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Non-manipulability by clones in bankruptcy problems

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ABSTRACT

We introduce non-manipulability by clones for bankruptcy problems, which entitles claimants to merge or split only when they are or become identical agents. We show that this weaker non-manipulability requirement, together with either claim monotonicity or claims continuity, allows for new characterizations of the proportional rule on the general class of bankruptcy problems.

1. Introduction

Bankruptcy problems (O'Neill, 1982) deal with situations where an amount of a perfectly divisible resource should be distributed among a group of agents presenting conflicting claims, that is, the total amount to divide is not enough to fulfill all demands. These problems are solved by rules proposing an allocation vector that takes into consideration the specifics of the agents.

An important topic in economics is the study of rules that are immune to the strategic behavior of the agents by misrepresenting their characteristics. For the bankruptcy problem, O'Neill (1982) introduces *non-manipulability* (or *strategy-proofness*) as the combination of *non-manipulability via merging and splitting*. A rule is *non-manipulable via merging* if no group of agents can benefit from consolidating claims and it is *non-manipulable via splitting* if no agent can benefit from dividing its claim into claims of a group of agents. A rule is *non-manipulable* if it is unaffected by both types of misrepresentations.

The *proportional* rule makes agents' payments proportional to their demands and it is one of the most commonly used proposals in real-life situations when a firm goes into bankruptcy. Due to its central role in both practice and theory, it has been extensively analyzed from an axiomatic viewpoint. O'Neill (1982) was the first to axiomatically characterize the proportional rule on the basis of non-manipulability, together with *budget balance*, *anonymity*, *continuity*, and the *dummy* axiom (or *null consistency*). Informally speaking, budget balance requires distributing the entire endowment; anonymity asserts that the only important feature of the agents is their claims; continuity imposes that small changes in both, the endowment and the claims, result in small changes in the final allocation; and the dummy axiom states that if an agent with zero claim leaves, then in the corresponding reduced problem the remaining agents get the same amount as initially. O'Neill's result was refined by Chun (1988) showing that the dummy axiom is redundant and de Frutos (1999) proves that anonymity and continuity in Chun's result can be replaced with *non-negativity*, which requires awards to be non-negative. More recently, Ju et al. (2007) strengthen de Frutos' result by relaxing non-negativity to *one-sided boundedness*, which states that payments are bounded from either above or below, and using a pairwise version of non-manipulability as defined in Ju (2003). Indeed, Ju (2003) introduces restrictions on a coalition formation, just permitting mergers or spin-offs by pairs, and characterizes a set of parametric rules (Young, 1987) that are either *non-manipulable via (pairwise) merging or splitting*.[1] On the other hand, Ju et al. (2007) investigate the relation between non-manipulability and proportionality in more general classes of allocation problems.

On the full domain of bankruptcy problems, Moreno-Ternero (2006) shows that non-manipulability is equivalent to *additivity of claims* (Curiel et al., 1987),[2] requiring that merging or splitting the agents' claims do not affect the amounts received by any other

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1 We refer readers to Ju et al. (2007) for formal definitions of the mentioned properties.

2 Moreno-Ternero (2006) renames this property as *strong non-manipulability*.

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agent involved in the problem. Related results can be found in Ju and Moreno-Ternero (2011) and Ju (2013). The former work connects non-manipulability to *progessivity* in taxation problems; the latter analyzes the implications of non-manipulability in networks.

Here, we limit our attention to splits and mergers involving identical agents, that is, with the same claim. It is quite usual in a real economy for agents (firms) with some common attributes to create a joint venture or for an agent to split into similar new spin-offs, although these practices involving very different agents are reproved. A natural and simple way to formally accommodate these ideas is to restrict the possibility of manipulating to symmetric agents or clones. We name this axiom *non-manipulability by clones*. Interestingly, we show that this substantially weaker form of non-manipulability is enough to uniquely determine the proportional rule for the realistic case in which all claims are zero o rational numbers (Theorem 1). We extend this result to the general domain of bankruptcy problems by adding either *claim monotonicity* (Theorem 2) or *claims continuity* (Theorem 3). While claims continuity enforces that small changes in the claims of the agents do not lead to large changes in the awards recommendation, claim monotonicity requires that if only one agent's claim increases, she should not be worse-off.

The rest of the paper is organized as follows. In Section 2 we introduce some notation and definitions. Section 3 contains the characterization results. Section 4 concludes with final remarks. The proofs of the lemmata are postponed to Appendix.

2. Preliminaries

Let $\mathbb{N} = \{1, 2, \dots\}$ (the set of natural numbers) represent the set of all potential agents (claimants) and let $\mathcal{N}$ be the collection of all non-empty finite subsets of $\mathbb{N}$. An element $N \in \mathcal{N}$ describes a finite set of agents where $|N| = n$. By $\mathbb{Q}_+ = \{a/b \mid a, b \in \mathbb{N}\}$ we denote the set of positive rational numbers.

