

Secure Domination in Rooted Product Graphs

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Abstract A secure dominating set of a graph G is a dominating set S satisfying that for every vertex $v \in V(G) \setminus S$ there exists a neighbour $u \in S$ of v such that $(S \cup \{v\}) \setminus \{u\}$ is a dominating set as well. The secure domination number, denoted by $\gamma_s(G)$, is the minimum cardinality among all secure dominating sets of G . This concept was introduced in 2005 by Cockayne et al. and studied further in a number of works. The problem of computing the secure domination number is NP-Hard. This suggests finding the secure domination number for special classes of graphs or obtaining tight bounds on this invariant. The aim of this work is to obtain closed formulas for the secure domination number of rooted product graphs. We show that for any graph G of order $n(G)$ and any graph H with root v , the secure domination number of the rooted product graph $G \circ_v H$ satisfies one of the following three formulas, $\gamma_s(G \circ_v H) = n(G)(\gamma_s(H) - 1) + \gamma(G)$, $\gamma_s(G \circ_v H) = n(G)(\gamma_s(H) - 1) + \gamma_s(G)$ or $\gamma_s(G \circ_v H) = n(G)\gamma_s(H)$, where $\gamma(G)$ denotes the domination number of G . We also characterize the graphs that satisfy each of these expressions. As a particular case of the study, we derive the corresponding results for corona graphs.

Keywords Secure domination · Rooted product graphs · Corona product graphs

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1 Introduction

The following approach to protection of a graph was described by Cockayne et al. [7]. Suppose that one or more guards are stationed at some of the vertices of a simple graph G and that a guard at a vertex can deal with a problem at any vertex in its closed neighbourhood. We say that G is protected if there is at least one guard available to handle a problem at any vertex.

The simplest form of protection of a graph is known under the name of domination. We say that $D \subseteq V(G)$ is a *dominating set* (DS) if $N(v) \cap D \neq \emptyset$ for every $v \in V(G) \setminus D$, where $N(v)$ is the open neighbourhood of v . Hence, if there is one guard stationed at each vertex belonging to the dominating set, then G is protected. The theory of domination in graphs has been studied extensively [8,9]. The *domination number* is defined to be

$$\gamma(G) = \min\{|D| : D \text{ is a DS of } G\}.$$

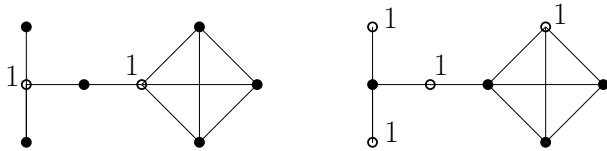


Fig. 1 Two placements of guards which correspond to a dominating set and a secure dominating set of the same graph. Notice that $\gamma(G) = 2$, while $\gamma_s(G) = 4$.

Consider the placement of guards shown in Figure 1 on the left. Although the graph is protected, if the guard stationed at the vertex of degree three moves to handle a problem at a vertex of degree one, then the graph becomes unprotected, as in such a case no guard is able to handle a problem at the other vertex of degree one. In order to avoid this drawback, Cockayne et al. in [7] proposed the protection of graphs under secure dominating sets, i.e., a *secure dominating set* (SDS) of G is a set $S \subseteq V(G)$ which satisfies the following two conditions:

- (*) S is a dominating set.
- (**) For every $v \in V(G) \setminus S$ there exists a vertex $u \in S \cap N(v)$ such that $(S \cup \{v\}) \setminus \{u\}$ is a dominating set.

In order to emphasize the role of u and v in condition (**), we will say that $v \in V(G) \setminus S$ is *protected with respect to S by vertex $u \in S \cap N(v)$* if and only if S and $(S \cup \{v\}) \setminus \{u\}$ are dominating sets.

The *secure domination number* is defined to be

$$\gamma_s(G) = \min\{|S| : S \text{ is an SDS of } G\}.$$

A secure dominating set of cardinality $\gamma_s(G)$ is called a $\gamma_s(G)$ -set. For instance, for the graph shown in Figure 1, on the right, a $\gamma_s(G)$ -set can place one guard at each white-coloured vertex.

This concept of protection has been studied further in the literature. Merouane and Chellali [3] proved that the secure dominating set problem is an NP-complete, even restricted to bipartite graphs and split graphs i.e., given a positive integer k and a graph G , the problem of deciding if G has a secure dominating set D of cardinality $|D| \leq k$ is NP-complete. This suggests finding the secure domination number for special classes of graphs or obtaining good bounds on this invariant. Cockayne et al. [7] provide exact values of $\gamma_s(G)$ for some families of graphs. Upper and lower bounds on $\gamma_s(G)$ have been established for various graph classes, see for instance [4, 5, 3, 11, 12, 14]. Some other properties of secure domination in graphs have been investigated in a number of works [1, 2, 6, 10, 13, 15]. In this paper, we obtain closed formulas for the secure domination number of rooted product graphs. As a particular case of the study, we derive the corresponding formulas for corona graphs.

Given a graph G of order $n(G)$ and a graph H with root vertex v , the rooted product $G \circ_v H$ is defined as the graph obtained from G and H by taking one copy of G and $n(G)$ copies of H and identifying the i^{th} vertex of G with the vertex v in the i^{th} copy of H for each $i \in \{1, \dots, n(G)\}$.

