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Review

A systematic review of the interactions of fuzzy set theory and option pricing

Jorge de Andrés-Sánchez

Universitat Rovira i Virgili, Social and Business Research Group, Business Management Department, Campus Bellissens, Reus 43204, Spain

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ABSTRACT

This paper makes a systematic bibliographical analysis of the contributions of fuzzy set theory (FST) on option pricing to state principal mainstream focuses and exposes the basic questions of the analytical foundation of the reviewed approaches. It performs a bibliographical analysis of journal articles and book chapters by applying PRISMA guidelines to the SCOPUS and WoS databases. We subsequently present a structured report of principal findings about research fields, outlets and authors of this topic. Once we have identified the ways in which FST has contributed to option pricing, we outline basics about their mathematic and conceptual grounds. We have identified four main approaches to fuzzy option pricing (FOP). The mainstream of papers, based on the so-called fuzzy-random approach, consists of fuzzifying option pricing formulas under the hypothesis that the parameters governing the stochastic movement of prices are not crisp but fuzzy numbers. The second stream of the literature also superposes FST to conventional option pricing models, but this is made by means of the distortion of neutral to risk probabilities with fuzzy measures. The third approach, so called fuzzy pay-off, is devoted to evaluating real options and uses strictly fuzzy number concepts. The fourth approach embeds tools such as fuzzy controllers or fuzzy neural networks, taking advantage of their capability to obtain good numerical approximations to any function from empirical data. Principal outlets of FOP are journals devoted to fuzzy mathematics and soft computing, and the evolution of contributions throughout time reveals that it has become a well-stablished topic in fuzzy mathematics. The bibliographical research developed in this paper provides a wide panoramic perspective about what the principal mainstreams of FOP research are, what is done, and thus, future research lines are suggested.

1. Introduction

A derivative is an asset whose price depends on a subjacent asset or variable (Hull, 2008). Examples of derivatives are futures, forward contracts and options, which are the objective of this paper. Call options give to the buyer the right to acquire an asset by a predefined price (strike price) at a concrete date (European style) or at any moment until the expiration date (American style). On the other hand, put options give the buyer the right to sell the subjacent asset by the strike price and can also be of European and American type (Hull, 2008).

Financial contracts are as old as human economic activities. At 2800 BCE, in ancient Mesopotamia, Hammurabi's code, which regulated credit activities, stipulated that crop failures due to natural disasters allowed cancelling yearly interest on a land loan (Ingersoll, 1989). Therefore, we can consider that Mesopotamian farmers were owners of a put option on their harvest whose strike price was yearly interests. It is usually agreed that the first equity options were traded in the Modern Era in the bourse circus of Amsterdam in the 17th century (Dotsis, 2020). At the end of the 19th century, they were common in the Paris, Berlin and London stock exchange, and thus, the first analytical approaches to option pricing (OP) arose in these years (Dotsis, 2020; Zimmermann & Hafner, 2007). Among these approaches, it is often outlined as the most relevant, the doctoral thesis by Bachelier and his application of arithmetical Brownian stochastic processes to model stock price fluctuations (Merton, 1998). However, pioneering contributions by Vincenz Bronzin, who provided an early formulation of call-put parity, and Leonard R. Higgins are also remarkable (Dotsis, 2020; Zimmermann & Hafner, 2007).

The birth of modern financial theory could be placed in the 1950s with the publication of portfolio selection models by Harry Markowitz, Russell Tobin or John Lintner (Merton, 1998). These authors introduced stochastic analysis and mathematical programming as ordinary mathematical tools for financial analysis, which fostered active research on warrant pricing in the 1960s. Some outputs of that research are OP

E-mail address: jorge.deandres@urv.cat.

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models such as those by Sprenkle (1961) and Samuelson (1965). These works were influenced by Bachelier's seminal findings, with the crucial difference that they supposed that the movements of stock prices followed a log-normal distribution instead of being Gaussian (Merton, 1998).

The option pricing model by Black and Scholes (Black & Scholes, 1973) and its slight generalization by Merton (Merton, 1973), BSM hereinafter, is commonly outlined as the keystone of option pricing theory (Hull, 2008) and a milestone of modern financial theory (Merton, 1998). A superficial examination of the BSM leads us to conclude that it is very close to those of the OP models developed in the 1960s. It seems logical, since all these formulas relies in a log-normal modelling of stock prices and thus, the price of the call (put) option is calculated as a weighted difference by standard Gaussian distribution values between the actual price of the stock (the strike price) and the strike price (the actual value of the stock). However, there are two key questions of the BSM pricing formula that supposed a revolution, not only in the OP field but also in the more general setting of financial analysis (Merton, 1998). BSM does not depend on any parameter that is difficult to estimate, such as the expected growth rate of the subjacent stock, or subjective, such as risk aversion. In contrast, that pricing formula is universal and easy to implement, and its parameters are "easy to fit". The second great contribution was the use of the so-called "arbitrage argument" to deduce the OP formula, which was subsequently generalized by the concepts of risk-neutral probabilities and the equivalent martingale measure (Broadie & Detemple, 2004). The use of equivalent risk-neutral probabilities allows pricing any contingent asset as an expected present value of their cash flows with parameters that do not depend on risk aversion (Merton, 1998).

Since the 1970s, OP has been applied not only to price options but also to value any asset that embeds optionally. Therefore, derivatives such as warrants or convertible bonds can be naturally priced as options. Of course, the BSM framework has allowed the development of formulas for new option types such as so-called "exotic" and new derivatives such as credit default swaps or catastrophe derivatives (Merton, 1998). However, despite not being derivatives, OP models can be adapted to price other financial assets such as insurance and loan guarantees (Merton, 1977), life equity-linked insurance policies (Brennan & Schwartz, 1976), and risk-free bonds (Cox, Ingersoll, & Ross, 1985). In fact, BSM was proposed to price options and warrants but also corporate liabilities (Black & Scholes, 1973) and to measure credit risk (Merton, 1974).

Option pricing theory is also actively used to guide strategic decisions in the capital budgeting context. That research field is known as real option theory (Trigeorgis & Reuer, 2017). Real option theory not only evaluates investment projects from their expected cash flows, initial investment and cost of capital, which are the classical inputs of the net present value and internal rate of return but also considers the optionality and flexibility features of some investment projects that increase their value. These traits are commonly known as real options. Some examples are deferring options, the possibility of rescaling, a grow option, the opportunity to switch, multiple interaction options or a right of abandonment (Trigeorgis, 1996).

Financial markets and entrepreneurship activity are developed in environments where different kinds of informational reliability coexist. Therefore, risk corresponds to a situation in which the probabilities of possible outcomes of a phenomenon are known. On the other hand, uncertainty implies that the probabilities of events are unknown and that no unique assignment of them can be obtained (Basili, 2001). This situation is common, for example, in illiquid markets or in incomplete markets. Likewise, although a given phenomenon such as option valuation could be modelled essentially by means of probability theory and the information about the parameters that govern the stochastic behaviour of the subjacent asset price may not be completely uncertain, it could still be affected by some degree of vagueness and imprecision. For example, even in the most active and liquid markets, the traded price of one asset within one session is not a crisp number but closer to a confidence interval (de Andres-Sanchez & Terceño, 2003) since for the same assets, there exists more than one price: bid and ask prices, maximum and minimum traded prices, average, opening and closing price, etc.

Fuzzy set theory (FST) has provided OP theory with useful tools to model nonprobabilistic uncertainty, such as fuzzy measures (Cherubini, 1997), fuzzy distribution functions (Viertl & Hareter, 2004) and fuzzy numbers and fuzzy regression (Muzzioli & De Baets, 2016). On the one hand, stochastic processes theory provides a solid analytical and theoretical framework to model asset price behaviour. From the pioneering works by Bachelier and Bronzin, OP models have been built on the basis of probability theory. On the other hand, fuzzy tools are a suitable complement to probability theory since they allow introducing different uncertainty sources to risk in the available information, such as vagueness, imprecision or ambiguity (Cherubini, 1997; Muzzioli, & De Baets, 2016).

