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From w-Domination in Graphs to Domination Parameters in Lexicographic Product Graphs

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Abstract

A wide range of parameters of domination in graphs can be defined and studied through a common approach that was recently introduced in [https://doi.org/10.26493/1855-3974.2318.fb9] under the name of w-domination, where $w = (w_0, w_1, \dots, w_l)$ is a vector of non-negative integers such that $w_0 \geq 1$. Given a graph G, a function $f : V(G) \longrightarrow \{0, 1, \dots, l\}$ is said to be a w-dominating function if $\sum_{u \in N(v)} f(u) \geq w_i$ for every vertex v with $f(v) = i$, where $N(v)$ denotes the open neighbourhood of $v \in V(G)$. The weight of f is defined to be $\omega(f) = \sum_{v \in V(G)} f(v)$, while the w-domination number of G, denoted by $\gamma_w(G)$, is defined as the minimum weight among all w-dominating functions on G. A wide range of well-known domination parameters can be defined and studied through this approach. For instance, among others, the vector $w = (1, 0)$ corresponds to the case of standard domination, $w = (2, 1)$ corresponds to double domination, $w = (2, 0, 0)$ corresponds to Italian domination, $w = (2, 0, 1)$ corresponds to quasi-total Italian domination, $w = (2, 1, 1)$ corresponds to total Italian domination, $w = (2, 2, 2)$ corresponds to total $\{2\}$-domination, while $w = (k, k-1, \dots, 1, 0)$ corresponds to $\{k\}$-domination. In this paper, we show that several domination parameters of lexicographic product graphs $G \circ H$ are equal to $\gamma_w(G)$ for some vector $w \in \{2\} \times \{0, 1, 2\}^l$ and $l \in \{2, 3\}$. The decision on whether

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the equality holds for a specific vector w will depend on the value of some domination parameters of H. In particular, we focus on quasi-total Italian domination, total Italian domination, 2-domination, double domination, total {2}-domination, and double total domination of lexicographic product graphs.

Keywords w-domination · (Total) Italian domination · Quasi-total Italian domination · 2-domination · Double domination · Lexicographic product graph

Mathematics Subject Classification 05C69 · 05C76

1 Introduction

The *lexicographic product* of two graphs G and H is the graph $G \circ H$ whose vertex set is $V(G \circ H) = V(G) \times V(H)$ and $(g, h)(g', h') \in E(G \circ H)$ if and only if $gg' \in E(G)$ or $g = g'$ and $hh' \in E(H)$. For simplicity, the neighbourhood of $(x, y) \in V(G) \times V(H)$ will be denoted by $N(x, y)$ instead of $N((x, y))$. Analogously, for any function f on $G \circ H$, the image of $(x, y) \in V(G) \times V(H)$ will be denoted by $f(x, y)$ instead of $f((x, y))$. For basic properties of the lexicographic product of two graphs, we cite the books [18, 23]. In particular, for results on domination theory of lexicographic product graphs we suggest the following works: standard domination [25, 26], Roman domination [27], weak Roman domination [6, 24, 29], total Roman domination [8, 12], total weak Roman domination [6, 11], rainbow domination [28], super domination [14], Italian domination [5], secure domination [6, 24], secure total domination [6, 11], double domination [9] and doubly connected domination [2].

In particular, the next theorem merges two results obtained in [27] and [30]. The result states that the domination number of $G \circ H$ equals the domination number of G whenever H has domination number equal to one, while the domination number of $G \circ H$ equals the total domination number of G for the remaining cases.

Theorem 1 ([27] and [30]) *For any graph G with no isolated vertex and any non-trivial graph H,*

$$\gamma(G \circ H) = \begin{cases} \gamma(G), & \text{if } \gamma(H) = 1, \\ \gamma_t(G), & \text{if } \gamma(H) \geq 2. \end{cases}$$

Another interesting result obtained in [11] concerns the case of total domination.

Theorem 2 [11] *For any graph G with no isolated vertex and any non-trivial graph H,*

$$\gamma_t(G \circ H) = \gamma_t(G).$$

These two theorems suggest to consider the following problem.

Problem 1 *Let G be a graph and let γ_y be a domination parameter well defined on $G \circ H$ for any non-trivial graph H. Determine if for each graph H, there exists a* *domination parameter* γ_x *such that*

$$\gamma_y(G \circ H) = \gamma_x(G).$$

We proceed to show other cases for which this problem has been solved. To this end, we need to formalize the notion of w-domination introduced in [5], where $w = (w_0, w_1, \ldots, w_l)$ is a vector of non-negative integers such that $w_0 \geq 1$. Given a graph G, a function $f : V(G) \longrightarrow \{0, 1, \ldots, l\}$ is said to be a w-*dominating function* if $\sum_{u \in N(v)} f(u) \geq w_i$ for every vertex v with $f(v) = i$, where $N(v)$ denotes the open neighbourhood of $v \in V(G)$. For every $i \in \{0, \ldots, l\}$, we define $V_i = \{v \in V(G) : f(v) = i\}$, and we will identify the function f with the subsets $V_0, \ldots, V_l$ associated with it. So, we will use the unified notation $f(V_0, \ldots, V_l)$ for the function and these associated subsets. The *weight* of f is defined to be $\omega(f) = \sum_{v \in V(G)} f(v)$, while the w-*domination number* of G, denoted by $\gamma_w(G)$, is defined as the minimum weight among all w-dominating functions on G. A w-dominating function of weight $\gamma_w(G)$ will be called a $\gamma_w(G)$-function.

