Discussiones Mathematicae
Graph Theory 42 (2022) 139–158
https://doi.org/10.7151/dmgt.2243

PROTECTION OF LEXICOGRAPHIC PRODUCT GRAPHS

Douglas J. Klein

Texas A&M University at Galveston
Foundational Sciences
Galveston, TX 77553, United States

e-mail: kleind@tamug.edu

and

Juan A. Rodríguez-Velázquez

Universitat Rovira i Virgili
Departament d'Enginyeria Informàtica i Matemàtiques
Av. Països Catalans 26, 43007 Tarragona, Spain

e-mail: juanalberto.rodriguez@urv.cat

Abstract

In this paper, we study the weak Roman domination number and the secure domination number of lexicographic product graphs. In particular, we show that these two parameters coincide for almost all lexicographic product graphs. Furthermore, we obtain tight bounds and closed formulas for these parameters.

Keywords: lexicographic product, weak Roman domination, secure domination, total domination, double total domination.

2010 Mathematics Subject Classification: 05C69, 05C76.

1. Introduction

The following approach to protection of a graph was described by Cockayne *et al.* [9]. Suppose that one or more guards are stationed at some of the vertices of a simple graph G and that a guard at a vertex can deal with a problem at any vertex in its closed neighbourhood. We say that G is protected if there is at least one guard available to handle a problem at any vertex. Consider a function $f : V(G) \longrightarrow \{0, 1, 2, \dots\}$ where $f(v)$ is the number of guards stationed at v, and let $V_i = \{v \in V(G) :\ f(v) = i\}$ for every $i \in \{0, 1, 2, \dots\}$. We will identify f with the partition of $V(G)$ induced by f and write $f(V_0, V_1, \dots)$. The weight of f is defined to be $w(f) = \sum_{v \in V(G)} f(v) = \sum_i i|V_i|$. A vertex $v \in V(G)$ is *undefended* with respect to f if $f(v) = 0$ and $f(u) = 0$ for every vertex u adjacent to v. We say that G is *protected* under the function f if G has no undefended vertices with respect to f. We now define the four particular subclasses of protected graphs considered in [9]. The functions in each subclass protect the graph according to a certain strategy.

• We say that $f(V_0, V_1)$ is a *dominating function* (DF) if G is protected under f. Obviously, $f(V_0, V_1)$ is a DF if and only if V_1 is a dominating set. The *domination number* is defined by

$$\gamma(G) = \min\{w(f) :\ f \text{ is a DF on } G\}.$$

This classical method of protection has been studied extensively [14, 15].

• A *Roman dominating function* (RDF) is a function $f(V_0, V_1, V_2)$ such that for every $v \in V_0$ there exists a vertex $u \in V_2$ which is adjacent to v. The *Roman domination number* is defined by

$$\gamma_R(G) = \min\{w(f) :\ f \text{ is a RDF on } G\}.$$

This concept of protection has historical motivation [22] and was formally proposed by Cockayne et al. in [10].

• A *weak Roman dominating function* (WRDF) is a function $f(V_0, V_1, V_2)$ such that for every $v \in V_0$ there exists a neighbour u of v such that $u \in V_1 \cup V_2$ and G does not have undefended vertices under the function $f' : V(G) \longrightarrow \{0, 1, 2\}$ defined by $f'(v) = 1$, $f'(u) = f(u) - 1$ and $f'(z) = f(z)$ for every $z \in V(G) \setminus \{u, v\}$. The *weak Roman domination number* is defined by

$$\gamma_r(G) = \min\{w(f) :\ f \text{ is a WRDF on } G\}.$$

A WRDF of weight $\gamma_r(G)$ is called a $\gamma_r(G)$-function. For instance, for the graph shown in Figure 1, on the left, a $\gamma_r(G)$-function can place 2 guards at the vertex of degree three and one guard at the other white-coloured vertex. This concept of protection was introduced by Henning and Hedetniemi [16] and studied further in [6, 8, 25].

• A *secure dominating function* is a WRDF $f(V_0, V_1, V_2)$ in which $V_2 = \emptyset$. In this case, it is convenient to define this concept of protected graph by the properties of V_1. Obviously, $f(V_0, V_1, \emptyset)$ is a secure dominating function if and only if V_1 is a dominating set and for every $v \in V_0$ there exists $u \in V_1$ which is adjacent to v and $(V_1 \setminus \{u\}) \cup \{v\}$ is a dominating set. In such a case, V_1 is said to be a *secure dominating set* (SDS). The *secure domination number* is defined by

$$\gamma_s(G) = \min\{|S| :\ S \text{ is a SDS of } G\}.$$

A secure dominating function of weight $\gamma_s(G)$ is called a $\gamma_s(G)$-function. Analogously, a secure dominating set of cardinality $\gamma_s(G)$ is called a $\gamma_s(G)$-set. For instance, for the graph shown in Figure 1, on the right, a $\gamma_s(G)$-function (set) can place one guard at each white-coloured vertex. This concept of protection was introduced by Cockayne *et al.* in [9], and studied further in [3, 4, 6, 8, 19, 26].

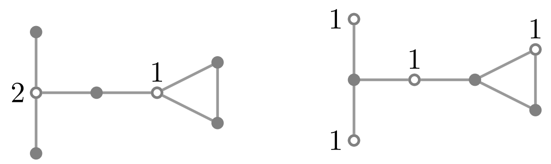

Figure 1. Two placements of guards which correspond to two different weak Roman dominating functions on the same graph. Notice that $2 = \gamma(G) < \gamma_r(G) < \gamma_s(G) = 4$.

The problem of computing $\gamma_r(G)$ is NP-hard, even when restricted to bipartite or chordal graphs [16], and the problem of computing $\gamma_s(G)$ is also NP-Hard, even when restricted to split graphs [3]. This suggests finding the weak Roman domination number and the secure domination number for special classes of graphs or obtaining good bounds on these invariants. This is precisely the aim of this work in which we show that these two parameters coincide for almost all lexicographic product graphs.

Let G and H be two graphs. The *lexicographic product* of G and H is the graph $G \circ H$ whose vertex set is $V(G \circ H) = V(G) \times V(H)$ and $(u, v)(x, y) \in E(G \circ H)$ if and only if $ux \in E(G)$ or $u = x$ and $vy \in E(H)$. Notice that for any $u \in V(G)$ the subgraph of $G \circ H$ induced by $\{u\} \times V(H)$ is isomorphic to H. For simplicity, we will denote this subgraph by H_u, and if a vertex of G is denoted by u_i, then the referred subgraph will be denoted by H_i.

For basic properties of the lexicographic product of two graphs we suggest the books [13, 18]. A main problem in the study of product of graphs consists of finding exact values or sharp bounds for specific parameters of the product of two graphs and express these in terms of invariants of the factor graphs. In particular, we cite the following works on domination theory of lexicographic product graphs: standard domination [20, 21], Roman domination [23], weak Roman domination [25], total weak Roman domination [5], rainbow domination [24], super domination [11] and doubly connected domination [1].

Throughout the paper, we will use the notation K_n, $K_{1,n-1}$, C_n, N_n and P_n for complete graphs, star graphs, cycle graphs, empty graphs and path graphs of order n, respectively. We use the notation $G \cong H$ if G and H are isomorphic graphs. For a vertex v of a graph G, $N(v)$ will denote the set of neighbours or *open neighbourhood* of v in G. The *closed neighbourhood* of v, denoted by $N[v]$, equals $N(v) \cup \{v\}$. A vertex $v \in V(G)$ such that $N[v] = V(G)$ is said to be an *universal vertex*. For the remainder of the paper, definitions will be introduced whenever a concept is needed.

2. Preliminaries and Tools

To begin this section we would emphasize the following inequality chains.

Proposition 1 [9]**.** *The following inequalities hold for any graph G.*

(i) $\gamma(G) \leq \gamma_r(G) \leq \gamma_R(G) \leq 2\gamma(G)$.