A *bankruptcy problem is a* triple (N, E, c) such that $N \in \mathcal{N}$, $c \in \mathbb{R}^N_+$, $E \geq 0$, and $\sum_{i\in N} c_i \geq E$. By $\mathcal{B}$ we denote the set of all bankruptcy problems. If $(N, E, c) \in \mathcal{B}$, then each agent i in the set of creditors N has a claim c_i to the net worth or estate $E \geq 0$ of a bankrupt firm. A *bankruptcy rule* is a function $\beta : \mathcal{B} \longrightarrow \bigcup_{N\in\mathcal{N}} \mathbb{R}^N$ that associates with every $(N, E, c) \in \mathcal{B}$ a unique vector $\beta(N, E, c) \in \mathbb{R}^N$ satisfying $\sum_{i\in N}\beta_i(N, E, c) = E$ (*budget balance* (BB)), that is, the sum of all payments should be equal to the estate.

Instances of well-studied bankruptcy rules are the *proportional* rule (P) and the *constrained equal awards* rule (CEA). The P rule makes awards proportional to the claims. Formally, for all $(N, E, c) \in \mathcal{B}$ and all $i \in N$, $P_i(N, E, c) = \lambda c_i$ where $\lambda \in \mathbb{R}_+$ is such that $\sum_{j\in N} \lambda c_j = E$. The CEA rule rewards all claimants equally subject to no one receiving more than her claim. Formally, for all $(N, E, c) \in \mathcal{B}$ and all $i \in N$, $CEA_i(N, E, c) = \min\{c_i, \lambda\}$ where $\lambda \in \mathbb{R}_+$ is such that $\sum_{j\in N} \min\{c_j, \lambda\} = E$. For a detailed analysis of bankruptcy rules we refer to Thomson (2019).

3. Axiomatizations of the proportional rule

In this part, we provide new axiomatic foundations of the proportional rule. In insolvency or liquidation proceedings of bankrupt firms, mergers or spin-offs can hide incentives to manipulate the final distribution of the firm's assets. A way to avoid this malpractice is to make use of allocation rules that prevent this strategic behavior. In a formal manner, a bankruptcy rule β satisfies

- *non-manipulability* (NM) if for all $(N, E, c), (N', E, c') \in \mathcal{B}$, if $N' \subset N$ and there is $m \in N'$ such that $c'_m = c_m + \sum_{k\in N\setminus N'} c_k$ and $c'_i = c_i$ for all $i \in N' \setminus \{m\}$, then $\beta_i(N', E, c') = \beta_i(N, E, c)$ for all $i \in N' \setminus \{m\}$.

NM imposes that agents not involved in the merging or splitting operations get the same amount as initially. Under BB, this formulation is equivalent to requiring that the payoffs of neither the agents merging or splitting vary, that is, $\beta_m(N', E, c') = \beta_m(N, E, c) + \sum_{j\in N\setminus N'} \beta_j(N, E, c)$, as defined in de Frutos (1999).[3]

A mild version of NM is obtained by restricting these operations to *symmetric* agents or *clones*, that is, agents who have the same claim. We refer to this new axiom as *non-manipulability by clones*, because it prevents equal agents taking advantage by merging or splitting claims.[4] Formally, a bankruptcy rule β satisfies

- *non-manipulability by clones* (NMC) if for all $(N, E, c), (N', E, c') \in \mathcal{B}$, if $N' \subset N$ and there is $m \in N'$ such that $c_i = \frac{c'_m}{|N\setminus N'|+1}$ for all $i \in N \setminus N' \cup \{m\}$ and $c'_i = c_i$ for all $i \in N' \setminus \{m\}$, then $\beta_i(N', E, c') = \beta_i(N, E, c)$ for all $i \in N' \setminus \{m\}$.

Note that, by BB, it also holds that $\beta_m(N'E, c') = \beta_m(N, E, c) + \sum_{j\in N\setminus N'} \beta_j(N, E, c)$.

Obviously, the proportional rule satisfies NMC as it satisfies the stronger NM. It also meets *equal treatment of equals*, a well-established axiom that requires that symmetric agents receive the same amount, and *no award for null*, which states that no amount is awarded to agents with zero claims. Formally, a bankruptcy rule β satisfies

- *equal treatment of equals* (ETE) if for all $(N, E, c) \in \mathcal{B}$ and all $i, j \in N$, if $c_i = c_j$ then $\beta_i(N, E, c) = \beta_j(N, E, c)$;
- *no award for null* (NAN) if for all $(N, E, c) \in \mathcal{B}$ and all $i \in N$, if $c_i = 0$ then $\beta_i(N, E, c) = 0$.

de Frutos (1999) shows that NM implies ETE. This result is strengthened in the first lemma establishing that ETE is a consequence of the weaker axiom of NMC.

Lemma 1. NMC *implies* ETE.

Ju et al. (2007) show that NM[5] implies NAN. The next lemma states that this logical relation holds when NM is replaced by NMC.

Lemma 2. NMC *implies* NAN.