It was shown in [5] that a wide range of well-known domination parameters can be defined and studied through this approach. For instance, the vector $w = (1, 0)$ corresponds to standard domination, $w = (1, 1)$ corresponds to total domination, $w = (2, 0, 0)$ corresponds to Italian domination, $w = (2, 0, 1)$ corresponds to quasi-total Italian domination, $w = (2, 1, 1)$ corresponds to total Italian domination, while $w = (k, k-1, \ldots, 1, 0)$ corresponds to $\{k\}$-domination.

As the next result shows, Problem 1 was solved for the case of the Italian domination number, which is a well-known parameter introduced in [13] under the name of Roman $\{2\}$-domination number. As mentioned above, in terms of w-domination, the Italian domination number of a graph G is defined as $\gamma_I(G) = \gamma_{(2,0,0)}(G)$.

Theorem 3 [5] *For any graph G with no isolated vertex and any non-trivial graph H,*

$$\gamma_I(G \circ H) = \begin{cases} \gamma_{(2,1,0)}(G) & \text{if } \gamma(H) = 1, \\ \gamma_{(2,2,0)}(G) & \text{if } \gamma_2(H) = \gamma(H) = 2, \\ \gamma_{(2,2,1)}(G) & \text{if } \gamma_2(H) > \gamma(H) = 2, \\ \gamma_{(2,2,2,0)}(G) & \text{if } \gamma_I(H) = \gamma(H) = 3, \\ \gamma_{(2,2,2)}(G) & \text{if } \gamma_I(H) \neq 3 \text{ and } \gamma(H) \geq 3. \end{cases}$$

In addition, Problem 1 was solved for the case of the $\{2\}$-domination number, which was introduced in [15]. In terms of w-domination, the $\{2\}$-*domination number* of a graph G is defined as $\gamma_{\{2\}}(G) = \gamma_{(2,1,0)}(G)$.

Theorem 4 [4] *For any graph G with no isolated vertex and any non-trivial graph H,*

$$\gamma_{\{2\}}(G \circ H) = \begin{cases} \gamma_{(2,1,0)}(G) & \text{if } \gamma(H) = 1, \\ \gamma_{(2,2,1)}(G) & \text{if } \gamma(H) = 2, \\ \gamma_{(2,2,2)}(G) & \text{if } \gamma(H) \geq 3. \end{cases}$$

We refer the reader to [5] for general results on w-domination, as well as for specific results on the domination parameters given in Theorems 3 and 4.

In this paper, we solve Problem 1 for the particular cases in which γ_y corresponds to the following parameters. Although we will use the standard notation for these parameters, we will define them in terms of w-domination.

- The *k-domination number* of a graph G, introduced in [16, 17], can be defined as $\gamma_k(G) = \gamma_{(k,0)}(G)$. In this paper, we are interested in the case $k = 2$, which is probably the most studied. In this case, if $f(V_0, V_1)$ is a $\gamma_{(2,0)}(G)$-function, then we will say that V_1 is a $\gamma_2(G)$-set.
- The *double domination number* of a graph G with no isolated vertex is defined to be $\gamma_{\times 2}(G) = \gamma_{(2,1)}(G)$. If $f(V_0, V_1)$ is a $\gamma_{(2,1)}(G)$-function, then we will say that V_1 is a $\gamma_{\times 2}(G)$-set. This parameter was introduced in two different papers [19, 20]. Moreover, the general version of this parameter, the *k-tuple domination number*, is defined to be $\gamma_{\times k}(G) = \gamma_{(k,k-1)}(G)$.
- The *double total domination number* of a graph G with minimum degree $\delta(G) \geq 2$ is defined to be $\gamma_{\times 2,t}(G) = \gamma_{(2,2)}(G)$. If $f(V_0, V_1)$ is a $\gamma_{(2,2)}(G)$-function, then we will say that V_1 is a $\gamma_{\times 2,t}(G)$-set. This domination parameter was introduced in [21], and its general version is the *k-tuple total domination number*, which is defined to be $\gamma_{\times k,t}(G) = \gamma_{(k,k)}(G)$.
- The *quasi-total Italian domination number* of a graph G, recently introduced in [7], is defined to be $\gamma_{I^*}(G) = \gamma_{(2,0,1)}(G)$. A $(2, 0, 1)$-dominating function of weight $\gamma_{I^*}(G)$ will be called a $\gamma_{I^*}(G)$-function.
- The *total Italian domination number* of a graph G with no isolated vertex is defined to be $\gamma_{tI}(G) = \gamma_{(2,1,1)}(G)$. This parameter was introduced in [3], and independently in [1], under the name of total Roman $\{2\}$-domination number. A $(2, 1, 1)$-dominating function of weight $\gamma_{tI}(G)$ will be called a $\gamma_{tI}(G)$-function.
- The total $\{2\}$-*domination number* of a graph G of minimum degree $\delta(G) \geq 2$ is defined as $\gamma_{\{2\},t}(G) = \gamma_{(2,2,2)}(G)$. This parameter was studied in [22].

