The k-metric dimension of a graph

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1 Introduction

The problem of uniquely determining the location of an intruder in a network was the principal motivation of introducing the concept of metric dimension in graphs by Slater in [9,10], where the metric generators were called *locating sets*. The concept of metric dimension of a graph was also introduced independently by Harary and Melter in [5], where metric generators were called *resolving sets*.

Nevertheless, the concept of metric generator, in its primary version, has a weakness related with the possible uniqueness of the vertex identifying a pair of vertices of the graph. Consider, for instance, some robots which are navigating, moving from node to node of a network. On a graph, however, there is neither the concept of direction nor that of visibility. We assume that robots have communication with a set of landmarks S (a subset of nodes) which provide them the distance to the landmarks in order to facilitate the navigation. In this sense, one aim is that each robot is uniquely determined by the landmarks. Suppose that in a specific moment there are two robots x, y (are located in v_{11} and v_{12} in the example of Figure 1 (a)) whose positions are only distinguished by one landmark $s \in S$ ($s = v_{13}$ and $S = \{v_{13}, v_{21}, v_{22}\}$ in the example of Figure 1 (a)). If the communication between robots x, y and s is "unexpectedly blocked", then the robots will get "lost'" in the sense that they can assume that they have the same position as we show in the example of Figure 1 (b). So, for a more realistic settings it could be desirable to consider a set of landmarks where each pair of nodes is distinguished by at least two landmarks, such that the set of landmarks be as "small" as possible.

A generator of a metric space is a set S of points in the space with the property that every point of the space is uniquely determined by its distances from the elements of S. Given a simple graph $G = (V, E)$ we consider the metric $d_G : V \times V \to \mathbb{N}$, where $d_G(x, y)$ represents the length of a shortest $a - b$ path. The pair (V, d_G) is clearly a metric space. It is said that a vertex

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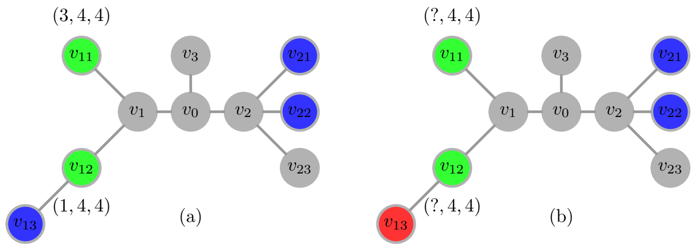

Fig. 1: Example of weakness of metric generator in robot navigation in networks.

$v \in V$ distinguish two different vertices $x, y \in V$, if $d_G(v, x) \neq d_G(v, y)$. A set $S \subseteq V$ is a *metric generator* for G if any pair of vertices of G is distinguished by some element of S. Such a name for S raises from the concept of *generator* of metric spaces, since a metric generator S has the property that every point of the space is uniquely determined by its "distances" from the elements of S. A metric generator of minimum cardinality is called a *metric basis*, and its cardinality the *metric dimension* of G, denoted by $\dim(G)$.

In this investigation we consider an extension of the concept of metric generators in the following way. Given a simple and connected graph $G = (V, E)$, a set $S \subseteq V$ is said to be a k-*metric generator* for G if and only if any pair of vertices of G is distinguished by at least k elements of S, *i.e.*, for any pair of different vertices $u, v \in V$, there exist at least k vertices $w_1, w_2, ..., w_k \in S$ such that

$$d_G(u, w_i) \neq d_G(v, w_i), \text{ for every } i \in \{1, ..., k\}. \quad (1)$$

A k-metric generator of minimum cardinality in G will be called a k-*metric basis* and its cardinality the k-*metric dimension* of G, which will be denoted by $\dim_k(G)$.

As an example we take a graph G obtained from the cycle graph C_5 and the path P_t of order $t \geq 2$, by identifying one of the vertices of the cycle, say u_1, and one of the extremes of P_t, as we show in Figure 2. Let $S_1 = \{v_1, v_2\}$, $S_2 = \{v_1, v_2, u_t\}$, $S_3 = \{v_1, v_2, v_3, u_t\}$ and $S_4 = \{v_1, v_2, v_3, v_4, u_t\}$. For $k \in \{1, 2, 3, 4\}$ the set S_k is k-metric basis of G.

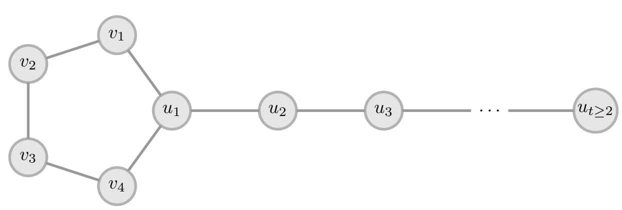

Fig. 2: For $k \in \{1, 2, 3, 4\}$, $\dim_k(G) = k + 1$.

The literature about metric dimension in graphs shows applications to the robot navigation in networks discussed in [7] and applications to chemistry in [6]. Several variations of metric generators including resolving dominating sets [1], independent resolving sets [2], and local metric sets [8] have been introduced and studied. It is therefore our goal to introduce this extension of metric generators in graphs as a possible future tool for other possible more general variations of the applications described above.

2 Objectives and results achieved

In this research we extend the concepts of metric basis and metric dimension of a graph and we study several problems related to k-metric basis and the k-metric dimension of a graph.

The first research line is to find the values of k for which there exists a k-metric generator in a connected graph G and the second research line is to calculate the value of the k-metric dimension of arbitrary graphs.

Given a connected graph G, the first problem is to determine if there exists an integer t such that G does not contain any k-metric generator for every $k > t$. According to that fact, a connected graph G is said to be a *k-metric dimensional graph*, if k is the largest integer such that there exists a k-metric basis for G.

In [11] we showed an algorithm that compute the integer k for which a graph of order n is k-metric dimensional in a time complexity of $O(n^3)$.

Since the problem of computing the value k' for which a given graph is k'-metric dimensional is polynomial, we studied in [11] the problem of deciding whether the k-metric dimension, $k \leq k'$, of G is less than r, for some $r \geq k+1$, *i.e.*, the following decision problem.

k-METRIC DIMENSION PROBLEM
INSTANCE: A connected k'-metric dimensional graph G of order n and integers k, r such that $k \leq k'$ and $r \geq k+1$.
PROBLEM: Deciding whether the k-metric dimension of G is less than r.

We proved that the k-METRIC DIMENSION PROBLEM is NP-complete by a reduction of our problem to 3-SAT problem. Taking into account this, we have obtained closed formulae and tight bounds for the k-metric dimension of some k'-metric dimensional graphs, where $1 \leq k \leq k'$. Many of these formulas and bounds were calculated from known parameters of graphs. In addition, we have given a result in [3] that allowed us to present an algorithm for a tree different from path in lineal time in [11]. In future works we will continue to obtain closed formulae and tight bounds for the k-metric dimension of arbitrary graphs. Also, we will analyse the complexity time of computing the k-metric dimension of specific family of graphs and we will intend to develop heuristics that allow us to compute or accurately estimate the k-metric dimension of some graphs.

Graphs are basic combinatorial structures, and product of graphs occur naturally in discrete mathematics as tools in combinatorial constructions. They give rise to important classes of graphs and deep structural problems. Thus, we are interested in the study of relationships between the k-metric dimension of Cartesian, strong, corona and rooted product graphs and the k-metric dimension of its corresponding factors. Recently, we have studied the particular case of Corona product of graphs in [4].

Finally we want to study some variants developed for the metric dimension in the k-metric dimension of graphs. Such would be the case of k-independent resolving sets, k-resolving dominating sets and local k-metric dimension.

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