This paper revises the literature that applies FST to OP, which we call fuzzy option pricing (FOP) with a twofold objective. The first aim is to provide a panoramic vision of the main focuses of FOP until 2022, i.e., 50 years after the publication of the BSM model. This is done by performing an ordered bibliographical search in the WoS and SCOPUS databases and presenting a structured assessment of this search. Subsequently, we describe the basic analytical questions of the main FOP approaches. Finally, we outline the principal conclusions of our review and suggest future research.

2. Bibliographical analysis

2.1. Methodology

We have developed this bibliographical assessment by using PRISMA guidelines (Belle & Zhao, 2023). Its implementation requires explicitly indicating the methods used to search the papers to review, the consulted databases, the criteria to be eligible, search strategies, etc. Subsequently, we have to explain how we compile bibliometric references. Third, we make a quantitative analysis to state the most relevant sources, authors and approaches to fuzzy option pricing and how this topic has been developed.

As far as eligible studies are concerned, we have only considered papers published in journals and book chapters until December 31, 2022. We have not taken into account grey literature such as conference contributions, working papers or documents in digital repositories. Usually, these contributions, with the improvement of a peer-review process, are finally published as articles or book chapters. Likewise, we have only analysed papers written in English. In regard to the databases, when the topic is not so wide, as is the case of FOP, it is advisable to combine more than one bibliographical source (Zupic & Čater, 2015). We have mixed SCOPUS and WoS, which are commonly used in bibliographical revisions.

In both databases, the search was performed over the titles, abstract and keywords by using the following search: ("option pricing") AND ("fuzzy sets" OR "fuzzy mathematics" OR "fuzzy numbers" OR "fuzzy measures" OR "fuzzy systems"). Fig. 1 graphically depicts how we proceeded.

There were 240 total documents reported by SCOPUS and 227 by WOS. When constraining our search to journal papers and book chapters written in English, we obtained 154 papers from SCOPUS and 158 covered by WoS. After taking into account overlapping reports, we had an initial universe of 194 documents, 40 were provided only by WoS, 36 exclusively by SCOPUS and 118 came from both databases. Regarding Meyer's index, which indicates the proportion of the coverage over a given set of contributions attributable to a given database (Meyer, Mehlman, Reeves, Origoni, Evans, & Sellers, 1983), SCOPUS registered a rate of 48.97%, and WoS registered a rate of 51.03%. The overlap degree, which measures the redundancy of a database, was 75.16% for

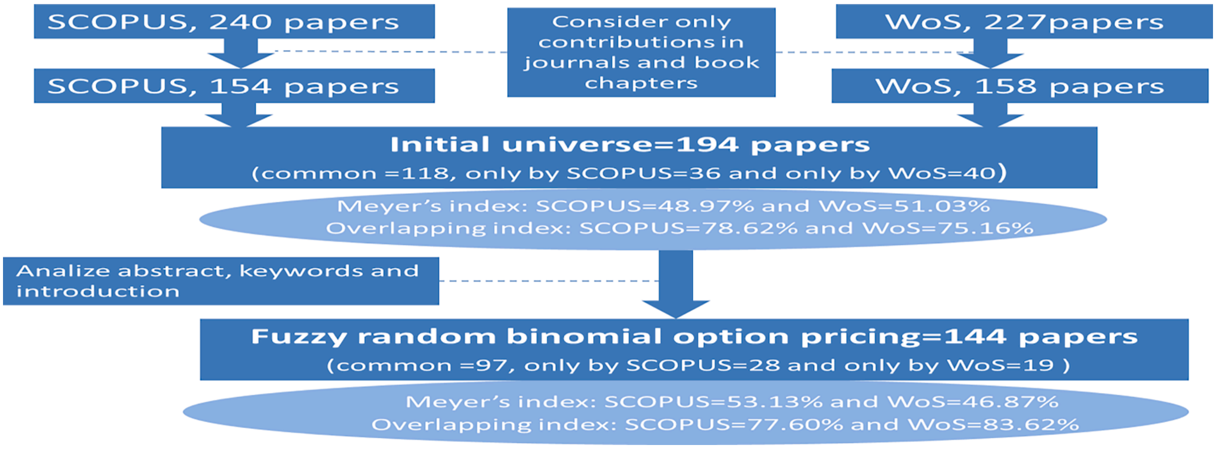


Fig. 1. Process followed to choose papers on fuzzy sets-option pricing to review.

WoS and 78.62% for SCOPUS.

To ensure that the papers were effectively devoted to option pricing with fuzzy set theory tools, we examined the title, abstract, keywords and, if necessary, the entire paper. At this stage, we removed 50 documents. Some of them were linked neither to option pricing nor to finance. However, within these papers, we found 24 contributions that developed option pricing by using the uncertainty theory by Liu (2013).

This mathematical instrument models uncertainty in such a way that it can be considered a "middle point" between probability and possibility theory. However, we have also removed these items since we were interested in the strict interaction between FST and OP.

We finally analysed 144 papers, 116 of which were included in WoS and 125 in SCOPUS. Likewise, 97 documents were common to SCOPUS and WoS, 29 were provided only by SCOPUS and 19 only by WoS. At this stage, Meyer's index was 53.13% for SCOPUS and 46.87% for WoS. Likewise, the overlap degree was 97/124 = 77.60% for SCOPUS and 97/116 = 83.62% for WoS.

2.2. *Classification*

2.2.1. *A map of the research on fuzzy option pricing*

Table 1 shows the taxonomy of the reviewed papers, which is ordered in a matrix representation. In the columns, we indicate the FOP framework in which the paper must be understood. Therefore, the first four conceptual grounds are provided by standard OP methodologies. FST is "superposed" on these conventional standard OP methods to model uncertainty sources in available information different from risk, such as vagueness or imprecision. That is, in these four backgrounds, FOP is applied as a complementary tool to probability theory to build up option pricing models. As is commonplace in option pricing theory, we have differentiated between continuous time models and discrete time methods. Within the continuous time models, we discern between those based on the application of the BSM formula, i.e., those grounded on the hypothesis of a geometric Brownian fluctuation of asset prices, and other continuous models, such as the jump-diffusion model (Merton, 1976) or Lévy processes (Carr & Wu, 2003). Within the discrete time models, we differentiate the binomial approach (Cox, Ross & Rubinstein, 1979), which is the mainstream in FOP in discrete time, and other discrete methods.

The last two groundworks (fifth and sixth) apply strictly fuzzy tools to price options. The fifth, the so-called fuzzy pay-off model, relies on the use of fuzzy numbers to model cash-flow uncertainty. The last set of methodologies prices options nonparametrically with instruments such as fuzzy controllers or fuzzy neural networks, taking advantage of their ability to be universal approximations of functions.

Rows discriminate nuances that may allow us to distinguish homogenous research streams within one of the frameworks identified by the columns. Therefore, whereas the first row contains all the reviewed papers, the subsequent rows (from the second to the eleventh) group these papers by common addressed topic. The second row outlines papers developed under the so-called fuzzy random approach, which is based on fuzzifying the parameters of the stochastic process followed by the subjacent asset. The third row points out the contributions that introduce traders' perception of uncertainty with fuzzy measures and prices options with Choquet's integral. Rows also indicate whether the paper contains a relevant empirical application to financial markets (fifth and sixth rows), if it is devoted to hedging strategies (seventh row), whether it evaluates non-European options or financial assets with some kind of optionality (from eighth and tenth row) or if the contribution deals with real options (eleventh row).