We will show that the above-mentioned domination parameters of lexicographic product graphs $G \circ H$ are equal to $\gamma_w(G)$ for some vector $w \in \{2\} \times \{0, 1, 2\}^l$ and $l \in \{2, 3\}$. The decision on whether the equality holds for a specific vector w will depend on the value of some domination parameters of H.

Notice that if G is a graph with no isolated vertex and H is a non-trivial graph, then the following domination chain is deduced by the definition of the parameters involved in it.

$$\gamma_I(G \circ H) \leq \gamma_{I^*}(G \circ H) \leq \gamma_2(G \circ H) \leq \gamma_{\times 2}(G \circ H) \leq \gamma_{\times 2,t}(G \circ H). \quad (1)$$

Furthermore, the equality $\gamma_{tI}(G \circ H) = \gamma_{\times 2}(G \circ H)$ was deduced in [9], while the equality $\gamma_{I^*}(G \circ H) = \gamma_2(G \circ H)$ will be proved in Sect. 2 and the equality $\gamma_{\{2\},t}(G \circ H) = \gamma_{\times 2,t}(G \circ H)$ will be proved in Sect. 4. Therefore, the following domination chain holds whenever G is a graph with no isolated vertex and H is a non-trivial graph.

$$\gamma_I(G \circ H) \begin{array}{l} \leq \gamma_{I^*}(G \circ H) = \gamma_2(G \circ H) \\ \leq \gamma_{tI}(G \circ H) = \gamma_{\times 2}(G \circ H) \\ \leq \gamma_{\{2\},t}(G \circ H) = \gamma_{\times 2,t}(G \circ H). \end{array} \quad (2)$$

2 Double Domination and Total Italian Domination

To get our results, we need to set up some tools and introduce some known results.

Lemma 1 *Let G be a graph with no isolated vertex and H a non-trivial graph. If $\gamma_{\times 2}(H) = 2$, then $\gamma_{\times 2}(G \circ H) \leq \gamma_{(2,1,0)}(G)$.*

Proof Let $S = \{v_1, v_2\}$ be a $\gamma_{\times 2}(H)$-set and $g(W_0, W_1, W_2)$ a $\gamma_{(2,1,0)}(G)$-function. Since $W = (W_1 \times \{v_1\}) \cup (W_2 \times S)$ is a double dominating set of $G \circ H$, we conclude that $\gamma_{\times 2}(G \circ H) \leq |W| = \omega(g) = \gamma_{(2,1,0)}(G)$. □

Notice that for any $u \in V(G)$ the subgraph of $G \circ H$ induced by $\{u\} \times V(H)$ is isomorphic to H. For simplicity, we will denote this subgraph by H_u.

Theorem 5 [9] *The following statements hold for any graph G with no isolated vertex and any non-trivial graph H.*

(i) $\gamma_{\times 2}(G \circ H) = \gamma_{tI}(G \circ H)$.
(ii) *If* $\gamma_2(H) \geq 3$ *and* $\gamma(H) = 1$, *then* $\gamma_{\times 2}(G \circ H) = \gamma_{tI}(G)$.
(iii) *There exists a* $\gamma_{\times 2}(G \circ H)$*-set S such that* $|S \cap V(H_u)| \leq 2$, *for every* $u \in V(G)$.

By Theorem 5 (i), we will restrict the proof of the next result to obtain the values of $\gamma_{\times 2}(G \circ H)$.

Theorem 6 *For any graph G with no isolated vertex and any non-trivial graph H,*

$$\gamma_{\times 2}(G \circ H) = \gamma_{tI}(G \circ H) = \begin{cases} \gamma_{(2,1,0)}(G) \text{ if } \gamma_{\times 2}(H) = 2, \\ \gamma_{(2,1,1)}(G) \text{ if } \gamma_2(H) \geq 3 \text{ and } \gamma(H) = 1, \\ \gamma_{(2,2,1)}(G) \text{ if } \gamma(H) = 2, \\ \gamma_{(2,2,2)}(G) \text{ if } \gamma(H) \geq 3. \end{cases}$$

Proof First, we assume that $\gamma(H) = 1$. Since $\gamma_I(G \circ H) \leq \gamma_{\times 2}(G \circ H)$, if $\gamma_{\times 2}(H) = 2$, then Theorem 3 and Lemma 1 lead to $\gamma_{(2,1,0)}(G) = \gamma_I(G \circ H) \leq \gamma_{\times 2}(G \circ H) \leq \gamma_{(2,1,0)}(G)$. Therefore, in this case we conclude that $\gamma_{\times 2}(G \circ H) = \gamma_{(2,1,0)}(G)$. Now, if $\gamma_2(H) \geq 3$, then Theorem 5 (ii) leads to $\gamma_{\times 2}(G \circ H) = \gamma_{tI}(G) = \gamma_{(2,1,1)}(G)$.

