On the perfect differential of a graph

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Abstract

Let G be a graph of order $\mathrm{n}(G)$ and vertex set $V(G)$. Given a set $S \subseteq V(G)$, we define the perfect neighbourhood of S as the set $N_p(S)$ of all vertices in $V(G) \setminus S$ having exactly one neighbour in S. The perfect differential of S is defined to be $\partial_p(S) = |N_p(S)| - |S|$. In this paper, we introduce the study of the perfect differential of a graph, which we define as $\partial_p(G) = \max\{\partial_p(S) : S \subseteq V(G)\}$. Among other results, we obtain general bounds on $\partial_p(G)$ and we prove a Gallai-type theorem, which states that $\partial_p(G) + \gamma_R^p(G) = \mathrm{n}(G)$, where $\gamma_R^p(G)$ denotes the perfect Roman domination number of G. As a consequence of the study, we show some classes of graphs satisfying a conjecture stated by Bermudo [Discrete Appl. Math. 232 (2017) 64-72].

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1. Introduction

The *open neighbourhood* of a vertex v of a graph G is defined to be $N(v) = \{u \in V(G) : u \text{ is adjacent to } v\}$. The *open neighbourhood of a set* $S \subseteq V(G)$ is defined as $N(S) = \cup_{v \in S} N(v)$, while the *external neighbourhood* of S, or boundary of S, is defined as $N_e(S) = N(S) \setminus S$. The *differential of a set* $S \subseteq V(G)$ is defined as $\partial(S) = |N_e(S)| - |S|$, while the *differential of a graph* G is defined to be

$$\partial(G) = \max\{\partial(S) : S \subseteq V(G)\}.$$

As described in [25], the definition of $\partial(G)$ was given by Hedetniemi about twenty-five years ago in an unpublished paper, and was also considered by Goddard and Henning [15]. After that, the differential of a graph has been studied by several authors, including [2, 3, 4, 25, 28, 31]

The *perfect neighbourhood of a set* $S \subseteq V(G)$ is defined to be $N_p(S) = \{v \in V(G) \setminus S : |N(v) \cap S| = 1\}$. We define the *perfect differential of a set* $S \subseteq V(G)$ as $\partial_p(S) = |N_p(S)| - |S|$. In this paper we introduce the study of the *perfect differential of a graph*, which we define as

$$\partial_p(G) = \max\{\partial_p(S) : S \subseteq V(G)\}.$$

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Among other results, we obtain general bounds on $\partial_p(G)$ and we prove a Gallai-type theorem, which states that $\partial_p(G)+\gamma_R^p(G)=\mathrm{n}(G)$, where $\gamma_R^p G)$ denotes the perfect Roman domination number of G. Therefore, we can see the theory of perfect differential in graphs as a new approach to the theory of perfect Roman domination. One of the advantages of this approach is that it allows us to study the perfect Roman domination number of a graph without the use of functions.

The remainder of the paper is organized as follows. In Section 2 we introduce some notation and tools needed to develop the remaining sections. Section 3 is devoted to provide general bounds on the perfect differential. We show that the bounds are tight and, in some cases, we characterize the graphs achieving the bounds. In Section 4 we prove the above mentioned Gallai-type theorem which states that $\partial_p(G)+\gamma_R^p(G)=\mathrm{n}(G)$. We derive some consequences of this result, including the fact that the problem of finding $\partial_p(G)$ is NP-hard. In Section 5 we obtain some results on the perfect differential of a graph and its complement and, as a consequence, we obtain some Nordhaus-Gaddum type relations. Finally, in Section 6 we discuss a conjecture given by Bermudo [2] which states that $\partial(G)\geq\gamma(G)$ for every graph of minimum degree at least three, where $\gamma(G)$ denotes the domination number of G. We prove the conjecture for many classes of graphs, concluding that the conjecture remains open only for some cases of graphs of order $\mathrm{n}(G)\geq 24$ and minimum degree three or four.

2. Notation, terminology and basic tools

Throughout the paper, we will use the notation $G\cong H$ if G and H are isomorphic graphs. The *closed neighbourhood* of a vertex v is defined as $N[v]=N(v)\cup\{v\}$. Given a set $S\subseteq V(G)$ and a vertex $v\in S$, the *external private neighbourhood* $\mathrm{epn}(v,S)$ of v with respect to S is defined to be $\mathrm{epn}(v,S)=\{u\in V(G)\setminus S: N(u)\cap S=\{v\}\}$.

We denote by $\deg(v)=|N(v)|$ the degree of vertex v, as well as $\delta(G)=\min_{v\in V(G)}\{\deg(v)\}$ the minimum degree of G, $\Delta(G)=\max_{v\in V(G)}\{\deg(v)\}$ the maximum degree of G and $\mathrm{n}(G)=|V(G)|$ the order of G. Given a set $S\subseteq V(G)$, $N[S]=N(S)\cup S$ and the subgraph of G induced by S will be denoted by $G[S]$.

A set $S\subseteq V(G)$ of vertices is a *dominating set* if $N(v)\cap S\neq\varnothing$ for every vertex $v\in V(G)\setminus S$. Let $\mathscr{D}(G)$ be the set of dominating sets of G. The *domination number* of G is defined to be,

$$\gamma(G)=\min\{|S|: S\in\mathscr{D}(G)\}.$$

Now, $S\subseteq V(G)$ is a *perfect dominating set* of G if every vertex in $V(G)\setminus S$ is adjacent to exactly one vertex in S. Let $\mathscr{D}^p(G)$ be the set of perfect dominating sets of G. The *perfect domination number* of G is defined to be,

$$\gamma^p(G)=\min\{|S|: S\in\mathscr{D}^p(G)\}.$$

Notice that $\mathscr{D}^p(G)\subseteq\mathscr{D}(G)$, which implies that $\gamma(G)\leq\gamma^p(G)$. We define a $\gamma^p(G)$-set as a set $S\in\mathscr{D}^p(G)$ with $|S|=\gamma^p(G)$. The same agreement will be assumed for optimal parameters associated to other characteristic sets defined in the paper. For instance, a $\gamma(G)$-set will be a set $S\in\mathscr{D}(G)$ with $|S|=\gamma(G)$, while a $\partial_p(G)$-set will be a set $S\subseteq V(G)$ such that $\partial_p(S)=\partial_p(G)$.

The domination number has been extensively studied. For instance, we cite the following books, [16, 17]. The theory of perfect domination was introduced by Livingston and Stout in [24] and has been studied by several authors, including [7, 9, 11, 12, 21, 23].

A set $S\subseteq V(G)$ is a *total dominating set* of a graph G without isolated vertices if every vertex $v\in V(G)$ is adjacent to at least one vertex in S. Let $\mathscr{D}_t(G)$ be the set of total dominating sets of

G. The theory of total domination has been extensively studied. For instance, we cite the book [20]. The *total domination number* of G is defined to be,

$$\gamma_t(G) = \min\{|S| : S \in \mathscr{D}_t(G)\}.$$

By definition, $\mathscr{D}_t(G) \subseteq \mathscr{D}(G)$, so that $\gamma(G) \leq \gamma_t(G)$. Furthermore, $\gamma_t(G) \leq 2\gamma(G)$.

A set $S \subseteq V(G)$ is a *packing* if $N[u] \cap N[v] = \varnothing$ for every pair of different vertices $u, v \in S$. We define

$$\wp(G) = \{S \subseteq V(G) : S \text{ is a packing of } G\}.$$

The *packing number* of G, denoted by $\rho(G)$, is defined to be

$$\rho(G) = \max\{|S| : S \in \wp(G)\}.$$

Obviously, $\gamma(G) \geq \rho(G)$. Notice that $\mathscr{D}(G) \cap \wp(G) \neq \varnothing$ if and only if there exists a $\gamma(G)$-set which is a $\rho(G)$-set. Furthermore, Meir and Moon [26] showed that $\gamma(T) = \rho(T)$ for every tree T. Even so, there are cases in which $\mathscr{D}(T) \cap \wp(T) = \varnothing$.

