

Double domination in lexicographic product graphs

A. Cabrera Martínez⁽¹⁾, S. Cabrera García⁽²⁾, J. A. Rodríguez-Velázquez⁽¹⁾

⁽¹⁾Universitat Rovira i Virgili

Departament d'Enginyeria Informàtica i Matemàtiques

Av. Països Catalans 26, 43007 Tarragona, Spain.

abel.cabrera@urv.cat, juanalberto.rodriguez@urv.cat

⁽²⁾Universitat Politècnica de Valencia

Departamento de Estadística e Investigación Operativa Aplicadas y Calidad

Camino de Vera s/n, 46022 Valencia, Spain.

suicabga@eio.upv.es

Abstract

In a graph G , a vertex dominates itself and its neighbours. A subset $S \subseteq V(G)$ is said to be a double dominating set of G if S dominates every vertex of G at least twice. The minimum cardinality among all double dominating sets of G is the double domination number. In this article, we obtain tight bounds and closed formulas for the double domination number of lexicographic product graphs $G \circ H$ in terms of invariants of the factor graphs G and H .

Keywords: Double domination; total domination; total Roman $\{2\}$ -domination; lexicographic product

1 Introduction

In a graph G , a vertex dominates itself and its neighbours. A subset $S \subseteq V(G)$ is said to be a *dominating set* of G if S dominates every vertex of G , while S is said to be a *double dominating set* of G if S dominates every vertex of G at least twice. A subset $S \subseteq V(G)$ is said to be a *total dominating set* of G if every vertex $v \in V(G)$ is dominated by at least one vertex in $S \setminus \{v\}$. The minimum cardinality among all dominating sets of G is the *domination number*, denoted by $\gamma(G)$. The *double domination number* and the *total domination number* of G are defined by analogy, and are denoted by $\gamma_{\times 2}(G)$ and $\gamma_t(G)$, respectively. The domination number and the total domination number have been extensively studied. For instance, we cite the following books [19, 20, 21]. The double domination number, which has been less studied, was introduced in [18] by Harary and Haynes, and studied further in a number of works including [4, 10, 15, 17, 23].

Let $f : V(G) \rightarrow \{0, 1, 2\}$ be a function. For any $i \in \{0, 1, 2\}$ we define the subsets of vertices $V_i = \{v \in V(G) : f(v) = i\}$ and we identify f with the three subsets of $V(G)$ induced by f .

Thus, in order to emphasize the notation of these sets, we denote the function by $f(V_0, V_1, V_2)$. Given a set $X \subseteq V(G)$, we define $f(X) = \sum_{v \in X} f(v)$, and the *weight* of f is defined to be $\omega(f) = f(V(G)) = |V_1| + 2|V_2|$.

A function $f(V_0, V_1, V_2)$ is a *total Roman dominating function* (TRDF) on a graph G if $V_1 \cup V_2$ is a total dominating set and $N(v) \cap V_2 \neq \emptyset$ for every vertex $v \in V_0$, where $N(v)$ denotes the *open neighbourhood* of v . This concept was introduced by Liu and Chang [24]. For recent results on total Roman domination in graphs we cite [1, 2, 7, 9].

A function $f(V_0, V_1, V_2)$ is a *total Roman $\{2\}$ -dominating function* (TR2DF) if $V_1 \cup V_2$ is a total dominating set and $f(N(v)) \geq 2$ for every vertex $v \in V_0$. This concept was recently introduced in [6]. Notice that $S \subseteq V(G)$ is a double dominating set of G if and only if there exists a TR2DF $f(V_0, V_1, V_2)$ such that $V_1 = S$ and $V_2 = \emptyset$.

The *total Roman domination number*, denoted by $\gamma_R(G)$, is the minimum weight among all TRDFs on G . By analogy, we define the *total Roman $\{2\}$ -domination number*, which is denoted by $\gamma_{\{R2\}}(G)$.

Notice that, by definition, $\gamma_{\times 2}(G) \geq \gamma_{\{R2\}}(G)$. As an example of graph G for which $\gamma_{\times 2}(G) > \gamma_{\{R2\}}(G)$ we consider a star graph $K_{1,r}$ for $r \geq 3$. In this case, $\gamma_{\times 2}(K_{1,r}) = r + 1 > 3 = \gamma_{\{R2\}}(K_{1,r})$. We would point out that the problem of characterizing all graphs with $\gamma_{\times 2}(G) = \gamma_{\{R2\}}(G)$ remains open. In this paper we show that the values of these two parameters coincide for any lexicographic product graph $G \circ H$ in which graph G has no isolated vertices and graph H is not trivial. Furthermore, we obtain tight bounds and closed formulas for $\gamma_{\times 2}(G \circ H)$ in terms of invariants of the factor graphs G and H .

1.1 Additional concepts, notation and tools

All graphs considered in this paper are finite and undirected, without loops or multiple edges. As usual, the *closed neighbourhood* of a vertex $v \in V(G)$ is denoted by $N[v] = N(v) \cup \{v\}$. We say that a vertex $v \in V(G)$ is a *universal vertex* of G if $N[v] = V(G)$. By analogy with the notation used for vertices, for a set $S \subseteq V(G)$, its *open neighbourhood* is the set $N(S) = \cup_{v \in S} N(v)$, and its *closed neighbourhood* is the set $N[S] = N(S) \cup S$. The subgraph induced by $S \subseteq V(G)$ will be denoted by $\langle S \rangle$, while the graph obtained from G by removing all the vertices in $S \subseteq V(G)$ (and all the edges incident with a vertex in S) will be denoted by $G - S$.

We will use the notation $K_n, K_{1,n-1}, C_n, N_n, P_n$ and $K_{n,n-r}$ for complete graphs, star graphs, cycle graphs, empty graphs, path graphs and complete bipartite graphs of order n , respectively. A double star S_{n_1, n_2} is the graph obtained by joining the center of two stars K_{1, n_1} and K_{1, n_2} with an edge.

Given two graphs G and H , the *lexicographic product* of G and H is the graph $G \circ H$ whose vertex set is $V(G \circ H) = V(G) \times V(H)$ and $(u, v)(x, y) \in E(G \circ H)$ if and only if $ux \in E(G)$ or $u = x$ and $vy \in E(H)$. Notice that for any vertex $u \in V(G)$ the subgraph of $G \circ H$ induced by $\{u\} \times V(H)$ is isomorphic to H . For simplicity, we will denote this subgraph by H_u . For basic properties of lexicographic product graphs we suggest the books [16, 22]. In particular, we cite the following works on domination theory of lexicographic product graphs: standard domination [25, 27, 31], Roman domination [28], total Roman domination [9], weak Roman domination [30], rainbow domination [29], k -rainbow independent domination [5], super domination [13], twin domination [26], power domination [14] and doubly connected domination [3].

For simplicity, for any $(u, v) \in V(G) \times V(H)$ and any TR2DF f on $G \circ H$ we write $N(u, v)$ and $f(u, v)$ instead of $N((u, v))$ and $f((u, v))$, respectively.

For the remainder of the paper, definitions will be introduced whenever a concept is needed.

Now we present some tools that will be very useful throughout the work.

Proposition 1.1. [6] *The following inequalities hold for any graph G with no isolated vertex.*

$$(i) \ \gamma_t(G) \leq \gamma_{t\{R2\}}(G) \leq \gamma_{tR}(G) \leq 2\gamma_t(G).$$

$$(ii) \ \gamma_{t\{R2\}}(G) \leq \gamma_{\times 2}(G).$$

A double dominating set of cardinality $\gamma_{\times 2}(G)$ will be called a $\gamma_{\times 2}(G)$ -set. A similar agreement will be assumed when referring to optimal sets (and functions) associated to other parameters used in the article.