From now on we assume that $\gamma(H) \geq 2$. Let S be a $\gamma_{\times 2}(G \circ H)$-set which satisfies Theorem 5 (iii). Let $f(X_0, X_1, X_2)$ be the function defined on G by $X_i = \{x \in V(G) : |S \cap V(H_x)| = i\}$ for every $i \in \{0, 1, 2\}$. Notice that $\gamma_{\times 2}(G \circ H) = |S| = \omega(f)$. We claim that f is a $\gamma_{(2,2,w)}(G)$-function, where $w \in \{1, 2\}$. In order to prove this claim and find the exact value of w, we differentiate the following two cases.

Case 1. $\gamma(H) = 2$. Assume that $x \in X_0 \cup X_1$. Since $\gamma(H) = 2$, there exists a vertex $z \in V(H)$ such that $(x, z) \notin S$ and $|S \cap N(x, z) \cap V(H_x)| = 0$. Hence, $|S \cap (N(x, z) \setminus V(H_x))| \geq 2$, which implies that $f(N(x)) \geq 2$. Now, assume that $x \in X_2$. In this case, there exists a vertex $y \in V(H)$ such that $|S \cap N(x, y) \cap V(H_x)| \leq 1$, and so $f(N(x)) \geq 1$. Therefore, f is a $(2, 2, 1)$-dominating function on G and, as a consequence, $\gamma_{\times 2}(G \circ H) = |S| = \omega(f) \geq \gamma_{(2,2,1)}(G)$.

Moreover, let $h(Y_0, Y_1, Y_2)$ be a $\gamma_{(2,2,1)}(G)$-function and $S = \{v_1, v_2\}$ a $\gamma(H)$-set. Notice that the set $Y = (Y_1 \times \{v_1\}) \cup (Y_2 \times S)$ is a double dominating set of $G \circ H$, which implies that $\gamma_{\times 2}(G \circ H) \leq |Y| = \omega(h) = \gamma_{(2,2,1)}(G)$.

Case 2. $\gamma(H) \geq 3$. Let $x \in V(G)$. Since $\gamma(H) \geq 3$, there exists $y \in V(H)$ such that $(x, y) \notin S$ and $|S \cap N(x, y) \cap V(H_x)| = 0$, which implies that $|S \cap (N(x, y) \setminus V(H_x))| \geq 2$, and so $f(N(x)) \geq 2$. Therefore, f is a (2, 2, 2)-dominating function on G and, as a consequence, $\gamma_{\times 2}(G \circ H) = |S| = \omega(f) \geq \gamma_{(2,2,2)}(G)$.

It remains to show that $\gamma_{\times 2}(G \circ H) \leq \gamma_{(2,2,2)}(G)$. To see this we only need to observe that for any $\gamma_{(2,2,2)}(G)$-function $g(W_0, W_1, W_2)$ and any pair of vertices $v_1, v_2 \in V(H)$, the set $W = (W_2 \times \{v_1, v_2\}) \cup (W_1 \times \{v_1\})$ is a double dominating set of $G \circ H$, which implies that $\gamma_{\times 2}(G \circ H) \leq |W| = \omega(g) = \gamma_{(2,2,2)}(G)$, as required. □

3 Quasi-total Italian Domination and 2-Domination

To begin this section, we will introduce some basic tools.

Lemma 2 *For any graph G with no isolated vertex and any non-trivial graph H with $\gamma(H) = 1$, there exists a $\gamma_2(G \circ H)$-set D satisfying that $|D \cap V(H_u)| \leq 2$ for every $u \in V(G)$.*

Proof Given a $\gamma_2(G \circ H)$-set D, we define the set $R_D = \{x \in V(G) : |D \cap V(H_x)| \geq 3\}$. Now, we assume that D is a $\gamma_2(G \circ H)$-set such that $|R_D|$ is minimum among all $\gamma_2(G \circ H)$-sets. Suppose that $|R_D| \geq 1$. Let v be a universal vertex of H and $u \in R_D$. Now, we take $u' \in N(u)$ and $v' \in N(v)$, and consider a set $D' \subseteq V(G) \times V(H)$ satisfying the following properties.

- $D' \cap V(H_u) = \{(u, v), (u, v')\}$;
- $|D' \cap V(H_{u'})| = \min\{2, |D \cap V(H_{u'})| + 1\}$;
- $D' \cap V(H_x) = D \cap V(H_x)$ for every $x \in V(G)\setminus\{u, u'\}$.

