Generating infinite links as periodic tilings of the da Vinci - Dürer's knots

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Introduction

The Austrian art historian Moritz Thausing 1838-1884 wrote: "There are six woodcuts of Albert Dürers' black discs, upon which a symmetrical and concentrically arranged arabesque scrollwork of ribbons or festoons stands out in relief. These wonderful ornamentations are commonly called Dürer's "Patterns for Embroidery"; he himself calls them in his Netherlands diary "Die sechs Knoten" –the Six Knots–. By adding four corner ornaments to the discs, Dürer gave these woodcuts an oblong shape. His monogram was not placed on them, in the centre, till the second impression. The very same patterns are to be found in some old Italian copper engravings, but on a white ground, and bearing in the centre the curious inscription "ACADEMIA LEONARDI VINCI"." [Thausing, 1882, p. 362]. Other authors directly claim that the Six Knots are by da Vinci, and that Dürer copied them [Richter, 1970, p. 297]. In the catalogue of the Herzog Anton Ulrich Museum, for instance, these works of art are registered as "da Vinci, Leonardo (Leonardo da Vinci) (Inventor); Dürer, Albrecht (publisher, draftsman)" [Virtuelles Kupferstichkabinett]. In this paper we do not attempt to determine who the real author is, so the "Six Knots" will be referred as: *da Vinci-Dürer's Six Knots*.

However, the *da Vinci-Dürer's Six Knots* are not really knots, since they are not one-component links. In actual fact, they are multi-knot links, so they should be called the *da Vinci-Dürer's Six Links.* Figure 1 shows the Six Links in the version of the six woodcuts of Albert Dürers' black discs. Apart from the 4 corner ornaments, there is a link inside each of the six discs. There is a total of six finite links, each having a finite number of knots. Next we will describe two of these six links; more specifically, the fourth and sixth links.

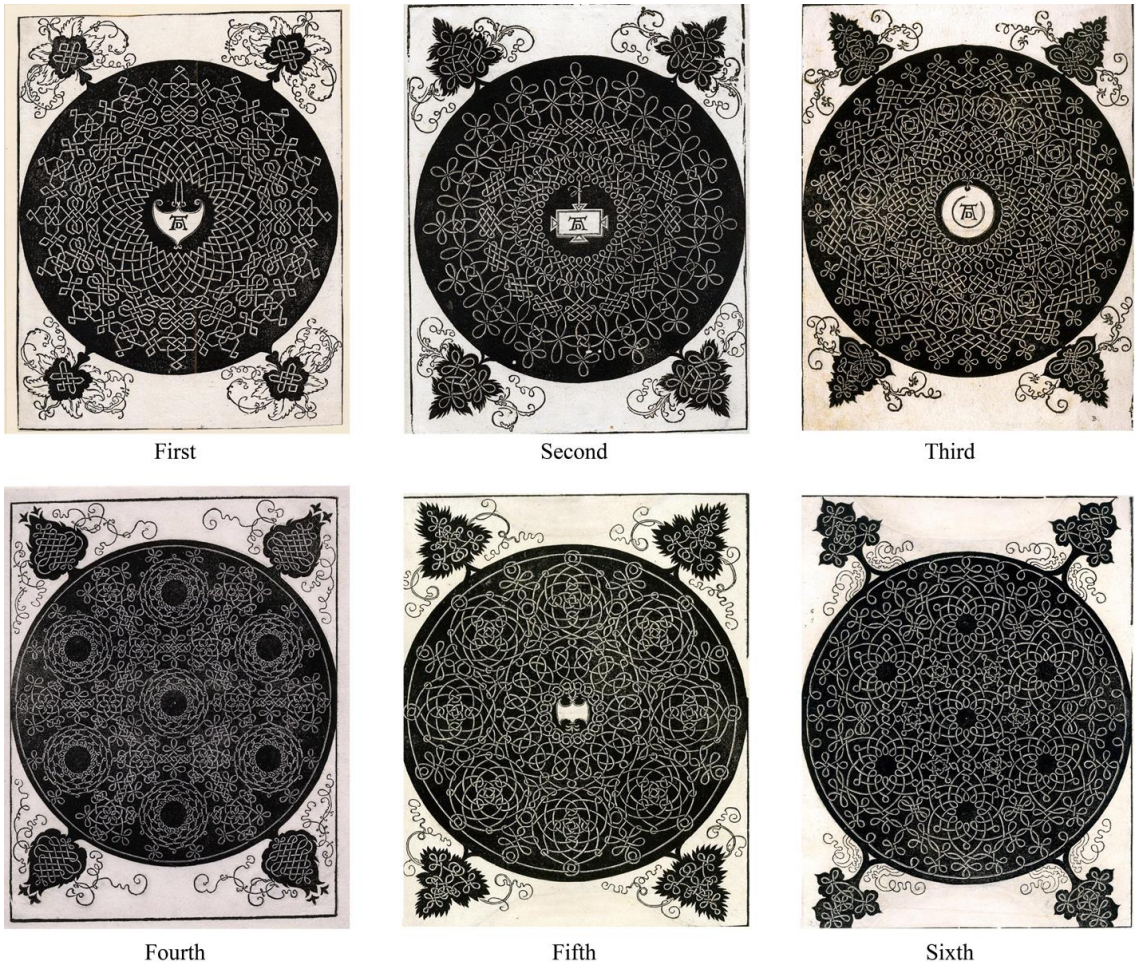

Figure 1. "Die sechs Knoten" –the Six Knots–. The *da Vinci-Dürer's Six Knots*.