For every $x \in V(G)$, H_x will denote the copy of H in $G \circ_v H$ containing x and H_x^- will denote the subgraph of $G \circ_v H$ induced by $V(H_x) \setminus \{x\}$. Analogously, if S is an SDS of $G \circ_v H$, then the intersection of S with $V(H_x)$ and $V(H_x^-)$ will be denoted by S_x and S_x^- , respectively. Notice that

$$V(G \circ_v H) = \bigcup_{x \in V(G)} V(H_x) = \bigcup_{x \in V(G)} V(H_x^-) \cup V(G)$$

and so, if S is a $\gamma_s(G \circ_v H)$ -set, then

$$\gamma_s(G \circ_v H) = \sum_{x \in V(G)} |S_x| = \sum_{x \in V(G)} |S_x^-| + |S \cap V(G)|.$$

Throughout the paper, we will use the notation K_t , N_t , $K_{1,t-1}$, C_t and P_t for complete graphs, empty graphs, star graphs, cycle graphs and path graphs of order t , respectively.

We will use the notation $G \cong H$ if G and H are isomorphic graphs. For a vertex v of a graph G , $N(v)$ will denote the set of neighbours or *open neighbourhood* of v in G . The *closed neighbourhood*, denoted by $N[v]$, equals $N(v) \cup \{v\}$. A vertex $v \in V(G)$ such that $N[v] = V(G)$ is said to be a *universal vertex*.

A leaf of a graph H is a vertex of degree one, while a support vertex of H is a vertex adjacent to at least one leaf. We denote the set of leaves of H by $\mathcal{L}(H)$ and the set of support vertices of H by $\mathcal{S}(H)$.

For the remainder of the paper, definitions will be introduced whenever a concept is needed.

2 Secure domination in rooted product graphs

To begin the analysis we need to establish some preliminary lemmas.

Lemma 1 *Let S be a $\gamma_s(G \circ_v H)$ -set. For any $x \in V(G)$, $|S_x| \geq \gamma_s(H) - 1$. Furthermore, if $|S_x| = \gamma_s(H) - 1$, then $x \notin S$.*

Proof Suppose that there exists $x \in V(G)$ such that $|S_x| \leq \gamma_s(H) - 2$. If $x \in S$ then S_x is an SDS of H_x and $|S_x| \leq \gamma_r(H_x) - 2$, yielding a contradiction. Now, if $x \notin S$, then the set $W = S_x \cup \{x\}$ is an SDS of H_x and $|W| \leq \gamma_s(H_x) - 1$, which is a contradiction. Therefore, $|S_x| \geq \gamma_s(H) - 1$ for every $x \in V(G)$.

Now, suppose that there exists $x \in V(G)$ such that $|S_x| = \gamma_s(H) - 1$. If $x \in S$, then S_x is an SDS of H_x and $|S_x| < \gamma_s(H_x)$, which is a contradiction. Hence, $x \notin S$.

Given a $\gamma_s(G \circ_v H)$ -set S , we define the sets

$$\mathcal{A}_S = \{x \in V(G) : |S_x| \geq \gamma_s(H)\}$$

and

$$\mathcal{B}_S = \{x \in V(G) : |S_x| = \gamma_s(H) - 1\}.$$

By Lemma 1, $V(G) = \mathcal{A}_S \cup \mathcal{B}_S$ and $\mathcal{A}_S \cap \mathcal{B}_S = \emptyset$. Notice that

$$\gamma_s(G \circ_v H) = \sum_{x \in \mathcal{A}_S} |S_x| + |\mathcal{B}_S|(\gamma_s(H) - 1). \quad (1)$$

Lemma 2 *If S is a $\gamma_s(G \circ_v H)$ -set, then every vertex in \mathcal{B}_S is adjacent to a vertex in $\mathcal{A}_S \cap S$.*

Proof By Lemma 1 we have that $\mathcal{B}_S \cap S = \emptyset$. Now, since S is a $\gamma_s(G \circ_v H)$ -set, if there exists $x \in \mathcal{B}_S$ such that $N(x) \cap V(G) \cap S = \emptyset$, then S_x is an SDS of H_x with $|S_x| = \gamma_s(H_x) - 1$, which is a contradiction. Therefore, every vertex $x \in \mathcal{B}_S$ is adjacent to some vertex belonging to $V(G) \cap S \subseteq \mathcal{A}_S \cap S$.

As a consequence of the above lemma, we obtain the following corollary.

Corollary 1 *If S is a $\gamma_s(G \circ_v H)$ -set, then \mathcal{A}_S is a dominating set of G .*

So, from Eq. (1) and Corollary 1 we deduce the following lower bound for $\gamma_s(G \circ_v H)$.

Theorem 1 *For any pair of graphs G and H ,*

$$\gamma_s(G \circ_v H) \geq n(G)(\gamma_s(H) - 1) + \gamma(G).$$

Proof Let S be a $\gamma_s(G \circ_v H)$ -set. By definition of \mathcal{A}_S and Eq. (1),

$$\gamma_s(G \circ_v H) \geq \sum_{x \in \mathcal{A}_S} \gamma_s(H_x) + |\mathcal{B}_S|(\gamma_s(H) - 1).$$

Now, by Corollary 1, $|\mathcal{A}_S| \geq \gamma(G)$. Hence,

$$\begin{aligned} \gamma_s(G \circ_v H) &\geq |\mathcal{A}_S|\gamma_s(H) + |\mathcal{B}_S|(\gamma_s(H) - 1) \\ &\geq |\mathcal{A}_S|(\gamma_s(H) - 1) + \gamma(G) + |\mathcal{B}_S|(\gamma_s(H) - 1) \\ &= (|\mathcal{A}_S| + |\mathcal{B}_S|)(\gamma_s(H) - 1) + \gamma(G). \\ &= n(G)(\gamma_s(H) - 1) + \gamma(G). \end{aligned}$$

Therefore, the result follows.