Observe that D' is a 2-dominating set of $G \circ H$ satisfying $|D'| \leq |D|$ and $|R_{D'}| < |R_D|$, which is a contradiction. Therefore, $R_D = \varnothing$, as required. □

Lemma 3 *Let G be a graph with no isolated vertex and H a non-trivial graph. If $\gamma_2(H) \geq 3$ and $\gamma(H) = 1$, then $\gamma_2(G \circ H) \geq \gamma_{(2,1,1)}(G)$.*

Proof Let D be a $\gamma_2(G \circ H)$-set which satisfies Lemma 2. Let $f(X_0, X_1, X_2)$ be the function defined on G by $X_i = \{x \in V(G) : |D \cap V(H_x)| = i\}$ for every $i \in \{0, 1, 2\}$. Notice that $\gamma_2(G \circ H) = |D| = \omega(f)$. We claim that f is a (2, 1, 1)-dominating function on G. Assume that $x \in X_0$. Since $D \cap V(H_x) = \varnothing$, we have that $|D \cap (N(x) \times V(H)| \geq 2$, which implies that $f(N(x)) \geq 2$. Now, assume that $x \in X_1 \cup X_2$. Since $|D \cap V(H_x)| \leq 2$ and $\gamma_2(H) \geq 3$, there exists $y \in V(H)$ such that $(x, y) \notin D$ and $|D \cap V(H_x) \cap N(x, y)| \leq 1$, which implies that $|D \cap (N(x) \times V(H))| \geq 1$, and so $f(N(x)) \geq 1$. Therefore, f is a (2, 1, 1)-dominating function on G and, as a consequence, $\gamma_2(G \circ H) = |D| = \omega(f) \geq \gamma_{(2,1,1)}(G)$. □

Theorem 7 *The following statements hold for any graph G with no isolated vertex and any non-trivial graph H.*

(i) $\gamma_{I^*}(G \circ H) = \gamma_2(G \circ H)$.
(ii) *If $\gamma(H) \geq 2$, then* $\gamma_{I^*}(G \circ H) = \gamma_I(G \circ H)$.

Proof By definition, $\gamma_{I^*}(G \circ H) \le \gamma_2(G \circ H)$. Hence, it remains to show that $\gamma_{I^*}(G \circ H) \ge \gamma_2(G \circ H)$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{I^*}(G \circ H)$-function such that $|V_2|$ is minimum among all $\gamma_{I^*}(G \circ H)$-functions. If $V_2 = \varnothing$, then V_1 is a 2-dominating set of $G \circ H$, and so $\gamma_2(G \circ H) \le |V_1| = \gamma_{I^*}(G \circ H)$. We assume that $V_2 \ne \varnothing$ and, in that case, we differentiate the next two cases for a fixed vertex $(u, v) \in V_2$. Obviously, $N(u, v) \cap (V_1 \cup V_2) \ne \varnothing$.

Case 1. $N(u, v) \cap (V_1 \cup V_2) \subseteq V(H_u)$. In this case, for any $(u', v') \in N(u) \times V(H)$ we define the function $f'(V_0', V_1', V_2')$ where $V_0' = V_0 \setminus \{(u', v')\}$, $V_1' = V_1 \cup \{(u, v), (u', v')\}$ and $V_2' = V_2 \setminus \{(u, v)\}$. Observe that $\omega(f') = \omega(f)$, every vertex in V_2' has a neighbour in $V_1' \cup V_2'$ and every vertex $w \in V_0' \subseteq V_0$ satisfies that $f'(N(w)) \ge 2$. Hence, f' is a $\gamma_{I^*}(G \circ H)$-function and $|V_2'| < |V_2|$, which is a contradiction.

Case 2. $(N(u) \times V(H)) \cap (V_1 \cup V_2) \ne \varnothing$. If $V(H_u) \subseteq V_1 \cup V_2$, then the function h, defined by $h(u, v) = 1$ and $h(x, y) = f(x, y)$ whenever $(x, y) \in V(G \circ H) \setminus \{(u, v)\}$, is a quasi-total Italian dominating function on $G \circ H$ with $\omega(h) < \omega(f)$, which is a contradiction. Hence, there exists $v' \in V(H)$ such that $(u, v') \in V_0$. In that case, let $f'(V_0', V_1', V_2')$ be a function defined by $V_0' = V_0 \setminus \{(u, v')\}$, $V_1' = V_1 \cup \{(u, v), (u, v')\}$ and $V_2' = V_2 \setminus \{(u, v)\}$. As in the previous case, $\omega(f') = \omega(f)$, every vertex in V_2' has a neighbour in $V_1' \cup V_2'$ and every vertex $w \in V_0' \subseteq V_0$ satisfies that $f'(N(w)) \ge 2$. Thus, f' is a $\gamma_{I^*}(G \circ H)$-function with $|V_2'| < |V_2|$, which is a contradiction again.

According to the two cases above, we deduce that $V_2 = \varnothing$, which implies that $\gamma_2(G \circ H) \le \gamma_{I^*}(G \circ H)$. Therefore, the proof of (i) is complete.