Theorem 1.2. *If $\gamma_{\times 2}(G) = \gamma_t(G)$, then for any $\gamma_{\times 2}(G)$ -set D there exists an integer $k \geq 1$ such that $\langle D \rangle \cong \cup_{i=1}^k K_2$.*

Proof. Let D be a $\gamma_{\times 2}(G)$ -set and suppose that $\langle D \rangle$ has a component G' which is not isomorphic to K_2 . Let $v \in V(G')$ be a vertex of minimum degree in G' . Notice that the set $D \setminus \{v\}$ is a total dominating set of G . Hence, $\gamma_t(G) \leq |D \setminus \{v\}| < |D| = \gamma_{\times 2}(G)$, which is a contradiction. Therefore, the result follows. \square

Theorem 1.3. [6] *The following statements are equivalent.*

- $\gamma_{t\{R2\}}(G) = 2\gamma_t(G)$.
- $\gamma_{t\{R2\}}(G) = \gamma_{tR}(G)$ and $\gamma_t(G) = \gamma(G)$.

The following theorem merges two results obtained in [6] and [18].

Theorem 1.4 ([6] and [18]). *The following statements are equivalent.*

- $\gamma_{t\{R2\}}(G) = 2$.
- $\gamma_{\times 2}(G) = 2$.
- G has at least two universal vertices.

It is readily seen that if G' is a spanning subgraph of G , then any $\gamma_{\times 2}(G')$ -set is a double dominating set of G . Therefore, the following result is immediate.

Theorem 1.5. *If G' is a spanning subgraph of G with no isolated vertex, then*

$$\gamma_{\times 2}(G) \leq \gamma_{\times 2}(G').$$

In Proposition 4.7 we will show some cases of lexicographic product graphs for which the equality above holds.

Remark 1.6. *For any integer $n \geq 3$,*

$$(i) \ \gamma_{\{R2\}}(P_n) \stackrel{[6]}{=} \gamma_{\times 2}(P_n) \stackrel{[4]}{=} \begin{cases} 2 \lceil \frac{n}{3} \rceil + 1, & \text{if } n \equiv 0 \pmod{3}, \\ 2 \lceil \frac{n}{3} \rceil, & \text{otherwise.} \end{cases}$$

$$(ii) \ \gamma_{\{R2\}}(C_n) \stackrel{[6]}{=} \gamma_{\times 2}(C_n) \stackrel{[18]}{=} \lceil \frac{2n}{3} \rceil.$$

The next theorem merges two results obtained in [28] and [31].

Theorem 1.7 ([28] and [31]). *For any graph G with no isolated vertex and any nontrivial graph H ,*

$$\gamma(G \circ H) = \begin{cases} \gamma(G), & \text{if } \gamma(H) = 1, \\ \gamma(G), & \text{if } \gamma(H) \geq 2. \end{cases}$$

Theorem 1.8. [8] *For any graph G with no isolated vertex and any nontrivial graph H ,*

$$\gamma(G \circ H) = \gamma(G).$$

2 Main results on lexicographic product graphs

Our first result shows that the double domination number and the total Roman $\{2\}$ -domination number coincide for lexicographic product graphs.

Theorem 2.1. *For any graph G with no isolated vertex and any nontrivial graph H ,*

$$\gamma_{\times 2}(G \circ H) = \gamma_{\{R2\}}(G \circ H).$$

Proof. Proposition 1.1 (ii) leads to $\gamma_{\times 2}(G \circ H) \geq \gamma_{\{R2\}}(G \circ H)$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{\{R2\}}(G \circ H)$ -function such that $|V_2|$ is minimum. Suppose that $\gamma_{\times 2}(G \circ H) > \gamma_{\{R2\}}(G \circ H)$. In such a case, $V_2 \neq \emptyset$ and we can differentiate two cases for a fixed vertex $(u, v) \in V_2$.

Case 1. $N(u, v) \cap (V_1 \cup V_2) \subseteq V(H_u)$. In this case, for any $(u', v') \in N(u) \times V(H)$ we define the function $g(V'_0, V'_1, V'_2)$ where $V'_0 = V_0 \setminus \{(u', v')\}$, $V'_1 = V_1 \cup \{(u, v), (u', v')\}$ and $V'_2 = V_2 \setminus \{(u, v)\}$. Observe that $V'_1 \cup V'_2$ is a total dominating set of $G \circ H$ and every vertex $w \in V'_0 \subseteq V_0$ satisfies that $g(N(w)) \geq 2$. Hence, g is a $\gamma_{\{R2\}}(G \circ H)$ -function and $|V'_2| = |V_2| - 1$, which is a contradiction.

Case 2. $N(u) \times V(H) \cap (V_1 \cup V_2) \neq \emptyset$. If $f(u, v') > 0$ for every vertex $v' \in V(H)$, then the function g , defined by $g(u, v) = 1$ and $g(x, y) = f(x, y)$ whenever $(x, y) \in V(G \circ H) \setminus \{(u, v)\}$, is a TR2DF on $G \circ H$ and $\omega(g) = \omega(f) - 1$, which is a contradiction. Hence, there exists a vertex $v' \in V(H)$ such that $f(u, v') = 0$. In this case, we define the function $g(V'_0, V'_1, V'_2)$ where $V'_0 = V_0 \setminus \{(u, v')\}$, $V'_1 = V_1 \cup \{(u, v), (u, v')\}$ and $V'_2 = V_2 \setminus \{(u, v)\}$. Notice that $V'_1 \cup V'_2$ is a total dominating set of $G \circ H$ and every vertex $w \in V'_0 \subseteq V_0$ satisfies that $g(N(w)) \geq 2$. Hence, g is a $\gamma_{\{R2\}}(G \circ H)$ -function and $|V'_2| = |V_2| - 1$, which is a contradiction again.

According to the two cases above, we deduce that $V_2 = \emptyset$. Therefore, V_1 is a $\gamma_{\times 2}(G \circ H)$ -set and so $\gamma_{\times 2}(G \circ H) = \gamma_{\{R2\}}(G \circ H)$. \square

From now on, the main goal is to obtain tight bounds or closed formulas for $\gamma_{\times 2}(G \circ H)$ and express them in terms of invariants of G and H .

A set $X \subseteq V(G)$ is called a *2-packing* if $N[u] \cap N[v] = \emptyset$ for every pair of different vertices $u, v \in X$, [20]. The *2-packing number* $\rho(G)$ is the maximum cardinality among all 2-packing sets of G . As usual, a 2-packing of cardinality $\rho(G)$ is called a $\rho(G)$ -set.

Theorem 2.2. *For any graph G with no isolated vertex and any nontrivial graph H ,*

$$\max\{\gamma_t(G), 2\rho(G)\} \leq \gamma_{\times 2}(G \circ H) \leq 2\gamma_t(G).$$

Proof. By Proposition 1.1 (i) and Theorem 1.8 we deduce that

$$\gamma_t(G) = \gamma_t(G \circ H) \leq \gamma_{\times 2}(G \circ H) \leq 2\gamma_t(G \circ H) = 2\gamma_t(G).$$

Now, for any $\rho(G)$ -set X and any $\gamma_{\times 2}(G \circ H)$ -set D we have that

$$\gamma_{\times 2}(G \circ H) = |D| = \sum_{u \in V(G)} |D \cap V(H_u)| \geq \sum_{u \in X} \sum_{w \in N[u]} |D \cap V(H_w)| \geq 2|X| = 2\rho(G).$$

Therefore, the proof is complete. \square

We would point out that the upper bound $\gamma_{\times 2}(G \circ H) \leq \min\{2\gamma_t(G), \gamma(G)\gamma_{\times 2}(H)\}$ was proposed in [12] for the particular case in which G and H are connected. Obviously, the connectivity is not needed, and the bound $\gamma_{\times 2}(G \circ H) \leq \gamma(G)\gamma_{\times 2}(H)$ also holds for any graph G (even if G is empty) and any graph H with no isolated vertices.