In order to deduce the trichotomy of the secure domination number of rooted product graphs (Theorem 2), we need to state the following three results.

Lemma 3 *If S is a $\gamma_s(G \circ_v H)$ -set such that $\mathcal{B}_S \neq \emptyset$, then $|S_x| = \gamma_s(H)$ for every $x \in \mathcal{A}_S$.*

Proof Let S be a $\gamma_s(G \circ_v H)$ -set and suppose that there exists $x \in \mathcal{A}_S$ such that $|S_x| \geq \gamma_s(H) + 1$. Let $u \in \mathcal{B}_S$ and define a set $W = (S \setminus S_x) \cup W_x^- \cup \{x\}$, where W_x^- is induced by S_u^- , that is to say, W_x^- is a copy of the vertices S_u^- of H_u^- , in the subgraph H_x^- . It is readily seen that W is SDS of $G \circ_v H$ and $|W| \leq |S| - 1 = \gamma_s(G \circ_v H) - 1$, which is a contradiction. Hence, $|S_x| = \gamma_s(H)$ for every $x \in \mathcal{A}_S$.

For any $\gamma_s(G \circ_v H)$ -set S , we define the sets

$$\mathcal{A}_S^0 = \{x \in \mathcal{A}_S : x \notin S \text{ and } |S_x| = \gamma_s(H)\}$$

and

$$\mathcal{A}_S^1 = \{x \in \mathcal{A}_S : x \in S \text{ and } |S_x| = \gamma_s(H)\}.$$

Lemma 4 *If there exists a $\gamma_s(G \circ_v H)$ -set S such that $\mathcal{B}_S \neq \emptyset$, then there exists a $\gamma_s(G \circ_v H)$ -set S' such that $\mathcal{B}_{S'} = \mathcal{B}_S$ and $\mathcal{A}_{S'} = \mathcal{A}_S^1$.*

Proof Let S be a $\gamma_s(G \circ_v H)$ -set with $\mathcal{B}_S \neq \emptyset$. Notice that, by Lemma 2, $\mathcal{A}_S^1 \neq \emptyset$, and by Lemma 3, $\mathcal{A}_S = \mathcal{A}_S^1 \cup \mathcal{A}_S^0$. Furthermore, if $\mathcal{A}_S^0 \neq \emptyset$, then we fix $y \in \mathcal{A}_S^1$ and we define a $\gamma_s(G \circ_v H)$ -set S' such that S'_x is induced by S_y for every $x \in \mathcal{A}_S^0$, and $S'_z = S_z$ for every $z \in V(G) \setminus \mathcal{A}_S^0$. In such a case, $\mathcal{B}_{S'} = \mathcal{B}_S$ and $\mathcal{A}_{S'} = \mathcal{A}_S^1$.

Proposition 1 *If there exists a $\gamma_s(G \circ_v H)$ -set S such that $\mathcal{B}_S \neq \emptyset$, then*

$$\gamma_s(G \circ_v H) \leq n(G)(\gamma_s(H) - 1) + \gamma_s(G).$$

Proof Let S be a $\gamma_s(G \circ_v H)$ -set such that $\mathcal{B}_S \neq \emptyset$. Let U be a $\gamma_s(G)$ -set and $x \in \mathcal{B}_S$. By Lemma 1, $x \notin S$, so that S_x^- is an SDS of H_x^- . Consider the set $W \subseteq V(G \circ_v H)$ such that for every vertex $u \in V(G)$, W_u^- is induced by S_x^- and $W \cap V(G) = U$. Thus, W is an SDS and $|W| = n(G)(\gamma_s(H) - 1) + \gamma_s(G)$, concluding that $\gamma_s(G \circ_v H) \leq n(G)(\gamma_s(H) - 1) + \gamma_s(G)$.

We are now ready to prove the following theorem.

Theorem 2 (Trichotomy) *For any graph G , any graph H and any $v \in V(H)$,*

$$\gamma_s(G \circ_v H) \in \{n(G)(\gamma_s(H) - 1) + \gamma(G), n(G)(\gamma_s(H) - 1) + \gamma_s(G), n(G)\gamma_s(H)\}.$$

Furthermore, the following statements hold for any pair of $\gamma_s(G \circ_v H)$ -sets S and S' .

- $\mathcal{B}_S = \emptyset$ if and only if $\mathcal{B}_{S'} = \emptyset$.
- $\gamma_s(G \circ_v H) = n(G)\gamma_s(H)$ if and only if $\mathcal{B}_S = \emptyset$.

Proof Let S be a $\gamma_s(G \circ_v H)$ -set. If $\mathcal{B}_S = \emptyset$, then Lemma 1 leads to $|S_x| \geq \gamma_s(H)$ for every $x \in V(G)$, which implies that $\gamma_s(G \circ_v H) \geq n(G)\gamma_s(H)$. Hence, $\gamma_s(G \circ_v H) = n(G)\gamma_s(H)$, as we always can construct an SDS W such that $|W_x| = \gamma_s(H)$ for every $x \in V(G)$.

From now on we consider the case $\mathcal{B}_S \neq \emptyset$, and so (by Lemma 4) we can assume that $\mathcal{A}_S = \mathcal{A}_S^1$. We differentiate two cases.

