EIGHT-CENTERED OVAL QUASI-EQUIVALENT TO KEPLER'S ELLIPSE OF PLANETS' TRAJECTORY

ÓVALO DE OCHO CENTROS QUASI-EQUIVALENTE A LA ELIPSE DE KEPLER DE LA TRAYECTORIA DE LOS PLANETAS

B. Herrera[a†] y A. Samper[b]

a) Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, Spain; blas.herrera@urv.net †
b) Unitat predepartamental d'Arquitectura, Universitat Rovira i Virgili, Spain.
† corresponding author

Recibido 9/5/2016; Aceptado 3/6/2016

This paper has academic nature, nevertheless it is an application from [9] that we think can be interesting to students and instructors of undergraduate level of physics. With low level of geometric and computational techniques the readers can reproduce the results of the present paper. The most important thing is that these results clearly show that it is not enough take numerical measures of the orbits of the bodies to decide which are the real equations and laws that govern them. We present an approximation of the Kepler's ellipse of the planets' trajectory by circular arcs, which is *quasi-equivalent*; that is: we present the approximation of the Kepler's ellipse $\mathcal{E}_b$ by the eight-centered oval $O_{\mathcal{E}_b}$ having the same center, axes, vertices, perimeter length and curvature at the vertices as $\mathcal{E}_b$, and also having practically negligible difference with respect to the surface area of $\mathcal{E}_b$, and also having barely distinguishable deformation error in relation to $\mathcal{E}_b$.

Este artículo, aunque tiene carácter académico en su naturaleza y presentación, consiste en una aplicación de las conclusiones geométricas de [9] la cual consideramos puede ser interesante para estudiantes y profesores en licenciaturas y grados de física. Con bajo nivel de geometría y técnicas computacionales, los lectores pueden reproducir los cálculos del presente trabajo. El aspecto más importante que se presenta aquí es que los resultados muestran que no es suficiente tomar medidas numéricas de las órbitas de cuerpos para poder determinar las ecuaciones y leyes reales que las gobiernan. Más concretamente, presentamos una aproximación, por arcos circulares, de la elipse de Kepler de las trayectorias de los planetas, la cual es *quasi-equivalente*. O sea, presentamos la aproximación de la elipse de Kepler $\mathcal{E}_b$ por el óvalo de ocho centros $O_{\mathcal{E}_b}$ el cual posee: mismo centro, ejes, vértices, perímetro, y curvatura en los vértices que $\mathcal{E}_b$, y además la diferencia entre las áreas de $\mathcal{E}_b$ y $O_{\mathcal{E}_b}$ es prácticamente nula y el error de deformación entre ambos es por añadidura prácticamente indistiguible.

PACS: 01.40.Fk Physics education, 01.50.H- Computers in education, 02.40.Dr Euclidean geometries, 95.10.Eg Orbit determination.

Approximating ellipses by circular arcs has been a classic subject of study by geometers. This has long been used for a wide range of applications, for instance in geometry, art, architecture. The reader can easily find a great deal of classical literature on these topics, in special for eight-centered ovals and four-centered ovals (also named *quadrarcs*). Moreover, because of its importance, this subject of study is continued in modern research papers as [1–7]. In astronomy this kind of approximation was classically considered, for example in [8].

This paper has academic nature, nevertheless it is an application from [9] that we think can be interesting to students and instructors of undergraduate level of physics. With low level of geometric and computational techniques the readers can reproduce the results of the present paper. The most important thing is that these results clearly show that it is not enough take numerical measures of the orbits of the bodies to decide which are the real equations and laws that govern them. Errors in measures, even if they are very small, can hide subtly the differences between different laws. Therefore in addition to taking measures, always depth studies of the physic which governs the motions of bodies are required. More specifically: here in this paper, we show the example of the planets' trajectory; because the geometric properties of two very different lines –ellipse and eight-centered oval– can be quasi-equivalent (having barely distinguishable differences), therefore computations of the orbits can hide the real physics laws. By contrast and as an added value, this paper also shows to students, and interested people, that they can change an ellipse by a quasi-equivalent eight-centered oval, if they consider that their negligible geometric differences are assumable in their concrete problems.