A set $S \subseteq V(G)$ is an *open packing*, if $N(u) \cap N(v) = \varnothing$ for every pair of different vertices $u, v \in S$. We define

$$\wp_o(G) = \{S \subseteq V(G) : S \text{ is an open packing of } G\}.$$

The *open packing number* of G, denoted by $\rho_o(G)$, is defined to be

$$\rho_o(G) = \max\{|S| : S \in \wp_o(G)\}.$$

Since $\wp(G) \subseteq \wp_o(G)$, we have that $\rho(G) \leq \rho_o(G)$ for every graph G. In addition, $\rho_o(G) \leq \gamma_t(G)$ for every graph G without isolated vertices. Now, if $S \in \wp_o(G)$, then every vertex of $G[S]$ has degree at most one, which implies that $S = S_0 \cup S_1$, where S_0 is the set of isolated vertices of $G[S]$ and $S_1 = S \setminus S_0$. Clearly, $S_1 = \varnothing$ if and only if $S \in \wp(G)$.

A graph G is an *efficient open domination graph* if there exists a set S, called an *efficient open dominating set*, for which $V(G) = \cup_{u \in S} N(u)$ and $N(u) \cap N(v) = \varnothing$ for every pair of distinct vertices $u, v \in S$. As shown in [22], if G is an efficient open domination graph with an efficient open dominating set S, then $\gamma_t(G) = |S|$. Hence, the following remark holds.

Remark 2.1. *A graph G is an efficient open domination graph if and only if there exists $S \in \mathscr{D}^p(G)$ such that $G[S] \cong \cup K_2$. In such a case, $|S| = \gamma_t(G) = \rho_o(G)$.*

Corollary 2.2. *If G is an efficient open domination graph, then $\gamma^p(G) \leq \gamma_t(G)$.*

A set S of vertices of G is a *vertex cover* if every edge of G is incident with at least one vertex in S. The *vertex cover number* of G, denoted by $\beta(G)$, is the minimum cardinality among all vertex covers of G. Recall that the largest cardinality of a set of vertices of G, no two of which are adjacent, is called the *independence number* of G and it is denoted by $\alpha(G)$. The following well-known result, due to Gallai, states the relationship between the independence number and the vertex cover number of a graph.

Theorem 2.3. [13](Gallai's theorem) *For any graph G,*

$$\alpha(G) + \beta(G) = \mathrm{n}(G).$$

Cockayne, Hedetniemi and Hedetniemi [8] defined a *Roman dominating function*, abbreviated RDF, on a graph G to be a function $f : V(G) \longrightarrow \{0,1,2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of f is defined to be

$$\omega(f) = \sum_{v \in V(G)} f(v).$$

For $X \subseteq V(G)$ we define the weight of X as $f(X) = \sum_{v \in X} f(v)$. The *Roman domination number*, denoted by $\gamma_R(G)$, is the minimum weight among all RDFs on G, *i.e.*,

$$\gamma_R(G) = \min\{\omega(f) : f \text{ is an RDF on } G\}.$$

An RDF of weight $\gamma_R(G)$ is called a $\gamma_R(G)$-function. Obviously, $\gamma_R(G) \leq 2\gamma(G)$ for every graph G. A *Roman graph* is a graph G with $\gamma_R(G) = 2\gamma(G)$.

Recently, a perfect version of Roman domination was introduced by Henning, Klostermeyer and MacGillivray [19]. They defined a *perfect Roman dominating function*, abbreviated PRDF, as an RDF f satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to exactly one vertex v for which $f(v) = 2$. The *perfect Roman domination number*, denoted by $\gamma_R^p(G)$, is the minimum weight among all perfect Roman dominating functions on G, *i.e.*,

$$\gamma_R^p(G) = \min\{\omega(f) : f \text{ is a PRDF on } G\}.$$

For recent results on perfect Roman domination in graphs we cite [1, 10, 18, 32].

A PRDF of weight $\gamma_R^p(G)$ is called a $\gamma_R^p(G)$-function. Obviously, $\gamma_R(G) \leq \gamma_R^p(G) \leq 2\gamma^p(G)$ for every graph G, and those graphs attaining the equality $\gamma_R^p(G) = 2\gamma^p(G)$ are called *perfect Roman graphs*. All perfect Roman trees were characterized in [30].

Figure 1 shows two copies of a graph G with $\gamma_R(G) = \gamma_R^p(G) = 4$. Notice that this graph is a perfect Roman tree and the labellings correspond to the positive weights of all $\gamma_R^p(G)$-functions.

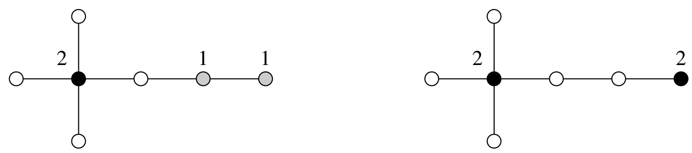

Figure 1: The labellings associated to the positive weights of all $\gamma_R^p(G)$-functions on the same graph.

Figure 2 shows a Roman graph G, namely, $\gamma_R(G) = 6 = 2\gamma(G)$. In this case, $\gamma^p(G) = 6$ and $\gamma_R^p(G) = 9$. The set of vertices of degree seven forms a $\partial_p(G)$-set and $\partial_p(G) = 6 < 9 = \partial(G)$.

We assume that the reader is familiar with the basic concepts, notation and terminology of domination in graph. If this is not the case, we suggest the textbooks [16, 17]. For the remainder of the paper, definitions will be introduced whenever a concept is needed.

3. General bounds

We next present tight bounds on the perfect differential of a graph. In some cases we provide classes of graphs achieving the bounds, while in other cases we characterize the graphs reaching the equalities.

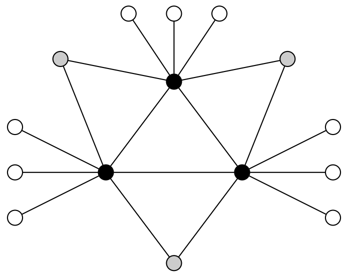

Figure 2: The set of (gray and black) coloured vertices forms a $\gamma^p(G)$-set and the set of black-coloured vertices forms a $\beta(G)$-set.

Proposition 3.1. *Given a nontrivial graph G, the following inequality chain holds,*

$$\mathrm{n}(G)-\gamma^p(G)-\min\{\beta(G),\gamma^p(G)\} \leq \partial_p(G) \leq \partial(G) \overset{[5]}{\leq} \frac{\mathrm{n}(G)(\Delta(G)-1)}{\Delta(G)+1}.$$

Proof. Let S be a $\gamma^p(G)$-set and D a $\beta(G)$-set. If $V(G)=S\cup D$, then $\partial_p(G)\geq\partial_p(\varnothing)=0=\mathrm{n}(G)-\gamma^p(G)-\beta(G)$. Assume $V(G)\setminus(S\cup D)\neq\varnothing$. Let $W=S\cap D$ and $x\in V(G)\setminus(S\cup D)$. If $N(x)\cap W=\emptyset$, then there exists a vertex $v\in N(x)\cap(S\setminus D)$, which is a contradiction, as D is a vertex cover. Thus, $N(x)\cap W\neq\emptyset$, which implies that $V(G)\setminus(S\cup D)\subseteq N_e(W)$. Now, since S is a perfect dominating set, $|N(x)\cap W|=1$ and, as a consequence, $V(G)\setminus(S\cup D)\subseteq N_p(W)$. Therefore,

$$\partial_p(G)\geq\partial_p(W)=|N_p(W)|-|W|\geq|V(G)\setminus(S\cup D)|-|W|\geq\mathrm{n}(G)-\gamma^p(G)-\beta(G).$$

To conclude the proof of the lower bound, we only need to observe that $N_p(S)=V(G)\setminus S$, which implies that $\partial_p(G)\geq\partial_p(S)=|N_p(S)|-|S|=\mathrm{n}(G)-2\gamma^p(G)$.