2.2.2. *Research streams in fuzzy option pricing*

Table 1 shows that the most fruitful research field in FOP has been the *fuzzy-random approach* to option prices. The uncertainty in the parameters is modelled by means of fuzzy numbers, which are often of triangular or trapezoidal type. However, other forms of fuzzy numbers, such as adaptive (Thiagarajah et al., 2007; Chrysafis, & Papadopoulos, 2009; Thavaneswaranet al., 2009; Thavaneswaran et al., 2013; Anzilli & Facchinetti, 2017; Anzilli et al., 2018), Gaussian (Chen et al., 2019), piecewise linear (Guerra et al., 2011; Guerra et al., 2017), and octagonal (Meenakshi & Kennedy, 2021b) fuzzy numbers, are also used. More residually, the literature has also applied more complex tools than fuzzy numbers to model vagueness, such as intuitionistic fuzzy numbers (Wu et al., 2016a; Wu et al., 2016b; Wu et al., 2018; Ersen et al., 2022) and Type-2 fuzzy numbers (Bandyopadhyay & Kar, 2019; Tolga, 2020; Wang et al., 2022).

The usual problem addressed by fuzzy-random developments is that often labelled as a direct problem (Muzzioli & De Baets, 2016). The parameters of option pricing formulas are modelled by means of fuzzy numbers, and thus, pricing formulas are evaluated by means of fuzzy arithmetic. The mainstream literature is concentrated on technical questions about how to implement OP models with fuzzy numbers, such as algorithms to fit α-cuts, defuzzification, and arithmetical issues, and propose solutions accordingly. While some of these contributions put FOP models to work with data from option markets (Han & Zheng, 2002; Muzzioli & Torricelli, 2004; Han & Zheng, 2005; Lee et al., 2005; Han & Chen, 2006; Han & Zhou, 2007; Liu & Jiang, 2010; Figa-Talamanca et al., 2012; de Andres-Sanchez, 2017; de Andres-Sanchez, 2018; Zhang & Watada, 2018a; Zhang & Watada, 2018b; Wu et al., 2020), others analyse hedging measures under uncertainty (Chrysafis, & Papadopoulos, 2009; Xu et al., 2010; Guerra et al., 2011; Jiang et al., 2012; Lin et al., 2016; Chen et al., 2019; Jafari, 2022).

Less common are empirical studies on the so-called inverse problem (Muzzioli & De Baets, 2016). Given the observed prices of the options,

Table 1
Contributions of fuzzy set theory to option pricing.

	Conventional option pricing models	
	Continuous time	
	Black-Scholes-Merton	Other models
Overall papers	Anzilli and Villani (2021); Anzilli and Villani (2022); Appadoo and Thavaneswaran (2013); Biancardi and Villani (2017); Buckley and Eslami (2008a); Buckley and Eslami (2008b); Capotorti and Figa-Talamanca (2013); Capotorti and Figà-Talamanca (2020); Carlsson and Fuller (2003); Carlsson, Heikkila and Fuller (2010); Chen, Hu and Yeh (2019); Cherubini and Della Lunga (2001a); Cherubini and Della Lunga (2001b); Cherubini and Mulinacci (2021); Chrysafis and Papadopoulos (2009); Collan, Carlsson and Majlender (2003); Dash, Panda and Panda (2022); de Andres-Sanchez (2017); de Andres-Sanchez (2018); Gao, Ding and Li (2018); Guerra and Sorini (2012); Guerra, Sorini and Stefanini (2011); Han and Chen (2006); Han and Zheng (2002); Han and Zheng (2005); Han and Zhou (2007); Haven (2005); Heng, Chen and Tan (2014); Jafari (2022); Jiang, Liu, Feng and Lai (2012); Kaino and Hirota (2007); Kim and Lee (2018); Lee, Tzeng and Wang (2005); Li, Ware, Di, Yuan, Swishchuk and Yuan (2018); Lin, Liu and Chen (2016); Liu and Jiang (2010); Liu (2009); Liu, Liu and Huang (2013); Liu, Liu, Huang, Chai and Chang (2013); Liu, Xu, Hou, Zhao and Sun (2013); Miyake, Inoue, Shi and Shimokawa (2014); Muzzioli, Gambarelli and De Baets (2018); Muzzioli, Gambarelli and De Baets (2020); Muzzioli, S., Ruggieri, A. and De Baets, B. (2015); Nguyen, Perfiljeva and Holcapek (2020); Rebiasz (2019); Tang, Cui, Zhang and Chen (2019); Teran (2006); Thavaneswaran, Appadoo and Frank (2013); Thavaneswaran, Appadoo and Paseka (2009); Thiagarajah, Appadoo and Thavaneswaran (2007); Tolga (2020); Wang, He and Li (2014); Wang, Kilgour and Hipel (2015); Wu (2004); Wu (2005); Wu (2007); Wu, Yang, Wu and Zhu (2020); Xu, Tan, Gao and Feng (2013); Xu, Xu, Li and Zhang (2010); Yoshida (2003a); Yoshida, Yasuda, Nakagami and Kurano (2006); Yu, Sun and Chen (2011); Zhang, Shi and Xiao (2011); Zhang, Xiao, Kong and Zhang (2015); Zmeskal (2001).	Bandyopadhyay and Kar (2019); Bian and Li (2021); Feng, Cheng, Liu and Jiang (2015); Figa-Talamanca, Guerra and Stefanini (2012); Ghasemalipour and Fathi-Vajargah (2019); Lia, Wareb and Swishchukb (2012); Liu et al. (2013); Liu, Xu, Chai, Liu and Wang (2013); Ma, Zhang, Liu and Fu (2012); Nowak (2011); Nowak and Romaniuk (2010); Nowak and Romaniuk (2013); Nowak and Romaniuk (2014); Nowak and Pawlowski (2017); Nowak and Romaniuk (2017); Nowak and Pawlowski (2019); Qin, Lin and Shang (2020); Wang and He (2016); Wang, Zhao and Song (2022); Xu, Wu, Xu and Li (2009); Zhang and Watada (2018a); Zhang and Watada (2018b); Zhang, Li, Liu and Zhang (2021); Zhang, Wang and Zhang (2022); Zhang, Zhang, Xu and Xiao (2012); Zhao, Wang, Xiang, Chen (2022); Liu and Li (2013).
Fuzzy-random approach	Anzilli and Villani (2021); Anzilli and Villani (2022); Appadoo and Thavaneswaran (2013); Biancardi and Villani (2017); Buckley and Eslami (2008a); Buckley and Eslami (2008b); Capotorti and Figa-Talamanca (2013); Capotorti and Figà-Talamanca (2020); Carlsson and Fuller (2003); Carlsson, et al.2010); Chen, et al. (2019); Chrysafis and Papadopoulos (2009); Collan, et al. (2003); Dash, et al. (2022); de Andres-Sanchez (2017); de Andres-Sanchez (2018); Gao, et al. (2018); Guerra and Sorini (2012); Guerra et al. (2011); Guerra et al. (2017); Heng et al. (2014); Jafari (2022); Kim and Lee (2018); Lee et al. (2005); Li, et al. (2018); Liu et al. (2013a); Liu et al. (2013b); Miyake et al. (2014); Muzzioli et al. (2015); Muzzioli et al. (2018); Muzzioli et al. (2020); Rebiasz (2019); Tang et al. (2019); Teran (2006); Thavaneswaran et al. (2013); Thavaneswaran, et al. (2009); Thiagarajah et al. (2007); Tolga (2020); Wang et al. (2015); Wang, et al. (2014); Wu (2004); Wu (2005); Wu (2007); Wu, et al. (2020); Xu, et al. (2010); Xu, et al. (2013); Yoshida (2003b); Yoshida, et al. (2006); Yu et al. (2011c); Zhang et al. (2011); Zhang et al. (2015); Zmeskal (2001)	Bandyopadhyay and Kar (2019); Bian and Li (2021); Feng et al. (2015); Figa-Talamanca et al. (2012); Ghasemalipour and Fathi-Vajargah (2019); Lia et al. (2012); Liu et al. (2013); Liu and Li (2013); Ma, Zhang, Liu and Fu (2012); Nowak (2011); Nowak and Romaniuk (2010); Nowak and Romaniuk (2013); Nowak and Romaniuk (2014); Nowak and Pawlowski (2017); Nowak and Romaniuk (2017); Nowak and Pawlowski (2019); Qin et al. (2020); Wang and He (2016); Wang et al. (2022); Xu et al. (2009); Zhang and Watada (2018a); Zhang and Watada (2018b); Zhang et al. (2021); Zhang et al. (2022); Zhang et al. (2012)
Fuzzy-measures	Kaino and Hirota (2007); Cherubini and Della Lunga (2001a); Cherubini and Della Lunga (2001b); Cherubini and Mulinacci (2021); Han and Zhou (2007); Han and Zheng (2002); Han and Chen (2006); Han and Zheng (2005); Jiang, et al. 2012); Liu (2009); Liu and Jiang (2010)	
Fuzzy numbers of higher order	Tolga (2020); Wu, et al. (2020)	Bandyopadhyay and Kar (2019); Wang et al. (2022)
The inverse problem	Capotorti and Figa-Talamanca (2013); Capotorti and Figà-Talamanca (2020); Muzzioli et al. (2015); Muzzioli et al. (2018); Muzzioli et al. (2020)	
Applications to financial markets	de Andres-Sanchez (2017); de Andres-Sanchez (2018); Han and Zheng (2002); Han and Chen (2006); Han and Zheng (2005); Liu and Jiang (2010); Wu, et al. (2020)	Figa-Talamanca et al. (2012); Zhang and Watada (2018a); Zhang and Watada (2018b)
Hedging	Chrysafis and Papadopoulos (2009); Chen et al., (2019); Guerra et al. (2011); Jafari (2022); Jiang et al. (2012); Lin et al. (2016); Xu et al., (2010)	
American Options	Liu et al., (2013b); Yoshida, et al. (2006); Zhang et al. (2011)	
Non plain vanilla options	Anzilli and Villani (2022); Anzilli and Villani (2021); Appadoo and Thavaneswaran (2013); Miyake et al. (2014); Thavaneswaran et al. (2013); Wang, et al. (2014); Xu, et al. (2013); Xu, et al. (2010); Yu et al. (2011c); Zhang et al. (2015)	Ma et al. (2012); Qin et al. (2020); Wang and He (2016); Zhao et al. (2022)
Other assets	Cherubini and Della Lunga (2001a); Cherubini and Della Lunga (2001b); Zmeskal (2001).	Nowak and Romaniuk (2017).
Real options	Anzilli and Villani (2022); Anzilli and Villani (2021); Biancardi and Villani (2017); Carlsson and Fuller (2003); Carlsson, et al. (2010); Collan et al. (2003); Gao, Ding and Li (2018); Heng et al.,(2014); Kim and Lee (2018); Rebiasz (2019); Tang et al. (2019); Tolga (2020); Wang et al. (2015)	
Other applications	Haven (2005); Lee et al. (2005); Lin et al. (2016); Nguyen et al. (2020); Wang et al. (2015)	Zhang et al. (2022)