In Theorem 2.4 we will show cases in which $\gamma_{\times 2}(G \circ H) = 2\gamma_t(G)$, while in Theorem 2.8 (i) and (ii) we will show cases in which $\gamma_{\times 2}(G \circ H) = 2\rho(G)$ or $\gamma_{\times 2}(G \circ H) = \gamma_t(G)$.

Corollary 2.3. *If $\gamma(G) = 1$, then for any nontrivial graph H ,*

$$2 \leq \gamma_{\times 2}(G \circ H) \leq 4.$$

In Section 3 we characterize the graphs with $\gamma_{\times 2}(G \circ H) \in \{2, 3\}$. Hence, by Corollary 2.3 the graphs with $\gamma_{\times 2}(G \circ H) = 4$ will be automatically characterized whenever $\gamma(G) = 1$.

Theorem 2.4. *If G is a graph with no isolated vertex and H is a nontrivial graph, then the following statements are equivalent.*

- (a) $\gamma_{\times 2}(G \circ H) = 2\gamma_t(G)$.
- (b) $\gamma_{\times 2}(G \circ H) = \gamma_R(G \circ H)$ and $(\gamma_t(G) = \gamma(G) \text{ or } \gamma(H) \geq 2)$.

Proof. Assume that $\gamma_{\times 2}(G \circ H) = 2\gamma_t(G)$. By Theorems 1.8 and 2.1 we deduce that

$$\gamma_{\{R2\}}(G \circ H) = \gamma_{\times 2}(G \circ H) = 2\gamma_t(G) = 2\gamma_t(G \circ H).$$

Hence, by Theorem 1.3 we have that $\gamma_{\times 2}(G \circ H) = \gamma_R(G \circ H)$ and $\gamma(G \circ H) = \gamma_t(G \circ H) = \gamma_t(G)$. Notice that $\gamma_t(G \circ H) = \gamma_t(G)$ if and only if $\gamma_t(G) = \gamma(G)$ or $\gamma(H) \geq 2$, by Theorem 1.7. Therefore, (b) follows.

Conversely, assume that (b) holds. By Theorem 2.1 we have that

$$\gamma_{\{R2\}}(G \circ H) = \gamma_{\times 2}(G \circ H) = \gamma_{IR}(G \circ H). \quad (1)$$

Now, if $\gamma_I(G) = \gamma(G)$ or $\gamma(H) \geq 2$, by Theorems 1.7 and 1.8 we deduce that

$$\gamma_I(G \circ H) = \gamma_I(G) = \gamma(G \circ H). \quad (2)$$

Hence, Theorem 1.3 and equations (1) and (2) lead to $\gamma_{\times 2}(G \circ H) = \gamma_{\{R2\}}(G \circ H) = 2\gamma_I(G \circ H) = 2\gamma_I(G)$, as required. \square

It was shown in [11] that for any connected graph G of order $n \geq 3$, $\gamma_I(G) \leq \frac{2n}{3}$. Hence, Proposition 1.1 (i) and Theorem 2.1 lead to the following result.

Theorem 2.5. *For any connected graph G of order $n \geq 3$ and any graph H ,*

$$\gamma_{\times 2}(G \circ H) \leq 2 \left\lfloor \frac{2n}{3} \right\rfloor.$$

In order to show that the bound above is tight, we consider the case of rooted product graphs. Given a graph G and a graph H with root $v \in V(H)$, the rooted product $G \bullet_v H$ is defined as the graph obtained from G and H by taking one copy of G and $|V(G)|$ copies of H and identifying the i^{th} vertex of G with vertex v in the i^{th} copy of H for every $i \in \{1, \dots, |V(G)|\}$. For instance, the graph $P_5 \bullet_v P_3$ where v is a leaf, is shown in Figure 1. Later, when we read Lemma 4.3, it will be easy to see that for $n = |V(G \bullet_v P_3)| = 3|V(G)|$ we have that $\gamma_{\times 2}((G \bullet_v P_3) \circ H) = 4|V(G)| = 2 \lfloor \frac{2n}{3} \rfloor$ whenever $\gamma(H) \geq 3$.

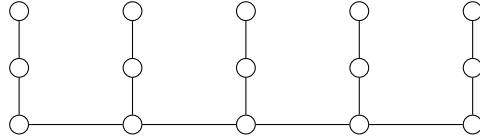


Figure 1: The graph $P_5 \bullet_v P_3$

Lemma 2.6. *For any graph G with no isolated vertex and any nontrivial graph H , there exists a $\gamma_{\times 2}(G \circ H)$ -set S such that $|S \cap V(H_u)| \leq 2$, for every $u \in V(G)$.*

Proof. Given a double dominating set S of $G \circ H$, we define the set $S_3 = \{x \in V(G) : |S \cap V(H_x)| \geq 3\}$. Let S be a $\gamma_{\times 2}(G \circ H)$ -set such that $|S_3|$ is minimum among all $\gamma_{\times 2}(G \circ H)$ -sets. If $|S_3| = 0$, then we are done. Hence, we suppose that there exists $u \in S_3$ and let $(u, v) \in S$. We assume that $|S \cap V(H_u)|$ is minimum among all vertices in S_3 . It is readily seen that if there exists $u' \in N(u)$ such that $|S \cap V(H_{u'})| \geq 2$, then $S' = S \setminus \{(u, v)\}$ is a double dominating set of $G \circ H$, which is a contradiction. Hence, if $u' \in N(u)$, then $|S \cap V(H_{u'})| \leq 1$, and in this case it is not difficult to check that for $(u', v') \notin S$ the set $S'' = (S \setminus \{(u, v)\}) \cup \{(u', v')\}$ is a $\gamma_{\times 2}(G \circ H)$ -set such that $|S''_3|$ is minimum among all $\gamma_{\times 2}(G \circ H)$ -sets. If $|S''_3| < |S_3|$, then we obtain a contradiction, otherwise $u \in S''_3$ and $|S'' \cap V(H_u)|$ is minimum among all vertices in S''_3 , so that we can successively repeat this process, until obtaining a contradiction. Therefore, the result follows. \square

Theorem 2.7. *Let G be a graph with no isolated vertex and let H be a nontrivial graph.*

- (i) *If $\gamma(H) = 1$, then $\gamma_{\times 2}(G \circ H) \leq \gamma_{t\{R2\}}(G)$.*
- (ii) *If H has at least two universal vertices, then $\gamma_{\times 2}(G \circ H) \leq 2\gamma(G)$.*
- (iii) *If H has exactly one universal vertex, then $\gamma_{\times 2}(G \circ H) = \gamma_{t\{R2\}}(G)$.*
- (iv) *If $\gamma(H) \geq 2$, then $\gamma_{\times 2}(G \circ H) \geq \gamma_{t\{R2\}}(G)$.*

Proof. Let f be a $\gamma_{t\{R2\}}(G)$ -function and let v be a universal vertex of H . Let f' be the function defined by $f'(u, v) = f(u)$ for every $u \in V(G)$ and $f'(x, y) = 0$ whenever $x \in V(G)$ and $y \in V(H) \setminus \{v\}$. It is readily seen that f' is a TR2DF on $G \circ H$. Hence, by Theorem 2.1 we conclude that $\gamma_{\times 2}(G \circ H) = \gamma_{t\{R2\}}(G \circ H) \leq \omega(f') = \omega(f) = \gamma_{t\{R2\}}(G)$ and (i) follows.