Now, let S' be a $\partial_p(G)$-set. Since $N_p(S')\subseteq N_e(S')$, we have that $\partial_p(G)=|N_p(S')|-|S'|\leq|N_e(S')|-|S'|\leq\partial(G)$.

Finally, as declared in the statement, the bound $\partial(G)\leq\frac{\mathrm{n}(G)(\Delta(G)-1)}{\Delta(G)+1}$ was obtained in [5]. □

The bounds above are tight. In Corollary 3.3 and Proposition3.4 we will show infinite families of graphs achieving $\partial_p(G)=\mathrm{n}(G)-2\gamma^p(G)$. Now we proceed to introduce an infinite family of graphs satisfying $\partial_p(G)=\mathrm{n}(G)-\gamma^p(G)-\beta(G)$. For any integer $k\geq3$ we define the graph G_k of order $\mathrm{n}(G_k)=5k$, which have $3k$ vertices of degree one, k vertices of degree two, and k vertices of degree seven. The subgraph induced by the set S_k of vertices of degree seven is a cycle, every vertex in S_k is adjacent to three vertices of degree one and every vertex of degree two is adjacent to two adjacent vertices belonging to S_k. For instance, G_3 is shown in Figure 2. Notice that S_k is the only $\partial_p(G_k)$-set and the only $\beta(G_k)$-set. Since $\gamma^p(G_k)=2k$, we have that $\partial_p(G_k)=2k=\mathrm{n}(G_k)-\gamma^p(G_k)-\beta(G_k)$. In Proposition 3.2 we give a characterization of all graphs with $\partial_p(G)=\frac{\mathrm{n}(G)(\Delta(G)-1)}{\Delta(G)+1}$.

Proposition 3.2. *Given a graph G,* $\partial_p(G)=\frac{\mathrm{n}(G)(\Delta(G)-1)}{\Delta(G)+1}$ *if and only if* $\gamma^p(G)=\frac{\mathrm{n}(G)}{\Delta(G)+1}$.

Proof. If $\gamma^p(G) = \frac{\mathrm{n}(G)}{\Delta(G)+1}$, then by Proposition 3.1 we have that

$$\begin{aligned}\frac{\mathrm{n}(G)(\Delta(G)-1)}{\Delta(G)+1} &= \mathrm{n}(G) - 2\left(\frac{\mathrm{n}(G)}{\Delta(G)+1}\right)\\ &= \mathrm{n}(G) - 2\gamma^p(G)\\ &\leq \partial_p(G)\\ &\leq \frac{\mathrm{n}(G)(\Delta(G)-1)}{\Delta(G)+1}.\end{aligned}$$

Therefore, $\partial_p(G) = \frac{\mathrm{n}(G)(\Delta(G)-1)}{\Delta(G)+1}$, as desired.

Conversely, assume $\partial_p(G) = \frac{\mathrm{n}(G)(\Delta(G)-1)}{\Delta(G)+1}$. Let S be a $\partial_p(G)$-set. Hence,

$$\mathrm{n}(G) \geq (|N_p(S)| - |S|) + 2|S| = \frac{\mathrm{n}(G)(\Delta(G)-1)}{\Delta(G)+1} + 2|S| = \mathrm{n}(G) - 2\left(\frac{\mathrm{n}(G)}{\Delta(G)+1}\right) + 2|S|,$$

which implies that $|S| \leq \frac{\mathrm{n}(G)}{\Delta(G)+1}$. Now, by using the fact that $|N_p(S)| \leq \Delta(G)|S|$, we deduce that

$$\frac{\mathrm{n}(G)(\Delta(G)-1)}{\Delta(G)+1} = \partial_p(G) = |N_p(S)| - |S| \leq \Delta(G)|S| - |S| \leq (\Delta(G)-1)|S| \leq \frac{\mathrm{n}(G)(\Delta(G)-1)}{\Delta(G)+1}.$$

Hence, $|S| = \frac{\mathrm{n}(G)}{\Delta(G)+1}$ and $|N_p(S)| = \mathrm{n}(G) - |S| = \mathrm{n}(G) - \frac{\mathrm{n}(G)}{\Delta(G)+1}$, which implies that S is a perfect dominating set of G of minimum cardinality, i.e., $\gamma^p(G) = \frac{\mathrm{n}(G)}{\Delta(G)+1}$. Therefore, the result follows. □

It is known that $\gamma(G) \geq \frac{\mathrm{n}(G)}{\Delta(G)+1}$ for every graph G, and a graph satisfies $\gamma(G) = \frac{\mathrm{n}(G)}{\Delta(G)+1}$ if and only if there exists a $\gamma(G)$-set S which is a packing and every vertex in S has degree $\Delta(G)$. Hence, the graphs satisfying $\gamma^p(G) = \frac{\mathrm{n}(G)}{\Delta(G)+1}$, also satisfy the following equality chain.

$$\mathrm{n}(G) - 2\gamma^p(G) = \partial_p(G) = \partial(G) = \frac{\mathrm{n}(G)(\Delta(G)-1)}{\Delta(G)+1} = (\Delta(G)-1)\gamma(G) = (\Delta(G)-1)\gamma^p(G).$$

Observe that, by the Gallai's theorem, the lower bound $\partial_p(G) \geq \mathrm{n}(G) - \gamma^p(G) - \beta(G)$ can be stated as $\partial_p(G) \geq \alpha(G) - \gamma^p(G)$.

Since $\mathscr{D}(G) \cap \wp(G) \subseteq \mathscr{D}^p(G)$ and every set in $\mathscr{D}(G) \cap \wp(G)$ is a $\gamma(G)$-set and also a $\rho(G)$-set, we deduce that $|S| = \rho(G) = \gamma(G) = \gamma^p(G)$, for every $S \in \mathscr{D}(G) \cap \wp(G)$. Therefore, the following corollary is a direct consequence of Proposition 3.1.

Corollary 3.3. *Let G be a graph. If $\mathscr{D}(G) \cap \wp(G) \neq \varnothing$ and $\partial(G) = \mathrm{n}(G) - 2\gamma(G)$, then*

$$\partial_p(G) = \partial(G) = \mathrm{n}(G) - 2\rho(G) = \mathrm{n}(G) - 2\gamma^p(G).$$

In order to show another class of graphs with $\partial_p(G) = \partial(G) = \mathrm{n}(G) - 2\gamma^p(G)$, we consider the case of corona graphs. Given two graphs G_1 and G_2, the *corona product graph* $G_1 \odot G_2$ is the graph obtained from G_1 and G_2, by taking one copy of G_1 and $\mathrm{n}(G_1)$ copies of G_2 and joining by an edge every vertex from the i^{th}-copy of G_2 with the i^{th}-vertex of G_1. For every $x \in V(G_1)$, the copy of G_2 in $G_1 \odot G_2$ associated to x will be denoted by $G_{2,x}$.