The reason to chose these two links is that, as stated in the title of this paper, we will therefrom generate two new infinite links –having an infinite number of knots– using the p6m periodic tiling technique. This is so because, as can be seen in Figure 1, in the center of the fourth and sixth *da Vinci-Dürer's Six Knots* there is an hexagonal lattice core which is invariant by symmetry and by $\frac{\pi}{3}$ radian rotations. We will show how the knots of these two initial *da Vinci-Dürer* finite links transform into the knots of the two new infinite links. However, as can also be seen in Figure 1, the remaining four *da Vinci-Dürer* links do not have a lattice core belonging to any of the 17 plane symmetry groups. For these reason, the four remaining links do not offer the possibility to generate new links using the periodic tiling technique. Those readers who are not familiar with the basic theory of periodic tilings may read, for instance, [Grünbaum and Shephard, 2016], which is accessible, for the most part, to high schoolers. There, they will learn about the concepts of *p6m periodic tiling* and *lattice*, and they will find a description of the *17 plane symmetry groups*. The letter *p* of "p6m" means that the tiling has a parallelogram which is a primitive cell, i.e. a repeated minimal region by the lattice translations, and the tiling is described with respect to the primitive cell axes. The lattice of a p6m tiling is hexagonal and the primitive cell consists of two equilateral triangles. The number *6* of "p6m" indicates the highest order of rotational symmetry, i.e. there is a rotation symmetry which is 1/6 of a revolution. The letter *m* of "p6m" indicates the existence of a mirror reflection.

Throughout the paper, we will always first refer to the sixth link and then to the fourth link; the reason is that the sixth link is visually simpler than the fourth link. In fact, this differentiation is not only visual, since, from the point of view of the knot design, the fourth link is more complex than the sixth link.

Description of the sixth and fourth *da Vinci-Dürer Links*

Figure 2 shows the embossed coloured redrawing of the sixth *da Vinci-Dürer Link* on a white background, hereinafter referred to as the *6-dVD-Link*.

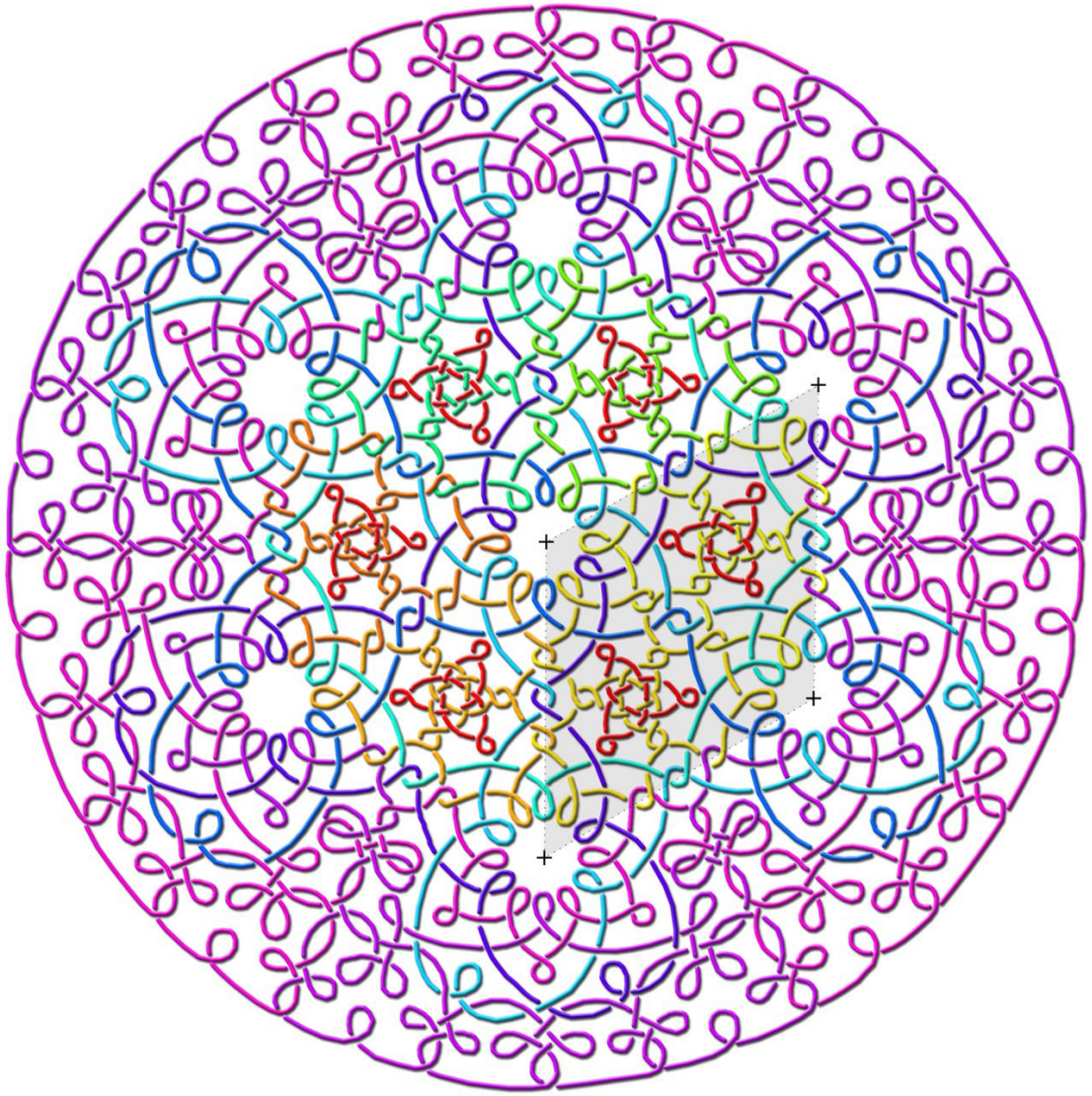

Figure 2. *6-dVD-Link.*

Figure 3 shows a simplified image of the six prime knots which will be mentioned in this paper. Those readers who are not familiar with the basic theory of knots may read, for instance, [Hoste et al., 1998] and [Adams, 1994], which are accessible, for the most part, to high schoolers. There, they will learn about the concepts of *Dowker-Thistlethwaite code*, *connected sum* (or *composition of knots*) and *prime knot*. Also, The Rolfsen Knot Table can be found in the knot data base *The Knot Atlas* [Knot Atlas] and also in [Adams, 1994].