It is well known that Johannes Kepler, in the 17th century, made a search for a better description of planetary motion and, in *Astronomia Nova* [10] (1609), he provided arguments for elliptical trajectory of the planets around the Sun. However, in 1675, Giovanni Domenico Cassini did not agree with Kepler and he tried to prove that the planetary orbits were Cassini's ovals [11]. Recently, Sivardiere [12] explored this question and concluded that the difference between the Kepler's ellipse and Cassini's oval is as distinguishable as that between the Kepler's ellipse and the circular orbit; therefore, if we discard the circle in favour of the ellipse, then, we also should discard the oval with the same argument. In the work [13], Morgado and Soares analyzed this possibility and they show that it is difficult to decide in favour of one of the two curves. In fact, Morgado an Soares calculated the following:

Table 1. Calculations of Morgado and Soares [13].

	Mercury	Venus	Earth	Mars
ϵ	0.205600	0.006700	0.016700	0.093500
$E(O,\mathcal{E}_b)$	0.021364	0.000022	0.000139	0.004381
$E(\mathcal{E}_b,C_b)$	0.021841	0.000022	0.000139	0.004400
$\frac{E(O,\mathcal{E}_b)}{b}$	0.021830	0.000022	0.000139	0.004400
$\frac{E(\mathcal{E}_b,C_b)}{b}$	0.022318	0.000022	0.000139	0.004419
	Jupiter	Saturn	Uranus	Neptune
ϵ	0.048900	0.056500	0.045700	0.011300
$E(O,\mathcal{E}_b)$	0.001196	0.001597	0.001045	0.000064
$E(\mathcal{E}_b,C_b)$	0.001198	0.001600	0.001046	0.000064
$\frac{E(O,\mathcal{E}_b)}{b}$	0.001198	0.001600	0.001046	0.000064
$\frac{E(\mathcal{E}_b,C_b)}{b}$	0.001199	0.001603	0.001047	0.000064

Let $\mathcal{E}_b$ be the Kepler's ellipse of the orbit of a planet, let A, A', B, B', be the four vertices of the ellipse $\mathcal{E}_b$ and let a, b, ϵ, M be the mayor semi-axis, the minor semi-axis, the eccentricity and the center, respectively, of $\mathcal{E}_b$. We can assume that $a = 1$. Let C_b be the Cassini's oval with same center, foci points and mayor semi-axis of the ellipse $\mathcal{E}_b$. Let b_C the distance between M and the point $B_C \in C_b$ such that $\frac{\pi}{2} = \measuredangle(A, M, B_C)$. Let $E(\mathcal{E}_b, C_b) = b - b_C$, and let $E(O, \mathcal{E}_b) = 1 - b$ *i.e.* for small eccentricities $E(\mathcal{E}_b, C_b)$ is the maximum difference between the positions given by the Kepler's ellipse and the Cassinian oval, and $E(O, \mathcal{E}_b)$ is the maximum difference between the positions given by the circular orbit O –of center M and radius 1– and the ellipse.

Morgado an Soares found, with $a = 1$, the results of Table1.

With Table 1, Morgado an Soares concluded in [13] that the maximum difference between the positions given by the circumference and the ellipse is of the same order of magnitude as that for the maximum difference between the positions given by the Cassinian oval and the ellipse. However, this difference is too small when compared with the values of the axis for all these curves to be observed with the naked eye, as was the case in Kepler's day. Kepler was probably unable to geometrically observe the difference between the ellipse and the circle unambiguously, and the difference between the ellipse and the Cassini's oval is also indistinguishable for the standards of his period. From the results of Morgado an Soares –in Table 1–, Kepler's ellipse and Cassini's oval are barely distinguishable when orbits with a small eccentricity are considered. This illustrates the ingenuity of Kepler in analyzing the observational data at his disposal. Finally, they call attention to Laplace's remark, found in his *Mécanique Céleste* [14], that only with Newton and his gravitation law will the ellipse be elected as the curve to better describe planetary motions, and all incompatibilities between theory and the real orbit are caused by the disturbance of another celestial body.