The last lemma says that, in the presence of NMC, if two agents have rational claims then the ratio between what they receive and what they claim remains constant.

Lemma 3. *Let β be a bankruptcy rule satisfying* NMC. *If $(N, E, c) \in \mathcal{B}$ and $i, j \in N$ are such that $c_i, c_j \in \mathbb{Q}_+$, then*

$$\frac{\beta_i(N, E, c)}{c_i} = \frac{\beta_j(N, E, c)}{c_j}. \tag{1}$$

Now, we have all the tools to prove that NMC uniquely determines the proportional rule for bankruptcy problems in which agents either have zero claims or their claims are expressed by rational numbers.

Theorem 1. *Let β be a bankruptcy rule satisfying* NMC. *If $(N, E, c) \in \mathcal{B}$ is such that, for all $i \in N$, c_i is either zero or a positive rational number, then $\beta(N, E, c) = P(N, E, c)$.*

[3] See Lemma 3.1. in Moreno-Ternero (2006).

[4] In more general settings, such as financial networks or multi-issue problems, the idea to restrict manipulations to agents with some common traits also plays a role in characterizing the extension of the proportional rule to these contexts (see, for instance, Csóka and Herings, 2021; Acosta-Vega et al., 2022).

[5] Ju et al. (2007) call this axiom *merging-splitting-proofness*.

Proof. Let $(N, E, c) \in \mathcal{B}$ be such that all claims are either zero or positive rational numbers. If $|\{i \in N \mid c_i = 0\}| \in \{n-1, n\}$ then, by NMC, that implies NAN, and BB, $\beta(N, E, c) = P(N, E, c)$. Otherwise, let $i, j \in N$ be such that $c_i, c_j \in \mathbb{Q}_+$. In this case, by Lemma 3 we have that $\beta_i(N, E, c)/c_i = \beta_j(N, E, c)/c_j$. Consequently, there exists a constant λ such that, for all $k \in N$, $\beta_k(N, E, c) = \lambda c_k$ since, by NAN, this also holds for players with zero claim. Hence, by BB, $\lambda = E/\sum_{j\in N} c_j \geq 0$, and thus $\beta(N, E, c) = P(N, E, c)$. □

A way to extend the above result to include non-rational claims is to impose, additionally, one of the following two widely accepted axioms. A bankruptcy rule β satisfies

- *claim monotonicity* (CM) if for all $(N, E, c), (N, E, c') \in \mathcal{B}$ such that $c'_i > c_i$ for some $i \in N$ and $c'_j = c_j$ for all $j \in N \setminus \{i\}$, then $\beta_i(N, E, c') \geq \beta_i(N, E, c)$;
- *claims continuity* (CC) if for each sequence of bankruptcy problems $\{(N, E, c^n)\}_{n\in\mathbb{N}}$ converging to (N, E, c), the sequence $\{\beta(N, E, c^n)\}_{n\in\mathbb{N}}$ converges to $\beta(N, E, c)$.

CM says that if an agent's claim increases, while the claims of the other agents and the amount to be distributed remain equal, her award should not decrease. CC imposes that small variations in the claims imply small variations in the resulting allocation vector. CM and CC are not related to each other.[6]

It is well known that the proportional rule satisfies both axioms. So, it remains to show that CM or CC in combination with NMC characterize it.

Theorem 2. *A bankruptcy rule satisfies* NMC *and* CM *if and only if it is the proportional rule.*

Proof. To show uniqueness, let β be a bankruptcy rule satisfying NMC and CM, and $(N, E, c) \in \mathcal{B}$. If c_i equals either zero or a rational number for all $i \in N$, by Theorem 1, $\beta(N, E, c) = P(N, E, c)$.

Otherwise, we use an induction argument on the number of agents with a non-rational and non-zero claim. Let us denote this set by $N_{\neg\mathbb{Q}_+}$. For $|N_{\neg\mathbb{Q}_+}| = 1$, let $N_{\neg\mathbb{Q}_+} = \{i^*\}$ with $c_{i^*} \notin \mathbb{Q}_+$, $c_{i^*} > 0$. Then, there exist two sequences of positive rational numbers $\{l^k\}_{k\in\mathbb{N}}$ and $\{r^k\}_{k\in\mathbb{N}}$ converging to c_{i^*} from left and right, respectively, and such that $l^k \leq l^{k+1} < c_{i^*} < r^{k+1} \leq r^k$ for all $k \in \mathbb{N}$. Let $\{(N, E, \underline{c}^k)\}_{k\in\mathbb{N}}$ and $\{(N, E, \bar{c}^k)\}_{k\in\mathbb{N}}$ be two associated sequences of bankruptcy problems converging to (N, E, c) where, for all $k \in \mathbb{N}$, $\underline{c}^k_i = \bar{c}^k_i = c_i$ for all $i \in N \setminus \{i^*\}$, $\underline{c}^k_{i^*} = l^k$, and $\bar{c}^k_{i^*} = r^k$. By Theorem 1 and CM, for all $k \in \mathbb{N}$,

$$\begin{aligned} P_{i^*}(N, E, \underline{c}^k) &= \beta_{i^*}(N, E, \underline{c}^k) \leq \beta_{i^*}(N, E, c) \leq \beta_{i^*}(N, E, \bar{c}^k) \\ &= P_{i^*}(N, E, \bar{c}^k). \end{aligned}$$