Finally, we proceed to prove (ii). By definition, $\gamma_I(G \circ H) \le \gamma_{I^*}(G \circ H)$. Thus, it remains to show that $\gamma_I(G \circ H) \ge \gamma_{I^*}(G \circ H)$ whenever $\gamma(H) \ge 2$. Let $g(W_0, W_1, W_2)$ be a $\gamma_I(G \circ H)$-function such that $|W_2|$ is the minimum among all $\gamma_I(G \circ H)$-functions. Obviously, if $W_2 = \varnothing$ or $N(u, v) \nsubseteq W_0$ for every $(u, v) \in W_2$, then g is a $\gamma_{I^*}(G \circ H)$-function and we are done. Suppose to the contrary that there exists a vertex $(u, v) \in W_2$ such that $N(u, v) \subseteq W_0$. Notice that $g(V(H_u)) \ge 3$, as $\gamma(H) \ge 2$. Thus, we differentiate the next two cases.

Case 1. $g(V(H_u)) \ge 4$. Let $u' \in N(u)$ and $v' \in V(H) \setminus \{v\}$. We define a function $g'(W_0', W_1', W_2')$ on $G \circ H$ as $g'(u, v) = g'(u, v') = g'(u', v) = g'(u', v') = 1$, $g'(V(H_u) \setminus \{(u, v), (u, v')\}) = g'(V(H_{u'}) \setminus \{(u', v), (u', v')\}) = 0$ and $g'(x, y) = g(x, y)$ for every $x \in V(G) \setminus \{u, u'\}$ and $y \in V(H)$. Notice that g' is an Italian dominating function on $G \circ H$ with $\omega(g') \le \omega(g)$ and $|W_2'| < |W_2|$, which is a contradiction.

Case 2. $g(V(H_u)) = 3$. In this case, since $\gamma(H) \ge 2$, we deduce that $\gamma_I(H) = 3$ and $\gamma(H) = 2$ by the minimality of W_2. Let $\{v_1, v_2\}$ be a $\gamma(H)$-set and $u' \in N(u)$. Consider the function $g'(W_0', W_1', W_2')$ defined as $g'(u, v_1) = g'(u, v_2) = 1$, $g'(u, v) = 0$ for every $v \in V(H) \setminus \{v_1, v_2\}$, $g'(V(H_{u'})) = 1$ and $g'(x, y) = g(x, y)$ for every $x \in V(G) \setminus \{u, u'\}$ and $y \in V(H)$. Notice that g' is an Italian dominating function on $G \circ H$ with $\omega(g') \le \omega(g)$ and $|W_2'| < |W_2|$, which is a contradiction.

Therefore, either $W_2 = \varnothing$ or every vertex in W_2 has a neighbour in $W_1 \cup W_2$, and so $\gamma_{I^*}(G \circ H) = \gamma_I(G \circ H)$. □

According to Theorem 7, we can restrict the proof of the next result to obtain the values of $\gamma_2(G \circ H)$.

Theorem 8 *For any graph G with no isolated vertex and any non-trivial graph H,*

$$\gamma_2(G \circ H) = \gamma_{I^*}(G \circ H) = \begin{cases} \gamma_{(2,1,0)}(G) & \text{if } \gamma_{\times 2}(H) = 2, \\ \gamma_{(2,1,1)}(G) & \text{if } \gamma_2(H) \geq 3 \text{ and } \gamma(H) = 1, \\ \gamma_{(2,2,0)}(G) & \text{if } \gamma_2(H) = \gamma(H) = 2, \\ \gamma_{(2,2,1)}(G) & \text{if } \gamma_2(H) > \gamma(H) = 2, \\ \gamma_{(2,2,2,0)}(G) & \text{if } \gamma_I(H) = \gamma(H) = 3, \\ \gamma_{(2,2,2)}(G) & \text{if } \gamma_I(H) \neq 3 \text{ and } \gamma(H) \geq 3. \end{cases}$$

Proof Since $\gamma_I(G \circ H) \leq \gamma_2(G \circ H)$, if $\gamma_{\times 2}(H) = 2$, then by Lemma 1 and Theorem 3 we have that $\gamma_{(2,1,0)}(G) = \gamma_I(G \circ H) \leq \gamma_2(G \circ H) \leq \gamma_{(2,1,0)}(G)$. Therefore, in this case we obtain $\gamma_2(G \circ H) = \gamma_{(2,1,0)}(G)$.

Now, since $\gamma_2(G \circ H) \leq \gamma_{\times 2}(G \circ H)$, if $\gamma_2(H) \geq 3$ and $\gamma(H) = 1$, then Lemma 3 and Theorem 5 (ii) lead to $\gamma_{(2,1,1)}(G) \leq \gamma_2(G \circ H) \leq \gamma_{\times 2}(G \circ H) = \gamma_{(2,1,1)}(G)$. Therefore, $\gamma_2(G \circ H) = \gamma_{(2,1,1)}(G)$.