In this paper we look for an approximation of the trajectory of the planets, *i.e.* the Kepler's ellipse of the planets, by circular arcs which can qualify as being a *quasi-equivalent* approximation.

In order to attain a similarity of results in the planets' trajectory, an approximation which is equivalent to the Kepler' ellipse must have exactly coinciding geometric parameters. An eight-centered oval which is *equivalent* to an ellipse should have the same: center, axes, vertices, perimeter length, curvature at the vertices and surface area. Also, it should have little deformation in relation to the ellipse. Unfortunately, an eight-centered oval with all these exactly coinciding geometric parameters does not exist; it can not have all the same geometric parameters and also the same surface area.

There are different methods of approximating curves –least squares, minimax, orthogonal family of polynomials–. However, our approach is different; we look for the exact coincidence of single geometric parameters. Present exact analytical formulae for approximations Kepler's ellipse trajectories by eight-centered ovals with some geometric parameters which coincide exactly. Further, we want to show the precise numerical calculations of these approximations. And, as a conclusion, we want to present not the "equivalent" approximation because it does not exist, but the approximation of the Kepler' ellipse orbit $\mathcal{E}_b$ of the planets by an eight-centered oval $O_{\mathcal{E}_b}$ having the same center, axes, vertices, perimeter length, and curvature at the vertices as the ellipse, and also having practically negligible difference with respect to the surface area of $\mathcal{E}_b$ and showing barely distinguishable deformation in relation to $\mathcal{E}_b$. We call this 8-centered oval $O_{\mathcal{E}_b}$ "quasi-equivalent" to the Kepler's ellipse $\mathcal{E}_b$.

Of course, approximating ellipses by circular arcs with four-centered ovals (quadrarcs) also has been a classic subject of study by geometers, but in paper [9] it was proved that the approximation of the ellipses with four-centered ovals is geometrically and numerically poor –lack of geometric similarities and not negligible deformation error– to be qualified as *quasi-equivalent*. Therefore in this present paper we focus attention in the eight-centered ovals.

In short, this paper is inspired on the two previous works: [9] and [13]. The authors of [13] show that the deformation error between the Kepler's ellipse and a circumference, also between the Kepler's ellipse and a Cassini's oval, is small (Table 1). Here, we show that the deformation error between the Kepler's ellipse and an 8-centered oval is even smaller (Table 2). The authors of [9] obtained and showed, among other mathematical equations, the formulae that we have used here; but they used the equations only for geometric theoretical questions. Here, we show an application of the formulae; it is an application which has educational implications and we think can be interesting to students of physics.

An oval is a curve resembling a flattened circle but, unlike the ellipse, it doesn't have a specific mathematical definition. Therefore, right now we must lay down the definitions and notations of this paper.

I. DEFINITIONS AND NOTATIONS

Let A, A', B, B', be the four vertices of an ellipse $\mathcal{E}$, where A, A' are the focal vertices and B, B' are the transverse vertices. Without loss of generality, in the affine Euclidean plane $\mathbb{E}^2$, we can consider a Cartesian coordinate system $\mathcal{R}$ such that $A = (1, 0)$, $A' = (-1, 0)$, $B = (0, b)$, $B' = (0, -b)$, $1 > b > 0$. We discard the cases $b = 1$ ($\mathcal{E}$ is a circle) and $b = 0$ ($\mathcal{E}$ is a straight segment), because the problem trivializes. In order to highlight the parameter b, we call the ellipse $\mathcal{E}_b$. Therefore, the parameter b is the hypothesis parameter which determines the problem.

We consider the infinite quantity of ovals the vertices of which are also the points A, A', B, B'. Amongst them, we focus on the family of the *eight-centered ovals*, which we denote by $O_{8,b}$. And finally, we consider the family of the four-centered ovals (quadrarcs) $O_{4,b}$. The second family is a sub-family of the first one.