By CC of the proportional rule,

$$\begin{aligned} \lim_{k\to\infty} P_{i^*}(N, E, \underline{c}^k) &= P_{i^*}(N, E, c) \leq \beta_{i^*}(N, E, c) \leq P_{i^*}(N, E, c) \\ &= \lim_{k\to\infty} P_{i^*}(N, E, \bar{c}^k), \end{aligned}$$

which leads to $\beta_{i^*}(N, E, c) = P_{i^*}(N, E, c)$.

It remains to see that $\beta_j(N, E, c) = P_j(N, E, c)$ for all $j \in N \setminus N_{\neg\mathbb{Q}_+}$.

If $|\{j \in N \setminus N_{\neg\mathbb{Q}_+} \mid c_j \neq 0\}| \leq 1$, by NMC, that implies NAN, and BB, $\beta(N, E, c) = P(N, E, c)$. Otherwise, there are at least two players in $N \setminus N_{\neg\mathbb{Q}_+}$ with rational claims. Thus, by Lemma 3 and NAN, a consequence of NMC (Lemma 2), there exists a constant λ such that, for all $j \in N \setminus N_{\neg\mathbb{Q}_+}$, $\beta_j(N, E, c) = \lambda c_j$. By BB,

$$E = \sum_{j\in N\setminus N_{\neg\mathbb{Q}_+}} \beta_j(N, E, c) + P_{i^*}(N, E, c) = \lambda \sum_{j\in N\setminus N_{\neg\mathbb{Q}_+}} c_j + \frac{c_{i^*}}{\sum_{j\in N} c_j} E,$$

which implies $\lambda = E/\sum_{j\in N} c_j \geq 0$, and hence $\beta(N, E, c) = P(N, E, c)$.

Induction hypothesis: if $|N_{\neg\mathbb{Q}_+}| = k$, for $1 \leq k \leq n-1$, then $\beta(N, E, c) = P(N, E, c)$.

Assume $|N_{\neg\mathbb{Q}_+}| = k+1$. Select an arbitrary agent $i^* \in N_{\neg\mathbb{Q}_+}$ and, as before, construct two sequences of bankruptcy problems converging to (N, E, c) obtained from two sequences of positive rational numbers converging to c_{i^*}, from left and right. For any of the bankruptcy problems in such sequences, the number of players with non-rational and non-zero claims equals k, and thus, by induction hypothesis, β coincides with the proportional rule. Now, exactly as before, applying CM, and CC of the proportional rule, we can conclude that $\beta_j(N, E, c) = P_j(N, E, c)$ for all $j \in N_{\neg\mathbb{Q}_+}$. From this point, using very similar arguments as for the case $|N_{\neg\mathbb{Q}_+}| = 1$ we reach that $\beta(N, E, c) = P(N, E, c)$. □

If, instead of CM, we require CC together with NMC, we obtain a new characterization.

Theorem 3. *A bankruptcy rule satisfies* NMC *and* CC *if and only if it is the proportional rule.*

Proof. Let β be a bankruptcy rule satisfying NMC and CC, and $(N, E, c) \in \mathcal{B}$. If, for all $i \in N$, $c_i \in \mathbb{Q}_+$ then, by Theorem 1, $\beta(N, E, c) = P(N, E, c)$. Otherwise, there exists a sequence $\{(N, E, c^k)\}_{k\in\mathbb{N}}$ of bankruptcy problems with $c^k_i \in \mathbb{Q}_+$ for all $i \in N$ and all $k \in \mathbb{N}$ converging to (N, E, c). By Theorem 1, $P(N, E, c^k) = \beta(N, E, c^k)$ for all $k \in \mathbb{N}$. Finally, by CC of β and P, we have that $P(N, E, c) = \lim_{k\to\infty} P(N, E, c^k) = \lim_{k\to\infty} \beta(N, E, c^k) = \beta(N, E, c)$. □

To conclude, in the following remark we show that the axioms in Theorems 2 and 3 are logically independent.

Remark 1.

The *CEA* rule satisfies CM and CC but clearly not NMC. Now, we define a bankruptcy rule β^* meeting NMC but neither CC nor CM. Let $(N, E, c) \in \mathcal{B}$.

- If $\sum_{k\in N\setminus N_{\neg\mathbb{Q}_+}} c_k \leq E$, then $\beta^*(N, E, c) = P(N, E, c)$.
- If $\sum_{k\in N\setminus N_{\neg\mathbb{Q}_+}} c_k > E$, then

  $$\beta^*_i(N, E, c) = \frac{c_i}{\sum_{k\in N\setminus N_{\neg\mathbb{Q}_+}} c_k} E \text{ for all } i \in N \setminus N_{\neg\mathbb{Q}_+}$$

  and

  $$\beta^*_i(N, E, c) = 0 \text{ for all } i \in N_{\neg\mathbb{Q}_+}.$$

  Note that β^* is well defined, that is, it meets BB.