	Conventional option pricing models		Fuzzy Sets	
	Discrete time			
	Binomial	Others	Fuzzy pay-off	Non parametrical approach
Overall papers	Anzilli and Facchinetti (2017); Anzilli, Facchinetti and Pirotti, T. (2018); Buckley and Eslami (2007); Buckley and Eslami (2008b); Chrysafis and Papadopoulos (2021); D`Amato, Zrobek, Bilozor, Walacik and Mercadante (2019); Ho and Liao (2011); Lee, Tzeng and Wan (2005); Meenakshi and Felbin (2019); Meenakshi and Kennedy (2021a); Meenakshi and Kennedy (2021b); Muzzioli and Reynaerts (2007); Muzzioli and Reynaerts (2008); Muzzioli and Torricelli (2004); Shang, Yang, Liu, Shang and Zhang (2020); Wang, Wang and Tang (2022); Xu, Liu and Yu (2018); Yoshida (2003a); Yu, Huarng, Li and Chen (2011b); Yu, Li, Huarng, Chen and Chen (2011a); Zhang and Yin (2021); Zmeskal (2010); Zmeskal, Dluhosova, Gurny and Kresta (2022)	Allenotor and Thulasiram (2011); Ersen, Tas and Ugurlu (2022); Gu, An, Chen and Zhang (2015); Wang, Hipel and Kilgour (2009); Wu, Liu, Wang and Zhuang (2016); Wu, Mei and Sun (2018); Wu, Zhuang and Li (2016)	Collan, Fuller and Mezei (2009); Collan, Fedrizzi and Luukka (2017); Hassanzadeh, Collan and Modarres (2012); Luukka and Collan (2015); Mintah, Higgins, Callanan and Wakefield (2017); Servati, Ghodsypour and Shirazi (2017); Stoklasa, Luukka and Collan 2021)	Gradojevic and Kukolj (2011); Hajizadeh (2020); Kakati 2008); Lee, Lin and Yang (2014); Leu, Lee and Hung (2011); Maciel, Lemos, Gomide and Ballini (2012); Magni, Mastroleo, Vignola and Facchinetti (2004); Sharma, Thulasiram, Thulasiraman and Buyya (2014); Tung and Quek (2011); Wang (2007); Yang, Leu and Lee (2014); Yen (2010a); Yen (2010b)
Fuzzy-random approach	Anzilli and Facchinetti (2017); Anzilli, Facchinetti and Pirotti, T. (2018); Buckley and Eslami (2007); Buckley and Eslami (2008b); Chrysafis and Papadopoulos (2021); D`Amato, Zrobek, Bilozor, Walacik and Mercadante (2019); Ho and Liao (2011); Lee, Tzeng and Wan (2005); Meenakshi and Felbin (2019); Meenakshi and Kennedy (2021a); Meenakshi and Kennedy (2021b); Muzzioli and Reynaerts (2007); Muzzioli and Reynaerts (2008); Muzzioli and Torricelli (2004); Shang, Yang, Liu, Shang and Zhang (2020); Wang, Wang and Tang (2022); Xu, Liu and Yu (2018); Yoshida (2003); Yu, Huarng, Li and Chen (2011b); Yu, Li, Huarng, Chen and Chen (2011a); Zhang and Yin (2021); Zmeskal (2010); Zmeskal, Dluhosova, Gurny and Kresta (2022)	Allenotor and Thulasiram (2011); Ersen, Tas and Ugurlu (2022); Wang, Hipel and Kilgour (2009); Wu, Liu, Wang and Zhuang (2016); Wu, Mei and Sun (2018); Wu, Zhuang and Li (2016)		
Fuzzy measures		Gu et al. (2015)		
Fuzzy numbers of higher order		Ersen et al. (2022); Wu et al. (2016a); Wu et al. (2016b); Wu et al. (2018).		
Inverse problem				Tung and Quek (2011)
Application to financial markets	Lee et al. (2005); Muzzioli and Torricelli (2004)			Abdollahi (2020); Gradojevic and Kukolj (2011); Hajizadeh (2020); Kakati 2008); Lee, et al. (2014); Leu et al. (2011); Maciel et al. (2012); Sharma et al. (2014); Tung and Quek (2011); Wang (2007); Yang, et al. (2014); Yen (2010a); Yen (2010b)
Hedging				Tung and Quek (2011)
American options	Yoshida (2003); Muzzioli and Reynaerts (2008); Zmeskal (2010); Meenakshi and Felbin (2019); Meenakshi and Kennedy (2021b)			
Not plain vanilla options	Xu et al. (2018).			
Other assets	Anzilli and Facchinetti (2017); Anzilli et al. (2018).	Gu, et al. (2015); Wu, et al. (2016a); Wu et al. (2016b); Wu, et al. (2018)		
Real options		Anzilli and Facchinetti (2017); Chrysafis and Papadopoulos (2021); D`Amato et al. (2019); Ho and Liao	Collan et al. (2009); Collan et al. (2017); et al. (2012); Luukka and Collan (2015); Mintah et al. (2017);	(Magni et al., 2004)