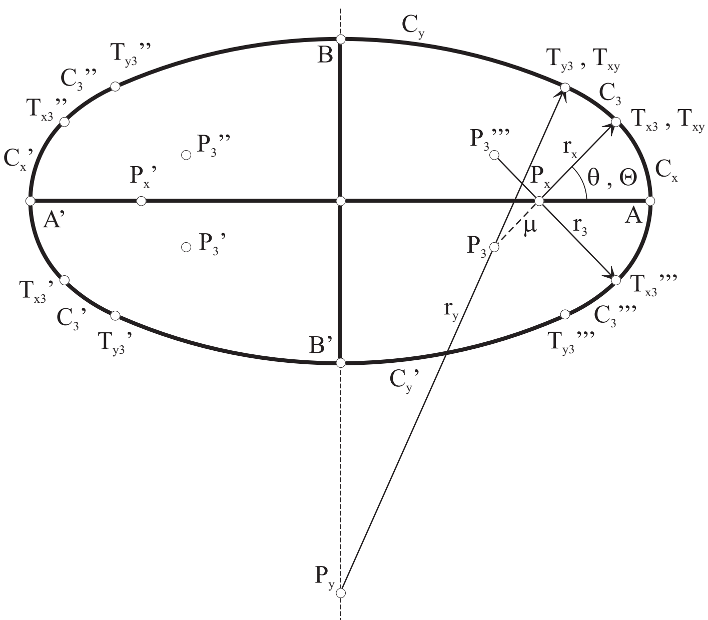

Figure 1. Elements of the 8-centered oval $O_{8,b}$ (quadrarc $O_{4,b}$ if $r_y = r_3$).

An oval $O_{8,b}$ is made up by 8 circle arcs which are tangent to each other such that, in the system $\mathcal{R}$, they have the following 8 centers (see Figure 1):

$$P_x = (x, 0)\,, \quad 0 < x < 1, \quad \text{with} \quad 1 - x < b, \quad P'_x = (-x, 0)\,,$$

$$P_y = (0, y)\,, \quad \text{with} \quad y \leq 0, \quad P'_y = (0, -y)\,,$$

$$P_3 = (x_3, y_3) \quad \text{with} \quad x_3 \geq 0,\ y_3 \leq 0,$$

$$P'_3 = (-x_3, y_3), \quad P''_3 = (-x_3, -y_3), \quad P'''_3 = (x_3, -y_3).$$

Moreover, the oval has radii r_x, r_y and r_3, where r_x is the radius of its arcs C_x, C'_x, with centers P_x, P'_x respectively; r_y is the radius of the arcs C_y, C'_y, with respective centers P_y, P'_y; and r_3 is the radius of the arcs C_3, C'_3, C''_3, and C'''_3 with respective centers P_3, P'_3, P''_3, and P'''_3. The curvatures of these arcs are inverse to their radii, $\frac{1}{r_x} = k_x$, $\frac{1}{r_y} = k_y$, $\frac{1}{r_3} = k_3$.

If $r_y = r_3$ then $P_y = P_3 = P'_3$ and $P'_y = P''_3 = P'''_3$. This special case of 8-centered oval $O_{8,b}$, noted as $O_{4,b}$, is called 4-centered oval. The segments AA', BB' are called major axis and minor axis of $O_{8,b}$, as well as at ellipse.

The oval $O_{8,b}$ has 8 contact points for its 8 arcs: point T_{x3} is the contact between C_x and C_3, point T_{y3} is the contact between C_y and C_3, points T'''_{x3}, T'''_{y3}, T''_{x3}, T''_{y3}, T'_{x3}, T'_{y3} are symmetrical to T_{x3}, T_{y3} with respect to the axes and center point and they are the contact between C'''_3, C''_3, C'_3 and the arcs having their centers on the x-axis and the y-axis, respectively (see Figure 1).

In the case of $O_{4,b}$, the 8 contact points are reduced to 4, then $T_{x3} = T_{y3}$ (we call it T_{xy}), and similarly: $T'''_{xy} = T'''_{x3} = T'''_{y3}$, $T''_{xy} = T''_{x3} = T''_{y3}$, $T'_{xy} = T'_{x3} = T'_{y3}$.