Case 1. There exists $x \in \mathcal{B}_S$ such that $S \cap N(x) \cap V(H_x) \neq \emptyset$. Let D be a $\gamma(G)$ -set and consider the set $W \subseteq V(G \circ_v H)$ where W_u^- is induced by S_x^- for every $u \in V(G)$ and $W \cap V(G) = D$. Notice that W_u^- is an SDS of H_u^- for every $u \in V(G)$. Moreover, since D is a dominating set of G and $W \cap N(u) \cap V(H_u) \neq \emptyset$ for every $u \in V(G)$, we conclude that every vertex $u \in V(G) \setminus D$ is protected with respect to W by some vertex in $D \subset W$. Hence, W is an SDS of $G \circ_v H$, which implies that $\gamma_s(G \circ_v H) \leq |W| = n(G)(\gamma_s(H) - 1) + \gamma(G)$, and by Theorem 1 we conclude that in fact this is an equality.

Case 2. $S \cap N(x) \cap V(H_x) = \emptyset$ for every $x \in \mathcal{B}_S$. Notice that in this case every vertex $x \in \mathcal{B}_S$ must be protected with respect to S by some vertex in \mathcal{A}_S . Furthermore, since $\mathcal{A}_S = \mathcal{A}_S^1$, we have that $V(G) \cap S = \mathcal{A}_S$ is an SDS of G , and so $|\mathcal{A}_S| \geq \gamma_s(G)$. Thus, by Lemma 3,

$$\begin{aligned} \gamma_s(G \circ_v H) &= n(G)|\mathcal{A}_S|\gamma_s(H) + |\mathcal{B}_S|(\gamma_s(H) - 1) \\ &= n(G)(\gamma_s(H) - 1) + |\mathcal{A}_S| \geq n(G)(\gamma_s(H) - 1) + \gamma_s(G). \end{aligned}$$

Hence, by Proposition 1 we conclude that $\gamma_s(G \circ_v H) = n(G)(\gamma_s(H) - 1) + \gamma_s(G)$.

Therefore, $\gamma_s(G \circ_v H) = n(G)\gamma_s(H)$ or $\gamma_s(G \circ_v H) = n(G)(\gamma_s(H) - 1) + \gamma(G)$ or $\gamma_s(G \circ_v H) = n(G)(\gamma_s(H) - 1) + \gamma_s(G)$. The remaining statements follow from the previous analysis.

We will characterize, in Theorem 3, the graphs with $\gamma_s(G \circ_v H) = n(G)\gamma_s(H)$. To this end, we need to prove the following two lemmas.

From now on, the subgraph of a graph H , obtained by removing vertex v , will be denoted by $H - \{v\}$. Notice that for any $\gamma_s(H - \{v\})$ -set S , the set $S \cup \{v\}$ is an SDS of H , which implies that the following lemma holds.

Lemma 5 *For any nontrivial graph H and any $v \in V(H)$,*

$$\gamma_s(H - \{v\}) \geq \gamma_s(H) - 1.$$

It is straightforward that for any empty graph $G \cong N_t$,

$$\gamma_s(G \circ_v H) = n(G)\gamma_s(H) = n(G)(\gamma_s(H) - 1) + \gamma_s(G).$$

Obviously, $\gamma_s(G) = n(G)$ if and only if G is an empty graph. In order to establish a necessary and sufficient condition to assure that $\gamma_s(G \circ_v H) = n(G)\gamma_s(H)$ when G is nonempty, we need to state the following lemma.

Lemma 6 *Let S be a $\gamma_s(G \circ_v H)$ -set. If $\mathcal{B}_S \neq \emptyset$, then*

$$\gamma_s(H - \{v\}) = \gamma_s(H) - 1.$$

Proof If there exists $x \in \mathcal{B}_S$, then $|S_x| = \gamma_s(H) - 1$ and $x \notin S$ (by Lemma 1), which implies that S_x^- is an SDS of H_x^- with $|S_x^-| = \gamma_s(H_x) - 1$, and so $\gamma_s(H - \{v\}) \leq \gamma_s(H) - 1$. By Lemma 5 we conclude the proof.

Theorem 3 *Let G be a nonempty graph, H a graph and $v \in V(H)$. The following statements are equivalent.*

- $\gamma_s(G \circ_v H) = n(G)\gamma_s(H)$.
- $\gamma_s(H - \{v\}) \geq \gamma_s(H)$.

Proof Suppose that $\gamma_s(H - \{v\}) < \gamma_s(H)$. In such a case, $\gamma_s(H - \{v\}) = \gamma_s(H) - 1$ by Lemma 5. Hence, from any $\gamma_s(H - \{v\})$ -set and any $\gamma_s(G)$ -set we can construct an SDS on $G \circ_v H$ of cardinality $n(G)(\gamma_s(H) - 1) + \gamma_s(G)$, concluding that $\gamma_s(G \circ_v H) \leq n(G)(\gamma_s(H) - 1) + \gamma_s(G) < n(G)\gamma_s(H)$. Therefore, if $\gamma_s(G \circ_v H) = n(G)\gamma_s(H)$, then $\gamma_s(H - \{v\}) \geq \gamma_s(H)$.

Now, assume that $\gamma_s(H - \{v\}) \geq \gamma_s(H)$ and let S be a $\gamma_s(G \circ_v H)$ -set. By Lemma 6 we have that $\mathcal{B}_S = \emptyset$, and so Theorem 2 leads to $\gamma_s(G \circ_v H) = n(G)\gamma_s(H)$.

We now discuss some particular cases in which $\gamma_s(G \circ_v H) = n(G)\gamma_s(H)$.

Theorem 4 *Given a graph G , a nontrivial graph H and a vertex $v \in V(H)$, $\gamma_s(G \circ_v H) = n(G)$ if and only if $H \cong K_t$, $t \geq 2$.*

Proof The result is trivially true if G is an empty graph. Assume that G is not empty. If $H \cong K_t$, where $t \geq 2$, then $\gamma_s(H - \{v\}) = \gamma_s(K_{t-1}) = 1$ for every vertex $v \in V(H)$. Hence, by Theorem 3 we have $\gamma_s(G \circ_v H) = n(G)\gamma_s(K_t) = n(G)$.