(ii) $\gamma(G) \leq \gamma_r(G) \leq \gamma_s(G)$.

The problem of characterizing the graphs with $\gamma_r(G) = \gamma(G)$ was solved by Henning and Hedetniemi [16]. The inequality chain (ii) has motivated the authors of [26] to obtain the following result, which shows that the problem of characterizing the graphs with $\gamma_s(G) = \gamma(G)$ is already solved.

Theorem 1 [26]**.** *Given a graph G,*

$$\gamma_r(G) = \gamma(G) \Leftrightarrow \gamma_s(G) = \gamma(G).$$

In particular, it is readily seen that the following remark follows.

Remark 2 [25]**.** Given a graph G of order n,

$$\gamma_r(G) = 1 \Leftrightarrow \gamma_s(G) = 1 \Leftrightarrow G \cong K_n.$$

The following remark will be useful in the next section.

Remark 3 [25]**.** Given a noncomplete graph G,

$$\gamma_r(G) = 2 \Leftrightarrow \gamma(G) = 1 \text{ or } \gamma_s(G) = 2.$$

Given a graph G and an edge $e \in E(G)$, the graph obtained from G by removing e will be denoted by $G - e$, i.e., $V(G - e) = V(G)$ and $E(G - e) = E(G)\backslash\{e\}$. As observed in [16], any $\gamma_r(G-e)$-function is a WRDF for G. Similarly, as observed in [26], any $\gamma_s(G-e)$-set is a secure dominating set for G. Therefore, the following basic result follows.

Proposition 4. *The following statement hold for any spanning subgraph H of a graph G.*

(i) [16] $\gamma_r(G) \leq \gamma_r(H)$.

(ii) [26] $\gamma_s(G) \leq \gamma_s(H)$.

A set $S \subseteq V(G)$ is a k-dominating set if $|N(v) \cap S| \geq k$ for every $v \in V(G) \setminus S$. The minimum cardinality among all k-dominating sets is called the k-domination number of G and it is denoted by $\gamma_k(G)$. It is readily seen that any 2-dominating set is a secure dominating set. Therefore, we can state the following result.

Theorem 2 [6]**.** *For any graph G,*

$$\gamma_s(G) \leq \gamma_2(G).$$

A *double total dominating set* of a graph G with minimum degree greater than or equal to two is a set S of vertices of G such that every vertex of G is adjacent to at least two vertices in S, [17]. The *double total domination number* of G, denoted by $\gamma_{2,t}(G)$, is the cardinality of a smallest double total dominating set, and we refer to such a set as a $\gamma_{2,t}(G)$-set. Since any $\gamma_{2,t}(G)$-set is a secure dominating set, we deduce the following result.

Theorem 3. *For any graph G of minimum degree greater than or equal to two,*

$$\gamma_s(G) \leq \gamma_{2,t}(G).$$

After Theorem 9 we show a family of lexicographic product graphs for which the bound above is achieved.

3. Some Cases Where $\gamma_s(G \circ H) = \gamma_r(G \circ H)$

From Proposition 1(ii) we learned that for any lexicographic product graph $\gamma_s(G \circ H) \geq \gamma_r(G \circ H)$. Furthermore, from Theorem 1 we have that $\gamma_r(G \circ H) = \gamma(G \circ H)$ if and only if $\gamma_s(G \circ H) = \gamma(G \circ H)$. In this paper we show that $\gamma_s(G \circ H) = \gamma_r(G \circ H)$ for almost all lexicographic product graphs.

Theorem 4. *For any graph G without isolated vertices and any graph H with $\gamma_s(H) \leq 2$ or $\gamma_r(H) \geq 3$,*

$$\gamma_s(G \circ H) = \gamma_r(G \circ H).$$

Proof. By Proposition 1, we only need to show that if G and H satisfy the premises, then $\gamma_s(G \circ H) \leq \gamma_r(G \circ H)$. To this end, let $f(W_0, W_1, W_2)$ be a $\gamma_r(G \circ H)$-function. If $W_2 = \emptyset$, then we are done, as W_1 is a secure dominating set, which implies that $\gamma_s(G \circ H) \leq |W_1| = w(f) = \gamma_r(G \circ H)$. From now on we suppose that $W_2 \neq \emptyset$.

We first consider the case $\gamma_s(H) \leq 2$. Let $\{h, h'\}$ be a secure dominating set of H and $D_2 = \{u \in V(G) : f(u, v) = 2$, for some $v \in V(H)\}$. The set $W' = W_1 \cup (D_2 \times \{h, h'\})$ is a secure dominating set, as $\{(u, h), (u, h')\}$ is a secure dominating set of H_u and every vertex outside $\{u\} \times V(H)$, adjacent to $(u,v) \in W_2$, is adjacent to (u,h) and also to (u,h'). Therefore, $\gamma_s(G \circ H) \leq |W'| \leq |W_1| + 2|W_2| = \gamma_r(G \circ H)$.

We now assume that $\gamma_r(H) \geq 3$. We differentiate two cases for $(u,v) \in W_2$ to show that there exists a WRDF $f_1\left(W_0^{(1)}, W_1^{(1)}, W_2^{(1)}\right)$ of weight $\omega(f_1) = \omega(f)$ such that $W_2^{(1)} = W_2 \setminus \{(u,v)\}$.

Case 1. $f(x,y) = 0$ for every $x \in N(u)$ and every $y \in V(H)$. In this case the restriction of f to $\{u\} \times V(H)$ is a WRDF on H_u. Thus, $D = \{h \in V(H) : f(u,h) > 0\}$ is a dominating set of H and, since $\gamma_r(H) \geq 3$, we can claim that $|D| \geq 2$. Now, we fix $x_0 \in N(u)$ and $y_0 \in V(H)$ and define the function $f_1\left(W_0^{(1)}, W_1^{(1)}, W_2^{(1)}\right)$ by $W_0^{(1)} = W_0 \setminus \{(x_0,y_0)\}$, $W_1^{(1)} = W_1 \cup \{(u,v),(x_0,y_0)\}$ and $W_2^{(1)} = W_2 \setminus \{(u,v)\}$. Obviously, $w(f_1) = w(f)$. Moreover, since every vertex in $N[u] \times V(H)$ is adjacent to at least two vertices in $\{(x_0,y_0)\} \cup (\{u\} \times D)$, the function f_1 is a WRDF on $G \circ H$.

Case 2. $f(x,y) > 0$ for some $x \in N(u)$ and $y \in V(H)$. Notice that if $f(u,y) > 0$ for every $y \in V(H)$, then the function $f_0\left(W_0^{(0)}, W_1^{(0)}, W_2^{(0)}\right)$ defined by $W_0^{(0)} = W_0$, $W_1^{(0)} = W_1 \cup \{(u,v)\}$ and $W_2^{(0)} = W_2 \setminus \{(u,v)\}$) is a WRDF on $G \circ H$, which is a contradiction, as $w(f_0) < w(f) = \gamma_r(G \circ H)$. Thus, there exists $v' \in V(H)$ such that $(u,v') \in W_0$ and we can define the function $f_1\left(W_0^{(1)}, W_1^{(1)}, W_2^{(1)}\right)$ by $W_0^{(1)} = W_0 \setminus \{(u,v')\}$, $W_1^{(1)} = W_1 \cup \{(u,v),(u,v')\}$ and $W_2^{(1)} = W_2 \setminus \{(u,v)\}$. Under the placement of guard stated by f_1, the movement of the guard stationed at (u,v') does not produce undefended vertices, as every vertex in $\{u\} \times V(H)$ is adjacent to (x,y) and, for any $u' \in N(u)$, every vertex in $\{u'\} \times V(H)$ is adjacent to (u,v). By similar arguments, the movement of the guard stationed at (u,v) does not produce undefended vertices. Hence, f_1 is a WRDF and $w(f_1) = w(f)$.