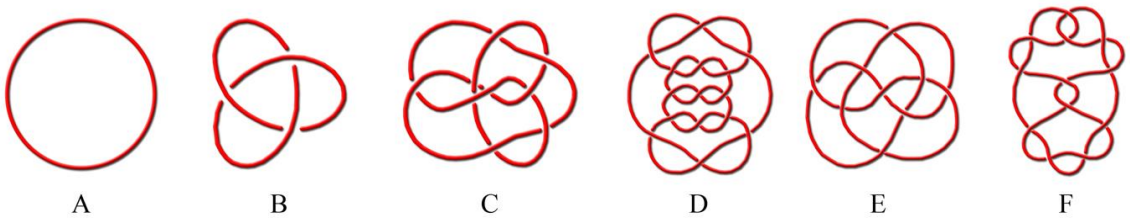

Figure 3. Prime Knots of *da Vinci-Dürer's Six Knots*.

Figure 4 shows the 18 knots which make up the *6-dVD-Link*, each knot being located in its corresponding position within the circle of Figure 2. The knots are as follows:

- Four unkots. They have zero crossings and their simplified image is Figure 3-A. Their exact locations within the circle are shown in Figure 4-III and Figure 4-IV.
- Six trefoil knots. They have three crossings, with Dowker-Thistlethwaite (DT) code: "4 6 2". Their simplified image is Figure 3-B. Their exact locations within the circle are shown in Figure 4-I.
- Six 9_{49} Rolfsen knots. They have nine crossings, with DT code: "6 -10 -14 12 -16 -2 18 -4 -8". Their simplified image is Figure 3-C. Their exact locations within the circle are shown in Figure 4-II.
- Two knots with 45 crossings, which we will call 45_{VD}. They are the connected sum of three 15-crossing prime knots, which we will call 15_{VD}, $45_{VD} = 15_{VD}\#15_{VD}\#15_{VD}$. These 15_{VD} knots have DT code: "18 16 20 24 22 26 30 28 2 6 4 8 12 10 14". Their simplified image is Figure 3-D. Readers will find information about knot 15_{VD} in the knot data base *Knotilus* [Knotilus], archive number 15x-1-1050. The exact locations of the 45_{VD} knots are shown in Figure 4-V. The bottom part shows a piece of one of the two 45_{VD} knots (the piece corresponding to one of its three summands 15_{VD}), and the top part shows the two knots in full.

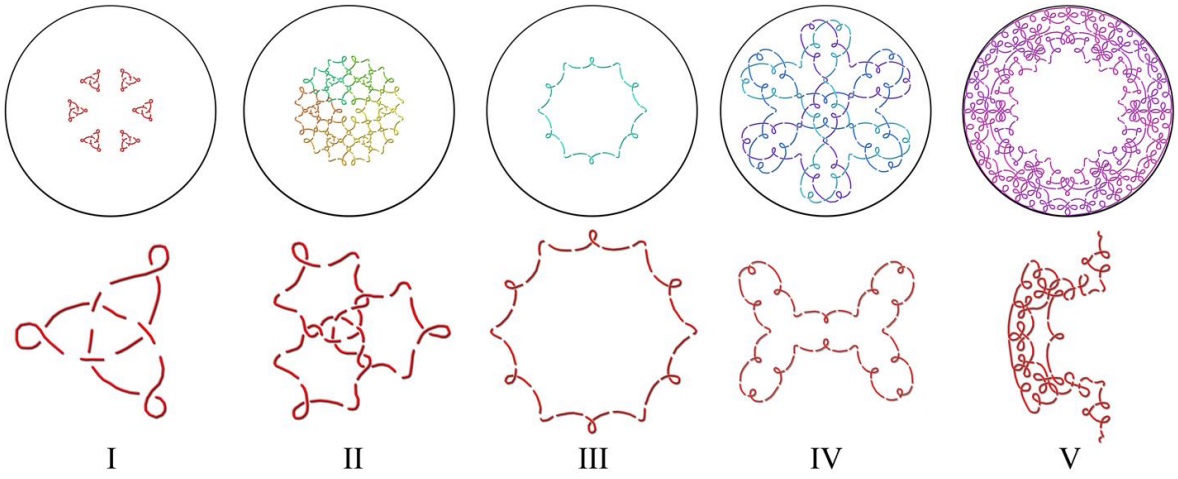

Figure 4. Knots of the *6-dVD-Link*.

Figure 5 shows the embossed coloured redrawing of the fourth *da Vinci-Dürer Link* on a white background, hereinafter referred to as the *4-dVD-Link*.