Proposition 3.4. *For any graph G_1 and any nontrivial graph G_2,*

$$\partial(G_1 \odot G_2) = \partial_p(G_1 \odot G_2) = \mathrm{n}(G_1)(\mathrm{n}(G_2) - 1).$$

Proof. Since every perfect dominating set of $G_1 \odot G_2$ has at least one vertex in $V(G_{2,x}) \cup \{x\}$ for every $x \in V(G_1)$, we have that $\gamma^p(G_1 \odot G_2) \geq \mathrm{n}(G_1)$. Thus, Proposition 3.1 leads to

$$\partial(G_1 \odot G_2) \geq \partial_p(G_1 \odot G_2) \geq \mathrm{n}(G_1)(\mathrm{n}(G_2)+1) - 2\gamma^p(G_1 \odot G_2) \geq \mathrm{n}(G_1)(\mathrm{n}(G_2) - 1).$$

Now, let S be a $\partial(G_1 \odot G_2)$-set and $S' = \{x \in V(G_1) : S \cap (\{x\} \cup V(G_{2,x})) \neq \varnothing\}$. Thus,

$$\mathrm{n}(G_1)(\mathrm{n}(G_2) - 1) \leq \partial(G_1 \odot G_2) \leq (\mathrm{n}(G_1) - |S'|) + \sum_{x \in S'} (\mathrm{n}(G_2) - 1) \leq \mathrm{n}(G_1)(\mathrm{n}(G_2) - 1).$$

Therefore, the result follows. □

Next we show some naive bounds for the perfect differential in terms of the order and the maximum degree of G. We also discuss the extreme cases.

Proposition 3.5. *Given a graph G of order $\mathrm{n}(G) \geq 3$, the following statements hold.*

(i) $\max\{0, \Delta(G) - 1\} \leq \partial_p(G) \leq \mathrm{n}(G) - 2$.

(ii) $\partial_p(G) = 0$ *if and only if* $\Delta(G) \leq 1$.

(iii) $\partial_p(G) = 1$ *if and only if either* $G \cong G_1$ *or* $G \cong G_1 \cup G_2$*, where* $G_1 \in \{C_3, C_4, C_5, P_3, P_4, P_5\}$ *and* $\Delta(G_2) \leq 1$.

(iv) $\partial_p(G) = \mathrm{n}(G) - 2$ *if and only if* $\Delta(G) = \mathrm{n}(G) - 1$.

(v) $\partial_p(G) = \mathrm{n}(G) - 3$ *if and only if* $\Delta(G) = \mathrm{n}(G) - 2$.

Proof. Since $\partial_p(\varnothing) = 0$, we have that $\partial_p(G) \geq 0$. Now, since $\partial_p(\{v\}) = \deg(v) - 1$ for every $v \in V(G)$, we conclude that $\partial_p(G) \geq \Delta(G) - 1$. Therefore, the lower bound follows. In order to obtain the upper bound, observe that for any set $S \subseteq V(G)$ we have that $N_p(S) = \cup_{v \in S} \mathrm{epn}(v, S)$, which implies that

$$\begin{aligned} \partial_p(G) &= \max_{S \subseteq V(G)} \{|N_p(S)| - |S|\} \\ &= \max_{S \subseteq V(G)} \{\sum_{v \in S} |\mathrm{epn}(v, S)| - |S|\} \\ &\leq \max_{\varnothing \neq S \subseteq V(G)} \{\mathrm{n}(G) - 2|S|\} \\ &\leq \mathrm{n}(G) - 2. \end{aligned}$$

Therefore, (i) follows.

On the other hand, if $\Delta(G) \leq 1$, then $|N_p(S)| \leq |S|$ for every $S \subseteq V(G)$, which implies that $\partial_p(G) = 0$. Conversely, if $\partial_p(G) = 0$, then $\deg(v) - 1 = \partial_p(\{v\}) \leq \partial_p(G) = 0$ for every $v \in V(G)$, which implies that $\Delta(G) \leq 1$. Therefore, (ii) follows.

Now, if $\partial_p(G) = 1$, then (i) and (ii) lead to $\Delta(G) = 2$. If G is connected, then for any vertex v of maximum degree, the set $\{v\}$ is a $\partial_p(G)$-set, which implies that v has eccentricity at most three, and so $G \in \{C_3, C_4, C_5, P_3, P_4, P_5\}$. If G is not connected, it is easy to see that $G \cong G_1 \cup G_2$, where

$G_1 \in \{C_3, C_4, C_5, P_3, P_4, P_5\}$ and $\Delta(G_2) \leq 1$. The other implication is straightforward. Thus, (iii) follows.

Now, if $\Delta(G) = n(G) - 1$, then (i) leads to $\partial_p(G) = n(G) - 2$. Conversely, assume $\partial_p(G) = n(G) - 2$. In this case, for any $\partial_p(G)$-set S we have that $n(G) - 2 = |N_p(S)| - |S|$. Thus, since $n(G) \geq 3$, we deduce that $S \neq \varnothing$ and so $n(G) - 2 = |N_p(S)| - |S| \leq n(G) - 2|S|$, which implies that $|S| = 1$. Hence, S is formed by a universal vertex, i.e., $\Delta(G) = n(G) - 1$, as required. Therefore, (iv) follows.

Assume $\Delta(G) = n(G) - 2$. By (i) and (iv) we deduce that $\partial_p(G) = n(G) - 3$. On the other side, assume that $\partial_p(G) = n(G) - 3$ and let S be a $\partial_p(G)$-set. As above, since $n(G) \geq 3$, we deduce that $S \neq \varnothing$ and so $n(G) - 3 = |N_p(S)| - |S| \leq n(G) - 2|S|$, which implies that $|S| = 1$. Hence, S is formed by a vertex of maximum degree, i.e., $\Delta(G) = n(G) - 2$. Therefore, (v) follows. □

The following remark is a consequence of Propositions 3.1 and 3.5.

Remark 3.6. *If $\gamma^p(G) = 2$, then either $\Delta(G) = n(G) - 2$ and so $\partial_p(G) = n(G) - 3$ or $\Delta(G) \leq n(G) - 3$ and $\partial_p(G) = n(G) - 4$.*

Proof. If $\gamma^p(G) = 2$, then Proposition 3.1 leads to $\partial_p(S) \geq n(G) - 4$. Notice also that $\gamma^p(G) = 2$ also leads to $\Delta(G) \leq n(G) - 2$. Now, if $\Delta(G) = n(G) - 2$, then $\partial_p(G) = n(G) - 3$, by Proposition 3.5. Finally, if $\Delta(G) \leq n(G) - 3$, then Proposition 3.5 leads to $\partial_p(G) \leq n(G) - 4$. Therefore, the result follows. □

From Propositions 3.1 and 3.5 we deduce the following result.

Proposition 3.7. *Let G be a graph. If $\partial(G) = \Delta(G) - 1$, then $\partial_p(G) = \Delta(G) - 1$.*

The converse of Proposition 3.7 does not hold. For instance, if G is the graph shown in Figure 2, then $\partial_p(G) = \Delta(G) - 1 = 6 < 9 = \partial(G)$.

We proceed to characterize all trees T with $\partial_p(T) = \Delta(T) - 1$. Given a vertex $v \in V(T)$ of maximum degree and any vertex $v_i \in N(v)$, we define $T_i(v)$ as the sub-tree of T with root v_i obtained from T by removing the edge $\{v, v_i\}$. Let $d_k(v)$ be the number of vertices $v_i \in N(v)$ such that $\Delta(T_i(v)) = k$. The eccentricity of a vertex v will be denoted by $\mathrm{ecc}(v)$ and the diameter of a graph G will be denoted by $\mathrm{diam}(G)$. With this notation in mind we can state the following result.