(*continued on next page*)

Table 1 (*continued*)

	Conventional option pricing models		Fuzzy Sets	
	Discrete time			
	Binomial	Others	Fuzzy pay-off	Non parametrical approach
		(2011); Shang et al. (2020); Zhang and Yin (2021); Zmeskal et al. (2022)	Servati et al. (2017); Stoklasa et al. (2021)	
Other applications				

which usually have some imprecision, the objective is fitting the parameters that govern the price movements of the subjacent asset, especially the implicit volatility that allows fitting traders' expectations about the future volatility of the subjacent asset. This problem is analysed with the help of coherent probability-possibility transformation methods (Dubois, Folloy, Mauris, & Prade, 2004) by Capotorti & Figa-Talamanca (2013), Capotorti & Figa-Talamanca (2020) and with the use of fuzzy regressions based on the minimum fuzziness principle (Chang & Ayyub, 2001) in the papers by Muzzioli et al. (2015), Muzzioli et al. (2018) and Muzzioli et al. (2020).

The main application of fuzzy-random contributions is on European financial options. However, they have been applied to price options of the American type (Yoshida, 2003a; Yoshida et al., 2006; Muzzioli & Reynaerts, 2008; Zmeskal, 2010; Zhang et al., 2011; Liu et al., 2013b; Meenakshi & Felbin, 2019; Meenakshi, & Kennedy, 2021b) and other types of options, such as exotics (Appadoo & Thavaneswaran, 2013; Thavaneswaran et al., 2013; Miyake et al., 2014; Zhang et al., 2015; Wang & He, 2016; Qin et al., 2020). This fuzzy random approach has also been applied to price other financial instruments such as foreign currency options by fuzzifying the adaptation of BSM to currency markets (Garman, & Kohlhagen, 1983) in (Xu et al., 2010; Yu et al., 2011c; Xu et al., 2013); to deposit insurance (Wu et al., 2020); in catastrophe bonds (Nowak, & Romaniuk, 2017); in minimum guarantees life insurance (Anzilli & Facchinetti, 2017; Anzilli et al., 2018); to credit default swaps (Wu et al., 2016a; Wu et al., 2016b; Wu et al., 2018) or to fit the firms' value (Zmeskal, 2001).

A fruitful application of FST on OP is the use of λ-*fuzzy measures* and *Choquet's integral*. In this case, fuzzification is not performed over the parameters but by distorting the risk-neutral distribution function with the help of λ-fuzzy measures. As shown in Table 1, this is usually done on the basis of the BSM framework. This modelling has been used to model traders' perception of the degree of market uncertainty (Han & Zhou, 2007) due to phenomena that cause frictions in markets, such as illiquidity (Cherubini & Della, 2001a; Jiang et al., 2012) and the possibility of issuers' insolvency (Cherubini & Della, 2001b; Han & Zheng, 2005).

FOP has modelled *real option pricing* by using a wide variety of approaches. A fruitful focus is the fuzzy-random approach with BSM (Carlsson & Fuller, 2003; Carlsson et al., 2010; Collan et al., 2003; Gao et al., 2018; Kim, & Lee, 2018; Rebiasz, 2019) and the adaptation of BSM to exchange options (Margrabe, 1978) under fuzziness (Biancardi & Villani, 2017; Anzilli, & Villani, 2021; Anzilli & Villani, 2022). Other contributions to this research stream come from the fuzzy-binomial methodology (Zmeskal, 2010; Ho & Liao, 2011; Anzilli & Facchinetti, 2017; D'Amato et al., 2019; Shang et al., 2020; Chrysafis & Papadopoulos, 2021; Zhang & Yin, 2021; Zmeskal et al., 2022) and by the fuzzy approach by Allenotor and Thulasiram (2011) to the trinomial method (Boyle, 1988). FOP has also developed real option pricing with the only use of fuzzy tools. This is the case for the *fuzzy pay-off* method (Collan et al., 2009; Hassanzadeh et al., 2012; Luukka & Collan, 2015; Collan et al., 2017; Mintah et al., 2017; Servati et al., 2017; Stoklasa et al., 2021) and *expert systems* (Magni et al., 2004).

The papers included in the row *other applications* do not try to capture alternative sources of informational uncertainty to risk to fit fuzzy option prices. In contrast, they use fuzzy tools to model imprecision on objectives of decision making, such as fuzzy programming (Li et al., 2018), fuzzy preferences axiomatic (Haven, 2005) or the concept of fuzzy events (Lee et al., 2005). Likewise, while Nguyen et al. (2020) develop a fuzzy-transform-based numerical method for solving ordinary differential equations with boundary conditions and apply it to the BSM model, Zhang et al. (2022) propose a homotopy transformation methodology for solving fuzzy fractional Black-Scholes pricing equations.

2.3. *Quantitative analysis of bibliometric search*

Fig. 2 shows the time trend of contributions in fuzzy option pricing. We can observe increasing interest from the early 2000s to 2011 when the number of papers reached a maximum (12). Since 2011, the number of papers published every year, with the exception of 2015 and 2016, has been at least 7. Thus, we can conclude that even though FOP is not a mainstream issue, it is a consolidated topic of fuzzy mathematics.

Table 2 shows the principal journals that have served as outlets. The main orientation of some of the most relevant journals is fuzzy sets theory and its applications (Fuzzy Sets and Systems with 15 papers; International Journal of Fuzzy Systems with 5 papers; IEEE Transactions of Fuzzy Systems with 4 papers). Papers on the most general topics of soft computing are also fruitful outlets (for example, Soft Computing with 5 papers; Expert Systems with Applications with 4 contributions and Applied Soft Computing with 3 papers). The last great source of contributions comes from journals in a more generalist field of operational research and computational mathematics (European Journal of Operational Research and Journal of Computational and Applied Mathematics with 6 contributions; Journal of Applied Mathematics with 4 papers; and International Journal of Applied Mathematics and Statistics that provides 2 papers).

Tables 3 and 4 show the most influential papers in both the SCOPUS and WoS databases. The most cited papers in SCOPUS are about the fuzzy random approach with the BSM framework (Zmeskal, 2001; Carlsson, & Fuller, 2003; Yoshida, 2003; Wu, 2004; Thiagarajah et al., 2007; Wu, 2007; Chrysafis, & Papadopoulos, 2009; Thavaneswaran et al., 2009) and with the binomial method (Muzzioli & Torricelli, 2004; Zmeskal, 2010). The exception is Collan et al. (2009), who propose an original fuzzy number approach to real option pricing. Within non-parametrical option pricing, we found (Maciel et al., 2012). Within the three four most relevant papers, two are about real option pricing (Carlsson, & Fuller, 2003; Collan et al., 2009).

In WoS, as in SCOPUS, the most cited contributions are about the fuzzy-random approach with BSM groundwork (Zmeskal, 2001; Carlsson & Fuller, 2003; Yoshida, 2003b; Wu, 2004; Thiagarajah et al., 2007; Wu, 2007; Chrysafis & Papadopoulos, 2009; Thavaneswaran et al., 2013; Thavaneswaran et al., 2009; Wang et al., 2014) and with a binomial lattice (Yoshida, 2003a; Muzzioli & Torricelli, 2004; Zmeskal, 2010). However, the comparison of Table 3 and Table 4 displays relevant differences. Although Collan et al. (2009) has a great influence on the fuzzy pricing of real options, it is not registered in the WoS database. Likewise, a very influential paper about fuzzy nonparametric pricing (Maciel et al., 2012), despite being covered by SCOPUS, is not covered by WoS. Thus, it reinforces the usefulness of combining SCOPUS and WoS in the bibliographical search.