We can repeat the procedure above for any vertex belonging to W_2 ($t = |W_2|$ times) until construct a WRDF $f_t\left(W_0^{(t)}, W_1^{(t)}, W_2^{(t)}\right)$ of weight $w(f_t) = w(f)$ such that $W_2^{(t)} = \emptyset$. Therefore, $\gamma_s(G \circ H) \leq w(f_t) = \gamma_r(G \circ H)$. ■

It was shown in [25] that $\gamma_r(G \circ K_{n'}) = \gamma_r(G)$, for every integer $n' \geq 1$. Therefore, by Theorem 4, the problem of computing $\gamma_s(G \circ K_{n'})$ is equivalent to the problem of computing $\gamma_r(G)$.

The problem of comparing $\gamma_s(G \circ H)$ and $\gamma_r(G \circ H)$ when $\gamma_s(H) \geq 3$ and $\gamma_r(H) = 2$ remains open. Notice that by Remark 3 we have that $\gamma_s(H) \geq 3$ and $\gamma_r(H) = 2$ if and only if $\gamma_s(H) \geq 3$ and $\gamma(H) = 1$. Hence, we can state the following open problem.

Problem 1. Characterize the graphs G and H where $\gamma_s(G \circ H) = \gamma_r(G \circ H)$ subject to the restrictions $\gamma_s(H) \geq 3$ and $\gamma(H) = 1$.

Some particular cases of this problem will be solved later. Notice that the class of graphs H with $\gamma_s(H) \geq 3$ and $\gamma(H) = 1$ contains the family of graphs having at least three vertices of degree one and exactly one universal vertex.

4. The Case Where $\gamma(G) = 1$

In this section we discuss the case in which G has universal vertices. To begin with, we consider the case $G \cong K_n$.

Proposition 5. *For any integer $n \geq 2$ and any noncomplete graph H,*

$$\gamma_s(K_n \circ H) = \gamma_r(K_n \circ H) \in \{2, 3\}.$$

Furthermore, $\gamma_s(K_n \circ H) = \gamma_r(K_n \circ H) = 2$ if and only if $\gamma_r(H) = 2$ or there exists a vertex a of H such that $\{a, b\}$ is a dominating set, for every $b \in V(H) \setminus N[a]$.

Proof. It was shown in [25] that $2 \leq \gamma_r(K_n \circ H) \leq 3$, and $\gamma_r(K_n \circ H) = 2$ if and only if $\gamma_r(H) = 2$ or there exists a vertex a of H such that $\{a, b\}$ is a dominating set, for every $b \in V(H) \setminus N[a]$. Hence, by Theorem 4 and Remarks 2 and 3 we only have to consider the case of noncomplete graphs H with $\gamma(H) = 1$. Now, since $2 \leq \gamma_r(K_n \circ H) \leq \gamma_s(K_n \circ H)$, we only need to show that $\gamma_s(K_n \circ H) \leq 2$. Notice that for any pair u_1, u_2 of different vertices of K_n and any universal vertex v of H, the set $W = \{(u_1, v), (u_2, v)\}$ is a 2-dominating set of $K_n \circ H$. Therefore, Theorem 2 leads to $\gamma_s(K_n \circ H) \leq |W| = 2$, as required. ∎

We now consider the case $G \not\cong K_n$.

Proposition 6. *The following statements hold for any noncomplete graph G with $\gamma(G) = 1$.*

(i) *If $\gamma_s(H) = 2$, then $\gamma_s(G \circ H) = \gamma_r(G \circ H) = 2$.*

(ii) *If $\gamma_s(H) > \gamma_r(H) = 2$ and G has more than one universal vertex, then $\gamma_s(G \circ H) = \gamma_r(G \circ H) = 2$.*

(iii) *If $\gamma_s(H) > \gamma_r(H) = 2$ and G has exactly one universal vertex, then $\gamma_s(G \circ H) = 3$ while $\gamma_r(G \circ H) = 2$.*

(iv) *If $\gamma_r(H) \geq 3$, then $\gamma_s(G \circ H) = \gamma_r(G \circ H) \in \{2, 3, 4\}$.*

Proof. First of all, notice that $\gamma_s(G \circ H) \geq \gamma_r(G \circ H) \geq 2$, as $G \circ H$ is not a complete graph. Let $v_1, v_2 \in V(H)$ be two different vertices of H, $u \in V(G)$ a universal vertex of G and $u' \in V(G) \setminus \{u\}$. Since $D = \{u, u'\} \times \{v_1, v_2\}$ is a 2-dominating set of $G \circ H$, by Theorem 2 we can conclude that $\gamma_s(G \circ H) \leq \gamma_2(G \circ H) \leq |D| = 4$. Therefore, by Theorem 4, (iv) follows. We proceed to study cases (i)–(iii) by separate.

Case (i). $\gamma_s(H) = 2$. Let $u \in V(G)$ be a universal vertex and let $S = \{v_1, v_2\}$ be a $\gamma_s(H)$-set. Since $H_u \cong H$, $W = \{u\} \times S$ is a secure dominating set of H_u and, since u is a universal vertex and $|W| = 2$, the movement of a guard from a vertex in W to any vertex outside $\{u\} \times V(H)$ does not produce undefended vertices, which implies that W is a secure dominating set of $G \circ H$. Thus, $\gamma_s(G \circ H) \leq |W| \leq 2$, and (i) follows.

Case (ii). $\gamma_s(H) > \gamma_r(H) = 2$ and $u, u' \in V(G)$ are two different universal vertices. By Remark 3, $\gamma(H) = 1$. Thus, $\gamma(G \circ H) = 1$, which implies that $\gamma_r(G \circ H) = 2$. Now, for any universal vertex $v \in V(H)$ we have that $W_1 = \{(u, v), (u', v)\}$ is a 2-dominating set of $G \circ H$. Hence, by Theorem 2, $\gamma_s(G \circ H) \leq \gamma_2(G \circ H) \leq |W_1| \leq 2$. Therefore, (ii) follows.

Case (iii). $\gamma_s(H) > \gamma_r(H) = 2$ and G has exactly one universal vertex denoted by u^*. Suppose that $\{(x, y), (a, b)\}$ is a secure dominating set of $G \circ H$. Since $\gamma_s(H) > 2$ and $H_x \cong H_a \cong H$, we can conclude that x and a have to be different, otherwise $\{y, b\}$ is a secure dominating set of H, which is a contradiction. We can assume that $a \neq u^*$. Let $a' \in V(G) \setminus N[a]$ and $v', v'' \in V(H)$ be two nonadjacent vertices. Since $\{(a', v'), (a', v'')\} \cap N(a, b) = \emptyset$ and $(a', v'') \notin N(a', v')$, when the guard stationed at (x, y) goes to manage any problem at (a', v') vertex (a', v'') is undefended, which is a contradiction. Hence, $\gamma_s(G \circ H) \geq 3$. To conclude that $\gamma_s(G \circ H) \leq 3$ we only need to observe that since $\gamma_s(H) > \gamma_r(H) = 2$, graph H has exactly one universal vertex $v^* \in V(H)$ and so for any $h \in V(H) \setminus \{v^*\}$ and any $g \in V(G) \setminus \{u^*\}$ the set $W' = \{(u^*, v^*), (u^*, h), (g, v^*)\}$ is a 2-dominating set of $G \circ H$. Hence, by Theorem 2 we have $\gamma_s(G \circ H) \leq \gamma_2(G \circ H) \leq |W'| \leq 3$. Therefore, (iii) follows. ■

By Propositions 5 and 6 (items (ii) and (iii)) we can conclude that Problem 1 is solved for any graph G with $\gamma(G) = 1$.

In order to give details for the case $K_{1,n} \circ H$ we need to state the following result obtained in [25].

Proposition 7 [25]. *Let H be a graph and let $n \geq 3$ be an integer. Then the following statements hold.*

- *If $\gamma_r(H) \in \{2, 3\}$, then $\gamma_r(K_{1,n} \circ H) = \gamma_r(H)$.*
- *If $\gamma_r(H) \geq 4$, then $3 \leq \gamma_r(K_{1,n} \circ H) \leq 4$.*
- *If $\gamma(H) \geq 4$, then $\gamma_r(K_{1,n} \circ H) = 4$.*

From Theorem 4 and Propositions 6 and 7 we derive the next result.