Let D be a $\gamma(G)$ -set and let y_1, y_2 be two universal vertices of H . It is not difficult to see that $S = D \times \{y_1, y_2\}$ is a double dominating set of $G \circ H$. Therefore, $\gamma_{\times 2}(G \circ H) \leq |S| = 2\gamma(G)$ and (ii) follows.

From now on, let S be a $\gamma_{\times 2}(G \circ H)$ -set that satisfies Lemma 2.6 and assume that either $\gamma(H) \geq 2$ or H has exactly one universal vertex. Let $g(V_0, V_1, V_2)$ be the function defined by $g(u) = |S \cap V(H_u)|$ for every $u \in V(G)$. We claim that g is a TR2DF on G . It is clear that every vertex in V_1 has to be adjacent to some vertex in $V_1 \cup V_2$ and, if $\gamma(H) \geq 2$ or H has exactly one universal vertex, then by Theorem 1.4 we have that $\gamma_{\times 2}(H) \geq 3$, which implies that every vertex in V_2 has to be adjacent to some vertex in $V_1 \cup V_2$. Hence, $V_1 \cup V_2$ is a total dominating set of G . Now, if $x \in V_0$, then $S \cap V(H_x) = \emptyset$, and so $|N(V(H_x)) \cap S| \geq 2$. Thus, $g(N(x)) \geq 2$, which implies that g is TR2DF on G and so $\gamma_{t\{R2\}}(G) \leq \omega(g) = |S| = \gamma_{\times 2}(G \circ H)$. Therefore, (iii) and (iv) follow. \square

The following result is a direct consequence of Theorems 2.2 and 2.7. Recall that $\gamma_{\times 2}(H) = 2$ if and only if H has at least two universal vertices (see Theorem 1.4).

Theorem 2.8. *Let G be a graph with no isolated vertex and let H be a nontrivial graph.*

- (i) *If $\gamma(G) = \rho(G)$ and $\gamma_{\times 2}(H) = 2$, then $\gamma_{\times 2}(G \circ H) = 2\gamma(G)$.*
- (ii) *If $\gamma_{t\{R2\}}(G) \in \{\gamma(G), 2\rho(G)\}$ and $\gamma(H) = 1$, then $\gamma_{\times 2}(G \circ H) = \gamma_{t\{R2\}}(G)$.*
- (iii) *If $\gamma_{t\{R2\}}(G) = 2\gamma(G)$ and $\gamma(H) \geq 2$, then $\gamma_{\times 2}(G \circ H) = \gamma_{t\{R2\}}(G)$.*

It is well known that $\gamma(T) = \rho(T)$ for any tree T . Hence, the following corollary is a direct consequence of Theorem 2.8.

Corollary 2.9. *For any tree T and any graph H with $\gamma_{\times 2}(H) = 2$,*

$$\gamma_{\times 2}(T \circ H) = 2\gamma(T).$$

A *double total dominating set* of a graph G is a set S of vertices of G such that every vertex in $V(G)$ is adjacent to at least two vertices in S [21]. The *double total domination number* of G , denoted by $\gamma_{2,t}(G)$, is the minimum cardinality among all double total dominating sets.

Theorem 2.10. [30] *If G is a graph of minimum degree greater than or equal to two, then for any graph H ,*

$$\gamma_{2,t}(G \circ H) \leq \gamma_{2,t}(G).$$

Theorem 2.11. *Let G be a graph of minimum degree greater than or equal to two and order n . The following statements hold.*

- (i) *For any graph H , $\gamma_{\times 2}(G \circ H) \leq \gamma_{2,t}(G)$.*
- (ii) *For any graph H , $\gamma_{\times 2}(G \circ H) \leq n$.*

Proof. Since every double total dominating set is a double dominating set, we deduce that $\gamma_{\times 2}(G \circ H) \leq \gamma_{2,t}(G \circ H)$. Hence, from Theorem 2.10 we deduce (i). Finally, since $\gamma_{2,t}(G) \leq n$, from (i) we deduce (ii). \square

The following family \mathcal{H}_k of graphs was shown in [30]. A graph G belongs to \mathcal{H}_k if and only if it is constructed from a cycle C_k and k empty graphs N_{s_1}, \dots, N_{s_k} of order s_1, \dots, s_k , respectively, and joining by an edge each vertex from N_{s_i} with the vertices v_i and v_{i+1} of C_k . Here we are assuming that v_i is adjacent to v_{i+1} in C_k , where the subscripts are taken modulo k . Figure 2 shows a graph G belonging to \mathcal{H}_k , where $k = 4$, $s_1 = s_3 = 3$ and $s_2 = s_4 = 2$.

Notice that $\gamma_{t\{R2\}}(G) = \gamma_{2,t}(G)$, for every $G \in \mathcal{H}_k$. Hence, from Theorems 2.7 (iv) and 2.11 (i) we deduce that $\gamma_{\times 2}(G \circ H) = \gamma_{2,t}(G)$ for any $G \in \mathcal{H}_k$ and any graph H such that $\gamma(H) \geq 2$.

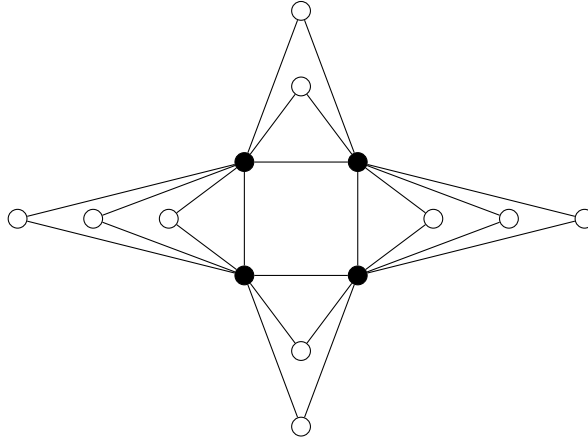


Figure 2: The set of black-coloured vertices is a $\gamma_{2,t}(G)$ -set.

3 Small values of $\gamma_{\times 2}(G \circ H)$

First, we characterize the graphs with $\gamma_{\times 2}(G \circ H) = 2$.

Theorem 3.1. *For any nontrivial graph G and any graph H , the following statements are equivalent.*

- (i) $\gamma_{\times 2}(G \circ H) = 2$.
- (ii) $\gamma(G) = \gamma(H) = 1$ and $(\gamma_{\times 2}(G) = 2$ or $\gamma_{\times 2}(H) = 2)$.