Proposition 3.8. *Let T be a nontrivial tree. Then $\partial_p(T) = \Delta(T) - 1$ if and only if the following conditions hold for every vertex $v \in V(T)$ with $\deg(v) = \Delta(T)$.*

(a) $\mathrm{ecc}(v) \leq 3$ *and* $\deg(u) \leq 3$ *for every* $u \in V(T) \setminus \{v\}$.

(b) *For any $v_i \in N(v)$, the number of vertices of $T_i(v)$ of degree three is at most one.*

(c) $2d_3(v) + d_2(v) \leq \Delta(T) - 1$.

(d) *If* $\mathrm{diam}(T_i(v)) \neq 2$ *for every* $v_i \in N(v)$ *such that* $\deg(v_i) = 3$, *then* $2d_3(v) + d_2(v) + d_1^*(v) \leq \Delta(T) - 1$, *where* $d_1^*(v) = \min\{1, d_1(v)\}$.

Proof. Assume $\partial_p(T) = \Delta(T) - 1$ and let $v \in V(T)$ with $\deg(v) = \Delta(T)$. First, we proceed to prove (a). If $\mathrm{ecc}(v) \geq 4$, then there exists a vertex $s \in V(T)$ such that $\deg(s) \geq 2$ and $d(s, v) = 3$. Notice that $\partial_p(\{v, s\}) \geq \Delta(T)$, which is a contradiction. Hence, $\mathrm{ecc}(v) \leq 3$ as desired. Now, suppose that there exists $u \in V(T) \setminus \{v\}$ such that $\deg(u) \geq 4$. It is easy to see that $\partial_p(\{v,u\}) \geq \Delta(T)$, which is again a contradiction. Therefore, (a) follows.

Now, we proceed to prove (b). Let $v_i \in N(v)$, and suppose that $T_i(v)$ has two vertices u, u' of degree three. By (a), these vertices are at distance two, and so $\partial_p(\{v,u,u'\}) \geq \Delta(T)$, which is a contradiction. Hence, (b) holds.

From now on, let S be a $\partial_p(T)$-set of minimum cardinality among all $\partial_p(T)$-sets, and let $S' \subseteq V(T) \setminus \{v\}$ be a set defined as follows. A vertex $u \in V(T_i(v))$ belongs to S' if and only if $\Delta(T_i(v)) \geq 2$ and u is the furthest vertex from v, among all those of maximum degree in $T_i(v)$.

To deduce (c) we only need to observe that $\Delta(T) - 1 = \partial_p(T) \geq \partial_p(S') = 2d_3(v) + d_2(v)$.

Now, assume that $\mathrm{diam}(T_i(v)) \neq 2$ for every $v_i \in N(v)$ such that $\deg(v_i) = 3$. If there exists $v_i \in N(v)$ such that $\Delta(T_i(v)) = 1$, then $\partial_p(S' \cup \{v_i\}) = 2d_3(v) + d_2(v) + 1$, which implies that $\Delta(T) - 1 = \partial_p(T) \geq \partial_p(S' \cup \{v_i\}) = 2d_3(v) + d_2(v) + 1 = 2d_3(v) + d_2(v) + d_1^*(v)$. Obviously, if $\Delta(T_i(v)) \neq 1$ for every $v_i \in N(v)$, then $d_1^*(v) = 0$, and so $\Delta(T) - 1 = \partial_p(T) \geq \partial_p(S') = 2d_3(v) + d_2(v) = 2d_3(v) + d_2(v) + d_1^*(v)$. Therefore, (d) follows.

Conversely, assume that for every $v \in V(T)$ with $\deg(v) = \Delta(T)$, the conditions (a), (b), (c) and (d) hold. As above, let $v \in V(T)$ be a vertex of maximum degree and recall that S is a $\partial_p(T)$-set of minimum cardinality among all $\partial_p(T)$-sets. Notice that $|S \cap V(T_i(v)| \leq 1$ for every $v_i \in N(v)$. Suppose that $|S| \geq 2$ and let $u \in S \setminus \{v\}$. If $v \in S$, then by (a) and (b) we deduce that the contribution of u to $\partial_p(S)$ is of one, and so $\partial_p(S \setminus \{u\}) \geq \partial_p(S) = \partial_p(T)$, which is a contradiction, as $|S \setminus \{u\}| < |S|$. Assume $v \notin S$. If there exists $v_i \in N(v)$ such that $\deg(v_i) = 3$ and $\mathrm{diam}(T_i(v)) = 2$, then $v_i \in S$, which implies that $\deg(v_j) = 3$ for every $v_j \in N(v) \cap S$. Thus, by (c) we obtain that $\Delta(T) - 1 \geq 2d_3(v) + d_2(v) \geq \partial_p(S) = \partial_p(T)$, which implies that $\partial_p(T) = \Delta(T) - 1$, by Proposition 3.5. Now, if $\mathrm{diam}(T_i(v)) \neq 2$ for every $v_i \in N(v)$ such that $\deg(v_i) = 3$, then by (d) we have that $\Delta(T) - 1 \geq 2d_3(v) + d_2(v) + d_1^*(v) \geq \partial_p(S) = \partial_p(T)$. Again Proposition 3.5 leads to $\partial_p(T) = \Delta(T) - 1$. Finally, it is clear that if $|S| = 1$ then $\partial_p(T) = \Delta(T) - 1$. Therefore, the result follows. □

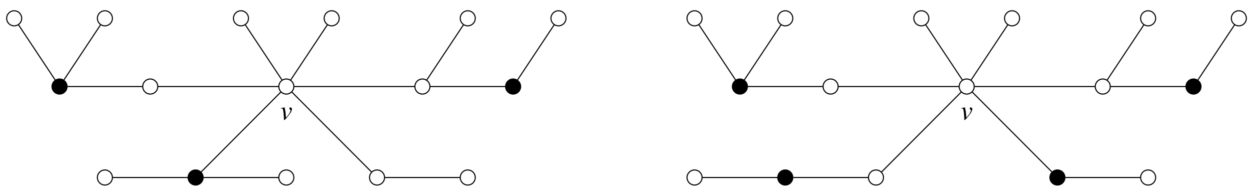

Figure 3: Two nontrivial examples of trees for which $\partial_p(T) = \Delta(T) - 1$.

Figure 3 shows two nontrivial examples of trees for which $\partial_p(T) = \Delta(T) - 1$. In both cases, the set of black-coloured vertices forms a $\partial_p(T)$-set.

Proposition 3.9. *For any graph G, the following statements hold.*

(i) $\partial_p(G) \geq \max_{S \in \wp_o(G)} \{\sum_{v \in S_0} (\deg(v) - 1) + \sum_{v \in S_1} (\deg(v) - 2)\}$.

(ii) $\partial_p(G) \geq \max_{S \in \wp(G)} \{\sum_{v \in S} (\deg(v) - 1)\}$.

Proof. Let $S = S_0 \cup S_1 \in \wp_o(G)$. Notice that $N_e(S) = N_p(S)$. Hence, $\partial_p(G) \geq \partial_p(S) = |N_p(S)| - |S| = |N_e(S)| - |S| = \sum_{v \in S_0} (\deg(v) - 1) + \sum_{v \in S_1} (\deg(v) - 2)$. Since the inequality holds for every $S \in \wp_o(G)$, we conclude that (i) follows.

Now, notice that $S \in \wp(G)$ if and only if $S_1 = \varnothing$. Thus, by analogy to the proof of (i) we deduce that (ii) follows. $\square$

Notice that from Proposition 3.9 we deduce that lower bound $\partial_p(G) \geq \Delta(G) - 1$ discussed previously. Furthermore, we would highlight the following particular cases of Proposition 3.9.