Proposition 8. *Let H be a graph and let $n \geq 3$ be an integer. Then the following statements hold.*

- *If $\gamma_s(H) = 2$, then $\gamma_s(K_{1,n} \circ H) = 2$.*
- *If $\gamma_s(H) > \gamma_r(H) = 2$, then $\gamma_s(K_{1,n} \circ H) = 3$.*
- *If $\gamma_r(H) = 3$, then $\gamma_s(K_{1,n} \circ H) = 3$.*
- *If $\gamma_r(H) \geq 4$, then $3 \leq \gamma_s(K_{1,n} \circ H) \leq 4$.*
- *If $\gamma(H) \geq 4$, then $\gamma_s(K_{1,n} \circ H) = 4$.*

5. THE CASE $G \cong P_n$

From Theorem 4 and the formula for $\gamma_r(P_n \circ H)$ obtained in [25], where $\gamma(H) \geq 4$, we derive the following result.

Proposition 9. *Let $n \geq 2$ be an integer and let H be a graph. If $\gamma(H) \geq 4$, then*

$$\gamma_s(P_n \circ H) = \gamma_r(P_n \circ H) \overset{[25]}{=} \begin{cases} n, & n \equiv 0 \pmod 4, \\ n+2, & n \equiv 2 \pmod 4, \\ n+1, & \textit{otherwise}. \end{cases}$$

Before considering the case $P_n \circ H$ where $\gamma(H) = 1$, we need to fix some notation. From now on, the vertex set of any path will be denoted by $V(P_n) = \{u_1, \ldots, u_n\}$, where u_i is adjacent to u_{i+1} for every $i \in \{1, \ldots, n-1\}$. For any graph H and any $\gamma_s(P_n \circ H)$-set S we define $S_i = S \cap (\{u_i\} \times V(H))$ for every $i \in \{1, \ldots, n\}$. Furthermore, for every $i \in \{2, \ldots, n-1\}$ we define $S_{\leq j} = \bigcup_{i=1}^{j} S_i$, while $P_{\leq j} \circ H$ will denote the subgraph of $P_n \circ H$ induced by $\{u_1, \ldots, u_j\} \times V(H)$. Notice that $P_{\leq j} \circ H \cong P_j \circ H$. Analogously, for every $j \in \{2, \ldots, n-1\}$ we define $S_{\geq j} = \bigcup_{i=j}^{n} S_i$, while $P_{\geq j} \circ H$ will denote the subgraph of $P_n \circ H$ induced by $\{u_j, \ldots, u_n\} \times V(H)$. With this notation in mind we can state the following results.

Lemma 5. *Let H be a noncomplete graph with $\gamma(H) = 1$. The following assertions hold for any integer $n \geq 2$.*

(i) *For any $\gamma_r(P_n \circ H)$-function f and $i \in \{1, n-1\}$, $f(\{u_i, u_{i+1}\} \times V(H)) \geq 2$.*

(ii) *If $\gamma_s(H) \geq 3$, then there exists a $\gamma_s(P_n \circ H)$-set S such that $S_i \neq \emptyset$ for every $i \in \{1, 2, n-1, n\}$.*

Proof. If f is a $\gamma_r(P_n \circ H)$-function such that $f(\{u_1, u_2\} \times V(H)) \leq 1$, then for any non-universal vertex $v' \in V(H)$, either (u_1, v') is undefended or the movement of a guard to (u_1, v') produces undefended vertices, which is a contradiction. Therefore, (i) follows.

On the other hand, if $\gamma(H) = 1$ and $\gamma_s(H) \geq 3$, then H has exactly one universal vertex. Let v be the universal vertex of H and let S be a $\gamma_s(P_n \circ H)$-set.

Suppose that $S_2 = \emptyset$. In this case, S_1 is a secure dominating set of $H_1 \cong H$, which implies that $|S_1| \geq 3$. Hence, $S' = (S \setminus S_1) \cup \{(u_1, v), (u_2, v)\}$ is a secure dominating set of $P_n \circ H$ and $|S'| < |S|$, which is a contradiction. Therefore, $S_2 \neq \emptyset$.

Suppose that $S_1 = \emptyset$. In such a case, $|S_2| \geq 2$, otherwise the movement of a guard to (u_1, v') produce undefended vertices whenever v' is a non-universal vertex. Moreover, if $S_3 = \emptyset$, then $|S_2| \geq 3$. Hence, if $S_3 \neq \emptyset$, then $S'' = (S \setminus S_2) \cup \{(u_1, v), (u_2, v)\}$ is a secure dominating set of $P_n \circ H$, and if $S_3 = \emptyset$, then $S'' = (S \setminus S_2) \cup \{(u_1, v), (u_2, v), (u_3, v)\}$ is a secure dominating set of $P_n \circ H$. In both cases S'' satisfies $S''_1 \neq \emptyset$ and $S''_2 \neq \emptyset$. Therefore, (ii) follows. ∎

Proposition 10. *Let H be a noncomplete graph with $\gamma(H) = 1$ and $n \geq 2$ an integer.*

- *If $n \equiv 0 \pmod 3$, then $\gamma_r(P_n \circ H) = \frac{2}{3}n$, while $\gamma_s(P_n \circ H) = \frac{2}{3}n + 1$ whenever $\gamma_s(H) \geq 3$ and $\gamma_s(P_n \circ H) = \frac{2}{3}n$ whenever $\gamma_s(H) = 2$.*
- *If $n \equiv 1 \pmod 3$, then $\gamma_s(P_n \circ H) = \gamma_r(P_n \circ H) = \frac{2}{3}n + \frac{4}{3}$.*
- *If $n \equiv 2 \pmod 3$, then $\gamma_s(P_n \circ H) = \gamma_r(P_n \circ H) = \frac{2}{3}n + \frac{2}{3}$.*

Proof. Let H be a noncomplete graph with $\gamma(H) = 1$. By Lemma 5(i) we deduce that $\gamma_r(P_2 \circ H) = 2$ and $\gamma_r(P_4 \circ H) = 4$. Moreover, since $P_3 \circ H$ is a noncomplete graph and it has a universal vertex, $\gamma_r(P_3 \circ H) = 2$. From now on we assume that $n \geq 5$.

For any $\gamma_r(P_{n-3} \circ H)$-function $f(U_0, U_1, U_2)$, we can define a WRDF $f'(U'_0, U'_1, U'_2)$ on $P_n \circ H$ by $U'_1 = U_1$ and $U'_2 = U_2 \cup \{(u_{n-1}, v)\}$, where v is a universal vertex of H. Thus, $\gamma_r(P_n \circ H) \leq w(f') = 2 + w(f) = 2 + \gamma_r(P_{n-3} \circ H)$.