And we use the following notation: θ is the the angle $\angle(\overrightarrow{P_xA}, \overrightarrow{P_xT_{x3}})$ in the oval $O_{8,b}$, Θ is the angle $\angle(\overrightarrow{P_xA}, \overrightarrow{P_xT_{xy}})$ in the oval $O_{4,b}$, and μ is the distance $d(P_x, P_3)$ between P_x and P_3.

II. AN EIGHT-CENTERED OVAL WHICH IS QUASI-EQUIVALENT TO THE ELLIPSE

There is no oval $O_{4,b}$ having the same center, axes and vertices as $\mathcal{E}_b$, and also having the same curvature at the vertices; but in paper [9] it was shown that:

Theorem 7 *There is only one oval $O^{c-l}_{8,b}$ sharing the vertices with $\mathcal{E}_b$, having the same curvature at the vertices, i.e., with $E^c_b(O^{c-l}_{8,b}) = 0$ in equation (5), and also the same perimeter length. For $O^{c-l}_{8,b}$ the analytical expressions for the circle centers $P_x = (x, 0)$, $P_y = (0, y)$, and for circle center P_3 $x = 1 - b^2$, $y = b - \frac{1}{b}$ and $P_3 = (x - \mu\cos\theta, -\mu\sin\theta)$ with μ given in equation (6) and θ is the zero of the function H^{c-l}_b given in equation (1) with equations (2), (3) and (4).*

$$\mathbb{H}^{c-l}_b(\theta) = L - \pi\,(1+b)\sum_{n=0}^{n=\infty}\left(\frac{\sqrt{\pi}}{n!(1-2n)\Gamma\left(\frac{1}{2}-n\right)}\left(\frac{1-b}{1+b}\right)^n\right)^2, \qquad (1)$$

$$L = 4b^2\theta + \frac{4}{b}\theta_y + 4\left(b^2 + \mu\right)\left(\frac{\pi}{2} - \theta - \theta_y\right), \qquad (2)$$

$$\theta_y = \arctan\frac{\lambda\left(1 - b^2 - \mu\cos\theta\right)}{-\lambda y - \lambda\mu\sin\theta}, \qquad (3)$$

$$\lambda = 1 + \frac{\mu + b^2}{\sqrt{y^2 + \mu^2 + (1-b^2)^2 + 2\mu\left(y\sin\theta - (1-b^2)\cos\theta\right)}}, \qquad (4)$$

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$$

The quadratic error $E^c_b(O_{8,b})$ between the curvatures $k_A(\mathcal{E}_b)$, $k_B(\mathcal{E}_b)$ of the ellipse $\mathcal{E}_b$ at the vertices A, B and the curvatures $k_A(O_{8,b})$, $k_B(O_{8,b})$ of an oval $O_{8,b}$ at the vertices A, B, is

$$E^c_b(O_{8,b}) = \left(k_A(\mathcal{E}_b) - k_A(O_{8,b})\right)^2 + \left(k_B(\mathcal{E}_b) - k_B(O_{8,b})\right)^2, \qquad (5)$$

The oval $O_{8,b}^{c-l}$ has center point $P_3(\theta)$ with $P_3(\theta) = (x-\mu\cos\theta, -\mu\sin\theta)$,

$$\mu = \frac{1}{2}\frac{b(b-1)}{b^2\cos\theta - b^2 + b\sin\theta + b\cos\theta - b + \sin\theta - 1}. \quad (6)$$

The oval $O_{8,b}^{c-l}$ has the following contact points, see Figure 1:

$$T_{x3} = \left(1 - b^2 + b^2\cos\theta, b^2\sin\theta\right), \quad (7)$$

$$T_{y3} = \left(\lambda\left(1 - b^2 - \mu\cos\theta\right), y - \lambda y - \lambda\mu\sin\theta\right). \quad (8)$$

Remark 9 of [9]. We point out that there is no 8-centered oval $O_{8,b}$ having the same vertices, the same surface area and the same perimeter length as the ellipse $\mathcal{E}_b$.