Proof. Notice that $G \circ H$ has at least two universal vertices if and only if $\gamma(G) = \gamma(H) = 1$, and also G has at least two universal vertices or H has at least two universal vertices. Hence, by Theorem 1.4 we conclude that (i) and (ii) are equivalent. \square

Next, we characterize the graphs that satisfying $\gamma_{\times 2}(G \circ H) = 3$. Before we shall need the following definitions. For a set $S \subseteq V(G \circ H)$ we define the following subsets of $V(G)$.

$$\mathcal{A}_S = \{v \in V(G) : |S \cap V(H_v)| \geq 2\};$$

$$\mathcal{B}_S = \{v \in V(G) : |S \cap V(H_v)| = 1\};$$

$$\mathcal{C}_S = \{v \in V(G) : S \cap V(H_v) = \emptyset\}.$$

Theorem 3.2. *For any nontrivial graphs G and H , $\gamma_{\times 2}(G \circ H) = 3$ if and only if one of the following conditions is satisfied.*

- (i) $G \cong P_2$ and $\gamma(H) = 2$.
- (ii) $G \not\cong P_2$ has at least two universal vertices and $\gamma(H) \geq 2$.
- (iii) G has exactly one universal vertex and either $\gamma(H) = 2$ or H has exactly one universal vertex.
- (iv) G has exactly one universal vertex, $\gamma_{2,t}(G) = 3$ and $\gamma(H) \geq 3$.
- (v) $\gamma(G) = 2$ and $\gamma_{2,t}(G) = 3$.
- (vi) $\gamma(G) = 2$, $\gamma_{\times 2}(G) = 3 < \gamma_{2,t}(G)$ and $\gamma(H) = 1$.

Proof. Notice that with the above premises, G does not have isolated vertices. Let S be a $\gamma_{\times 2}(G \circ H)$ -set that satisfies Lemma 2.6 and assume that $|S| = 3$. By Theorems 1.8 and 1.2 we have that $3 = \gamma_{\times 2}(G \circ H) > \gamma_t(G \circ H) = \gamma_t(G) \geq 2$, which implies that $\gamma_t(G) = 2$ and so $\gamma(G) \in \{1, 2\}$. We differentiate two cases.

Case 1. $\gamma(G) = 1$. In this case, Theorem 3.1 leads to $\gamma_{\times 2}(H) \geq 3$. Now, we consider the following subcases.

Subcase 1.1. $G \cong P_2$. Notice that Theorem 3.1 leads to $\gamma(H) \geq 2$. Suppose that $\gamma(H) \geq 3$ and let $V(G) = \{u, w\}$. Observe that $S \cap V(H_u) \neq \emptyset$ and $S \cap V(H_w) \neq \emptyset$. Without loss of generality, let $S \cap V(H_u) = \{(u, v_1), (u, v_2)\}$ and $|S \cap V(H_w)| = 1$. Since $\gamma(H) \geq 3$, we have that $\{v_1, v_2\}$ is not a dominating set of H , which implies that no vertex in $\{u\} \times (V(H) \setminus (N(v_1) \cup N(v_2)))$ has two neighbours in S , which is a contradiction. Hence $\gamma(H) = 2$. Therefore, (i) follows.

Subcase 1.2. $G \not\cong P_2$ has at least two universal vertices. In this case, $\gamma_{\times 2}(G) = 2$ and by Theorem 3.1 we deduce that $\gamma(H) \geq 2$. Thus, (ii) follows.

Subcase 1.3. G has exactly one universal vertex. If $\gamma(H) \leq 2$, then by Theorem 3.1 we deduce that either $\gamma(H) = 2$ or H has exactly one universal vertex, so that (iii) follows. Assume

that $\gamma(H) \geq 3$. Recall that $|S \cap V(H_x)| \leq 2$ for every $x \in V(G)$. Now, if there exist two vertices $u, w \in V(G)$ and two vertices $v_1, v_2 \in V(H)$ such that $S \cap V(H_u) = \{(u, v_1), (u, v_2)\}$ and $|S \cap V(H_w)| = 1$, then we deduce that no vertex in $\{u\} \times (V(H) \setminus (N(v_1) \cup N(v_2)))$ has two neighbours in S , which is a contradiction. Therefore, $\mathcal{A}_S = \emptyset$ and \mathcal{B}_S has to be a $\gamma_{2,t}(G)$ -set, as every vertex $x \in V(G)$ satisfies $|N(x) \cap \mathcal{B}_S| \geq 2$. Therefore, (iv) follows.

Case 2. $\gamma(G) = 2$. In this case, Theorem 1.4 leads to $\gamma_{\times 2}(G) \geq 3$. If there exist two vertices $u, w \in V(G)$ such that $\mathcal{A}_S = \{u\}$ and $\mathcal{B}_S = \{w\}$, then $\{u, w\}$ is a $\gamma_t(G)$ -set, and so for any $x \in N(w) \setminus N[u]$ we have that no vertex in $V(H_x)$ has two neighbours in S , which is a contradiction. Therefore, $\mathcal{A}_S = \emptyset$ and $|\mathcal{B}_S| = 3$, which implies that \mathcal{B}_S is a $\gamma_{\times 2}(G)$ -set. Notice that either $\langle \mathcal{B}_S \rangle \cong C_3$ or $\langle \mathcal{B}_S \rangle \cong P_3$. In the first case, \mathcal{B}_S is a $\gamma_{2,t}(G)$ -set and (v) follows. Now, assume that $\langle \mathcal{B}_S \rangle \cong P_3$. If $\gamma(H) \geq 2$, then for any vertex x of degree one in $\langle \mathcal{B}_S \rangle$ we have that $V(H_x)$ have vertices which do not have two neighbours in S , which is a contradiction. Therefore, $\gamma(H) = 1$ and if $\gamma_{\times 2}(G) = \gamma_{2,t}(G)$, then G satisfies (v), otherwise G satisfies (vi), by Theorem 2.11.

Conversely, notice that if G and H satisfy one of the six conditions above, then Theorem 3.1 leads to $\gamma_{\times 2}(G \circ H) \geq 3$. To conclude that $\gamma_{\times 2}(G \circ H) = 3$, we proceed to show how to define a double dominating set D of $G \circ H$ of cardinality three for each of the six conditions.

- (i) Let $\{v_1, v_2\}$ be a $\gamma(H)$ -set and $V(G) = \{u, w\}$. In this case, $D = \{(u, v_1), (u, v_2), (w, v_1)\}$.
- (ii) Let $u, w \in V(G)$ be two universal vertices, $z \in V(G) \setminus \{u, w\}$ and $v \in V(H)$. In this case, $D = \{(u, v), (w, v), (z, v)\}$.
- (iii) Let u be a universal vertex of G and $w \in V(G) \setminus \{u\}$. If $\{v_1, v_2\}$ is a $\gamma(H)$ -set or v_1 is a universal vertex of H and $v_2 \in V(H) \setminus \{v_1\}$, then we set $D = \{(u, v_1), (u, v_2), (w, v_1)\}$.
- (iv) Let X be a $\gamma_{2,t}(G)$ -set and $v \in V(H)$. In this case, $D = X \times \{v\}$.
- (v) Let X be a $\gamma_{2,t}(G)$ -set and $v \in V(H)$. In this case, $D = X \times \{v\}$.
- (vi) Let X be a $\gamma_{\times 2}(G)$ -set and v a universal vertex of H . In this case, $D = X \times \{v\}$.

It is readily seen that in all cases D is a double dominating set of $G \circ H$. Therefore, $\gamma_{\times 2}(G \circ H) = 3$. \square

The following result, which is a direct consequence of Theorems 2.2, 3.1 and 3.2, shows the cases when G is isomorphic to a complete graph or a star graph.

Proposition 3.3. *Let H be a nontrivial graph. For any integer $n \geq 3$, the following statements hold.*

- (i) $\gamma_{\times 2}(K_n \circ H) = \begin{cases} 2 & \text{if } \gamma(H) = 1, \\ 3 & \text{otherwise.} \end{cases}$
- (ii) $\gamma_{\times 2}(K_{1,n-1} \circ H) = \begin{cases} 2 & \text{if } \gamma_{\times 2}(H) = 2, \\ 3 & \text{if } \gamma_{\times 2}(H) \geq 3 \text{ and } \gamma(H) \leq 2, \\ 4 & \text{otherwise.} \end{cases}$

We now consider the cases in which G is a double star graph or a complete bipartite graph. The following result is a direct consequence of Theorems 2.2, 3.1 and 3.2.