We proceed to show that $\gamma_r(P_n \circ H) \geq 2 + \gamma_r(P_{n-3} \circ H)$. To this end, let g be a $\gamma_r(P_n \circ H)$-function and g_1 the restriction of g to $\{u_1, \ldots, n_{n-3}\} \times V(H)$. If g_1 is a WRDF on $P_{\leq n-3} \circ H$, then Lemma 5(i) leads to

$$\gamma_r(P_n \circ H) = w(g) = w(g_1) + g(\{u_{n-2}, u_{n-1}, u_n\} \times V(H)) \geq \gamma_r(P_{n-3} \circ H) + 2.$$

Notice that if $g(\{u_{n-2}\} \times V(H)) = 0$, then g_1 is a WRDF on $P_{\leq n-3} \circ H$, and we are done. From now on we can assume that $g(\{u_{n-2}\} \times V(H)) = \xi \geq 1$. Notice that, by the minimality of $w(g)$, $\xi \leq 2$. Notice also that if $g(u_{n-4} \times V(H)) \geq 2$, then g_1 is a WRDF on $P_{\leq n-3} \circ H$, and we are done. Now, if $g(u_{n-4} \times V(H)) \leq 1$, then for any $v'' \in V(H)$ such that $g(u_{n-4}, v'') = 0$, we can construct a WRDF on $P_{\leq n-3} \circ H$, say g_2, defined from g_1 by $g_2(u_{n-4}, v'') = \xi$ and $g_2(u_i, y) = g_1(u_i, y)$ for every $(u_i, y) \in \{u_1, \ldots, n_{n-3}\} \times V(H) \setminus \{(u_{n-4}, v'')\}$. Thus, $\gamma_r(P_{\leq n-3} \circ H) \leq$

$w(g_2) = w(g_1) + \xi$, which implies that

$$\begin{aligned}\gamma_r(P_n \circ H) &= w(g) = w(g_1) + \xi + g(\{u_{n-1}, u_n\} \times V(H)) \\ &= w(g_2) + g(\{u_{n-1}, u_n\} \times V(H)) \geq \gamma_r(P_{n-3} \circ H) + 2.\end{aligned}$$

We have shown that $\gamma_r(P_n \circ H) = 2 + \gamma_r(P_{n-3} \circ H)$ where $\gamma_r(P_i \circ H) = i$ for $i \in \{2, 4\}$ and $\gamma_r(P_3 \circ H) = 2$. By solving this linear recurrence equation we have that for any $n \geq 2$,

$$\gamma_r(P_n \circ H) = \frac{2}{3} - \frac{2}{3}\cos\frac{2n\pi}{3} + \frac{2}{3\sqrt{3}}\sin\frac{2n\pi}{3} + \frac{2n}{3}. \tag{1}$$

From (1) we deduce the formulas for $\gamma_r(P_n \circ H)$ and also the formula for $\gamma_s(P_n \circ H)$ when $\gamma_s(H) = 2$, as in this case Theorem 4 leads to $\gamma_s(P_n \circ H) = \gamma_r(P_n \circ H)$.

We now assume that $\gamma_s(H) \geq 3$. By Lemma 5(ii) we deduce that $\gamma_s(P_n \circ H) = n$ for every $n \in \{2, 3, 4\}$. From now on we assume that $n \geq 5$. By Lemma 5(ii), there exists a $\gamma_s(P_{n-3} \circ H)$-set S such that $S_{n-3} \neq \emptyset$ and $S_{n-4} \neq \emptyset$. Let v be the universal vertex of H. It is readily seen that $S' = S \cup \{(u_{n-1}, v), (u_n, v)\}$ is a secure dominating set of $P_n \circ H$, and so $\gamma_s(P_n \circ H) \leq 2 + |S| = 2 + \gamma_s(P_{n-3} \circ H)$.

We proceed to show that $\gamma_s(P_n \circ H) \geq 2 + \gamma_s(P_{n-3} \circ H)$. To this end, let X be a $\gamma_s(P_n \circ H)$-set such that $X_{n-1} \neq \emptyset$ and $X_n \neq \emptyset$. If $X_{n-2} = \emptyset$, then $X_{\leq n-3}$ is a secure dominating set of $P_{\leq n-3} \circ H$, which implies that

$$\gamma_s(P_n \circ H) = |X| = |X_{\leq n-3}| + 2 \geq \gamma_s(P_{\leq n-3} \circ H) + 2 = \gamma_s(P_{n-3} \circ H) + 2.$$

Assume that $X_{n-2} \neq \emptyset$. Suppose that $|X_{\leq n-3}| \leq \gamma_s(P_{\leq n-3} \circ H) - 2$. In such a case, there exists $l = \max\{i \leq n-3 : \ X_i = \emptyset\}$. Since $X_l = \emptyset$, we conclude that $X_{\leq l-1}$ has to be a secure dominating set of $P_{\leq l-1} \circ H$, and so $Y = X_{\leq n-3} \cup \{(u_l, v)\}$ is a secure dominating set of $P_{\leq n-3} \circ H$ whose cardinality is $|Y| = |X_{\leq n-3}| + 1 \leq \gamma_s(P_{\leq n-3} \circ H) - 1$, which is a contradiction. Thus, $|X_{\leq n-3}| \geq \gamma_s(P_{\leq n-3} \circ H) - 1$, and so

$$\gamma_s(P_n \circ H) = |X| \geq |X_{\leq n-3}| + 3 \geq \gamma_s(P_{\leq n-3} \circ H) + 2 = \gamma_s(P_{n-3} \circ H) + 2.$$

We have shown that $\gamma_s(P_n \circ H) = 2 + \gamma_s(P_{n-3} \circ H)$, where $\gamma_s(P_i \circ H) = i$ for $i \in \{2, 3, 4\}$. By solving this linear recurrence equation we have that for any $n \geq 2$,

$$\gamma_s(P_n \circ H) = 1 + \frac{2}{3\sqrt{3}}\sin\frac{2n\pi}{3} + \frac{2n}{3}. \tag{2}$$

From (2) we complete the proof. ∎

Theorem 6 [25]**.** *For any tree T and any noncomplete graph H,*

$$\gamma_r(T \circ H) \geq 2\gamma(T).$$

Proposition 11. *If H is a graph with $\gamma_s(H) = \gamma(H) = 2$, then for any integer $n \geq 2$,*

$$\gamma_s(P_n \circ H) = \gamma_r(P_n \circ H) = 2\left\lfloor \frac{n+2}{3} \right\rfloor.$$

Proof. By Theorem 4, if $\gamma_s(H) = \gamma(H) = 2$, then $\gamma_s(P_n \circ H) = \gamma_r(P_n \circ H)$. Thus, by Theorem 6, $\gamma_s(P_n \circ H) = \gamma_r(P_n \circ H) \geq 2\gamma(P_n) = 2\left\lfloor \frac{n+2}{3} \right\rfloor$. Now, by Theorem 8 we conclude that $\gamma_s(P_n \circ H) = \gamma_r(P_n \circ H) \leq 2\gamma(P_n) = 2\left\lfloor \frac{n+2}{3} \right\rfloor$. ■

We would emphasize that the problem of computing $\gamma_s(P_n \circ H)$ when $\gamma_s(H) \geq 3$ and $\gamma(H) \in \{2, 3\}$ remains open.

6. The Case $H \cong C_n$

As shown in [25], if $\gamma(H) \geq 4$, then $\gamma_r(C_n \circ H) = n$, so that Theorem 4 leads to the following result.

Proposition 12. *Let $n \geq 3$ be an integer and let H be a graph. If $\gamma(H) \geq 4$, then*

$$\gamma_s(C_n \circ H) = \gamma_r(C_n \circ H) \overset{[25]}{=} n.$$

We now consider the case of noncomplete graphs with $\gamma(H) = 1$.

Proposition 13. *If H is a noncomplete graph with $\gamma(H) = 1$, then for any integer $n \geq 3$,*

$$\gamma_s(C_n \circ H) = \gamma_r(C_n \circ H) = \left\lceil \frac{2n}{3} \right\rceil.$$

Proof. Let f be a $\gamma_r(C_n \circ H)$-function and $V(C_n) = \{u_1, \ldots, u_n\}$ where u_i is adjacent to u_{i+1} for every i (the subscripts are taken modulo n). If there exists $u_i \in V(C_n)$ such that $\sum_{j=i-1}^{i+1} f(\{u_j\} \times V(H)) = 1$, then for any non-universal vertex $v' \in V(H)$, the movement of the corresponding guard to (u_i, v') produces undefended vertices. Thus, $\sum_{j=i-1}^{i+1} f(\{u_j\} \times V(H)) \geq 2$ for every $i \in \{1, \ldots, n\}$, which implies that

$$3\gamma_r(C_n \circ H) = 3w(f) = \sum_{i=1}^{n} \sum_{j=i-1}^{i+1} f(\{u_j\} \times V(H)) \geq 2n.$$

Hence, $\gamma_r(C_n \circ H) \geq \left\lceil \frac{2n}{3} \right\rceil$.