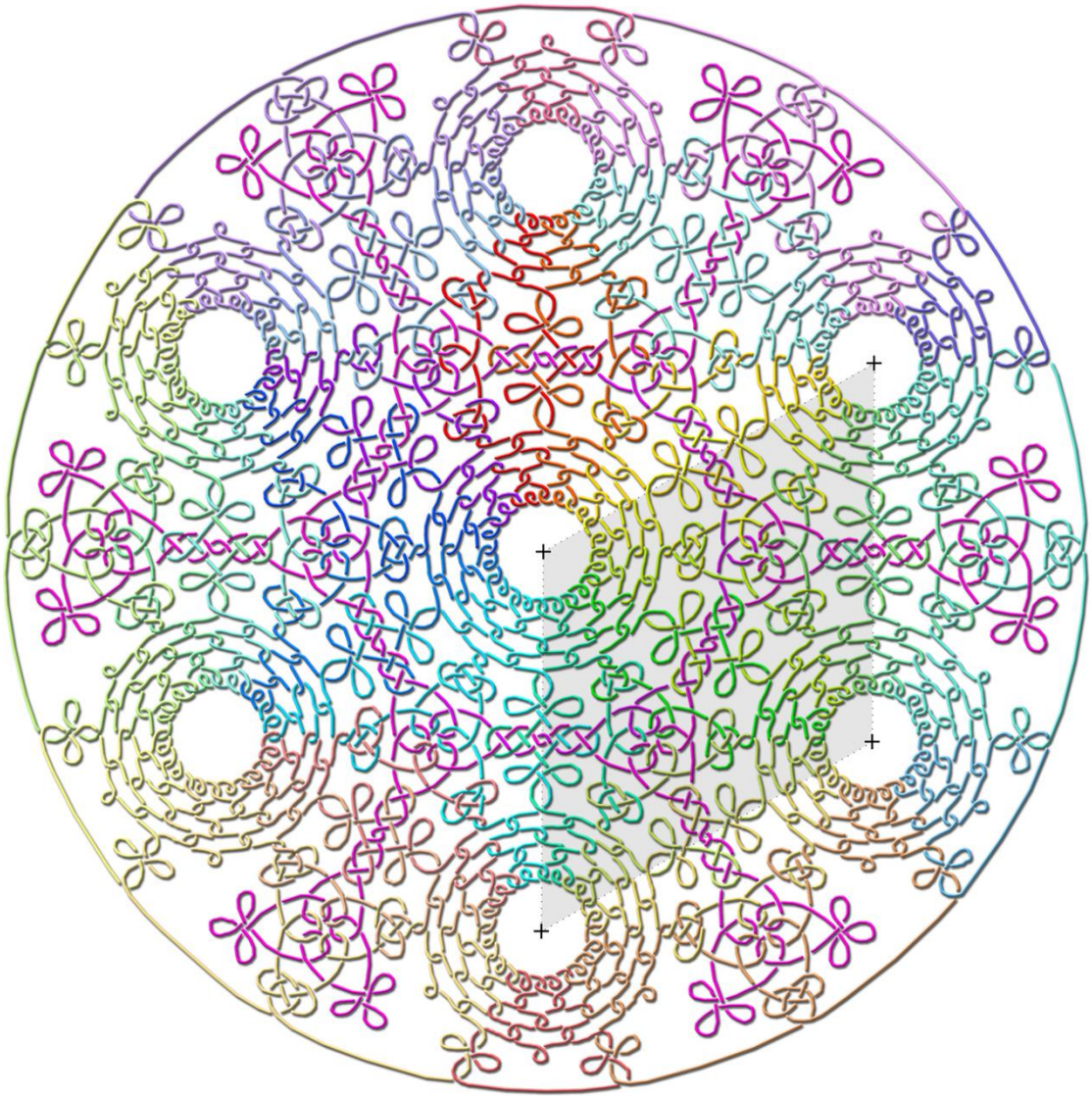

Figure 5. *4-dVD-Link.*

Figure 6 shows the 38 knots which make up the *4-dVD-Link*, each knot being located in its corresponding position within the circle of Figure 5. The knots are as follows:

- Twenty-six unkots. Their exact locations within the circle are shown in Figure 6-VI, Figure 6-VII and Figure VIII (twenty-four of them are shown on the top part of Figure 6-VIII). The bottom part of Figure 6-VI shows an enlarged image of a piece of the unknot which can be found in full on the top part of Figure 6-VI.
- Six 12-crossing prime knots, which we will call 12_{VD} knots. These 12_{VD} knots have DT code: "-12 -8 -20 -4 -16 -2 -24 -10 -22 -6 -18 -14". Their simplified image is Figure 3-E. Readers will find information about knot 12_{VD} in the knot data base *Knotilus* [Knotilus], archive number 12x-1-313. The exact locations of these knots within the circle are shown in Figure 6-X.
- Six knots with 33 crossings, which we will call 33_{VD} knots. They are the connected sum of two 12_{VD} knots and a 9_{40} Rolfsen knot, $33_{VD} = 12_{VD}\#9_{40}\#12_{VD}$. Their exact locations within the circle are shown in Figure 6-IX. The 9_{40} knots have DT code: "6 16 14 12 4 2 18 10 8". Their simplified image is Figure 3-F.

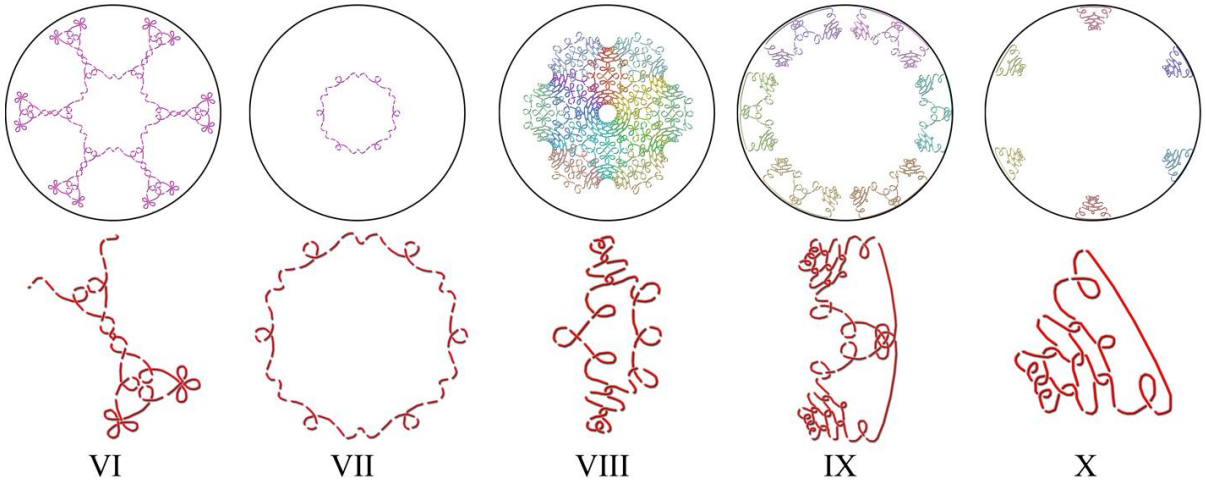

Figure 6. Knots of the *4-dVD-Link*.