Proposition 3.4. *Let H be a graph. For any integers $n_2 \geq n_1 \geq 2$, the following statements hold.*

- (i) $\gamma_{\times 2}(S_{n_1, n_2} \circ H) = 4$.
- (ii) $\gamma_{\times 2}(K_{n_1, n_2} \circ H) = \begin{cases} 3 & \text{if } n_1 = 2 \text{ and } \gamma(H) = 1; \\ 4 & \text{otherwise.} \end{cases}$

4 All cases where $G \cong P_n$ or $G \cong C_n$

4.1 Cases where $\gamma(H) = 1$

Proposition 4.1. *Let $n \geq 3$ be an integer and let H be a nontrivial graph. If $\gamma(H) = 1$, then*

$$\gamma_{\times 2}(P_n \circ H) = \begin{cases} 2 \lceil \frac{n}{3} \rceil + 1, & \text{if } \gamma_{\times 2}(H) \geq 3 \text{ and } n \equiv 0 \pmod{3}, \\ 2 \lceil \frac{n}{3} \rceil, & \text{otherwise.} \end{cases}$$

Proof. If $\gamma_{\times 2}(H) = 2$, then by Corollary 2.9 we deduce that $\gamma_{\times 2}(P_n \circ H) = 2\gamma(P_n)$. Now, if $\gamma_{\times 2}(H) \geq 3$, then H has exactly one universal vertex and by Theorem 2.7 (iii) we deduce that $\gamma_{\times 2}(G \circ H) = \gamma_{\{R2\}}(P_n)$. \square

From now on we assume that $V(C_n) = \{u_1, \dots, u_n\}$, where the subscripts are taken modulo n and consecutive vertices are adjacent.

Proposition 4.2. *Let $n \geq 3$ be an integer and let H be a graph. If $\gamma(H) = 1$, then*

$$\gamma_{\times 2}(C_n \circ H) = \left\lceil \frac{2n}{3} \right\rceil.$$

Proof. If H is a trivial graph, then we are done, by Remark 1.6. From now on we assume that H has at least two vertices. If $\gamma(H) = 1$, then by combining Theorem 2.7 (i) and Remark 1.6 (ii), we deduce that $\gamma_{\times 2}(C_n \circ H) \leq \lceil \frac{2n}{3} \rceil$.

Now, let S be a $\gamma_{\times 2}(C_n \circ H)$ -set. Notice that for any $i \in \{1, \dots, n\}$ we have that

$$\left| S \cap \left(\bigcup_{j=0}^2 V(H_{u_{i+j}}) \right) \right| \geq 2.$$

Hence,

$$3\gamma_{\times 2}(C_n \circ H) = 3|S| = \sum_{i=1}^n \left| S \cap \left(\bigcup_{j=0}^2 V(H_{u_{i+j}}) \right) \right| \geq 2n.$$

Therefore, $\gamma_{\times 2}(C_n \circ H) \geq \lceil \frac{2n}{3} \rceil$, and the result follows. \square

4.2 Cases where $\gamma(H) = 2$

To begin this subsection we need to state the following four lemmas.

Lemma 4.3. *Let G be a nontrivial connected graph and let H be a graph. The following statements hold for every $\gamma_{\times 2}(G \circ H)$ -set S that satisfies Lemma 2.6.*

(i) *If $\gamma(H) \geq 2$ and $x \in \mathcal{B}_S \cup \mathcal{C}_S$, then $\sum_{u \in N(x)} |S \cap V(H_u)| \geq 2$.*

(ii) *If $\gamma(H) = 2$ and $x \in \mathcal{A}_S$, then $\sum_{u \in N(x)} |S \cap V(H_u)| \geq 1$.*

(iii) *If $\gamma(H) \geq 3$ and $x \in V(G)$, then $\sum_{u \in N(x)} |S \cap V(H_u)| \geq 2$.*

Proof. First, we suppose that $\gamma(H) = 2$. If there exists either a vertex $x \in \mathcal{B}_S \cup \mathcal{C}_S$ such that $\sum_{u \in N(x)} |S \cap V(H_u)| \leq 1$ or a vertex $x \in \mathcal{A}_S$ such that $\sum_{u \in N(x)} |S \cap V(H_u)| = 0$, then there exists a vertex in $V(H_x) \setminus S$ which does not have two neighbours in S . Therefore, (ii) follows, and (i) follows for $\gamma(H) = 2$. Now, let $x \in V(G)$. Since S satisfies Lemma 2.6, if $\gamma(H) \geq 3$, then there exists a vertex in $V(H_x) \setminus S$ which does not have neighbours in $S \cap V(H_x)$, which implies that $\sum_{u \in N(x)} |S \cap V(H_u)| \geq 2$ and so (i) and (iii) follows. Therefore, the proof is complete. \square

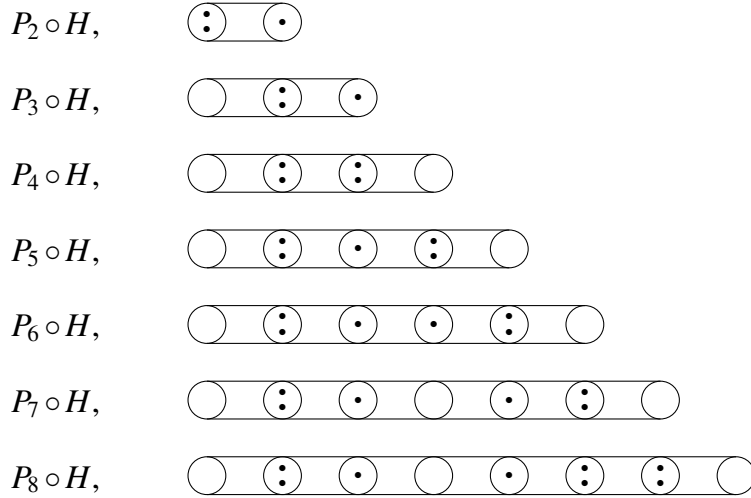


Figure 3: The scheme used in the proof of Lemma 4.4.

Lemma 4.4. *For any integer $n \geq 3$ and any graph H with $\gamma(H) = 2$,*

$$\gamma_{\times 2}(P_n \circ H) \leq \begin{cases} n - \lfloor \frac{n}{7} \rfloor + 1 & \text{if } n \equiv 1, 2 \pmod{7}, \\ n - \lfloor \frac{n}{7} \rfloor & \text{otherwise.} \end{cases}$$

Proof. In Figure 3 we show how to construct a double dominating set S of $P_n \circ H$ for $n \in \{2, \dots, 8\}$. In this scheme, the circles represent the copies of H in $P_n \circ H$, two dots in a circle represent two vertices belonging to S , which form a dominating set of the corresponding copy of H , while a single dot in a circle represents one vertex belonging to S .

We now proceed to describe the construction of S for any $n = 7q + r$, where $q \geq 1$ and $0 \leq r \leq 6$. We partition $V(P_n) = \{u_1, \dots, u_n\}$ into q sets of cardinality 7 and for $r \geq 1$ one additional set of cardinality r , in such a way that the subgraph induced by all these sets are paths. For any $r \neq 1$, the restriction of S to each of these q paths of length 7 corresponds to the scheme associated with $P_7 \circ H$ in Figure 3, while for the path of length r (if any) we take the scheme associated with $P_r \circ H$. The case $r = 1$ and $q \geq 2$ is slightly different, as for the first $q - 1$ paths of length 7 we take the scheme associated with $P_7 \circ H$ and for the path associated with the last 8 vertices of P_n we take the scheme associated with $P_8 \circ H$.