We now proceed to construct a secure dominating set of $C_n \circ H$ of cardinality $\lceil \frac{2n}{3} \rceil$. Let $v \in V(H)$ be a universal vertex of H. For $n \equiv 0 \pmod 3$ we set $X = \{u_1, u_2, u_4, u_5, \ldots, u_{n-2}, u_{n-1}\} \times \{v\}$, for $n \equiv 1 \pmod 3$ we set $X = \{u_1, u_2, u_4, u_5, \ldots, u_{n-3}, u_{n-2}, u_n\} \times \{v\}$, and for $n \equiv 2 \pmod 3$ we set $X = \{u_1, u_2, u_4, u_5, \ldots, u_{n-1}, u_n\} \times \{v\}$. Since in every case X is a 2-dominating set of $C_n \circ H$, by Theorem 2 we have $\gamma_s(C_n \circ H) \le \gamma_2(C_n \circ H) \le |X| = \lceil \frac{2n}{3} \rceil$, as required. ■

Proposition 14. *If H is a graph with $\gamma_s(H) = \gamma(H) = 2$, then for any integer $n \ge 3$,*

$$\gamma_s(C_n \circ H) = \gamma_r(C_n \circ H) = 2 \left\lfloor \frac{n+2}{3} \right\rfloor.$$

Proof. By Theorem 4, if $\gamma_s(H) = 2$, then $\gamma_s(C_n \circ H) = \gamma_r(C_n \circ H)$. On the other hand, by Remark 4 and Proposition 11 we have that $\gamma_s(C_n \circ H) \le \gamma_s(P_n \circ H) = 2 \lfloor \frac{n+2}{3} \rfloor$.

We proceed to show that $\gamma_s(C_n \circ H) \ge 2 \lfloor \frac{n+2}{3} \rfloor$. To this end, let W be a $\gamma_s(C_n \circ H)$-set and $V(C_n) = \{u_1, \ldots, u_n\}$ where u_i is adjacent to u_{i+1} for every i (the subscripts are taken modulo n). As in our previous results, let $W_i = W \cap (\{u_i\} \times V(H))$. Since $\gamma_s(H) = 2$, we deduce that

$$\sum_{j=i-1}^{i+1} |W_j| \ge 2 \text{ for every } i \in \{1, \ldots, n\}. \tag{3}$$

Hence,

$$3\gamma_s(C_n \circ H) = 3|W| = \sum_{i=1}^{n} \sum_{j=i-1}^{i+1} |W_j| \ge 2n,$$

which implies that $\gamma_s(C_n \circ H) \ge \lceil \frac{2n}{3} \rceil$. Now, if $n \equiv 0, 2 \pmod 3$, then we are done, as in these cases $2 \lfloor \frac{n+2}{3} \rfloor = \lceil \frac{2n}{3} \rceil$. From now on, suppose that $n \equiv 1 \pmod 3$. By the minimality of $|W|$ we deduce that $|W_i| \le 2$, for every $u_i \in V(C_n)$. Notice that, by (3), if there exists $u_i \in V(C_n)$ such that $\sum_{j=i-1}^{i+1} |W_j| \ge 4$, or $u_i, u_l \in V(C_n)$ such that $\sum_{j=i-1}^{i+1} |W_j| = \sum_{j=l-1}^{l+1} |W_j| = 3$, then

$$3\gamma_r(C_n \circ H) = 3|W| = \sum_{i=1}^{n} \sum_{j=i-1}^{i+1} |W_j| \ge 2n + 2. \tag{4}$$

In such a case, $\gamma_r(C_n \circ H) \ge \lceil \frac{2n+2}{3} \rceil = 2 \lfloor \frac{n+2}{3} \rfloor$, as $n \equiv 1 \pmod 3$.

To conclude the proof, we differentiate the following three cases. Symmetric cases or cases where it is obvious that (4) holds are omitted.

Case 1. There exists $u_i \in V(C_n)$ such that $|W_{i-1}| \le 1$, $|W_i| = 0$ and $|W_{i+1}| \le 1$. By (3), the case $|W_{i-1}| = 0$ and $|W_{i+1}| = 1$ is not possible. Hence, we can assume that $|W_{i-1}| = 1$ and $|W_{i+1}| = 1$. Again by (3), we deduce that $|W_{i-2}| \geq 1$ and $|W_{i+2}| \geq 1$. If $|W_{i-2}| \geq 2$ or $|W_{i+2}| \geq 2$, then (4) holds and we are done. Suppose that $|W_{i-2}| = 1$ and $|W_{i+2}| = 1$. If $|W_{i-3}| = 0$ and $|W_{i+3}| = 0$, then the movement of a guard from H_{i+1} (or H_{i-1}) to H_i produces undefended vertices, which is a contradiction. Hence, without loss of generality, we assume that $|W_{i+3}| \geq 1$. Now, if $|W_{i+3}| = 1$ and $|W_{i+4}| = 0$, then the movement of a guard from H_{i+2} to H_{i+3} produces undefended vertices in H_{i+1}, which is a contradiction. Thus, $|W_{i+3}| \geq 2$ or $|W_{i+4}| \geq 1$ and, in both cases we deduce that (4) holds and so $\gamma_r(C_n \circ H) \geq \left\lceil \frac{2n+2}{3} \right\rceil = 2\left\lfloor \frac{n+2}{3} \right\rfloor$.

Case 2. There exists $u_i \in V(C_n)$ such that $|W_{i-1}| = 1$, $|W_i| = 0$ and $|W_{i+1}| = 2$. By (3), $|W_{i-2}| \geq 1$. Now, if $|W_{i-2}| = 2$, then (4) holds and we are done. Suppose that $|W_{i-2}| = 1$. If $|W_{i-3}| \geq 1$ or $|W_{i-4}| \geq 2$, then (4) holds and we are done. Finally, if $|W_{i-3}| = 0$ and $|W_{i-4}| = 1$, then we apply Case 1 to $|W_{i-4}| = 1$, $|W_{i-3}| = 0$ and $|W_{i-2}| = 1$, and we are done.

Case 3. $|W_i| \in \{0, 2\}$ for every $u_i \in V(C_n)$. In this case, $D = \{u_i : |W_i| \neq 0\}$ has to be a dominating set of C_n, and so

$$\gamma_s(C_n \circ H) = |W| = \sum_{i=1}^{n} |W_i| = 2|D| \geq 2\gamma(C_n) = 2\left\lfloor \frac{n+2}{3} \right\rfloor.$$

According to the three cases above, the proof is complete. ■

We would emphasize that the problem of computing $\gamma_s(C_n \circ H)$ when $\gamma_s(H) \geq 3$ and $\gamma(H) \in \{2, 3\}$ remains open.

7. General Bounds

To continue our analysis we would point out the following two results, which are direct consequence of Proposition 4(ii).

Remark 15. Let G be a connected graph of order n and let H be a nonempty graph. For any spanning subgraph G_1 of G,

$$\gamma_s(K_n \circ H) \leq \gamma_s(G \circ H) \leq \gamma_s(G_1 \circ H).$$

In particular, if G is a Hamiltonian graph, then

$$\gamma_s(G \circ H) \leq \gamma_s(C_n \circ H).$$

Remark 16. Let G be a graph and let H be a graph of order $n' \geq 2$. For any spanning subgraph H_1 of H,

$$\gamma_s(G \circ K_{n'}) \leq \gamma_s(G \circ H) \leq \gamma_s(G \circ H_1).$$

In particular, if H is a Hamiltonian graph, then

$$\gamma_s(G \circ H) \leq \gamma_s(G \circ C_{n'}).$$

A *total dominating set* of a graph G with no isolated vertex is a set $S \subseteq V(G)$ such that every vertex of G is adjacent to at least one vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the cardinality of a smallest total dominating set, and we refer to such a set as a $\gamma_t(G)$-set. Notice that for any graph G with no isolated vertex,

$$\gamma_r(G) \leq 2\gamma(G) \leq 2\gamma_t(G). \tag{5}$$

The reader is referred to the book [17] for details on total domination in graphs. This book provides and explores the fundamentals of total domination in graphs.