Generation of two new infinite links using the p6m periodic tiling technique.

Figure 7 left shows rhombus C_6 from Figure 2, which contains a portion of the *6-dVD-Link* with the corresponding pieces of knots. Similarly, Figure 7 right shows rhombus C_4 from Figure 5, which contains a portion of the *4-dVD-Link* with the corresponding pieces of knots.

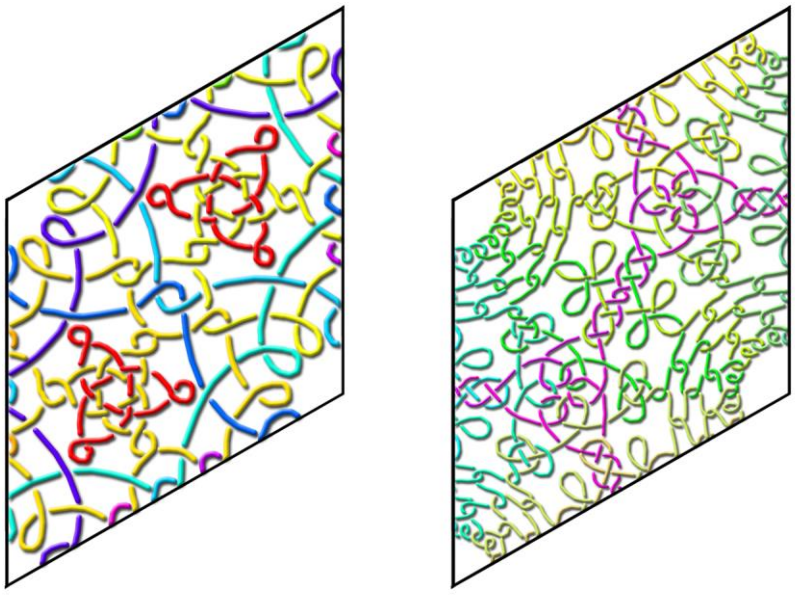

Figure 7. Primitive cells (left C_6, right C_4) and fundamental regions (center).

These rhombuses C_6 and C_4, together with their inner knot patterns, can be regarded as primitive cells which, by means of translations, generate two p6m periodic tilings. But keep in mind, we could just as easily have used the hexagon at the center of the original woodcut and tiled the plane using the hexagonal tiling. Or we could have used one of the two regular triangles of the rhombus and tiled the plane with copies using the triangle tiling. The only advantage to the rhombus is that the fundamental domain of the p6m tiling is a rhombus.

Those readers who are not familiar with the basic theory of periodic tilings may read, for instance, [Grünbaum and Shephard, 2016]. There, they will learn about the concepts of *fundamental region* and *primitive cell*.

Figure 8 shows the p6m periodic tiling generated in the way described above. It is a new link and it has an infinite number of knots. We will call it *6-dVD-TilingLink*. To give the reader a clear vision of which original knot pieces of the *6-dVD-Link* make up the final knot pieces of the *6-dVD-TilingLink*, the colours used for the knot pieces in Figure 8 are the same as those used for the original knots of the *6-dVD-Link* in Figure 2.