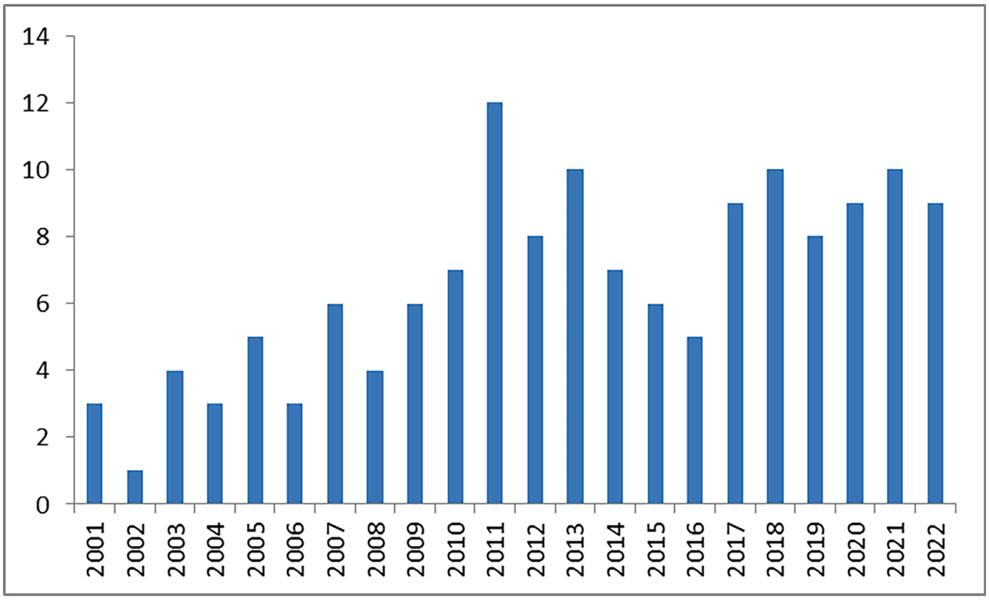


Fig. 2. Number of papers on fuzzy option pricing published in SCOPUS and WoS as journal articles and book chapters.

Table 2
Journals/Books that published at least two papers on fuzzy option pricing.

Journal	Number of items
Fuzzy Sets and Systems	15
European Journal of Operational Research	
Journal of Computational and Applied Mathematics	
6	
International Journal of Fuzzy Systems	
Soft Computing	5
Expert Systems with Applications	
IEEE Transactions of Fuzzy Systems	
Journal of Applied Mathematics	4
Applied Soft Computing	
International Journal of Applied Mathematics and Statistics	
3	
Computers & Mathematics with Applications	
Discrete Dynamics in Nature And Society	
Information Sciences	
Insurance Mathematics & Economics	
International Journal of Approximate Reasoning	
International Journal of Information Technology & Decision Making	
International Journal of Intelligent Systems	
Journal of Applied Mathematics and Decision Sciences	
Journal of Intelligent & Fuzzy Systems	
Journal of Theoretical And Applied Information Technology	
Mathematical and Computer Modelling	
Perception-Based Data Mining and Decision Making in Economics And Finance (Book)	2

Table 5 outlines the most prolific authors on FOP and their nationality (at least three contributions). The most represented country is Italy (8 authors), followed by China (4 authors). The most productive authors are P. Nowak (7 contributions), followed by S. Muzzioli (6 items) and L. Anzili, A. Thavaneswaran, M. Collan, S. Liu, L.Y. Han and M. Romaniuk with 4 papers.

3. Fuzzy-random approach to option pricing

3.1. *Fuzzy-random option pricing in continuous time*

3.1.1. *Fuzzy random Black-Scholes-Merton formula*

Table 1 shows that the main approach of fuzzy-random option pricing in continuous time consists of adapting the BSM formula to the use of parameters estimated with fuzzy numbers. However, other stochastic frameworks on subjacent asset prices, such as that by Heston (1993) in Figa-Talamanca et al. (2012); jump-diffusion processes (Xu et al., 2009; Zhang et al., 2012; Liu et al., 2013; Liu & Li, 2013; Meenakshi and Kennedy, 2021a); fractional Brownian motion (Ghasemalipour & Fathi-Vajargah, 2019; Bian & Li, 2021; Zhang et al., 2021; Zhao et al., 2022) and Levy processes (Nowak & Romaniuk, 2010; Nowak, 2011; Nowak & Romaniuk, 2013; Feng et al., 2015; Wang & He, 2016; Nowak & Romaniuk, 2014; Nowak & Pawlowski, 2017; Nowak & Pawlowski, 2019), have been used to complement random modelling of prices with imprecision in parameters governing that stochastic behaviour.

In the following, we constrain our exposition to fuzzy BSM without dividend payment of the subjacent asset. BSM supposes that the price of the asset follows a geometric Brownian motion that we can express as $dS_t = mS_tdt + \sigma S_tdW_t$. Therefore, S_t is the price at *t*, *m is* the expected annual instantaneous growth rate, σ is the standard deviation of that growth and dW_t is a standard Wiener process. The dynamic of a European call option price at t (C_t) with exercise price K and maturity τ can be deduced by Ito's lemma as:

$$dC_t = \frac{\partial C_t}{\partial S_t}dS_t + \frac{\partial C_t}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}\sigma^2 S_t^2 dt. \tag{1}$$

Although dC_t is random since dS_t is so, it is possible to fit nonrandom prices to C_t. BSM supposes that in complete markets, it is possible to build up a continuously hedged portfolio of the subjacent asset with a short position on a call option with value $\Pi_t = \frac{\partial C_t}{\partial S_t}S_t - C_t$. This portfolio has no risk, and thus, to avoid arbitrage, its profit within an interval time dt must be that provided by the risk-free asset. Thus,

$$d\Pi_t = \frac{\partial C_t}{\partial S_t}dS_t - dC_t = r\Pi_t dt = r\left(\frac{\partial C_t}{\partial S_t}S_t - C_t\right)dt. \tag{2}$$

Therefore, by substituting (1) in (2), we have the well-known heat partial differential equation:

$$\frac{\partial C_t}{\partial t} + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}\sigma^2 S_t^2 dt + \frac{\partial C_t}{\partial S_t} rS_t - rC_t = 0. \tag{3}$$

By stating the boundary condition at $t = \tau$, $C_\tau = Max\{S_\tau - K, 0\}$, we obtain the price at $t = 0$.

Table 3
Table of the fifteen most cited papers in SCOPUS.

Cites	Reference	Tittle	Year	Journal
191	(Carlsson, & Fuller, 2003)	A fuzzy approach to real option valuation	2003	Fuzzy Sets and Systems
120	(Yoshida, 2003b).	The valuation of European options in uncertain environment	2003	European Journal of Operational Research
106	(Wu, 2004)	Pricing European options based on the fuzzy pattern of Black-Scholes formula	2004	Computers and Operations Research
102	(Collan et al., 2009)	A Fuzzy Pay-Off Method for Real Option Valuation	2009	Journal of Applied Mathematics and Decision Sciences
76	(Yoshida et al., 2006)	A new evaluation of mean value for fuzzy numbers and its application to American put option under uncertainty	2006	Fuzzy Sets and Systems
73	(Wu, 2007)	Using fuzzy sets theory and Black-Scholes formula to generate pricing boundaries of European options	2007	Applied Mathematics and Computation
72	(Zmeskal, 2001)	Application of the fuzzy-stochastic methodology to appraising the firm value as a European call option	2001	European Journal of Operational Research
58	(Muzzioli & Torricelli, 2004)	A multiperiod binomial model for pricing options in a vague world	2004	Journal of Economic Dynamics and Control
54	(Lee et al., 2005)	A new application of fuzzy set theory to the Black-Scholes option pricing model	2005	Expert Systems with Applications
53	(Thavaneswaran et al., 2009)	Weighted possibilistic moments of fuzzy numbers with applications to GARCH modelling and option pricing	2009	Mathematical and Computer Modelling
51	(Zmeskal, 2010)	Generalized soft binomial American real option pricing model (fuzzy-stochastic approach)	2010	European Journal of Operational Research
50	(Chrysafis, & Papadopoulos, 2009)	On theoretical pricing of options with fuzzy estimators	2009	Journal of Computational and Applied Mathematics
50	(Thiagarajah et al., 2007)	Option valuation model with adaptive fuzzy numbers	2007	Computers and Mathematics with Applications
47	(Maciel et al., 2012).	Evolving fuzzy systems for pricing fixed income options	2012	Evolving Systems
44	(Wu, 2005)	European option pricing under fuzzy environments	2005	International Journal of Intelligent Systems

Table 4
Table of the fifteen most cited papers in WoS.