As shown in [25] if G is a graph with no isolated vertex, then for any graph H we have $\gamma_r(G \circ H) \leq 2\gamma_t(G)$. Thus, by Theorem 4 we have that if $\gamma(H) \geq 2$, then $\gamma_s(G \circ H) = \gamma_r(G \circ H) \leq 2\gamma_t(G)$. Our next result shows that this bound holds for every graph without isolated vertices.

Theorem 7. *If G is a graph with no isolated vertex, then for any nontrivial graph H,*

$$\gamma_s(G \circ H) \leq 2\gamma_t(G).$$

Proof. Let G be a graph with no isolated vertex and H a nontrivial graph. Let S be a $\gamma_t(G)$-set and let $h', h'' \in V(H)$ be two different vertices of H. We claim that $W = S \times \{h', h''\}$ is a 2-dominating set of $G \circ H$. Let $(g, h) \in \overline{W}$. Since S is a total dominating set of G, there exists $g' \in S \cap N(g)$. Thus, $\{(g', h'), (g', h'')\} \subseteq W \cap N(g, h)$, which implies that W is a 2-dominating set of $G \circ H$. Hence, $\gamma_2(G \circ H) \leq |W| = 2\gamma_t(G)$. Finally, by Theorem 2 we have $\gamma_s(G \circ H) \leq \gamma_2(G \circ H) \leq 2\gamma_t(G)$ ■

As shown in [25], there are several families of graphs with $\gamma_r(G \circ H) = 2\gamma_t(G)$, which implies that the bound above is tight.

We have learned from [7] that $\gamma_t(G) \leq \frac{2}{3}n$ for any connected graph of order $n \geq 3$. Hence, Theorem 7 leads to the following result.

Corollary 17. *For any connected graph G of order $n \geq 3$ and any nontrivial graph H,*

$$\gamma_s(G \circ H) \leq 2\left\lfloor \frac{2n}{3} \right\rfloor.$$

To see that the bound above is tight we can take the family T_n of trees defined in [25] where $\gamma_r(T_n \circ H) = 2\left\lfloor \frac{2n}{3} \right\rfloor$ for any graph H with $\gamma(H) > 4$.

As stated by Goddard and Henning [12], if G is a planar graph with diameter two, then $\gamma_t(G) \leq 3$. Hence, as an immediate consequence of Theorem 7, we have the following result.

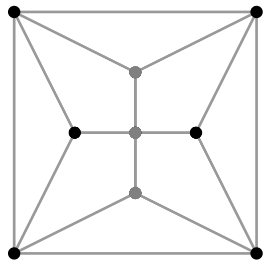

Figure 2. A planar graph of diameter two.

Corollary 18. *If G is a planar graph of diameter two, then for any nonempty graph H,*

$$\gamma_s(G \circ H) \leq 6.$$

The bound above is achieved, for instance, for the planar graph G shown in Figure 2 and any graph H with $\gamma(H) \geq 4$. An optimum placement of guards in $G \circ H$ can be done by assigning two guards to the copies of H corresponding to the gray-coloured vertices of G.

Theorem 8. *For any graph without isolated vertices G and any noncomplete graph H,*

$$\gamma_s(G \circ H) \leq \gamma(G) \min\{4, \gamma_s(H)\}.$$

Proof. We first show that $\gamma_s(G \circ H) \leq 4\gamma(G)$. It is well known that for every graph G with no isolated vertex, $\gamma_t(G) \leq 2\gamma(G)$ (see, for instance, [2]). Hence, by Theorem 7 we have $\gamma_s(G \circ H) \leq 4\gamma(G)$, as required.

We now show that $\gamma_s(G \circ H) \leq \gamma(G)\gamma_s(H)$. Let S_1 be a $\gamma(G)$-set and S_2 a $\gamma_s(H)$-set. Notice that $\gamma_s(H) \geq 2$, as H is not complete. It is readily seen that $W = S_1 \times S_2$ is a dominating set of $G \circ H$. To see that it is a secure dominating set we only need to observe that for any $u \in V(G)$ the restriction W_u of W to H_u is a secure domination set and, since $|W_u| \geq 2$, the movement of a guard from H_u to $H_{u'}$, where $u \in S_1$ and $u' \in \overline{S_1} \cap N(u)$, does not produce undefended vertices. Hence, $\gamma_s(G \circ H) \leq |W| = \gamma(G)\gamma_s(H)$. Therefore, the result follows. ∎

Theorem 9. *For any graph G of minimum degree greater than or equal to two and any graph H,*

$$\gamma_s(G \circ H) \leq \gamma_{2,t}(G).$$

Proof. As shown in [25], $\gamma_{2,t}(G \circ H) \leq \gamma_{2,t}(G)$. Therefore, from Theorem 3 we deduce the result $\gamma_s(G \circ H) \leq \gamma_{2,t}(G \circ H) \leq \gamma_{2,t}(G)$. ∎

In order to show an example of graphs where $\gamma_s(G \circ H) = \gamma_{2,t}(G)$, we consider the family $\mathcal{G}$ defined in [25] as follows. A graph $G_{k,l} = (V, E)$ belongs to $\mathcal{G}$ if and only if there exit two positive integers k, l such that $V = \{x_1, x_2, x_3, y_1, y_2, \ldots, y_k, z_1, z_2, \ldots, z_l\}$ and $E = \{x_1y_i : 1 \le i \le k\} \cup \{x_1z_i : 1 \le i \le l\} \cup \{x_2y_i : 1 \le i \le k\} \cup \{x_3z_i : 1 \le i \le l\} \cup \{x_2x_3\}$. Figure 3 shows the graph $G_{4,4}$.

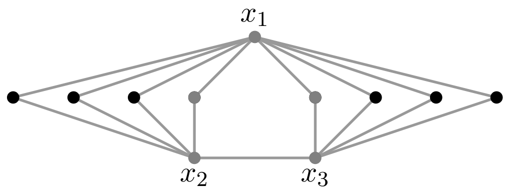

Figure 3. The set of gray-coloured vertices is a double total dominating set of $G_{4,4}$.

It is not difficult to check that for any graph $G_{k,l} \in \mathcal{G}$ and any graph H with $\gamma(H) \ge 3$ we have $\gamma_r(G_{k,l} \circ H) = \gamma_s(G_{k,l} \circ H) = \gamma_{2,t}(G_{k,l} \circ H) = 5 = \gamma_{2,t}(G_{k,l})$.

Corollary 19. *For any graph H and any graph G of order n and minimum degree greater than or equal to two,*

$$\gamma_s(G \circ H) \le n.$$

By Proposition 12, the bound above is tight.

A set $X \subseteq V(G)$ is called a 2-*packing* if $N[u] \cap N[v] = \emptyset$ for every pair of different vertices $u, v \in X$. The 2-*packing number* $\rho(G)$ is the cardinality of any largest 2-packing of G. A 2-packing of cardinality $\rho(G)$ is called a $\rho(G)$-set.

Theorem 10 [25]**.** *For any graph G without isolated vertices and any noncomplete graph H,*

$$\gamma_r(G \circ H) \ge \max\{\gamma_r(G), \gamma_t(G), 2\rho(G)\}.$$

Theorem 10 suggests to ask if $\gamma_s(G \circ H) \ge \gamma_s(G)$ for any graph G and any noncomplete graph H. In general, this inequality does not hold. For instance, if we take any graph G with $\gamma_s(G) > 2\gamma_t(G)$, then Theorem 7 leads to $\gamma_s(G \circ H) \le 2\gamma_t(G) < \gamma_s(G)$.