On the other hand, if $H \not\cong K_t$, then $\gamma_s(H) \geq 2$, and by Theorem 1 we have $\gamma_s(G \circ_v H) \geq n(G)(\gamma_s(H) - 1) + \gamma(G) > n(G)$. Therefore, the result follows.

We next provide sufficient conditions to assure that $\gamma_s(G \circ_v H) = n(G)\gamma_s(H)$. The following theorem considers the case in which the root of H is a support vertex.

Theorem 5 *Let G be a graph and H a connected graph. If $v \in \mathcal{S}(H)$ then $\gamma_s(G \circ_v H) = n(G)\gamma_s(H)$.*

Proof The result is trivially true if G is an empty graph. Assume that G is not empty. By Theorem 3, it is enough to show that $\gamma_s(H - \{v\}) \geq \gamma_s(H)$. Let S be a $\gamma_s(H - \{v\})$ -set and $u \in \mathcal{L}(H) \cap N(v)$. Notice that $u \in S$, and so S is an SDS of H , which implies that $\gamma_s(H) \leq |S| = \gamma_s(H - \{v\})$.

Theorem 6 *Let G be a graph, H a graph and $v \in V(H)$. If $v \notin S$ for every $\gamma_s(H)$ -set S , then $\gamma_s(G \circ_v H) = n(G)\gamma_s(H)$.*

Proof As above, we can assume that G is a nonempty graph. Assume that $v \notin S$ for every $\gamma_s(H)$ -set S , and suppose to the contrary that $\gamma_s(G \circ_v H) \neq n(G)\gamma_s(H)$. By Lemma 5 and Theorem 3 we have that $\gamma_s(H - \{v\}) = \gamma_s(H) - 1$. Now, for any $\gamma_s(H - \{v\})$ -set X , we have that $Y = X \cup \{v\}$ is an SDS of H and $|Y| = \gamma_s(H)$, which is a contradiction, as $v \in Y$. Therefore, $\gamma_s(G \circ_v H) = n(G)\gamma_s(H)$.

Notice that if $v \in V(H)$ is an isolated vertex, then $\gamma_s(G \circ_v H) = n(G)(\gamma_s(H) - 1) + \gamma_s(G)$. The following result concerns a case in which v is not an isolated vertex.

Theorem 7 *Let G be a graph and H a graph. If $v \in V(H)$ is not an isolated vertex and $v \in S$ for every $\gamma_s(H)$ -set S , then $\gamma_s(G \circ_v H) = n(G)\gamma_s(H)$.*

Proof As above, we only need to consider the case in which G is a nonempty graph. Let v be a nonisolated vertex such that $v \in S$ for every $\gamma_s(H)$ -set S . Suppose to the contrary that $\gamma_s(G \circ_v H) \neq n(G)\gamma_s(H)$. By Lemma 5 and Theorem 3 we have that $\gamma_s(H - \{v\}) = \gamma_s(H) - 1$. Let X be a $\gamma_s(H - \{v\})$ -set. Since v is not isolated, there exists $u \in V(H) \cap N(v)$. If $u \notin X$, then $Y = X \cup \{u\}$ is a $\gamma_s(H)$ -set and $v \notin Y$, which is a contradiction. Now, if $N(v) \subset X$, observe that there exists $w \in N(u)$ for some $u \in N(v)$ such that $w \notin X$ and w is protected with respect to X by u , otherwise X would be a $\gamma_s(H)$ -set, which is a contradiction. Notice that in this case $Y' = X \cup \{w\}$ is a $\gamma_s(H)$ -set and $v \notin Y'$, which is a contradiction. Therefore, $\gamma_s(G \circ_v H) = n(G)\gamma_s(H)$.

We now discuss the cases in which $\gamma_s(G \circ_v H) \neq n(G)\gamma_s(H)$. From Lemma 5 and Theorems 2 and 3 we deduce the following result.

Theorem 8 *Let G be a nonempty graph, H a graph and $v \in V(H)$. The following statements are equivalent.*

- $\gamma_s(G \circ_v H) \in \{n(G)(\gamma_s(H) - 1) + \gamma_s(G), n(G)(\gamma_s(H) - 1) + \gamma(G)\}$.
- $\gamma_s(H - \{v\}) = \gamma_s(H) - 1$.

We now focus on the case of graphs G with $\gamma_s(G) \neq \gamma(G)$. The next theorem characterizes the graphs with $\gamma_s(G \circ_v H) = n(G)(\gamma_s(H) - 1) + \gamma(G)$.

Theorem 9 *Let G be a graph such that $\gamma_s(G) > \gamma(G)$, and let H be a graph and $v \in V(H)$. The following statements are equivalent.*

- $\gamma_s(G \circ_v H) = n(G)(\gamma_s(H) - 1) + \gamma(G)$.
- $\gamma_s(H - \{v\}) = \gamma_s(H) - 1$ and there exists a $\gamma_s(H - \{v\})$ -set U such that $N(v) \cap U \neq \emptyset$.