Corollary 3.10. *Given a graph G, the following statements hold.*

(i) $\partial_p(G) \geq \rho_o(G)(\delta(G) - 2)$.

(ii) $\partial_p(G) \geq \rho(G)(\delta(G) - 1)$.

The bounds above are tight. For instance, the first bound is achieved by the Hamming graph $H_{2,r} = K_2 \Box K_r$, for $r \geq 3$, as $\partial_p(H_{2,r}) = 2(r-2) = \rho_o(H_{2,r})(\delta(H_{2,r}) - 2)$. The second bound is achieved by the 3-cube, Q_3, as $\partial_p(Q_3) = 4 = \rho(Q_3)(\delta(Q_3) - 1)$. In fact, Q_3 is an example of graph satisfying the next proposition, which is a direct consequence of combining Proposition 3.1 and Corollary 3.10.

Proposition 3.11. *If G is a k-regular graph with $\gamma(G) = \rho(G)$, then*

$$\partial_p(G) = \partial(G) = (k-1)\gamma(G) = \frac{\mathrm{n}(G)(k-1)}{k+1}.$$

As we proceed to show, for the case of efficient domination graphs, Proposition 3.9 leads to the following bound on $\partial_p(G)$.

Theorem 3.12. *If G is an efficient domination graph, then $\partial_p(G) \geq \mathrm{n}(G) - 2\gamma_t(G)$.*

Proof. Let G be an efficient domination graph. We already know from Remark 2.1 that there exists a $\rho_o(G)$-set S with $G[S] = \cup K_2$ and $|S| = \gamma_t(G)$. Since, $\sum_{v \in S} \deg(v) = \mathrm{n}(G)$, we have that $\sum_{v \in S}(\deg(v) - 2) = \mathrm{n}(G) - 2|S| = \mathrm{n}(G) - 2\gamma_t(G)$. Therefore, Proposition 3.9-(i) leads to $\partial_p(G) \geq \mathrm{n}(G) - 2\gamma_t(G)$. $\square$

The bound above is tight. Now we proceed to show an infinite family of connected efficient open domination graphs G_r^* achieving the bound. For any integer $r \geq 2$ we take the corona graph $P_4 \odot N_r$, where P_4 is the path of order 4 and N_r is an empty graph of order r. Then G_r^* is obtained from $P_4 \odot N_r$ by subdividing twice the edge $\{2,3\}$ of the subgraph P_4. For instance, Figure 4 shows the efficient open domination graph G_3^*. Notice that $\mathrm{n}(G_r^*) = 4(r+1) + 2$, $\gamma_t(G_r^*) = 4$ and $\partial_p(G_r^*) = 4r - 2$. Hence, $\partial_p(G_r^*) = \mathrm{n}(G_r^*) - 2\gamma_t(G_r^*)$.

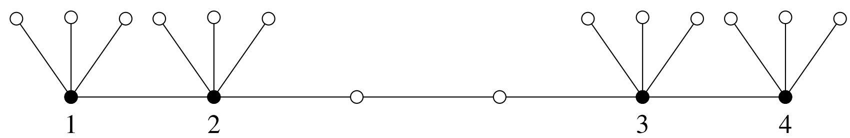

Figure 4: A graph G with $\partial_p(G) = \mathrm{n}(G) - 2\gamma_t(G) = 10$. The set of black-coloured vertices forms a $\partial_p(G)$-set.

4. A Gallai-type theorem

A Gallai-type theorem has the form $a(G)+b(G)=\mathrm{n}(G)$, where $a(G)$ and $b(G)$ are parameters defined on G. This terminology comes from Theorem 2.3, which was stated in 1959 by the Hungarian mathematician Tibor Gallai (15 July 1912 - 2 January 1992).

The following basic theorem was obtained by Bermudo, Fernau and Sigarreta in [3].

Theorem 4.1 (Gallai-type theorem for the differential and the Roman domination number, [3]). *For any graph G,*

$$\gamma_R(G)+\partial(G)=\mathrm{n}(G).$$

Next we establish a Gallai-type theorem which states the relationship between the perfect differential and the perfect Roman domination number.

Theorem 4.2 (Gallai-type theorem for the perfect differential and the perfect Roman domination number). *For any graph G,*

$$\gamma_R^p(G)+\partial_p(G)=\mathrm{n}(G).$$

Proof. Since $\partial_p(S)=|N_p(S)|-|S|$ for every $S\subseteq V(G)$,

$$\begin{aligned}\gamma_R^p(G) &= \min_{S\subseteq V(G)}\{2|S|+|V(G)\setminus(N_p(S)\cup S)|\}\\ &= \min_{S\subseteq V(G)}\{2|S|+\mathrm{n}(G)-(|N_p(S)|+|S|)|\}\\ &= \min_{S\subseteq V(G)}\{\mathrm{n}(G)-\partial_p(S)\}\\ &= \mathrm{n}(G)-\max_{S\subseteq V(G)}\{\partial_p(S)\}\\ &= \mathrm{n}(G)-\partial_p(G).\end{aligned}$$

Therefore, the result follows. □

From Theorems 4.1 and 4.2 we immediately have the following result.

Proposition 4.3. *Given a graph G, $\partial_p(G)=\partial(G)$ if and only if $\gamma_R^p(G)=\gamma_R(G)$.*

The following result shows that the $\partial_p(G)$-sets play an important role in the theory of perfect Roman domination.

Proposition 4.4. *Given a graph G, a function $f:V(G)\longrightarrow\{0,1,2\}$ is a $\gamma_R^p(G)$-function if and only if $V_2=\{v\in V(G): f(v)=2\}$ is a $\partial_p(G)$-set.*

Proof. Let f be a $\gamma_R^p(G)$-function and $V_i=\{v\in V(G): f(v)=i\}$, where $i\in\{1,2\}$. Since $V_1=V(G)\setminus(N_p(V_2)\cup V_2)$,

$$\begin{aligned}\gamma_R^p(G) &= 2|V_2|+|V_1|\\ &= 2|V_2|+|V(G)\setminus(N_p(V_2)\cup V_2)|\\ &= 2|V_2|+\mathrm{n}(G)-(|N_p(V_2)|+|V_2|)|\\ &= \mathrm{n}(G)-\partial_p(V_2).\end{aligned}$$

Hence, by Theorem 4.2, $\partial_p(V_2)=\partial_p(G)$, which implies that V_2 is a $\partial_p(G)$-set.

Conversely, let S be a $\partial_p(G)$-set, and define a function $g : V(G) \longrightarrow \{0,1,2\}$ by $g(v) = 2$ whenever $v \in S$ and $g(v) = 1$ whenever $v \in V(G) \setminus (N_p(S) \cup S)$. It is readily seen that g is a PRDF of weight $\omega(g) = \mathrm{n}(G) - \partial_p(S) = \mathrm{n}(G) - \partial_p(G)$. Thus, Theorem 4.2 implies that g is a $\gamma_R^p(G)$-function. Therefore, the result follows. □

Theorem 4.2 allows us to derive results on the perfect Roman domination number from results on the perfect differential and vice versa. For instance, the next result is a direct consequence of combining Theorem 4.2 and Proposition 3.1.

Proposition 4.5. *Given a graph G, the following inequality chain holds,*

$$\frac{2\,\mathrm{n}(G)}{\Delta(G)+1} \leq \gamma_R^p(G) \leq \beta(G) + \gamma^p(G).$$

Figure 2 shows a graph with $\gamma_R^p(G) = \gamma^p(G) + \beta(G)$. The next result, which is a characterization of the graphs with $\gamma_R^p(G) = \frac{2\,\mathrm{n}(G)}{\Delta(G)+1}$, is a consequence of Theorem 4.2 and Proposition 3.2.