Notice that, for $n \equiv 1, 2 \pmod{7}$, we have that $\gamma_{\times 2}(P_n \circ H) \leq |S| = 6q + r + 1 = n - \lfloor \frac{n}{7} \rfloor + 1$, while for $n \not\equiv 1, 2 \pmod{7}$ we have $\gamma_{\times 2}(P_n \circ H) \leq |S| = 6q + r = n - \lfloor \frac{n}{7} \rfloor$. Therefore, the result follows. \square

Lemma 4.5. *Let $P_7 = w_1, \dots, w_7$ be a subgraph of C_n . Let H be a graph such that $\gamma(H) = 2$ and $W = \{w_1, \dots, w_7\} \times V(H)$. If S is a double dominating set of $C_n \circ H$ which satisfies Lemma 2.6, then*

$$|S \cap W| \geq 6.$$

Proof. By Lemma 4.3 (i) and (ii) we have that $|S \cap (\{w_1, w_2, w_3\} \times V(H))| \geq 2$ and $|S \cap (\{w_4, w_5, w_6, w_7\} \times V(H))| \geq 3$. If $|S \cap (\{w_1, w_2, w_3\} \times V(H))| \geq 3$, then we are done. Hence, we assume that $|S \cap (\{w_1, w_2, w_3\} \times V(H))| = 2$. In this case, and by applying again Lemma 4.3 (i) and (ii) we deduce that $|S \cap (\{w_4, w_5, w_6, w_7\} \times V(H))| \geq 4$, which implies that $|S \cap W| \geq 6$, as desired. Therefore, the proof is complete. \square

Lemma 4.6. *For any integer $n \geq 3$ and any graph H with $\gamma(H) = 2$,*

$$\gamma_{\times 2}(C_n \circ H) \geq \begin{cases} n - \lfloor \frac{n}{7} \rfloor + 1 & \text{if } n \equiv 1, 2 \pmod{7}, \\ n - \lfloor \frac{n}{7} \rfloor & \text{otherwise.} \end{cases}$$

Proof. It is easy to check that $\gamma_{\times 2}(C_n \circ H) = n$ for every $n \in \{3, 4, 5, 6\}$. Now, let $n = 7q + r$, with $0 \leq r \leq 6$ and $q \geq 1$. Let S be a $\gamma_{\times 2}(C_n \circ H)$ -set that satisfies Lemma 2.6.

If $r = 0$, then by Lemma 4.5 we have that $|S| \geq 6q = n - \lfloor \frac{n}{7} \rfloor$. From now on we assume that $r \geq 1$. By Theorem 1.5 and Lemma 4.4 we deduce that $\gamma_{\times 2}(C_n \circ H) \leq \gamma_{\times 2}(P_n \circ H) < n$, which implies that $\mathcal{A}_S \neq \emptyset$, otherwise there exists $u \in V(C_n)$ such that $N(u) \cap \mathcal{C}_S \neq \emptyset$ and so $|N(u) \cap \mathcal{B}_S| \leq 1$, which is a contradiction. Let $x \in \mathcal{A}_S$ and, without loss of generality, we can label the vertices of C_n in such a way that $x = u_1$, and $u_2 \in \mathcal{A}_S \cup \mathcal{B}_S$ whenever $r \geq 2$. We partition $V(C_n)$ into $X = \{u_1, \dots, u_r\}$ and $Y = \{u_{r+1}, \dots, u_n\}$. Notice that Lemma 4.5 leads to $|S \cap (Y \times V(H))| \geq 6q$.

Now, if $r \in \{1, 2\}$, then $|S \cap (X \times V(H))| \geq r + 1$, which implies that $|S| \geq r + 1 + 6q = n - \lfloor \frac{n}{7} \rfloor + 1$. Analogously, if $r = 3$, then $|S \cap (X \times V(H))| \geq r$ and so $|S| \geq r + 6q = n - \lfloor \frac{n}{7} \rfloor$.

Finally, if $r \in \{4, 5, 6\}$, then by Lemma 4.3 (i) and (ii) we deduce that $|S \cap (X \times V(H))| \geq r$, which implies that $|S| \geq r + 6q = n - \lfloor \frac{n}{7} \rfloor$. \square

The following result is a direct consequence of Theorem 1.5 and Lemmas 4.4 and 4.6.

Proposition 4.7. For any integer $n \geq 3$ and any graph H with $\gamma(H) = 2$,

$$\gamma_{\times 2}(C_n \circ H) = \gamma_{\times 2}(P_n \circ H) = \begin{cases} n - \lfloor \frac{n}{7} \rfloor + 1 & \text{if } n \equiv 1, 2 \pmod{7}, \\ n - \lfloor \frac{n}{7} \rfloor & \text{otherwise.} \end{cases}$$

4.3 Cases where $\gamma(H) \geq 3$

To begin this subsection we need to recall the following well-known result.

Remark 4.8. [21] For any integer $n \geq 3$,

$$\gamma_t(P_n) = \gamma_t(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{4}, \\ \frac{n+1}{2} & \text{if } n \equiv 1, 3 \pmod{4}, \\ \frac{n}{2} + 1 & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

Lemma 4.9. Let $P_n = u_1 u_2 \dots u_n$ be a path of order $n \geq 6$, where consecutive vertices are adjacent, and let H be a graph. If $\gamma(H) \geq 3$, then there exists a $\gamma_{\times 2}(P_n \circ H)$ -set S such that $u_n, u_{n-3} \in \mathcal{C}_S$ and $u_{n-1}, u_{n-2} \in \mathcal{A}_S$.

Proof. Let S be a $\gamma_{\times 2}(P_n \circ H)$ -set that satisfies Lemma 2.6 such that $|\mathcal{A}_S|$ is maximum. First, we observe that $u_{n-1} \in \mathcal{A}_S$ by Lemma 4.3. Now, by applying again Lemma 4.3, we have that $|S \cap V(H_{u_n})| + |S \cap V(H_{u_{n-2}})| \geq 2$. Hence, without loss of generality we can assume that $u_{n-2} \in \mathcal{A}_S$ and $u_n \in \mathcal{C}_S$ as $|\mathcal{A}_S|$ is maximum. If $u_{n-3} \in \mathcal{C}_S$, then we are done. On the other hand, if $u_{n-3} \notin \mathcal{C}_S$, then as every vertex of $V(H_{u_{n-3}})$ has two neighbours in $S \cap V(H_{u_{n-2}})$, we can redefine S by replacing the vertices in $S \cap V(H_{u_{n-3}})$ with vertices in $V(H_{u_{n-4}}) \cup V(H_{u_{n-5}})$ and obtain a new $\gamma_{\times 2}(P_n \circ H)$ -set S satisfying that $u_{n-3} \in \mathcal{C}_S$, as desired. Therefore, the result follows. \square

Proposition 4.10. Let $n \geq 3$ be an integer and let H be a graph. If $\gamma(H) \geq 3$, then

$$\gamma_{\times 2}(P_n \circ H) = 2\gamma_t(P_n) = \begin{cases} n & \text{if } n \equiv 0 \pmod{4}, \\ n+1 & \text{if } n \equiv 1, 3 \pmod{4}, \\ n+2 & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