Proof Assume that $\gamma_s(G \circ_v H) = n(G)(\gamma_s(H) - 1) + \gamma(G)$. By Theorem 8, $\gamma_s(H - \{v\}) = \gamma_s(H) - 1$. Suppose to the contrary that $N(v) \cap U = \emptyset$ for every $\gamma_s(H - \{v\})$ -set U . Let S be a $\gamma_s(G \circ_v H)$ -set. Since $\gamma(G) < \gamma_s(G) \leq n(G)$, we have that $\gamma_s(G \circ_v H) = n(G)(\gamma_s(H) - 1) + \gamma(G) < n(G)\gamma_s(H)$, concluding that $\mathcal{B}_S \neq \emptyset$ by Theorem 2. Notice that by Lemma 4 we can assume that $\mathcal{A}_S = \mathcal{A}_S^1$. Now, by Lemma 1 we have that $S \cap \mathcal{B}_S = \emptyset$, which implies that S_x^- is a $\gamma(H_x^-)$ -set for every $x \in \mathcal{B}_S$. Hence, by assumption, $N(x) \cap V(H_x) \cap S = N(x) \cap S_x^- = \emptyset$ for every $x \in \mathcal{B}_S$. Therefore, as in the proof of Theorem 2 (Case 2), we have that $\gamma_s(G \circ_v H) = n(G)(\gamma_s(H) - 1) + \gamma_s(G)$, which is a contradiction as $\gamma(G) < \gamma_s(G)$.

Conversely, assume that $\gamma_s(H - \{v\}) = \gamma_s(H) - 1$ and there exists a $\gamma_s(H - \{v\})$ -set U such that $N(v) \cap U \neq \emptyset$. Let X be a $\gamma(G)$ -set and consider the set

$W \subseteq V(G \circ_v H)$ such that $W \cap V(G) = X$ and W_x^- is induced by U for every vertex $x \in V(G)$. Notice that W is SDS of $G \circ_v H$ and $|W| = n(G)(\gamma_s(H) - 1) + \gamma(G)$, which implies that $\gamma_s(G \circ_v H) \leq n(G)(\gamma_s(H) - 1) + \gamma(G)$. Thus, by Theorem 8 (or by Theorem 2) we conclude that $\gamma_s(G \circ_v H) = n(G)(\gamma_s(H) - 1) + \gamma(G)$.

As a consequence of Theorems 8 and 9 we characterize the graphs with $\gamma_s(G \circ_v H) = n(G)(\gamma_s(H) - 1) + \gamma_s(G)$.

Theorem 10 *Let G be a graph such that $\gamma_s(G) > \gamma(G)$, and let H be a graph and $v \in V(H)$. The following statements are equivalent.*

- $\gamma_s(G \circ_v H) = n(G)(\gamma_s(H) - 1) + \gamma_s(G)$.
- $\gamma_s(H - \{v\}) = \gamma_s(H) - 1$ and $N(v) \cap S = \emptyset$ for every $\gamma_s(H - \{v\})$ -set S .

The particular case in which $\gamma_s(H) = 1$ was discussed in Theorem 4. We now consider the particular case in which $\gamma_s(H) = 2$.

Theorem 11 *Let G be a nonempty graph, H a connected graph and $v \in V(H)$. If $\gamma_s(H) = 2$, then the following statements hold.*

- (i) *If $H - \{v\} \not\cong K_t$, then $\gamma_s(G \circ_v H) = 2n(G)$.*
- (ii) *If $H - \{v\} \cong K_t$, then $\gamma_s(G \circ_v H) = n(G) + \gamma(G)$.*

Proof If $H - \{v\} \not\cong K_t$ then $\gamma_s(H - \{v\}) \geq 2 = \gamma_s(H)$. Hence, Theorem 3 leads to $\gamma_s(G \circ_v H) = n(G)\gamma_s(H) = 2n(G)$.

On the other hand, if $H - \{v\} \cong K_t$ then $\gamma_s(H - \{v\}) = 1 = \gamma_s(H) - 1$. Now, since H is connected and $H - \{v\} = K_t$, any $\{y\} \subseteq N(v)$ is a $\gamma_s(H - \{v\})$ -set. Therefore, by Theorem 9 (and by Theorem 8 for the case $\gamma_s(G) = \gamma(G)$) we conclude that $\gamma_s(G \circ_v H) = n(G)(\gamma_s(H) - 1) + \gamma(G) = n(G) + \gamma(G)$.

3 Secure domination in corona product graphs

Given two graphs G and H , the corona product $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G and $n(G)$ copies of H and joining by an edge each vertex of the i^{th} copy of H with the i^{th} vertex of G for each $i \in \{1, \dots, n(G)\}$.

The join $G + H$ is defined as the graph obtained from disjoint graphs G and H by taking one copy of G and one copy of H and joining by an edge each vertex of G with each vertex of H . Notice that the corona product graph $K_1 \odot H$ is isomorphic to the join graph $K_1 + H$. Furthermore, any corona product graph $G \odot H$ can be seen as a rooted product, i.e.,

$$G \odot H \cong G \circ_v (K_1 + H), \quad (2)$$

where v is the vertex of K_1 .

The following lemmas allow us to provide a closed formula for the secure domination number of corona product graphs.

Lemma 7 For any graph H ,

$$\gamma(H) \leq \gamma_s(K_1 + H) \leq \gamma(H) + 1.$$

Proof Let D be a $\gamma(H)$ -set, $V(K_1) = \{x\}$ and $X = D \cup \{x\}$. Notice that for every $u \in V(K_1 + H) \setminus X = V(H) \setminus D$ there exists a vertex $v \in N(u) \cap D$ and $(X \setminus \{v\}) \cup \{u\}$ is a dominating set of $K_1 + H$, which implies that X is an SDS of $K_1 + H$. Therefore, $\gamma_s(K_1 + H) \leq |X| = \gamma(H) + 1$.