Finally, if $\gamma(H) \geq 2$, then Theorem 7 leads to $\gamma_2(G \circ H) = \gamma_{I^*}(G \circ H) = \gamma_I(G \circ H)$ and so we complete the proof by Theorem 3. $\square$

4 Double Total Domination and Total {2}-Domination

Although in general, $\gamma_{\{2\},t}(G) \leq \gamma_{\times 2,t}(G)$, we show below that for the case of lexicographic product graphs these parameters always coincide.

Theorem 9 *For any graph G with no isolated vertex and any non-trivial graph H,*

$$\gamma_{\{2\},t}(G \circ H) = \gamma_{\times 2,t}(G \circ H).$$

Proof By definition, $\gamma_{\{2\},t}(G \circ H) \leq \gamma_{\times 2,t}(G \circ H)$. Hence, it remains to show that $\gamma_{\{2\},t}(G \circ H) \geq \gamma_{\times 2,t}(G \circ H)$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{\{2\},t}(G \circ H)$-function such that $|V_2|$ is minimum among all $\gamma_{\{2\},t}(G \circ H)$-functions. If $V_2 = \emptyset$, then V_1 is a double total dominating set of $G \circ H$, and so $\gamma_{\times 2,t}(G \circ H) \leq |V_1| = \gamma_{\{2\},t}(G \circ H)$, as required. We assume that $V_2 \neq \emptyset$ and, in that case, we differentiate the next two cases for a fixed vertex $(u, v) \in V_2$. Obviously, $N(u, v) \cap (V_1 \cup V_2) \neq \emptyset$.
Case 1. $N(u, v) \cap (V_1 \cup V_2) \subseteq V(H_u)$. In this case, for any $(u', v), (u', v') \in N(u) \times V(H)$ we define the function $f'(V_0', V_1', V_2')$ where $V_0' = V_0 \setminus \{(u', v), (u', v')\}$, $V_1' = V_1 \cup \{(u', v), (u', v')\}$ and $V_2' = V_2 \setminus \{(u, v)\}$. Observe that $\omega(f') = \omega(f)$ and every vertex $(x, y) \in V(G \circ H)$ satisfies that $f'(N(x, y)) \geq 2$. Hence, f' is a $\gamma_{\{2\},t}(G \circ H)$-function and $|V_2'| < |V_2|$, which is a contradiction.
Case 2. $(N(u) \times V(H)) \cap (V_1 \cup V_2) \neq \emptyset$. If $V(H_u) \subseteq V_1 \cup V_2$, then the function h, defined by $h(u, v) = 1$ and $h(x, y) = f(x, y)$ whenever $(x, y) \in V(G \circ H) \setminus \{(u, v)\}$, is a double total dominating function on $G \circ H$ with $\omega(h) < \omega(f)$, which is a contradiction. Hence, there exists $v' \in V(H)$ such that $(u, v') \in V_0$. In that case, let $f'(V_0', V_1', V_2')$ be a function defined by $V_0' = V_0 \setminus \{(u, v')\}$, $V_1' = V_1 \cup \{(u, v), (u, v')\}$ and $V_2' = V_2 \setminus \{(u, v)\}$. Notice that $\omega(f') = \omega(f)$ and every vertex $(x, y) \in V(G \circ H)$ satisfies that $f'(N(x, y)) \geq 2$. Thus, f' is a $\gamma_{\{2\},t}(G \circ H)$-function with $|V_2'| < |V_2|$, which is a contradiction again.

According to the two cases above, we deduce that $V_2 = \varnothing$, which implies that V_1 is a double total dominating set of $G \circ H$, and so $\gamma_{\times 2,t}(G \circ H) \leq |V_1| = \gamma_{\{2\},t}(G \circ H)$, as required. Therefore, the proof is complete. □

We are now in a position to formalize the tools which will allow us to calculate $\gamma_{\times 2,t}(G \circ H)$.

Lemma 4 *For any graph G with no isolated vertex and any non-trivial graph H, there exists a $\gamma_{\times 2,t}(G \circ H)$-set S satisfying that $|S \cap V(H_x)| \leq 2$ for every $x \in V(G)$.*

Proof Given a $\gamma_{\times 2,t}(G \circ H)$-set S, we define the set $R_S = \{x \in V(G) : |S \cap V(H_x)| \geq 3\}$. Assume that S is a $\gamma_{\times 2,t}(G \circ H)$-set such that R_S has minimum cardinality among all $\gamma_{\times 2,t}(G \circ H)$-sets. Suppose that $R_S \neq \varnothing$ and let $x, y \in V(G)$ be two adjacent vertices with $x \in R_S$. Let $S_x = S \cap V(H_x)$ and take $(x, v_1), (x, v_2) \in S_x$. Hence, there exists a set $S' \subseteq V(G \circ H)$ satisfying the following properties.

- $S' \cap V(H_x) = \{(x, v_1), (x, v_2)\}$.
- $|S' \cap V(H_y)| = \min\{2, |S \cap V(H_y)| + |S_x| - 2\}$.
- $S' \cap V(H_z) = S \cap V(H_z)$ for every $z \in V(G) \setminus \{x, y\}$.

