((g,h),(g',h')) \neq |d_{G\circ H}((g,h),(z,h)) - d_{G\circ H}((z,h),(g',h'))|$. Hence, $(z,h) \notin \mathcal{L}_{G\circ H}\{(g,h),(g',h')\}$, which is a contradiction, and so $\mathcal{L}_G\{g,g'\}$ has to be universal. Now, suppose that $d_G(g,g') \geq 2$. In this case, for any $y \in V(H) \setminus \{h\}$,

$$d_{G\circ H}((g,h),(g',h')) = d_{G\circ H}((g,y),(g',h')) = d_G(g,g') \geq 2 \geq d_{G\circ H}((g,h),(g,y)).$$

Hence, $(g,y) \notin \mathcal{L}_{G\circ H}\{(g,h),(g',h')\}$, which is a contradiction, and so $d_G(g,g') = 1$. Thus, if there exists $y \in N_H(h)$, then

$$d_{G\circ H}((g,h),(g',h')) = d_{G\circ H}((g,y),(g',h')) = d_G(g,g') = 1 = d_{G\circ H}((g,h),(g,y)),$$

and so $(g,y) \notin \mathcal{L}_{G\circ H}\{(g,h),(g',h')\}$, which is a contradiction again. This implies that $\deg_H(h) = 0$ and, by analogy, $\deg_H(h') = 0$. Therefore, (b) follows. □

As in Section 5, we shall use the notation $\mathcal{G}_u$ for the family of graphs such that $G \in \mathcal{G}_u$ if and only if there exist exactly two vertices $x,y \in V(G)$ such that $N_G(x) = N_G(y) = V(G) \setminus \{x,y\}$.

Theorem 6.3 (Uniqueness)**.** *Let G be a connected graph and H a nontrivial graph. Then $\mu(G\circ H) = 1$ if and only if one of the following conditions holds.*

(i) *G has exactly one universal vertex and $H \in \mathcal{G}_u$.*

(ii) *$\mu_a(G) = 1$ and H has exactly one isolated vertex.*

Proof. By Theorem 6.2 we differentiate the following two cases for G and H.

Case 1. G has a universal vertex and $H \in \mathcal{G}$. Since for any universal vertex $g \in V(G)$ and any pair $h, h' \in V(H)$ such that $N_H(h) = N_H(h') = V(H) \setminus \{h, h'\}$, the line $\mathcal{L}_{G \circ H}\{(g,h),(g,h')\}$ is universal, we conclude that $\mu(G \circ H) = 1$ if and only if (i) holds.

Case 2. $\mu_a(G) \geq 1$ and $\delta(H) = 0$. Since for any universal line $\mathcal{L}_G\{g, g'\}$ with $g \sim g'$ and any $h \in V(H)$ such that $\deg_H(h) = 0$, the line $\mathcal{L}_{G \circ H}\{(g,h),(g',h)\}$ is universal, we conclude that $\mu(G \circ H) = 1$ if and only if (ii) holds.

Observe that (i) and (ii) do not occur simultaneously for a graph H. □

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