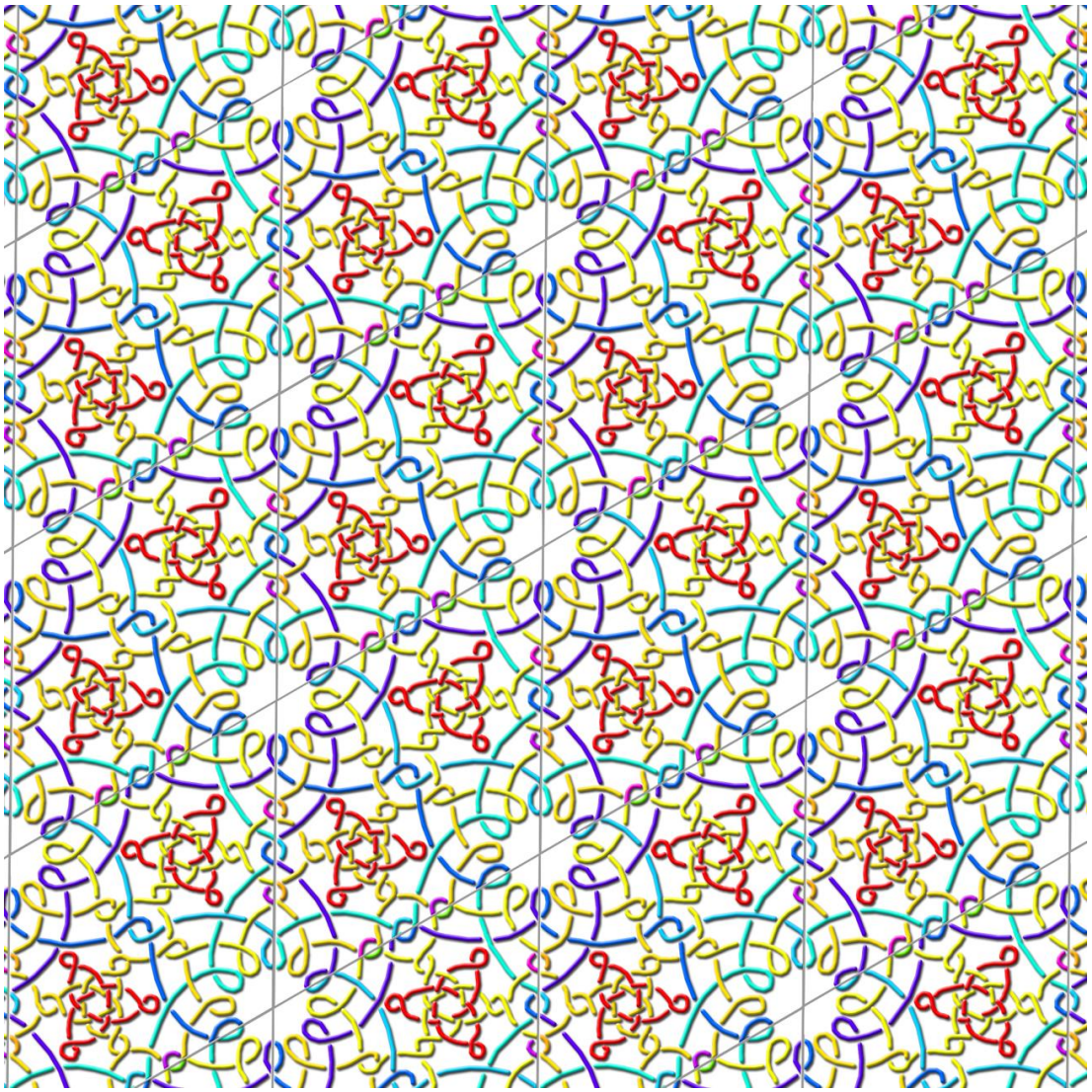

Figure 8. *6-dVD-TilingLink.*

Figure 9 shows the p6m periodic tiling generated in the same way as described above for the *6-dVD-TilingLink*. It is a new link and it has an infinite number of knots. We will call it *4-dVD-TilingLink.* Again, to give the reader a clear vision of which original knot pieces of the *4-dVD-Link* make up the final knot pieces of the *4-dVD-TilingLink*, the colours used for the knot pieces in Figure 9 are the same as those used for the original knots of the *4-dVD-Link* in Figure 5.

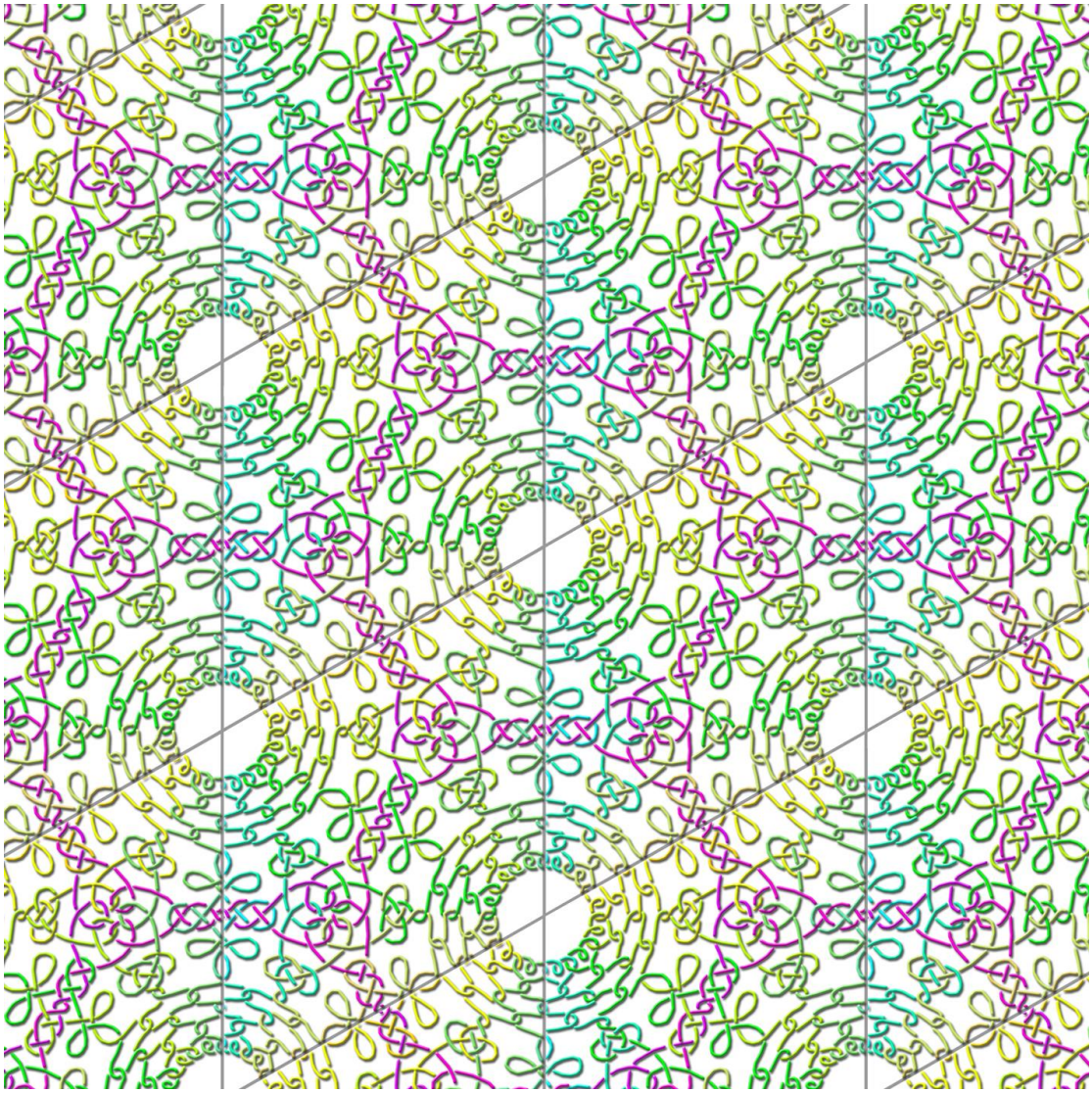

Figure 9. *4-dVD-TilingLink.*

In the next section we describe the knots of the two new infinite links obtained, namely: the *6-dVD-TilingLink* and the *4-dVD-TilingLink*.