Observe that S' is a double total dominating set of $G \circ H$ with $|S'| \leq |S|$ and $|R_{S'}| < |R_S|$, which is a contradiction. Therefore, the result follows. □

Proposition 1 *For any graph G with no isolated vertex and any non-trivial graph H,*

$$\gamma_{\times 2,t}(G \circ H) \leq \gamma_{(2,2,2)}(G).$$

Furthermore, if H has isolated vertex or $\gamma_t(H) \geq 3$, then the equality holds.

Proof The proof of the inequality is straightforward, as we only need to observe that for any $\gamma_{(2,2,2)}(G)$-function $g(W_0, W_1, W_2)$ and any pair of vertices $v_1, v_2 \in V(H)$, the set $W = (W_2 \times \{v_1, v_2\}) \cup (W_1 \times \{v_1\})$ is a double total dominating set of $G \circ H$, which implies that $\gamma_{\times 2,t}(G \circ H) \leq |W| = \omega(g) = \gamma_{(2,2,2)}(G)$.

From now on, assume that either H has isolated vertex or $\gamma_t(H) \geq 3$. Notice that these assumptions imply that for any set $S \subseteq V(H)$ of cardinality at most two, there exists a vertex $v \in V(H)$ such that $N(v) \cap S = \varnothing$.

Now, let D be a $\gamma_{\times 2,t}(G \circ H)$-set satisfying Lemma 4. Since $|D \cap V(H_x)| \leq 2$ for every $x \in V(G)$, from the assumptions above we have that there exists a vertex $v \in V(H)$ such that $N(x, v) \cap D \cap V(H_x) = \varnothing$. Thus, $|(N(x) \times V(H)) \cap D| \geq 2$ for every $x \in V(G)$, which implies that any function $f : V(G) \longrightarrow \{0, 1, 2\}$ such that $f(V(H_x)) = |D \cap V(H_x)|$, is a $(2, 2, 2)$-dominating function on G. Therefore, $\gamma_{(2,2,2)}(G) \leq \omega(f) = |D| = \gamma_{\times 2,t}(G \circ H)$, as required. □

According to Theorem 9, in the proof of the following result we can restrict ourselves to determining the value of $\gamma_{\times 2,t}(G \circ H)$.

Theorem 10 *For any graph G with no isolated vertex and any non-trivial graph H,*

$$\gamma_{\times 2,t}(G \circ H) = \gamma_{\{2\},t}(G \circ H) = \begin{cases} \gamma_{(2,2,1)}(G) & \text{if } \gamma_t(H) = 2, \\ \gamma_{(2,2,2)}(G) & \text{otherwise.} \end{cases}$$

Proof First we assume that $\gamma_t(H) = 2$. Let $h(Y_0, Y_1, Y_2)$ be a $\gamma_{(2,2,1)}(G)$-function and let $S = \{v_1, v_2\}$ be a $\gamma_t(H)$-set. Notice that the set $Y = (Y_1 \times \{v_1\}) \cup (Y_2 \times S)$ is a double total dominating set of $G \circ H$, which implies that $\gamma_{\times 2,t}(G \circ H) \leq |Y| = \omega(h) = \gamma_{(2,2,1)}(G)$. Now, let S be a $\gamma_{\times 2,t}(G \circ H)$-set which satisfies Lemma 4 and let $f(X_0, X_1, X_2)$ be the function defined on G by $X_i = \{x \in V(G) : |S \cap V(H_x)| = i\}$ for every $i \in \{0, 1, 2\}$. Notice that $\gamma_{\times 2,t}(G \circ H) = |S| = \omega(f)$. We claim that f is a $(2, 2, 1)$-dominating function on G.

Let $x \in X_0 \cup X_1$. Since $\gamma_t(H) = 2$, there exists a vertex $z \in V(H)$ such that $(x, z) \notin S$ and $|S \cap N(x, z) \cap V(H_x)| = 0$. Hence, as S is a $\gamma_{\times 2,t}(G \circ H)$-set, $|S \cap (N(x, z) \setminus V(H_x))| \geq 2$, and so $f(N(x)) \geq 2$.

Now, let $x \in X_2$. Since $\gamma_t(H) = 2$ implies $\gamma_{\times 2,t}(H) \geq 3$, we have that there exists a vertex $y \in V(H)$ such that $|S \cap V(H_x) \cap N(x, y)| \leq 1$, which leads to $|S \cap (N(x, z) \setminus V(H_x))| \geq 1$, as S is a $\gamma_{\times 2,t}(G \circ H)$-set, and so $f(N(x)) \geq 1$.

Therefore, f is a $(2, 2, 1)$-dominating function on G and, as a consequence, $\gamma_{\times 2,t}(G \circ H) = |S| = \omega(f) \geq \gamma_{(2,2,1)}(G)$, concluding that $\gamma_{\times 2,t}(G \circ H) = \gamma_{(2,2,1)}(G)$.

Finally, if $\gamma_t(H) \geq 3$ or H has isolated vertex, then by Proposition 1 we have $\gamma_{\times 2,t}(G \circ H) = \gamma_{(2,2,2)}(G)$. □

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