A rigorous definition of the deformation error between the ellipse and the oval is the following: For each point $p \in \mathcal{E}_b$ let $q_p \in O_{8,b}$ be the point, which is closest to p among all points of intersection between $O_{8,b}$ and the straight line perpendicular to the ellipse at p. The maximum value of the distance $d(p, q_p)$, when p moves along the ellipse $\mathcal{E}_b$, is called the *deformation error* $E(\mathcal{E}_b, O_{8,b})$ between the two curves.

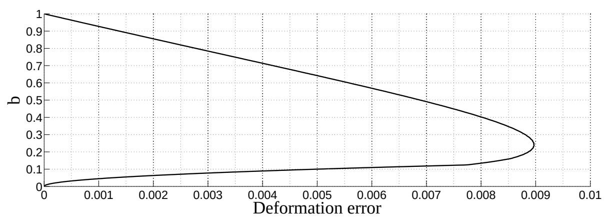

Figure 2. Graph of $\mathcal{A}(O_{\mathcal{E}_b}) - \mathcal{A}(\mathcal{E}_b) = \Delta\mathcal{A}_b$.

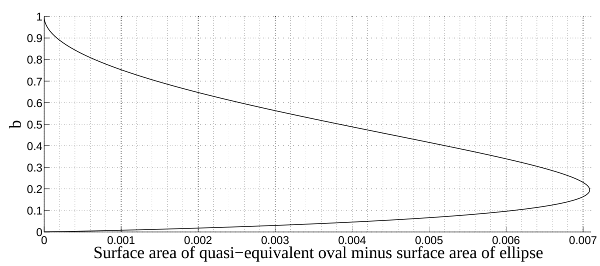

Figure 3. Graph of $\mathcal{A}(O_{\mathcal{E}_b}) - \mathcal{A}(\mathcal{E}_b) = \Delta\mathcal{A}_b$.

In [9] it was calculated, by making a software using the above equations (1) to (6), the deformation error $E(\mathcal{E}_b, O_{8,b}^{c-l}) = E_b$ for all values of parameter b, and we have shown it in the graph of Figure 2.

Furthermore, in [9] it was calculated the difference $\mathcal{A}(O_{8,b}^{c-l}) - \mathcal{A}(\mathcal{E}_b) = \Delta\mathcal{A}_b$ of their surface areas, and we have shown it, for all values of parameter b, in the graph of Figure 3. And it was proved that the eight-centered oval $O_{8,b}^{c-l}$ has not only the same vertices, perimeter length, curvature and curvature at the vertices as $\mathcal{E}_b$, but also only a small difference of the surface areas and a small deformation error. This qualifies to the oval $O_{8,b}^{c-l}$ "quasi-equivalent" to the ellipse $\mathcal{E}_b$, in short, we will call it $O_{\mathcal{E}_b}$.

III. EIGHT-CENTERED OVAL QUASI-EQUIVA- LENT TO THE KEPLER'S ELLIPSE OF PLANETS' TRAJECTORY

Applying the above formulae to the ellipse of the planets' trajectory, we have the following results:

Table 2. Calculations for $O_{\mathcal{E}_b}$ of planets' trajectory.

	Mercury	Venus	Earth	Mars
ϵ	0.205600	0.006700	0.016700	0.093500
b	0.978636	0.999978	0.999861	0.995619
θ	0.451296	0.445567	0.445597	0.446742
E_b	0.000290	0.000002	0.000002	0.000059
$\Delta\mathcal{A}_b$	0.000008	0.000000	0.000000	0.000001
	Jupiter	Saturn	Uranus	Neptune
ϵ	0.048900	0.056500	0.045700	0.011300
b	0.998804	0.998403	0.998955	0.999936
θ	0.445878	0.445984	0.445838	0.445579
E_b	0.000017	0.000021	0.000013	0.000002
$\Delta\mathcal{A}_b$	0.000000	0.000000	0.000000	0.000000

All the geometric elements of $O_{\mathcal{E}_b}$ are determined by eccentricity ϵ of Table 2 and the formulae of Section 3.