Description of the two new infinite *da Vinci-Dürer Tiling Links*:

Figure 10 shows again the new infinite *6-dVD-TilingLink* which is generated from the sixth *da Vinci-Dürer Link*.

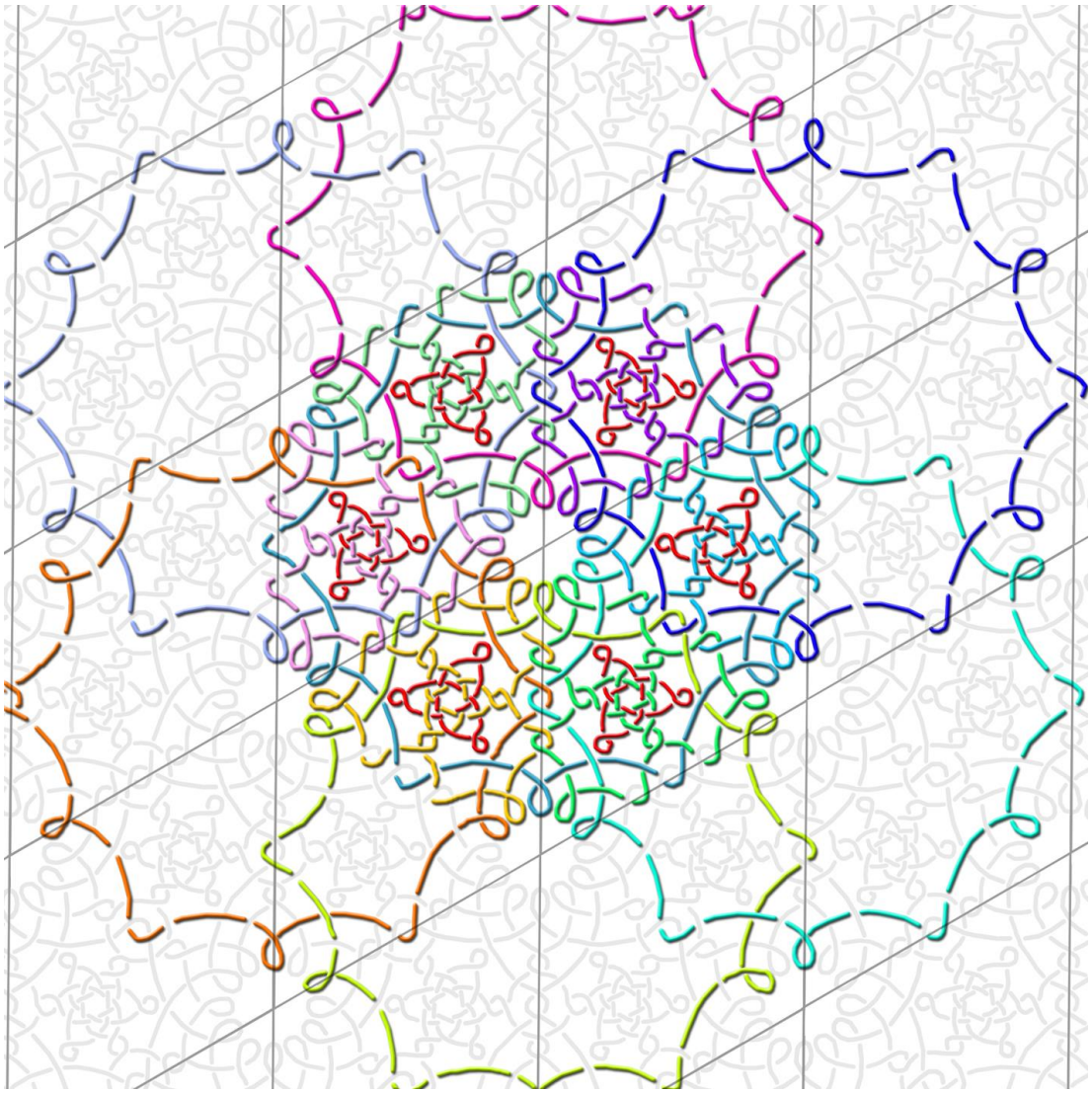

Figure 10. Core Link B_6.

However, even if it is the same link as shown in Figure 8, in order to give the reader a clear vision of how the *6-dVD-TilingLink* is made up, Figure 10 shows in grey the infinite knots, and in other colours the 19 knots making up the core link, which we will call B_6. By translating this knot core B_6 in the two translation directions of the p6m periodic tiling, we obtain the infinite number of knots making up the *6-dVD-TilingLink*.

The knots in the B_6 core are as follows: seven unkots, six trefoil knots and six 9_{49} Rolfsen knots having the same geometric shapes as in Figure 4-III, Figure 4-I and Figure 4-II, respectively. Their simplified images are Figure 3-A, Figure 3-B and Figure 3-C, respectively.

By comparing Figures 2 and 10, the reader may see that, surprisingly, the unkots shown in Figure 4-IV are what enable the replication of the unknot shown in Figure 4-III, thus creating the six unknots surrounding the core B_6 of the *6-dVD-TilingLink*. Similarly, the two 45_{VD} knots shown in Figure 4-V are what enable the replication of the six 9_{49} knots shown in Figure 4-II, thus creating the six 9_{49} knots located in the B_6 core of the *6-dVD-TilingLink*.

Figure 11 shows again the new infinite *4-dVD-TilingLink* which is generated from the fourth *da Vinci-Dürer Link*.