  To show that β^* satisfies NMC it is enough to observe that when a group of agents with equal rational (irrational) claims merge, in the new problem the representative agent of the merger will have a rational (irrational) claim as well. In the same way, when an agent with rational (irrational) claim splits into a group of clones, all of them will present a rational (irrational) claim. Technically, the product of a rational (irrational) number and a natural number results in a rational (irrational) number, and the division of a rational (irrational) number by a natural number results in a rational (irrational) number. Hence, whenever an agent splits or a merger occurs, if initially $\sum_{k\in N\setminus N_{\neg\mathbb{Q}_+}} c_k \leq E$ (or the contrary), the condition will hold afterwards. Since $\beta^* \neq P$, from Theorems 2 and 3 we can conclude that β^* does not satisfy either CM or CC.

6 For a discussion see Thomson (2019).

4. Concluding remarks

In the context of bankruptcy problems, we have obtained new characterizations for the proportional rule imposing NMC, a weak form of non-manipulability that entitles agents to merge or split only when they are symmetric, together with BB (included in the definition of a rule) and a standard axiom referring monotonicity or continuity on claims. These characterizations allow for a comparison with those provided by de Frutos (1999) and Ju et al. (2007). With respect to de Frutos' result, non-manipulability is weakened into NMC, while non-negativity and either CM or CC are not comparable. Regarding Ju et al.'s result, one-sided boundedness is weaker than CC. Clearly, one-sided boundedness does not imply CM but the reverse implication remains open. On the other hand, β^* as defined in Remark 1 satisfies NMC but not pairwise non-manipulability, which indicates that the former is not stronger than the latter. It has not been determined yet whether pairwise non-manipulability implies NMC. In future research, it would be interesting to study whether a pairwise version of NMC suffices to distinguish the proportional rule.

Data availability

No data was used for the research described in the article.

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Appendix

Proof (*Lemma 1*).
Let β be a bankruptcy rule satisfying NMC, $\varepsilon_0 = (N^0, E, c^0) \in \mathcal{B}$, and $i, j \in N^0$ such that $c_i^0 = c_j^0 = \bar{c}$. We distinguish two cases:

Case 1: $|N^0| \geq 3$. Suppose, w.l.o.g., that

$$\beta_i(\varepsilon_0) > \beta_j(\varepsilon_0). \tag{2}$$

We will show that this assumption leads to a contradiction. The proof of this case is done in six steps.

Step 1: From ε_0 player i splits into clones i and i' defining the bankruptcy problem $\varepsilon_1 = (N^1, E, c^1)$, being $N^1 = N^0 \cup \{i'\}$, $c_k^1 = c_k^0$ for all $k \in N^1 \setminus \{i, i'\}$, and $c_i^1 = c_{i'}^1 = \bar{c}/2$. By NMC, for all $k \in N^1 \setminus \{i, i'\}$,

$$\beta_k(\varepsilon_0) = \beta_k(\varepsilon_1), \tag{3}$$

which implies, by BB,

$$\beta_i(\varepsilon_0) = \beta_i(\varepsilon_1) + \beta_{i'}(\varepsilon_1). \tag{4}$$

Step 2: From ε_1 player j splits into clones j and j' defining the bankruptcy problem $\varepsilon_2 = (N^2, E, c^2)$, being $N^2 = N^1 \cup \{j'\}$, $c_k^2 = c_k^1$ for all $k \in N^2 \setminus \{j, j'\}$, and $c_j^2 = c_{j'}^2 = \bar{c}/2$. Note that $c_j^2 = c_i^2 = c_{j'}^2 = c_{i'}^2 = \bar{c}/2$. By NMC, for all $k \in N^2 \setminus \{j, j'\}$,

$$\beta_k(\varepsilon_1) = \beta_k(\varepsilon_2), \tag{5}$$

which implies, by BB,

$$\beta_j(\varepsilon_1) = \beta_j(\varepsilon_2) + \beta_{j'}(\varepsilon_2). \tag{6}$$

Hence,

$$\beta_i(\varepsilon_2) + \beta_{i'}(\varepsilon_2) \underset{(5)}{=} \beta_i(\varepsilon_1) + \beta_{i'}(\varepsilon_1) \underset{(4)}{=} \beta_i(\varepsilon_0) \underset{(2)}{>} \beta_j(\varepsilon_0) \underset{(3)}{=} \beta_j(\varepsilon_1) \underset{(6)}{=} \beta_j(\varepsilon_2) + \beta_{j'}(\varepsilon_2).$$

That is,

$$\beta_i(\varepsilon_2) + \beta_{i'}(\varepsilon_2) > \beta_j(\varepsilon_2) + \beta_{j'}(\varepsilon_2).$$