For example, for Mars we have:

$b \simeq 0.995619$, $\theta \simeq 0.446724$, $P_x \simeq (0.008742, 0)$,

$P_y \simeq (0, -0.008781)$, $P_3 \simeq (0.002833, -0.002831)$,

$T_{x3} \simeq (0.902725, 0.428237)$, $T_{y3} \simeq (0.431808, 0.898061)$.

The calculation steps for this example are as follows:

With equation of eccentricity

$\epsilon = 0.093500$ with $a = 1 \Rightarrow 1 \simeq b^2 + 0.093500^2 \Rightarrow$

$\Rightarrow b \simeq 0.995619$.

With Theorem 7 of [9] in Section 3 of this paper

$$P_x = \left(1 - b^2, 0\right) \simeq \left(1 - 0.995619^2, 0\right) \simeq (0.008742, 0),$$

$$P_y = \left(0, b - \tfrac{1}{b}\right) \simeq \left(0, 0.995619 - \tfrac{1}{0.995619}\right) \simeq$$

$\simeq (0, -0.008781)$, $y = -0.008781$.

With equation (1)

$$\xi = \pi(1+b)\sum_{n=0}^{n=\infty}\left(\frac{\sqrt{\pi}}{n!(1-2n)\Gamma\left(\frac{1}{2}-n\right)}\left(\frac{1-b}{1+b}\right)^n\right)^2 \simeq$$

$$\simeq \pi(1+0.995619)\sum_{n=0}^{n=\infty}\left(\frac{\sqrt{\pi}}{n!(1-2n)\Gamma\left(\frac{1}{2}-n\right)}\left(\frac{1-0.995619}{1+0.995619}\right)^n\right)^2,$$

then $\xi \simeq 6.269430$.

Now, using an iterative numerical method we calculate the implicit value θ such that:

with equation (6)

$$\mu = \frac{1}{2}\frac{b(b-1)}{b^2\cos\theta - b^2 + b\sin\theta + b\cos\theta - b + \sin\theta - 1};$$

and with equation (4)

$$\lambda = 1 + \frac{\mu + b^2}{\sqrt{y^2 + \mu^2 + (1-b^2)^2 + 2\mu\left(y\sin\theta - (1-b^2)\cos\theta\right)}};$$

and with equation (3)

$\theta_y = \arctan \frac{\lambda\left(1-b^2-\mu\cos\theta\right)}{-\lambda y-\lambda\mu\sin\theta}$;

and with equation (2) we have that

$L = 4b^2\theta + \frac{4}{b}\theta_y + 4\left(b^2+\mu\right)\left(\frac{\pi}{2}-\theta-\theta_y\right) \simeq 6.269430.$

So, we find, with an iterative numerical method, that $\theta \simeq 0.446724$ and also $\mu \simeq 0.006552$, $\lambda \simeq 152.405404$, $\theta_y \simeq 0.444\,399$.

We think it could be a good exercise for physics undergraduate students to make a software of a numerical method to reproduce the calculations.

Finally, with Theorem 7 of [9] in Section 3 of this paper

$P_3 = (x-\mu\cos\theta, -\mu\sin\theta) \simeq (0.002833\ -0.002\,831)$;

with equation (7)

$T_{x3} = \left(1-b^2+b^2\cos\theta, b^2\sin\theta\right) \simeq (0.902\,725, 0.428\,237\,)$;

and with equation (8)

$T_{y3} = \left(\lambda\left(1-b^2-\mu\cos\theta\right), y-\lambda y-\lambda\mu\sin\theta\right) \simeq$

$\simeq (0.431\,808\,0.898\,061\,)$.

Table 2 shows that the maximum difference –deformation error– $E(O,\mathcal{E}_b)$ between the positions given by the circumference O and the Kepler's ellipse $\mathcal{E}_b$ –Table 1– is greater than the maximum difference –deformation error– $E(\mathcal{E}_b, O_{\mathcal{E}_b})$ between the positions given by the quasi-equivalent oval $O_{\mathcal{E}_b}$ and the Kepler's ellipse $\mathcal{E}_b$. Also the deformation error $E(\mathcal{E}_b, C_b)$ between the positions given by the Cassinian oval C_b and the Kepler's ellipse $\mathcal{E}_b$ –Table 1– is greater than the deformation error $E(\mathcal{E}_b, O_{\mathcal{E}_b})$. For Mercury and Mars, with greater eccentricity, this maximum difference –deformation error $E(\mathcal{E}_b, O_{\mathcal{E}_b})$– is two order of magnitude lower than in the case of circumference O and Cassinian oval C_b; and for the rest of planets' deformation error $E(\mathcal{E}_b, O_{\mathcal{E}_b})$ is one or two order of magnitude lower.