Now, let S be a $\gamma_s(K_1 + H)$ -set. If $x \notin S$, then $\gamma(H) \leq \gamma_s(K_1 + H)$, as $S \subseteq V(H)$ is a dominating set of H . From now on, suppose that $x \in S$. If there exists $v \in V(H)$ such that v is protected by x with respect to S , then $(S \setminus \{x\}) \cup \{v\}$ is a dominating set of H , concluding that $\gamma(H) \leq |(S \setminus \{x\}) \cup \{v\}| = |S| = \gamma_s(K_1 + H)$. Finally, if x does not protect any vertex with respect to S , then $S \setminus \{x\}$ is a dominating set of H , obtaining that $\gamma(H) \leq |S \setminus \{x\}| < \gamma_s(K_1 + H)$.

Lemma 8 For any graph H ,

$$\gamma_s(K_1 + H) \leq \gamma_s(H).$$

Proof Let S be a $\gamma_s(H)$ -set and $V(K_1) = \{x\}$. To conclude that S is an SDS of $K_1 + H$ we only need to observe that $(S \setminus \{v\}) \cup \{x\}$ is a dominating set of $K_1 + H$ for any $v \in S$. Therefore, $\gamma_s(K_1 + H) \leq |S| = \gamma_s(H)$.

We next provide a closed formula for the secure domination number of corona product graphs.

Theorem 12 For any nonempty graph G and any graph H ,

$$\gamma_s(G \odot H) = n(G)\gamma_s(K_1 + H) = \begin{cases} n(G)\gamma(H), & \text{if } \gamma_s(K_1 + H) = \gamma(H); \\ n(G)(\gamma(H) + 1), & \text{otherwise.} \end{cases}$$

Proof From Eq. (2), Theorem 3 and Lemma 8 we immediately conclude that $\gamma_s(G \odot H) = n(G)\gamma_s(K_1 + H)$. Therefore, applying Lemma 7 we conclude the proof.

We now proceed to consider some particular cases of Theorem 12.

Theorem 13 If $\gamma_s(H) = \gamma(H)$, then for any graph G ,

$$\gamma_s(G \odot H) = n(G)\gamma(H)$$

Proof The result is trivially true if G is an empty graph. Assume that G is a nonempty graph. By Lemmas 7 and 8 we have that $\gamma(H) \leq \gamma_s(K_1 + H) \leq \gamma_s(H)$. So, if $\gamma_s(H) = \gamma(H)$ then $\gamma_s(K_1 + H) = \gamma(H)$. Therefore, by Theorem 12 we conclude the proof.

The next result is an important tool to derive some particular cases of Theorem 12. Recall that given a set $S \subseteq V(H)$ and a vertex $v \in S$ the *set of external private neighbours of v with respect to S* is defined to be

$$epn(v, S) = \{u \in V(H) \setminus S : N(u) \cap S = \{v\}\}.$$

Notice that if S is an SDS, then for any $v \in S$ we have that $epn(v, S) = \emptyset$ or the subgraph induced by $epn(v, S)$ is a clique.

Theorem 14 *Let H be a graph. The following statements are equivalent.*

- $\gamma_s(K_1 + H) = \gamma(H)$.
- There exists a $\gamma(H)$ -set D and a vertex $v \in D$ such that $epn(v, D) = \emptyset$ or the subgraph induced by $epn(v, D)$ is a clique.

Proof It is clear that if H is a complete graph, then the result follows. From now on we can assume that $\gamma_s(H) \geq 2$. Let $V(K_1) = \{x\}$ and suppose that (ii) holds. We will show that $S = (D \setminus \{v\}) \cup \{x\}$ is an SDS of $K_1 + H$. Since x is adjacent to every vertex of H , S is a dominating set of $K_1 + H$. Let $u \in V(K_1 + H) \setminus S$. If $u \notin epn(v, D) \cup \{v\}$, then there exists a vertex $w \in S \setminus \{x\}$ which is adjacent to u . So, the set $(S \setminus \{w\}) \cup \{u\}$ is a dominating set of $K_1 + H$. If $u \in epn(v, D) \cup \{v\}$, as $(D \setminus \{v\}) \cup \{u\}$ is a dominating set of H , $(S \setminus \{x\}) \cup \{u\}$ is a dominating set of $K_1 + H$. Hence, S is an SDS of $K_1 + H$, concluding that $\gamma_s(K_1 + H) \leq |S| = |D| = \gamma(H)$. Therefore, $\gamma_s(K_1 + H) = \gamma(H)$ by Lemma 7.

Now, suppose that $\gamma_s(K_1 + H) = \gamma(H)$ and let S' be an SDS of $K_1 + H$. If $x \notin S'$, then S' is an SDS of H and so (ii) follows. From now on we assume that $x \in S'$. Notice that $epn(x, S') \neq \emptyset$, otherwise $S' \setminus \{x\}$ would be a dominating set of H , contradicting that $\gamma_s(K_1 + H) = \gamma(H)$. Notice also that $epn(x, S')$ is a clique since S' is an SDS of $K_1 + H$. Hence, for any $v \in epn(x, S')$ the set $D' = (S' \setminus \{x\}) \cup \{v\}$ is a $\gamma(H)$ -set and $epn(v, D') = epn(x, S') \setminus \{v\}$, concluding that $epn(v, D') = \emptyset$ or the subgraph induced by $epn(v, D')$ is a clique. Therefore, (ii) follows.

To conclude the analysis, we consider some particular cases of graph H .