Assume, w.l.o.g., that

$$\beta_i(\varepsilon_2) > \beta_j(\varepsilon_2). \tag{7}$$

Step 3: From ε_2 players i, i' and j' merge under the name of i' defining the bankruptcy problem $\varepsilon_3 = (N^3, E, c^3)$ being $N^3 = N^2 \backslash \{i, j'\}$, $c_k^3 = c_k^2$ for all $k \in N^3 \backslash \{i'\}$, and $c_{i'}^3 = c_i^2 + c_{i'}^2 + c_{j'}^2 = 3\,\bar{c}/2$. Note that $c_j^3 = c_j^2 = \bar{c}/2$. By NMC, for all $k \in N^3 \setminus \{i'\}$,

$$\beta_k(\varepsilon_3) = \beta_k(\varepsilon_2), \tag{8}$$

which implies, by BB,

$$\beta_{i'}(\varepsilon_3) = \beta_i(\varepsilon_2) + \beta_{i'}(\varepsilon_2) + \beta_{j'}(\varepsilon_2) = E - \beta_j(\varepsilon_2) - \sum_{k \in N^2 \backslash \{i,j,i',j'\}} \beta_k(\varepsilon_2). \tag{9}$$

Step 4: From ε_2 players i', j and j' merge under the name of i' defining the bankruptcy problem $\varepsilon_4 = (N^4, E, c^4)$ being $N^4 = N^2 \setminus \{j, j'\}$, $c_k^4 = c_k^2$ for all $k \in N^4 \setminus \{i'\}$, and $c_{i'}^4 = c_{i'}^2 + c_j^2 + c_{j'}^2 = 3\,\bar{c}/2$. By NMC, for all $k \in N^4 \setminus \{i'\}$,

$$\beta_k(\varepsilon_4) = \beta_k(\varepsilon_2), \tag{10}$$

which implies, by BB,

$$\beta_{i'}(\varepsilon_4) = \beta_{i'}(\varepsilon_2) + \beta_j(\varepsilon_2) + \beta_{j'}(\varepsilon_2) = E - \beta_i(\varepsilon_2) - \sum_{k \in N^2 \backslash \{i,j,i',j'\}} \beta_k(\varepsilon_2). \tag{11}$$

Hence,

$$\beta_{i'}(\varepsilon_4) \underset{(7)}{<} E - \beta_j(\varepsilon_2) - \sum_{k \in N^2 \backslash \{i,j,i',j'\}} \beta_k(\varepsilon_2) \underset{(9)}{=} \beta_{i'}(\varepsilon_3). \tag{12}$$

Moreover, by BB,

$$E = \beta_i(\varepsilon_4) + \beta_{i'}(\varepsilon_4) + \sum_{k \in N^4 \backslash \{i,i'\}} \beta_k(\varepsilon_4) \underset{(10)}{=} \beta_i(\varepsilon_4) + \beta_{i'}(\varepsilon_4) + \sum_{k \in N^4 \backslash \{i,i'\}} \beta_k(\varepsilon_2), \tag{13}$$

and

$$\begin{aligned} E &= \beta_{i'}(\varepsilon_3) + \beta_j(\varepsilon_3) + \textstyle\sum_{k \in N^3 \backslash \{i',j\}} \beta_k(\varepsilon_3) \\ &\underset{(8)}{=} \beta_{i'}(\varepsilon_3) + \beta_j(\varepsilon_3) + \textstyle\sum_{k \in N^3 \backslash \{i',j\}} \beta_k(\varepsilon_2) \\ &\underset{N^3 \backslash \{i',j\} = N^4 \backslash \{i,i'\}}{=} \beta_{i'}(\varepsilon_3) + \beta_j(\varepsilon_3) + \textstyle\sum_{k \in N^4 \backslash \{i,i'\}} \beta_k(\varepsilon_2). \end{aligned} \tag{14}$$

From (13) and (14),

$$\beta_i(\varepsilon_4) + \beta_{i'}(\varepsilon_4) = \beta_{i'}(\varepsilon_3) + \beta_j(\varepsilon_3)$$

and from (12) we can conclude that

$$\beta_i(\varepsilon_4) > \beta_j(\varepsilon_3). \tag{15}$$

Step 5: From ε_4 player i splits into clones i and j defining the bankruptcy problem $\varepsilon_5 = (N^5, E, c^5)$, being $N^5 = N^4 \cup \{j\}$, $c^5_k = c^4_k$ for all $k \in N^5 \setminus \{i, j\}$, and $c^5_i = c^5_j = c^4_i/2 = c^2_i/2 = \bar{c}/4$. Note that, $c^5_{i'} = c^4_{i'} = 3\bar{c}/2$. By NMC, for all $k \in N^5 \setminus \{i, j\}$, $\beta_k(\varepsilon_5) = \beta_k(\varepsilon_4)$ which implies, by BB,

$$\beta_i(\varepsilon_4) = \beta_i(\varepsilon_5) + \beta_j(\varepsilon_5). \quad (16)$$

Step 6: From ε_3 player j splits into clones i and j defining the bankruptcy problem $\varepsilon_6 = (N^6, E, c^6)$, being $N^6 = N^3 \cup \{i\}$, $c^6_k = c^3_k$ for all $k \in N^6 \setminus \{i, j\}$, and $c^6_i = c^6_j = c^3_j/2 = c^2_j/2 = \bar{c}/4$. Note that, $c^6_{i'} = c^3_{i'} = 3\bar{c}/2$.