From Proposition 1(ii) and Theorems 9 and 10 we deduce the following result.

Theorem 11. *If $\gamma_{2,t}(G) = \max\{\gamma_r(G), 2\rho(G)\}$, then*

$$\gamma_s(G \circ H) = \gamma_r(G \circ H) \overset{[25]}{=} 2\gamma_{2,t}(G).$$

From Proposition 1(ii) and Theorems 7 and 10 we deduce the following result.

Theorem 12. *Let G be a graph without isolated vertices and let H be a noncomplete graph. If $\gamma_t(G) = \max\{\frac{1}{2}\gamma_r(G), \rho(G)\}$, then*

$$\gamma_s(G \circ H) = \gamma_r(G \circ H) \overset{[25]}{=} 2\gamma_t(G).$$

Theorem 13. *If G is a graph without isolated vertices, then for any graph H with $\gamma(H)=1$,*

$$\gamma_s(G \circ H) \leq 3\gamma(G).$$

Proof. Let $S \subseteq V(G)$ be a $\gamma(G)$-set and let $v_1, v_2 \in V(H)$, where v_1 is a universal vertex. Let $S' \subseteq V(G)$ such that $|S'[\leq |S|$ and for each $u \in S$ there exists $u' \in N(u) \cap S'$. Notice that $W = (S \times \{v_1, v_2\}) \cup (S' \times \{v_1\})$ is a 2-dominating set and $|W| \leq 3\gamma(G)$. Hence, by Theorem 2 we have $\gamma_s(G \circ H) \leq \gamma_2(G \circ H) \leq |W| \leq 3\gamma(G)$. Therefore, the result follows. ■

As we have shown in Proposition 8, if $\gamma_s(H) > \gamma_r(H) = 2$, then $\gamma_s(K_{1,n} \circ H) = 3 = 3\gamma(K_{1,n})$. Hence, the bound above is tight. In addition, in Figure 4 we show a graph G such that $\gamma_s(G \circ K_{1,l}) = 3\gamma(G)$ for every $l \geq 3$.

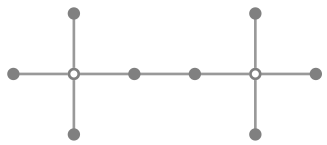

Figure 4. It is not difficult to check that if G is the graph above, then $\gamma_r(G \circ H) = 4 < 6 = \gamma_s(G \circ K_{1,l})$ for any $l \geq 3$.

Acknowledgements

This research was supported in part by the Spanish government under the grants MTM2016-78227-C2-1-P and PRX17/00102.

References

[1] B.H. Arriola and S.R. Canoy, Jr., *Doubly connected domination in the corona and lexicographic product of graphs*, Appl. Math. Sci. **8** (2014) 1521–1533. https://doi.org/10.12988/ams.2014.4136

[2] B. Bollobás and E.J. Cockayne, *Graph-theoretic parameters concerning domination, independence, and irredundance*, J. Graph Theory **3** (1979) 241–249. https://doi.org/10.1002/jgt.3190030306

[3] H. Boumediene Merouane and M. Chellali, *On secure domination in graphs*, Inform. Process. Lett. **115** (2015) 786–790. https://doi.org/10.1016/j.ipl.2015.05.006

[4] A.P. Burger, M.A. Henning and J.H. van Vuuren, *Vertex covers and secure domination in graphs*, Quaest. Math. **31** (2008) 163–171. https://doi.org/10.2989/QM.2008.31.2.5.477

[5] A. Cabrera Martínez, L.P. Montejano and J.A. Rodríguez-Velázquez, *Total weak Roman domination in graphs*, Symmetry **11(6)** (2019) #831. https://doi.org/10.3390/sym11060831

[6] M. Chellali, T.W. Haynes and S.T. Hedetniemi, *Bounds on weak Roman and 2-rainbow domination numbers*, Discrete Appl. Math. **178** (2014) 27–32.
https://doi.org/10.1016/j.dam.2014.06.016

[7] E.J. Cockayne, R.M. Dawes and S.T. Hedetniemi, *Total domination in graphs*, Networks **10** (1980) 211–219.
https://doi.org/10.1002/net.3230100304

[8] E.J. Cockayne, O. Favaron and C.M. Mynhardt, *Secure domination, weak Roman domination and forbidden subgraphs*, Bull. Inst. Combin. Appl. **39** (2003) 87–100.

[9] E.J. Cockayne, P.J.P. Grobler, W.R. Gründlingh, J. Munganga and J.H. van Vuuren, *Protection of a graph*, Util. Math. **67** (2005) 19–32.

[10] E.J. Cockayne, P.A. Dreyer Jr., S.M. Hedetniemi and S.T. Hedetniemi, *Roman domination in graphs*, Discrete Math. **278** (2004) 11–22.
https://doi.org/10.1016/j.disc.2003.06.004

[11] M. Dettlaff, M. Lemańska, J.A. Rodríguez-Velázquez and R. Zuazua, *On the super domination number of lexicographic product graphs*, Discrete Appl. Math. **263** (2019) 118–129.
https://doi.org/10.1016/j.dam.2018.03.082

[12] W. Goddard and M.A. Henning, *Domination in planar graphs with small diameter*, J. Graph Theory **40** (2002) 1–25.
https://doi.org/10.1002/jgt.10027

[13] R. Hammack, W. Imrich and S. Klavžar, Handbook of Product Graphs (CRC Press, 2011).
https://doi.org/10.1201/b10959

[14] T. Haynes, S. Hedetniemi and P. Slater, Domination in Graphs: Volume 2: Advanced Topics (CRC Press, New York, 1998).

[15] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (CRC Press, New York, 1998).

[16] M.A. Henning and S.T. Hedetniemi, *Defending the Roman Empire—A new strategy*, Discrete Math. **266** (2003) 239–251.
https://doi.org/10.1016/S0012-365X(02)00811-7

[17] M.A. Henning and A. Yeo, Total Domination in Graphs (Springer, New York, 2013).
https://doi.org/10.1007/978-1-4614-6525-6

[18] W. Imrich and S. Klavžar, Product Graphs: Structure and Recognition (Wiley-Interscience, New York, 2000).

[19] W.F. Klostermeyer and C.M. Mynhardt, *Secure domination and secure total domination in graphs*, Discuss. Math. Graph Theory **28** (2008) 267–284.
https://doi.org/10.7151/dmgt.1405

[20] J. Liu, X. Zhang and J. Meng, *Domination in lexicographic product digraphs*, Ars Combin. **120** (2015) 23–32.

[21] R.J. Nowakowski and D.F. Rall, *Associative graph products and their independence, domination and coloring numbers*, Discuss. Math. Graph Theory **16** (1996) 53–79. https://doi.org/10.7151/dmgt.1023

[22] I. Stewart, *Defend the Roman Empire!*, Sci. Amer. **281** (1999) 136–138. https://doi.org/10.1038/scientificamerican1299-136

[23] T.K. Šumenjak, P. Pavlič and A. Tepeh, *On the Roman domination in the lexicographic product of graphs*, Discrete Appl. Math. **160** (2012) 2030–2036. https://doi.org/10.1016/j.dam.2012.04.008

[24] T.K. Šumenjak, D.F. Rall and A. Tepeh, *Rainbow domination in the lexicographic product of graphs*, Discrete Appl. Math. **161** (2013) 2133–2141. https://doi.org/10.1016/j.dam.2013.03.011

[25] M. Valveny, H. Pérez-Rosés and J.A. Rodríguez-Velázquez, *On the weak Roman domination number of lexicographic product graphs*, Discrete Appl. Math. **263** (2019) 257–270. https://doi.org/10.1016/j.dam.2018.03.039

[26] M. Valveny and J.A. Rodríguez-Velázquez, *Protection of graphs with emphasis on Cartesian product graphs*, Filomat **33** (2019) 319–333. https://doi.org/10.2298/FIL1901319V

Received 21 January 2019
Revised 23 August 2019
Accepted 23 August 2019