Proof. Since Proposition 1.1 leads to $\gamma_{\times 2}(P_n \circ H) \leq 2\gamma_t(P_n)$, we only need to prove that $\gamma_{\times 2}(P_n \circ H) \geq 2\gamma_t(P_n)$. We proceed by induction on n . By Propositions 3.3 and 3.4 we obtain that $\gamma_{\times 2}(P_n \circ H) = 2\gamma_t(P_n)$ for $n = 3, 4$. By Lemma 4.3 it is easy to see that $\gamma_{\times 2}(P_5 \circ H) = 2\gamma_t(P_5)$. This establishes the base case. Now, we assume that $n \geq 6$ and that $\gamma_{\times 2}(P_k \circ H) \geq 2\gamma_t(P_k)$ for $k < n$. Let S be a $\gamma_{\times 2}(P_n \circ H)$ -set that satisfies Lemma 4.9. Let $D = V(P_n \circ H) \setminus (\cup_{i=0}^3 V(H_{u_{n-i}}))$. Notice that $S \cap D$ is a double dominating set of $(P_n \circ H) - D \cong P_{n-4} \circ H$. Hence, by applying the induction hypothesis,

$$\gamma_{\times 2}(P_n \circ H) \geq \gamma_{\times 2}(P_{n-4} \circ H) + 4 \geq 2\gamma_t(P_{n-4}) + 4 \geq 2\gamma_t(P_n),$$

as desired. To conclude the proof we apply Remark 4.8. \square

Proposition 4.11. *Let $n \geq 3$ be an integer and let H be a graph. If $\gamma(H) \geq 3$, then*

$$\gamma_{\times 2}(C_n \circ H) = n.$$

Proof. From Theorem 2.11 we know that $\gamma_{\times 2}(C_n \circ H) \leq n$. We only need to prove that $\gamma_{\times 2}(C_n \circ H) \geq n$. Let S be a $\gamma_{\times 2}(G \circ H)$ -set that satisfies Lemma 2.6. Since $\gamma(H) \geq 3$, by Lemma 4.3 (iii) we deduce that

$$2\gamma_{\times 2}(C_n \circ H) = 2|S| = \sum_{x \in V(C_n)} \sum_{u \in N(x)} |S \cap V(H_u)| \geq 2n.$$

Therefore, the result follows. □

References

- [1] H. Abdollahzadeh Ahangar, M. A. Henning, V. Samodivkin, I. G. Yero, Total Roman domination in graphs, *Appl. Anal. Discrete Math.* **10** (2016) 501–517.
- [2] J. Amjadi, S. M. Sheikholeslami, M. Soroudi, On the total Roman domination in trees, *Discuss. Math. Graph Theory* **39** (2019) 519–532.
- [3] B. H. Arriola, S. R. Canoy, Jr., Doubly connected domination in the corona and lexicographic product of graphs, *Appl. Math. Sci.* **8** (29-32) (2014) 1521–1533.
- [4] M. Blidia, M. Chellali, T. W. Haynes, M. A. Henning, Independent and double domination in trees, *Util. Math.* **70** (2006) 159–173.
- [5] S. Brezovnik, T.K. Šumenjak, Complexity of k-rainbow independent domination and some results on the lexicographic product of graphs. *Appl. Math. Comput.* **349** (2019) 214–220.
- [6] S. Cabrera García, A. Cabrera Martínez, F. A. Hernández Mira, I.G. Yero, Total Roman $\{2\}$ -domination in graphs, *Quaestiones Mathematicae*. In press with DOI: 10.2989/16073606.2019.1695230
- [7] A. Cabrera Martínez, S. Cabrera García, A. Carrión García, Further results on the total Roman domination in graphs, *Mathematics* **8(3)** (2020) 349.
- [8] A. Cabrera Martínez and J.A. Rodríguez-Velázquez, Total protection of lexicographic product graphs. Submitted.
- [9] N. Campanelli, D. Kuziak, Total Roman domination in the lexicographic product of graphs, *Discrete Appl. Math.* **263** (2019) 88–95.
- [10] M. Chellali, T. Haynes, On paired and double domination in graphs, *Util. Math.* **67** (2005) 161–171.
- [11] E. J. Cockayne, R. M. Dawes, S. T. Hedetniemi, Total domination in graphs, *Networks* **10** (3) (1980) 211–219.

- [12] A. M. Cuivillas, S. R., Jr. Canoy, Double domination in graphs under some binary operations, *Appl. Math. Sci.* **8** (2014) 2015–2024.
- [13] M. Dettlaff, M. Lemańska, J. A. Rodríguez-Velázquez, R. Zuazua, On the super domination number of lexicographic product graphs, *Discrete Appl. Math.* **263** (2019) 118–129.
- [14] P. Dorbec, M. Mollard, S. Klavžar, S. Špacapan, Power domination in product graphs, *SIAM J. Discrete Math.* **22** (2) (2008) 554–567.
- [15] M. Hajiana, N. J. Rad, A new lower bound on the double domination number of a graph, *Discrete Appl. Math.* **254** (2019) 280–282.
- [16] R. Hammack, W. Imrich, S. Klavzar, Handbook of product graphs, Discrete Mathematics and its Applications, 2nd ed., CRC Press, 2011.
- [17] J. Harant, M. A. Henning, On double domination in graphs, *Discuss. Math. Graph Theory* **25** (2005) 29–34.
- [18] F. Harary, T.W. Haynes, Double domination in graphs, *Ars. Combin.* **55** (2000) 201–213.
- [19] T. Haynes, S. Hedetniemi, P. Slater, Domination in Graphs: Volume 2: Advanced Topics, Chapman & Hall/CRC Pure and Applied Mathematics, Taylor & Francis, 1998.
- [20] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, Fundamentals of Domination in Graphs, Chapman and Hall/CRC Pure and Applied Mathematics Series, Marcel Dekker, Inc. New York, 1998.
- [21] M.A. Henning, A. Yeo, Total domination in graphs. Springer, New York, USA, 2013.
- [22] W. Imrich, S. Klavzar, Product graphs, structure and recognition, Wiley-Interscience series in discrete mathematics and optimization, Wiley, 2000.
- [23] S. Khelifi, M. Chellali, Double domination critical and stable graphs upon vertex removal, *Discuss. Math. Graph Theory* **32** (4) (2012) 643–657.
- [24] C.-H. Liu, G. J. Chang, Roman domination on strongly chordal graphs. *J. Comb. Optim.* **26** (2013) 608–619.
- [25] J. Liu, X. Zhang, J. Meng, Domination in lexicographic product digraphs, *Ars Combin.* **120** (2015) 23–32.
- [26] H. Ma, J. Liu, The twin domination number of lexicographic product of digraphs. *J. Nat. Sci. Hunan Norm. Univ.* **39** (6) (2016) 80–84.
- [27] R. J. Nowakowski, D. F. Rall, Associative graph products and their independence, domination and coloring numbers, *Discuss. Math. Graph Theory* **16** (1996) 53–79.
- [28] T. K. Šumenjak, P. Pavlič, A. Tepeh, On the Roman domination in the lexicographic product of graphs, *Discrete Appl. Math.* **160** (13) (2012) 2030–2036.

- [29] T. K. Šumenjak, D. F. Rall, A. Tepeh, Rainbow domination in the lexicographic product of graphs, *Discrete Appl. Math.* **161** (13-14) (2013) 2133–2141.
- [30] M. Valveny, H. Pérez-Rosés, J. A. Rodríguez-Velázquez, On the weak Roman domination number of lexicographic product graphs, *Discrete Appl. Math.* **263** (2019) 257–270.
- [31] X. Zhang, J. Liu, J. Meng, Domination in lexicographic product graphs, *Ars Combin.* **101** (2011) 251–256.