Since $\varepsilon_6 = \varepsilon_5$, we have that

$$\beta_j(\varepsilon_3) \underset{(15)}{<} \beta_i(\varepsilon_4) \underset{(16)}{=} \beta_i(\varepsilon_5) + \beta_j(\varepsilon_5) \underset{\varepsilon_5=\varepsilon_6}{=} \beta_i(\varepsilon_6) + \beta_j(\varepsilon_6),$$

in contradiction with NMC. Thus, we conclude that

$$\beta_i(\varepsilon_0) = \beta_j(\varepsilon_0).$$

Hence, β satisfies ETE for $|N^0| \geq 3$.

Case 2: $|N^0| = 2$. The proof of this case is done in two steps.

Step 1: From ε_0 player i splits into clones i and i' defining the bankruptcy problem $\varepsilon_1 = (N^1, E, c^1)$, being $N^1 = \{i, i', j\}$, $c^1_i = c^1_{i'} = \bar{c}/2$, and $c^1_j = c^0_j = \bar{c}$. Since $|N^1| = 3$, by Case 1 we know that β satisfies ETE. Hence, $\beta_i(\varepsilon_1) = \beta_{i'}(\varepsilon_1)$ and, by NMC,

$$\beta_j(\varepsilon_0) = \beta_j(\varepsilon_1). \quad (17)$$

By BB,

$$\beta_i(\varepsilon_0) = \beta_i(\varepsilon_1) + \beta_{i'}(\varepsilon_1) \underset{\text{ETE}}{=} 2\,\beta_i(\varepsilon_1). \quad (18)$$

Step 2: From ε_1 player j splits into clones j and j' defining the bankruptcy problem $\varepsilon_2 = (N^2, E, c^2)$, being $N^2 = \{i, i', j, j'\}$, $c^2_j = c^2_{j'} = c^1_j/2 = \bar{c}/2$, $c^2_i = c^1_i = \bar{c}/2$, and $c^2_{i'} = c^1_{i'} = \bar{c}/2$. Since $|N^1| \geq 3$, by Case 1 we know hat β satisfies ETE. Hence, $\beta_i(\varepsilon_2) = \beta_j(\varepsilon_2) = \beta_{i'}(\varepsilon_2) = \beta_{j'}(\varepsilon_2)$. By NMC,

$$\beta_i(\varepsilon_2) = \beta_i(\varepsilon_1), \;\; \beta_{i'}(\varepsilon_2) = \beta_{i'}(\varepsilon_1), \quad (19)$$

and, by BB,

$$\beta_j(\varepsilon_1) = \beta_j(\varepsilon_2) + \beta_{j'}(\varepsilon_2) \underset{\text{ETE}}{=} 2\,\beta_j(\varepsilon_2). \quad (20)$$

Finally, $\beta_j(\varepsilon_0) \underset{(17)}{=} \beta_j(\varepsilon_1) \underset{(20)}{=} 2\,\beta_j(\varepsilon_2)$ and $\beta_i(\varepsilon_0) \underset{(18)}{=} 2\,\beta_i(\varepsilon_1) \underset{(19)}{=} 2\,\beta_i(\varepsilon_2) \underset{\text{ETE}}{=} 2\,\beta_j(\varepsilon_2)$. Hence, $\beta_j(\varepsilon_0) = \beta_i(\varepsilon_0)$ which concludes the proof. □

Proof (*Lemma 2*).

Let β be a bankruptcy rule satisfying NMC, $\varepsilon_0 = (N^0, E, c^0) \in \mathcal{B}$, and $i \in N^0$ such that $c^0_i = 0$. The proof is done in two steps.

Step 1: From ε_0 player i splits into clones i and i' defining the bankruptcy problem $\varepsilon_1 = (N^1, E, c^1)$, being $N^1 = N^0 \cup \{i'\}$, $c^1_k = c^0_k$ for all $k \in N^1 \setminus \{i, i'\}$, and $c^1_i = c^1_{i'} = 0$. By NMC, for all $k \in N^1 \setminus \{i, i'\}$, $\beta_k(\varepsilon_0) = \beta_k(\varepsilon_1)$, which implies, by BB, $\beta_i(\varepsilon_0) = \beta_i(\varepsilon_1) + \beta_{i'}(\varepsilon_1)$. By ETE, a consequence of NMC (Lemma 1), we have

$$\beta_i(\varepsilon_1) = \beta_{i'}(\varepsilon_1) = \frac{\beta_i(\varepsilon_0)}{2}. \quad (21)$$