Cites	Authors	Title	Year	Source Title
167	(Carlsson, & Fuller, 2003)	A fuzzy approach to real option valuation	2003	Fuzzy Sets and Systems
117	(Yoshida, 2003b).	The valuation of European options in uncertain environment	2003	European Journal of Operational Research
102	(Wu, 2004)	Pricing European options based on the fuzzy pattern of Black-Scholes formula	2004	Computers & Operations Research
81	(Zmeskal, 2001)	Application of the fuzzy-stochastic methodology to appraising the firm value as a European call option	2001	European Journal of Operational Research
79	(Wu, 2007)	Using fuzzy sets theory and Black-Scholes formula to generate pricing boundaries of European options	2007	Applied Mathematics and Computation
70	(Yoshida et al., 2006)	A new evaluation of mean value for fuzzy numbers and its application to American put option under uncertainty	2006	Fuzzy Sets and Systems
63	(Muzzioli & Torricelli, 2004)	A multiperiod binomial model for pricing options in a vague world	2004	Journal of Economic Dynamics & Control
60	(Zmeskal, 2010)	Generalized soft binomial American real option pricing model (fuzzy-stochastic approach)	2010	European Journal of Operational Research
50	(Thavaneswaran et al., 2009)	Weighted possibilistic moments of fuzzy numbers with applications to GARCH modelling and option pricing	2009	Mathematical And Computer Modelling
49	(Chrysafis, & Papadopoulos, 2009)	On theoretical pricing of options with fuzzy estimators	2009	Journal of Computational And Applied Mathematics
49	(Lee et al., 2005)	A new application of fuzzy set theory to the Black-Scholes option pricing model	2005	Expert Systems with Applications
48	(Thiagarajah et al., 2007)	Option valuation model with adaptive fuzzy numbers	2007	Computers & Mathematics with Applications
44	(Wu, 2005)	European option pricing under fuzzy environments	2005	International Journal of Intelligent Systems
44	(Yoshida, 2003a)	A discrete-time model of American put option in an uncertain environment	2003	European Journal of Operational Research
33	(Thavaneswaran et al., 2013)	Binary option pricing using fuzzy numbers	2013	Applied Mathematics Letters

Table 5
Authors with at least three contributions.

Author	Items	Author	Items
Nowak, P. (Poland)	7	Eslami, E. (Iran)	3
Muzzioli, S. (Italy)	6	Figa-Talamanca (Italy)	3
Anzili, L. (Italy)	4	Fachinetti, G. (Italy)	3
Collan, M. (Finland)	4	Fuller, R. (Hungary)	3
Liu, S. (China)	4	Guerra, M.L. (Italy)	3
Han, L.Y. (China)	4	Meenakshi, K. (India)	3
Romaniuk, M. (Poland)	4	Sorini, L. (Italy)	3
Thavaneswaran, A. (Canada)	4	Villani, G. (Italy)	3
Appadoo, S.S. (Canada)	3	Wu, H.C. (Taiwan)	3
Buckley, J.J. (United States)	3	Wu, L. (China)	3
Carlsson, C. (Finland)	3	Yoshida, Y. (Japan)	3
Cherubibi, U. (Italy)	3	Zhang, W.G. (China)	3
De Baets, B. (Belgium)	3	Zmeskal, Z. (Czech Republic)	3

$$C_0(S_0,K,r,\sigma,\tau)=S_0\Phi\left(\frac{ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right)-e^{-r\tau}K\Phi\left(\frac{ln\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right),\tag{4}$$

where $\Phi(\bullet)$ is the cumulative standard Gaussian function.

The fuzzy-random approach on the framework provided by (3) and its boundary condition states that some of the parameters cannot be quantified precisely but by means of fuzzy numbers. Of course, which of those parameters are fuzzy and which are crisp depends on the setting in which the BSM must be applied. It has no sense supposing uncertainty in the strike price or in the expiration date for the options traded in financial markets, since these variables are clearly defined in the contract. However, it does not follow on real options. In this setting, the strike price could be a future outflow to exercise optionality that at $t=0$ is not known precisely (Carlsson & Fuller, 2003; Collan et al., 2003; Carlsson et al., 2010). Real options are not linked to standard assets but to investment projects, and thus, many times, the maturity of the optionality is not known with precision, and therefore, the date of exercise may also be uncertain. As a general setting, we will suppose that the five parameters of (4) could be fuzzy numbers. Therefore, we symbolize the price of the subjacent asset as $\widetilde{S_t}$; the strike price with $\widetilde{K}$; the risk-free rate with $\widetilde{r}$; volatility as $\widetilde{\sigma}$ and expiration date $\widetilde{\tau}$ as their α-cuts are $S_{0\alpha}=\left[S_{0\alpha -},\overline{S_{0\alpha}}\right]$, $K_\alpha=\left[K_{\alpha -},\overline{K_\alpha}\right]$, $r_\alpha=\left[r_{\alpha -},\overline{r_\alpha}\right]$, $\sigma_\alpha=\left[\sigma_{\alpha -},\overline{\sigma_\alpha}\right]$ and $\tau_\alpha=\left[\tau_{\alpha -},\overline{\tau_\alpha}\right]$, respectively.

In the fuzzy-random approach setting of the FOP, the differential Eq. (3) has fuzzy parameters and a fuzzy boundary condition $\widetilde{C_\tau}=Max\left\{\widetilde{S_\tau}-\widetilde{K},0\right\}$. Commonplace in the literature consists of obtaining a fuzzy price of the call option $\widetilde{C_t}$ by evaluating the crisp function (4) with fuzzy numbers with the rules proposed by Buckley & Qu (1990). This procedure is supported by the concept of the solution of differential equations with fuzzy coefficients and fuzzy boundary conditions (Buckley & Feuring, 2000), in such a way that $\widetilde{C_0}=C_0\left(\widetilde{S_0},\widetilde{K},\widetilde{r},\widetilde{\sigma},\widetilde{\tau}\right)$. Thus, under the hypothesis $S_{0\alpha -}\geq 0$, $K_{\alpha -}\geq 0$ the α-cuts $C_{0\alpha}=\left[C_{0\alpha -},\overline{C_{0\alpha}}\right]$ are evaluated as:

$C_{0\alpha}=\{x|x=C_0(S_0,K,r,\sigma,\tau),\ S_0\in S_{0\alpha},K\in K_\alpha,r\in r_\alpha,\sigma\in\sigma_\alpha,\tau\in\tau_\alpha\}$, and given that $\frac{\partial C_0}{\partial S_0}\geq 0$, $\frac{\partial C_0}{\partial K}\leq 0,\frac{\partial C_0}{\partial r}\geq 0,\frac{\partial C_0}{\partial\sigma}\geq 0$ and $\frac{\partial C_0}{\partial\tau}\geq 0$ (Hull, 2008), then:

$$\begin{aligned}C_{0\alpha}&=\left[C_{0\alpha -},\overline{C_{0\alpha}}\right]\\&=\left[C_0\left(S_{0\alpha -},\overline{K_\alpha},r_{\alpha -},\sigma_{\alpha -},\tau_{\alpha -}\right),C_0\left(\overline{S_{0\alpha}},K_{\alpha -},\overline{r_\alpha},\overline{\sigma_\alpha},\overline{\tau_\alpha}\right)\right].\end{aligned}\tag{5}$$