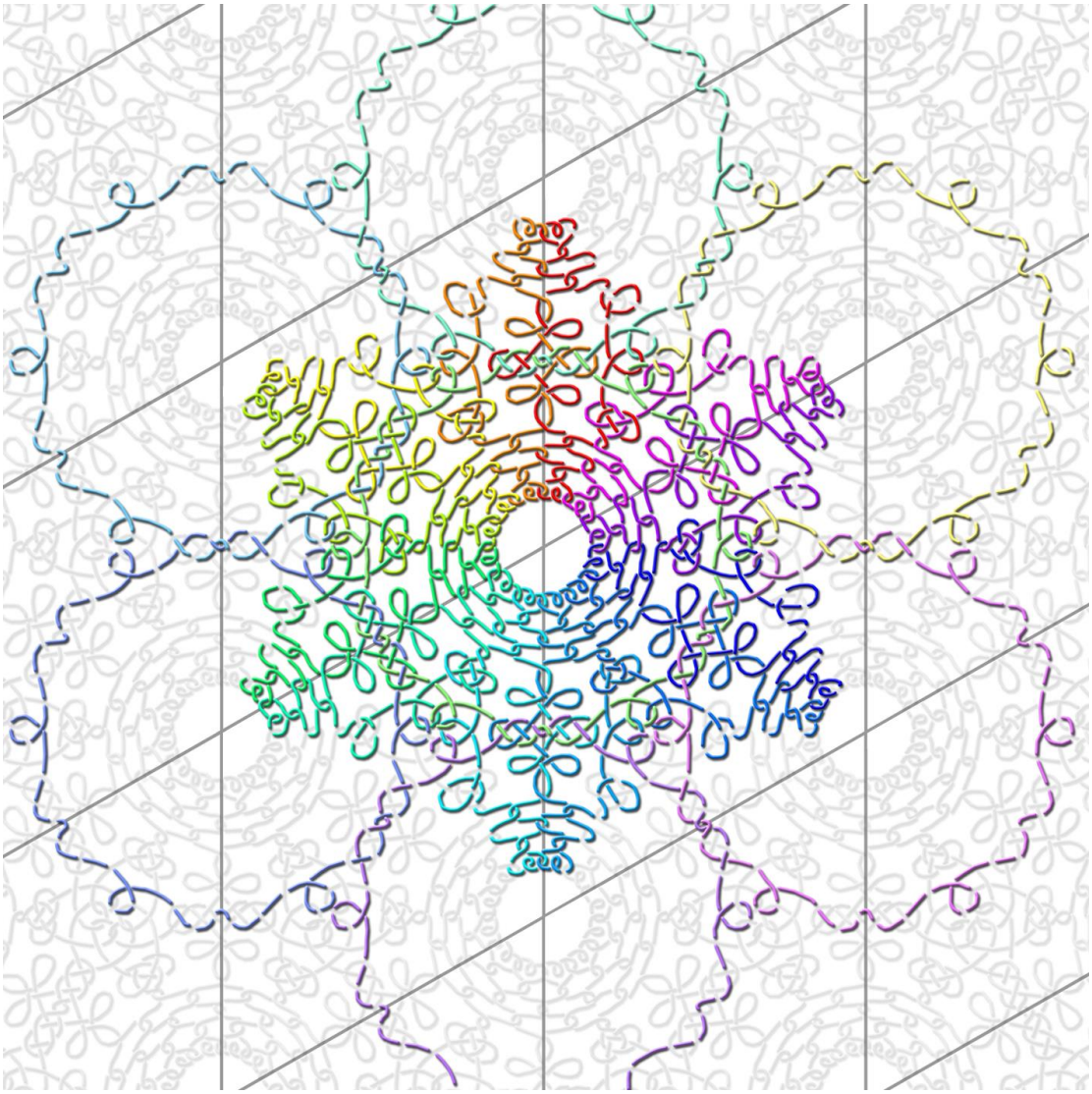

Figure 11. Core Link B_4.

However, even if it is the same link as shown in Figure 9, in order to give the reader a clear vision of how the *4-dVD-TilingLink* is made up, Figure 11 shows in grey the infinite knots, and in other colours the 19 knots making up the core link, which we will call B_4. By translating this knot core B_4 in the two translation directions of the p6m periodic tiling, we obtain the infinite number of knots making up the *4-dVD-TilingLink*.

The knots in the B_4 core are as follows: seven unkots and twelve 12_{VD} knots having the same geometric shapes as in Figure 6-VII and Figure 6-VIII, respectively. Their simplified images are Figure 3-A and Figure 3-E, respectively.

By comparing Figures 5 and 11, the reader may see that, surprisingly, the unkot shown in Figure 6-VI is what enables the replication of the unknot shown in Figure 6-VII, thus creating the six unknots surrounding the B_4 core of the *4-dVD-TilingLink*.

In summary, we believe it is beautiful to see how the da Vinci-Dürer links are broken down into their knots –and their prime knots–, and see how they transform into infinite links using the p6m periodic tiling technique, even though this transformation is only applicable in two of the six original da Vinci-Dürer links.

References

Thausing, M. 1882. *Alber Dürer, his life and works*. London: J. Murray.

Richter J.P. 1970. *The Notebooks of Leonardo da Vinci, Compiled and Edited from the Original Manuscripts, Vol. I.* New York: Dover Publications, Inc.

Virtuelles Kupferstichkabinett: http://www.virtuelles-kupferstichkabinett.de/

Grünbaum, B. and Shephard G. C. 2016. *Tilings and Patterns (second edition)*. Mineola, New York: Dover Publications.

Hoste, J., Thistlethwaite, M. and Weeks J. 1998. The First 1,701,936 Knots. *The Mathematical Intelligencer* **20**(4): 33–48.

Adams, C. C. 1994. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*. Providence: American Mathematical Soc.

Knot Atlas: http://katlas.org/wiki/The_Rolfsen_Knot_Table/

Knotilus: http://knotilus.math.uwo.ca/