Proposition 2 *The following statements hold for any graph G .*

- $\gamma_s(G \odot N_t) = n(G)t$.
- $\gamma_s(G \odot K_t) = n(G)$.
- $\gamma_s(G \odot K_{1,l}) = 2n(G)$ whenever $l \geq 2$.
- $\gamma_s(G \odot K_{l,r}) = \begin{cases} 3n(G), & \text{if } l \geq r \geq 3; \\ 2n(G), & \text{if } l = 2 \leq r. \end{cases}$
- $\gamma_s(G \odot P_t) = \begin{cases} n(G)(\frac{t}{3} + 1), & \text{if } t \equiv 0 \pmod{3}; \\ n(G)\lceil \frac{t}{3} \rceil, & \text{otherwise.} \end{cases}$
- $\gamma_s(G \odot C_t) = \begin{cases} n(G)(\frac{t}{3} + 1), & \text{if } t \equiv 0 \pmod{3}, t \geq 4; \\ n(G)\lceil \frac{t}{3} \rceil, & \text{otherwise.} \end{cases}$

Proof Since $\gamma_s(K_1 + N_t) = \gamma_s(K_{1,t}) = t = \gamma(N_t)$, $\gamma_s(K_1 + K_t) = \gamma_s(K_{t+1}) = 1 = \gamma(K_t)$ and $\gamma_s(K_1 + K_{1,l}) = \gamma_s(K_2 + N_l) = 2 > 1 = \gamma(K_{1,l})$, from Theorem 12 we deduce (i), (ii) and (iii).

Notice that if $l, r \geq 2$, then $\gamma(K_{l,r}) = 2$. Let $\{\mathcal{L}, \mathcal{R}\}$ be the bipartition of $V(K_{l,r})$ with $|\mathcal{L}| = l$ and $|\mathcal{R}| = r$. Observe that for every $x_1 \in \mathcal{L}$ and $x_2 \in \mathcal{R}$, the set $X = \{x_1, x_2\}$ is a $\gamma(K_{l,r})$ -set. Now, if $l, r \geq 3$, then the subgraphs induced by $epn(x_1, X) = \mathcal{R} \setminus \{x_2\}$ and $epn(x_2, X) = \mathcal{L} \setminus \{x_1\}$ are not cliques and these sets are not empty. Hence, by Theorems 12 and 14, we deduce that $\gamma_s(G \odot K_{l,r}) = 3n(G)$ whenever $l, r \geq 3$. On the other hand, if $\mathcal{L} = \{u_1, u_2\}$, then $epn(u_1, \mathcal{L}) = epn(u_2, \mathcal{L}) = \emptyset$ and \mathcal{L} is a $\gamma(K_{2,r})$ -set. Hence, by Theorems 12 and 14 we deduce that if $l = 2 \leq r$, then $\gamma_s(G \odot K_{l,s}) = 2n(G)$. Therefore, (iv) follows.

Let $V(P_t) = \{u_1, u_2, \dots, u_t\}$ and assume that consecutive vertices are adjacent. It is well known that $\gamma(P_t) = \lceil \frac{t}{3} \rceil$. If $t \equiv 0 \pmod{3}$, the only $\gamma(P_t)$ -set is $S = \{u_k : k \equiv 2 \pmod{3}\}$ and for every $k \equiv 2 \pmod{3}$, $epn(u_k, S) = \{u_{k-1}, u_{k+1}\}$, so the graph induced by $epn(u_k, S)$ is not a clique, and by Theorems 12 and 14 we deduce that $\gamma_s(G \odot P_t) = n(G)(\gamma(P_t) + 1)$. Now, if $t \not\equiv 0 \pmod{3}$, then $S = \{u_k : k \equiv 1 \pmod{3}\}$ is a $\gamma(P_t)$ -set and $epn(u_1, S) = \{u_2\}$. Therefore, in this case Theorems 12 and 14 lead to $\gamma_s(G \odot P_t) = n(G)\gamma(P_t)$. Therefore, (v) follows.

Finally, for $t \geq 4$ the proof of (vi) is analogous to the proof of (v), where we use the fact that $\gamma(C_t) = \lceil \frac{t}{3} \rceil$. Notice that the case $t = 3$ was previously discussed in (ii).

4 Conclusions.

This article is a contribution to the theory of protection of graphs. In particular, it is devoted to study the secure domination number of a graph. We obtain closed formulas for the secure domination number of rooted product graphs and, as a particular case of the study, we derive the corresponding formulas for corona graphs. Among our main contributions we highlight the following.

We first show that the secure domination number of any rooted product graph satisfies the following trichotomy (Theorem 2):

$$\gamma_s(G \circ_v H) \in \{n(G)(\gamma_s(H) - 1) + \gamma(G), n(G)(\gamma_s(H) - 1) + \gamma_s(G), n(G)\gamma_s(H)\}$$

Then, we characterize the graphs with $\gamma_s(G \circ_v H) = n(G)\gamma_s(H)$ (Theorem 3), the graphs with $\gamma_s(G \circ_v H) = n(G)(\gamma_s(H) - 1) + \gamma(G)$ (Theorems 8 and 9) and the graphs with $\gamma_s(G \circ_v H) = n(G)(\gamma_s(H) - 1) + \gamma_s(G)$ (Theorems 8 and 10).

As a consequence of the study, we derive the following formula for corona product graphs $G \odot H$, where G is a nonempty graph (Theorem 12):

$$\gamma_s(G \odot H) = n(G)\gamma_s(K_1 + H) = \begin{cases} n(G)\gamma(H), & \text{if } \gamma_s(K_1 + H) = \gamma(H); \\ n(G)(\gamma(H) + 1), & \text{otherwise.} \end{cases}$$

Finally, we characterize the graphs H satisfying $\gamma_s(K_1 + H) = \gamma(H)$ (Theorem 14).

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