Moreover Table 2 shows that difference $\mathcal{A}(O_{\mathcal{E}_b}) - \mathcal{A}(\mathcal{E}_b)$ of their surface areas: for Mercury and Mars are barely distinguishable; and for the rest of planets are practically negligible.

Table 1 -Morgado and Soares [13]- shows that, for Mars, Cassini's oval is barely distinguishable from Kepler's ellipse (deformation error 0.004400), but it is distinguishable near to the minor semi-axis for Mercury with high eccentricity (deformation error 0.021841). Figure 2 of [13] shows visually this deformation error for Mercury. However, Table 2 shows that, even for Mercury, a figure can not display visually the deformation error between Kepler's ellipse $\mathcal{E}_b$ and the quasi-equivalent oval $O_{\mathcal{E}_b}$ (deformation error for Mercury is 0.000290).

IV. CONCLUSION

With the results of this paper, we have an approximation of the Kepler's ellipse of the planets' trajectory by circular arcs, which is *quasi-equivalent*, that is: we have presented the approximation of the Kepler's ellipse $\mathcal{E}_b$ by the eight-centered oval $O_{\mathcal{E}_b}$ having the same center, axes, vertices, perimeter length and curvature at the vertices as $\mathcal{E}_b$, and also having practically negligible difference with respect to the surface area $\mathcal{E}_b$, and showing barely indistinguishable deformation error in relation to $\mathcal{E}_b$.

Then the oval $O_{\mathcal{E}_b}$ and the corresponding ellipse $\mathcal{E}_b$ are numerically barely distinguishable. Therefore, we think it could be a good exercise for undergraduate level physics to discuss these ovals as quasi-equivalent curves related to the Kepler's ellipse of planets' trajectory because the results show that it is not enough to consider only numerical measures to decide which are the actual laws governing the planetary movement. This paper discusses differences and similarities among ovals and ellipses by solving a set of algebraic equations, we think it could be a good exercise for physics undergraduate students to make a software of a numerical method to reproduce the calculations. We think this discussion could be interesting to the readers because it is well known that before to decide by the ellipse as the best curve to describe planet's orbit, Kepler tried to fit different ovals to the Tycho Brahe's astronomical data.

REFERENCES

[1] T. Banchoff, and P. Giblin, Am. Math. Mon. 101, 403 (1994).

[2] A. A. Mazzotti, Nexus Netw. J. 16, 389 (2014).

[3] A. A. Mazzotti, Nexus Netw. J. 16, 345 (2014).

[4] P. L. Rosin, Math. Intell. 23, 58 (2001).

[5] P. L. Rosin, Comput. Aided Geom. Des. 16, 269 (1999).

[6] P. L. Rosin, Graph. Mod. Im. Proc.60, 209(1998).

[7] P. L. Rosin, M. L. V. Pitteway, Math. Gazette. 85, 13 (2001).

[8] F. R. Honey, Pop. Astronomy. 16, 617 (1908).

[9] B. Herrera, and A. Samper, J. Geom. Graph. 19, 257 (2015).

[10] J. Kepler and translator W. H. Donahue, *The New Astronomy*, (Cambridge University Press, Cambridge, 1992).

[11] I. B. Cohen, J. Hist. Med. Allied Sci. 17, 72 (1962).

[12] J. Sivardiere, Eur. J. Phys. 15, 62 (1994).

[13] B. Morgado, and V. Soares, Eur. J. Phys. 35, 025009 (2004).

[14] P. S. M. de Laplace, *Mécanique Céleste, vol 1*, (Hilliard, Gray, Little and Wilkins, Boston, 1829).