Step 2: From ε_1 player i' splits into clones i' and i'' defining the bankruptcy problem $\varepsilon_2 = (N^2, E, c^2)$, being $N^2 = N^1 \cup \{i''\}$, $c^2_k = c^1_k$ for all $k \in N^2 \setminus \{i', i''\}$, and $c^2_{i'} = c^2_{i''} = 0$. By NMC, for all $k \in N^2 \setminus \{i', i''\}$,

$$\beta_k(\varepsilon_1) = \beta_k(\varepsilon_2), \quad (22)$$

which implies, by BB, $\beta_{i'}(\varepsilon_1) = \beta_{i'}(\varepsilon_2) + \beta_{i''}(\varepsilon_2)$. Hence, by ETE and (21),

$$\beta_{i'}(\varepsilon_2) \underset{\text{ETE}}{=} \beta_{i''}(\varepsilon_2) = \frac{\beta_{i'}(\varepsilon_1)}{2} \underset{(21)}{=} \frac{\beta_i(\varepsilon_0)}{4}. \quad (23)$$

Since $c^2_i = c^1_i = 0$, by ETE, $\beta_i(\varepsilon_2) = \beta_{i'}(\varepsilon_2)$. Hence, from (21) and (22), we obtain $\beta_{i'}(\varepsilon_2) = \beta_i(\varepsilon_2) \underset{(22)}{=} \beta_i(\varepsilon_1) \underset{(21)}{=} \frac{\beta_i(\varepsilon_0)}{2}$ and, from (23), $\beta_{i'}(\varepsilon_2) = \frac{\beta_i(\varepsilon_0)}{4}$, which implies that $\beta_i(\varepsilon_0) = 0$, concluding the proof. □

Proof (*Lemma 3*).

Let $(N, E, c) \in \mathcal{B}$ and $i, j \in N$ such that $c_i, c_j \in \mathbb{Q}_+$, that is, $c_i = p_i/q_i$ and $c_j = p_j/q_j$ for some $p_i, q_i, p_j, q_j \in \mathbb{N}$. Then, $b\,c_i = a\,c_j$ being $a = p_i q_j$ and $b = p_j q_i$. Assume, w.l.o.g., that $a < b$. The proof is done in two steps.

Step 1: From (N, E, c) define the bankruptcy problem (N^1, E, c^1) where player j splits into b identical players $j, j^1, \ldots, j^{b-1}$ being $N^1 = N \cup \{j^1, \ldots, j^{b-1}\}$, $c^1_j = c^1_{j^1} = \cdots = c^1_{j^{b-1}} = \frac{c_j}{b}$, and $c^1_l = c_l$ for all $l \in N \setminus \{j\}$; in particular, $c^1_i = c_i = \frac{a}{b} c_j$. By NMC, which implies ETE, and BB we have

$$\beta_i(N, E, c) \underset{\text{NMC}}{=} \beta_i(N^1, E, c^1) \quad (24)$$

and

$$\beta_j(N, E, c) \underset{\text{NMC+BB}}{=} \sum_{k \in \{j, j^1, \ldots, j^{b-1}\}} \beta_k(N^1, E, c^1) \underset{\text{ETE}}{=} b\beta_j(N^1, E, c^1). \quad (25)$$

Step 2: From (N^1, E, c^1) define the bankruptcy problem (N^2, E, c^2) where player i splits into a identical players $i, i^1, \ldots, i^{a-1}$ being $N^2 = N^1 \cup \{i^1, \ldots, i^{a-1}\}$, $c^2_i = c^2_{i^1} = \cdots = c^2_{i^{a-1}} = \frac{c^1_i}{a} = \frac{c_j}{b}$, and $c^2_l = c^1_l$ for all $l \in N^1 \setminus \{i\}$; in particular, $c^2_j = c^1_j = \frac{c_j}{b}$.

By NMC, ETE, and BB we have

$$\beta_j(N^2, E, c^2) \underset{\text{NMC}}{=} \beta_j(N^1, E, c^1) \quad (26)$$

and

$$\beta_i(N^1, E, c^1) \underset{\text{NMC+BB}}{=} \sum_{k \in \{i, i^1, \ldots, i^{a-1}\}} \beta_k(N^2, E, c^2) \underset{\text{ETE}}{=} a\beta_i(N^2, E, c^2). \quad (27)$$

Finally, since $c^2_i = c^2_j = \frac{c_j}{b}$, we can apply ETE again to obtain

$$\frac{\beta_i(N, E, c)}{a} \underset{(24)}{=} \frac{\beta_i(N^1, E, c^1)}{a} \underset{(27)}{=} \beta_i(N^2, E, c^2) \underset{\text{ETE}}{=} \beta_j(N^2, E, c^2) \underset{(26)}{=} \beta_j(N^1, E, c^1) \underset{(25)}{=} \frac{\beta_j(N, E, c)}{b}$$

which, in view of $c_i = p_i/q_i$, $c_j = p_j/q_j$, $a = p_i q_j$, and $b = p_j q_i$, finishes the proof. □

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