Proposition 4.6. *Given a graph G,* $\gamma_R^p(G) = \frac{2\,\mathrm{n}(G)}{\Delta(G)+1}$ *if and only if* $\gamma^p(G) = \frac{\mathrm{n}(G)}{\Delta(G)+1}$.

Next we show some interesting results on the perfect differential that can be derived from known results on the perfect Roman domination number. For instance, it was shown in [19] that if T is a tree of order at least three, then $\gamma_R^p(T) \leq \frac{4}{5}\,\mathrm{n}(T)$. Hence, Theorem 4.2 leads to the following result.

Proposition 4.7. *If T is a tree of order at least three, then* $\partial_p(T) \geq \frac{1}{5}\,\mathrm{n}(T)$.

The reader is referred to [19] for a characterization of all trees with $\gamma_R^p(T) = \frac{4}{5}\,\mathrm{n}(T)$. Obviously, the same characterization works for trees with $\partial_p(T) = \frac{1}{5}\,\mathrm{n}(T)$.

We learned from Proposition 3.1 that $\partial_p(G) \geq \mathrm{n}(G) - 2\gamma^p(G)$ for every graph G. A characterization of all trees with $\gamma_R^p(T) = 2\gamma^p(T)$ can be found in [30]. Thus, the same characterization applies to the trees with $\partial_p(T) = \mathrm{n}(G) - 2\gamma^p(T)$.

In general, $\gamma_R^p(G)$ and $\gamma^p(G)$ are not comparable for arbitrary graphs. It was shown in [29] that for the case of trees the picture is quite different. In particular, the authors obtained the following result.

Proposition 4.8. [29] *For every nontrivial tree T,* $\gamma_R^p(T) \geq \gamma^p(T) + 1$.

From the result above and Theorem 4.2 we derive the following result.

Proposition 4.9. *For every nontrivial tree T,* $\partial_p(T) \leq \mathrm{n}(T) - \gamma^p(T) - 1$.

The reader is referred to [29] for a characterization of all trees with $\gamma_R^p(T) = \gamma^p(T) + 1$, which is also the characterization of all trees with $\partial_p(T) = \mathrm{n}(T) - \gamma^p(G) - 1$.

Given a graph G and a positive integer k, the *perfect Roman domination problem* is to decide whether there is a perfect Roman dominating function f on G such that $\omega(f)$ is at most k. It was shown in [1] that the perfect Roman domination problem is *NP*-complete for chordal graphs, planar graphs, and bipartite graphs. In the same paper the authors provided polynomial time algorithms for computing a perfect Roman dominating function with minimum weight in block graphs, cographs, series-parallel graphs, and proper interval graphs. Therefore, by Theorem 4.2 we immediately obtain analogous results for the perfect differential. In particular, we would highlight the following one.

Proposition 4.10. *Given a graph G and a positive integer k, the problem of deciding if there exists a set $S \subseteq V(G)$ such that $\partial_p(S) \geq k$ is NP-complete, even for chordal graphs, planar graphs, and bipartite graphs.*

5. Perfect differential of a graph and its complement

Nordhaus and Gaddum [27] in 1956 proposed lower and upper bounds, in terms of the order of the graph, on the sum and the product of the chromatic number of a graph and its complement. Since then, several inequalities of a similar type have been proposed for other graph parameters. From the following consequence of Proposition 3.5, we can derive some Nordhaus-Gaddum type relations for the perfect differential of a graph.

Proposition 5.1. *For a graph G of order $\mathrm{n}(G) \geq 3$, the following statements hold.*

(i) $\mathrm{n}(G) + \Delta(G) - \delta(G) - 3 \leq \partial_p(G) + \partial_p(G^c) \leq 2\,\mathrm{n}(G) - 5$.

(ii) *If $\delta(G) \geq 2$ and $\Delta(G) \leq \mathrm{n}(G) - 3$, then $\mathrm{n}(G) + \Delta(G) - \delta(G) - 3 \leq \partial_p(G) + \partial_p(G^c) \leq 2\,\mathrm{n}(G) - 8$.*

(iii) *$\partial_p(G) + \partial_p(G^c) = 2\,\mathrm{n}(G) - 5$ if and only if either $\Delta(G) = \mathrm{n}(G) - 1$ and $\delta(G) = 1$ or $\Delta(G) = \mathrm{n}(G) - 2$ and $\delta(G) = 0$.*

Proof. By the lower bound obtained in Proposition 3.5 we have that

$$\partial_p(G) + \partial_p(G^c) \geq \Delta(G) + \Delta(G^c) - 2 = \Delta(G) + (\mathrm{n}(G) - 1 - \delta(G)) - 2 = \mathrm{n}(G) + \Delta(G) - \delta(G) - 3.$$

On the other hand, by Proposition 3.5, $\partial_p(G) \leq \mathrm{n}(G) - 2$ and $\partial_p(G) = \mathrm{n}(G) - 2$ if and only if $\Delta(G) = \mathrm{n}(G) - 1$. Now, if $\Delta(G) = \mathrm{n}(G) - 1$, then $\delta(G^c) = 0$, which implies that $\gamma(G^c) \geq 2$ and, as a result, $\partial_p(G^c) \leq \mathrm{n}(G) - 3$. Therefore,

$$\partial_p(G) + \partial_p(G^c) \leq (\mathrm{n}(G) - 2) + (\mathrm{n}(G) - 3) = 2\,\mathrm{n}(G) - 5,$$

which completes the proof of (i).

Now, if $\delta(G) \geq 2$, then $\Delta(G^c) \leq \mathrm{n}(G) - 3$. Hence, from Proposition 3.5 we deduce that $\partial_p(G) \leq \mathrm{n}(G) - 4$ and $\partial_p(G^c) \leq \mathrm{n}(G) - 4$. Therefore, (ii) follows.

On the other side, by Proposition 3.5, $\partial_p(G) + \partial_p(G^c) = 2\,\mathrm{n}(G) - 5$ if and only if $\partial_p(G) = \mathrm{n}(G) - 2$ and $\partial_p(G^c) = \mathrm{n}(G) - 3$ or vice versa. Assume that $\partial_p(G) = \mathrm{n}(G) - 2$ and $\partial_p(G^c) = \mathrm{n}(G) - 3$. In this case, $\Delta(G) = \mathrm{n}(G) - 1$ and so $\delta(G^c) = 0$. Thus, there exists a graph G' such that $G^c \cong K_1 \cup G'$. Now, since $\partial_p(G^c) = \partial_p(G') = \mathrm{n}(G) - 3 = \mathrm{n}(G') - 2$, we have that $\Delta(G^c) = \Delta(G') = \mathrm{n}(G') - 1 = \mathrm{n}(G) - 2$, which implies that $\delta(G) = \mathrm{n}(G) - 1 - \Delta(G^c) = 1$, as required.

Conversely, if $\Delta(G) = \mathrm{n}(G) - 1$ and $\delta(G) = 1$ or $\Delta(G) = \mathrm{n}(G) - 2$ and $\delta(G) = 0$, then (i) leads to $\partial_p(G) + \partial_p(G^c) = 2\,\mathrm{n}(G) - 5$. Therefore, (ii) follows. □

The lower bound is achieved by the self-complementary graph shown in Figure 5, and also for the graphs with $\Delta(G) = \mathrm{n}(G) - 1$ and $\delta(G) = 1$ or $\Delta(G) = \mathrm{n}(G) - 2$ and $\delta(G) = 0$.

Observe that the Nordhaus-Gaddum type relation derived from the result above is the following one.

Corollary 5.2. *Given a graph G of order $\mathrm{n}(G) \geq 3$, the following statements hold.*

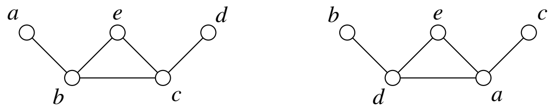

Figure 5: A self-complementary graph.

(i) $\mathrm{n}(G)-3 \leq \partial_p(G)+\partial_p(G^c) \leq 2\,\mathrm{n}(G)-5$.

(ii) *If* $\delta(G) \geq 2$ *and* $\Delta(G) \leq \mathrm{n}(G)-3$*, then* $\mathrm{n}(G)-3 \leq \partial_p(G)+\partial_p(G^c) \leq 2\,\mathrm{n}(G)-8$.

In particular, for $\gamma^p(G)=2$ we deduce the following Nordhaus-Gaddum type relations.

Proposition 5.3. *Given a graph G with* $\gamma^p(G)=2$*, the following statements hold.*

(i) *If* $\delta(G)=0$*, then* $\partial_p(G)+\partial_p(G^c)=2\,\mathrm{n}(G)-5$.

(ii) *If* $\delta(G)=1$ *and* $\Delta(G)=\mathrm{n}(G)-2$*, then* $\partial_p(G)+\partial_p(G^c)=2\,\mathrm{n}(G)-6$.

(iii) *If* $\delta(G)=1$ *and* $\Delta(G)\leq \mathrm{n}(G)-3$ *or* $\delta(G)\geq 2$ *and* $\Delta(G)=\mathrm{n}(G)-2$*, then* $\partial_p(G)+\partial_p(G^c)=2\,\mathrm{n}(G)-7$.

(iv) *If* $\delta(G)\geq 2$ *and* $\Delta(G)\leq \mathrm{n}(G)-3$*, then* $\partial_p(G)+\partial_p(G^c)=2\,\mathrm{n}(G)-8$.

Proof. Since $\gamma^p(G)=2$, if $\delta(G)=0$, then $\Delta(G)=\mathrm{n}(G)-2$ and $\Delta(G^c)=\mathrm{n}(G)-1$. Hence, Proposition 3.5 leads to $\partial_p(G)=\mathrm{n}(G)-3$ and $\partial_p(G^c)=\mathrm{n}(G)-2$. Therefore, $\partial_p(G)+\partial_p(G^c)=2\,\mathrm{n}(G)-5$.

Now, if $\gamma^p(G)=2$ and $\delta(G)\geq 1$, then any $\gamma^p(G)$-set S is a $\gamma^p(G^c)$-set and from Remark 3.6 we have the following statements.

- Either $\Delta(G)=\mathrm{n}(G)-2$ and $\partial_p(G)=\mathrm{n}(G)-3$ or $\Delta(G)\leq \mathrm{n}(G)-3$ and $\partial_p(G)=\mathrm{n}(G)-4$.
- Either $\delta(G)=1$ and $\partial_p(G^c)=\mathrm{n}(G)-3$ or $\delta(G)\geq 2$ and $\partial_p(G^c)=\mathrm{n}(G)-4$.

Therefore, by combining both items, we conclude the proof. □

6. Partial solution to a conjecture of Bermudo

The following conjecture, stated by Bermudo [2] in 2017, is one of the 73 conjectures proposed by Gera, Haynes and Hedetniemi in [14].

Conjecture 6.1 (Bermudo [2])**.** *Let G be a graph. If* $\delta(G)\geq 3$*, then* $\partial(G)\geq\gamma(G)$.

In the following proposition we describe several cases in which the conjecture above is solved. Notice that the result obtained in the last item is stronger than the one proposed in the conjecture, as $\gamma_t(G)\geq\gamma(G)$.

First we need to define the following parameter,

$$\Omega(G)=\max\left\{\frac{\mathrm{n}(G)}{3},\Delta(G)-1,(\delta(G)-1)\rho(G),(\delta(G)-2)\rho_o(G)\right\}.$$

Proposition 6.2. *The following statements hold for a connected graph G.*

(i) *If* $\gamma(G) \leq \frac{\mathrm{n}(G)}{3}$*, then* $\partial(G) \geq \gamma(G)$.

(ii) *If* $\delta(G) \geq 5$*, then* $\partial(G) \geq \gamma(G)$.

(iii) *If* $\mathrm{n}(G) \geq 3$ *and* $\delta(G) \geq 3$*, then the following statements hold.*

- *If* $\mathrm{n}(G) \leq 23$ *or* $\Omega(G) \geq \gamma(G)$*, then* $\partial(G) \geq \gamma(G)$.
- *If* $\mathrm{n}(G) \leq 23$ *or* $\Omega(G) \geq \frac{3\,\mathrm{n}(G)}{8}$*, then* $\partial(G) \geq \gamma(G)$.

(iv) *If* $\delta(G) \geq 3$ *and* G *is an efficient open domination graph, then* $\partial(G) \geq \gamma_t(G)$.

Proof. We proceed to discuss the statements separately.

(i) It is well known that $\partial(G) \geq \mathrm{n}(G) - 2\gamma(G)$. Therefore, $\gamma(G) \leq \frac{\mathrm{n}(G)}{3}$ leads to $\partial(G) \geq \gamma(G)$.

(ii) As shown in [6], if $\delta(G) = 5$, then $\gamma(G) \leq \frac{\mathrm{n}(G)}{3}$. Hence, if $\delta(G) \geq 5$, then for any spanning subgraph G' of G, with $\delta(G') = 5$, we have that $\gamma(G) \leq \gamma(G') \leq \frac{\mathrm{n}(G)}{3}$. Therefore, by (i) we conclude that $\delta(G) \geq 5$ leads to $\partial(G) \geq \gamma(G)$.

(iii) It is known that if $\delta(G) \geq 3$ and G is connected, then $\gamma(G) \leq \frac{3\,\mathrm{n}(G)}{8}$, [17]. Hence, if $\frac{\mathrm{n}(G)}{3} + 1 \leq \gamma(G) \leq \frac{3\,\mathrm{n}(G)}{8}$, then $\mathrm{n}(G) \geq 24$. Therefore, $\delta(G) \geq 3$ and $\mathrm{n}(G) \leq 23$ lead to $\partial(G) \geq \gamma(G)$.

Now, if $\gamma(G) \leq \Delta(G) - 1$, then by Proposition 3.5 we have that $\partial(G) \geq \partial(G)_p \geq \Delta(G) - 1 \geq \gamma(G)$.

On the other hand, if $(\delta(G) - 1)\rho(G) \geq \gamma(G)$, then Corollary 3.10-(ii) leads to $\partial(G) \geq \partial_p(G) \geq (\delta(G) - 1)\rho(G) \geq \gamma(G)$.

Finally, if $\rho_o(G)(\delta(G) - 2) \geq \gamma(G)$, then Corollary 3.10-(i) leads to $\partial(G) \geq \partial_p(G) \geq (\delta(G) - 2)\rho_o(G) \geq \gamma(G)$, which completes the proof of (iii).

(iv) Let G be an efficient domination graph. By Remark 2.1, $\rho_o(G) = \gamma_t(G)$. Hence, if $\delta(G) \geq 3$, then Proposition 3.9-(i) leads to $\partial(G) \geq \gamma_t(G)(\delta(G) - 2) \geq \gamma_t(G)$. □

It remains to prove the following particular case of Conjecture 6.1.

Conjecture 6.3. *Let* G *be a connected graph with* $\mathrm{n}(G) \geq 24$ *and either* $\delta(G) = 3$ *or* $\delta(G) = 4$*. If* $\gamma(G) \geq \Omega(G) + 1$*, then* $\partial(G) \geq \gamma(G)$.

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