Once the price of the call option has been obtained, it is straightforward to adjust an expression for the price of a put option with the same strike price by means of put-call parity (Stoll, 1969). At any moment of time t, the price of a European call,C_t, and that of a put option, P_t, with the same strike price K, must accomplish the equilibria equation:

$$C_t+Ke^{r(\tau-t)}=S_t+P_t,\tag{6}$$

from (6), we can state that P_0 also depends on S_0,K,r,σ and τ in such a way that

$$P_0(S_0,K,r,\sigma,\tau)=e^{-r\tau}K\Phi\left(-\frac{ln\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right)-S_0\Phi\left(-\frac{ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right).\tag{7}$$

In the case of having fuzzy estimates of the parameters of (7) $\widetilde{S}_0,\widetilde{K},\widetilde{r}$, $\widetilde{\sigma}$ and $\widetilde{\tau}$, the put-call parity equation turns into an equation with fuzzy parameters $\widetilde{C_0}+\widetilde{K}e^{\widetilde{r\bullet\widetilde{\tau}}}=\widetilde{S}_0+\widetilde{P}_0$. In this regard, the concept of solving equations with fuzzy coefficients (Buckley & Qu, 1991; Buckley, 1992) justifies evaluating (7) as $\widetilde{P}_0=P_0\left(\widetilde{S}_0,\widetilde{K},\widetilde{r},\widetilde{\sigma},\widetilde{\tau}\right)$. Therefore, $P_{0\alpha}$ is:

$P_{0\alpha}=\{x|x=P_0(S_t,K,r,\sigma,\tau),\ S_0\in S_{0\alpha},K\in K_\alpha,r\in r_\alpha,\sigma\in\sigma_\alpha,\tau\in\tau_\alpha\}$, and $\frac{\partial P_0}{\partial S_0}\leq 0$, $\frac{\partial P_0}{\partial K}\geq 0,\frac{\partial P_0}{\partial r}\leq 0$ and $\frac{\partial P_0}{\partial\sigma}\geq 0$, see (Hull, 2008). Regarding $\frac{\partial P_0}{\partial\tau}$, despite usually being ≥ 0, it could be negative for in-the-money European put options on nondividend-paying stocks (Hull, 2008). Then, $P_{0\alpha}=\left[P_{0\alpha -},\overline{P_{0\alpha}}\right]$, where:

$$\begin{aligned}P_{0\alpha -}&=\ \text{minimize}P_0\left(\overline{S_{0\alpha}},K_{\alpha -},\overline{r_\alpha},\sigma_{\alpha -},\tau\right),\quad\text{subject to}\ \tau_{\alpha -}\leq\tau\leq\overline{\tau_\alpha},\\\overline{P_{0\alpha}}&=\text{maximize}P_0\left(S_{0\alpha -},\overline{K_\alpha},r_{\alpha -},\overline{\sigma_\alpha},\tau\right),\ \text{subject to}\ \tau_{\alpha -}\leq\tau\leq\overline{\tau_\alpha}.\end{aligned}$$

3.1.2. Numerical application

Let us suppose that the uncertainty of the parameters in the BSM is modelled by means of triangular fuzzy numbers (TFNs). A TFN $\widetilde{a}$ will be denoted as $\widetilde{a}=(a_1,a_2,a_3)$, where a_2 is the core and $[a_1,a_3]$ the support in such a way that $a_\alpha=\left[a_{\alpha -},\overline{a_\alpha}\right]=[a_1+(a_2-a_1)\alpha,a_3-(a_3-a_2)\alpha]$. When parameters are given by TFNs, the result is a fuzzy number that admits a good approximation by a linear fuzzy number with the same support and core (de Andres-Sanchez, 2018); therefore, we can state that $\widetilde{C}_0\approx\widetilde{c}_0=(c_{01},c_{02},c_{03})$ and $\widetilde{P}_0\approx\widetilde{p}_0=(p_{01},p_{02},p_{03})$.

Let be a fuzzy observation of the price of an asset $\widetilde{S_0}=(103, 104, 105)$, and its annual volatility is $\widetilde{\sigma}$=(0.35, 0.4, 0.45). We will price European call and put options on that asset with a maturity (in years) $\widetilde{\tau}=(1, 1, 1)$, strike option, $\widetilde{K}$=(108, 109, 110) and with a risk-free rate, $\widetilde{r}$=(0.005, 0.006, 0.007). In both cases, $S_{0\alpha}=[103+\alpha,105-\alpha]$, $\sigma_\alpha=[0.35+0.05\alpha,0.45-0.05\alpha]$, $\tau_\alpha=[1,1]$, $K_\alpha=[108+\alpha,110-\alpha]$ and $\sigma_\alpha=[0.005+0.001\alpha,0.007-0.001\alpha]$.

The results of our application are displayed in Table 6. The triangular approximates to $\widetilde{C}_0$ and $\widetilde{P}_0$ with TFNs are very accurate since the deviations in the extremes of α-cuts are never above 1%. Therefore, to obtain a reasonably good estimate for the shape of the price, it is not necessary to calculate many α-cuts, which implies making 9 calculations in the case of α = 0, 0.25, 0.5, 0.75, 1. It is sufficient to evaluate α = 0 and α = 1.

Table 6
Alpha-cuts of call and put fuzzy prices with Black-Scholes-Merton formula and $\widetilde{S_0} = (103, 104, 105)$, $\widetilde{K}$=(108, 109, 110), $\widetilde{\sigma}$=(0.35, 0.4, 0.45), $\widetilde{r}$=(0.005, 0.006, 0.007) and $\widetilde{\tau} = (1, 1, 1)$.

	$\widetilde{C_0}$		$\widetilde{c_0} = (9.75, 14.74, 20.67)$		Error in $C_{0\alpha-}$ and $\overline{C_{0\alpha}}$	
α	$C_{0\alpha-}$	$\overline{C_{0\alpha}}$	$c_{0\alpha-}$	$\overline{c_{0\alpha}}$	Error in $C_{0\alpha-}$	Error in $\overline{C_{0\alpha}}$
1	14.74	14.74	14.74	14.74	0.00%	0.00%
0.75	14.01	15.48	14.02	15.48	0.04%	0.02%
0.5	13.28	16.22	13.29	16.22	0.06%	0.03%
0.25	12.55	16.96	12.56	16.96	0.05%	0.02%
0	11.83	17.71	11.83	17.71	0.00%	0.00%
	$\widetilde{P_0}$		$\widetilde{p_0} = (15.19, 19.09, 23.17)$		Error in $P_{0\alpha-}$ and $\overline{P_{0\alpha}}$	
α	$P_{0\alpha-}$	$\overline{P_{0\alpha}}$	$p_{0\alpha-}$	$\overline{p_{0\alpha}}$	Error in $P_{0\alpha-}$	Error in $\overline{P_{0\alpha}}$
1	19.09	19.09	19.09	19.09	0.00%	0.00%
0.75	18.29	19.89	18.29	19.89	0.01%	0.00%
0.5	17.49	20.70	17.49	20.70	0.01%	0.01%
0.25	16.69	21.50	16.69	21.50	0.01%	0.00%
0	15.89	22.31	15.89	22.31	0.00%	0.00%

Note: The errors are fitted as the relative difference between the actual extremes of α-cuts and those provided by the triangular approximate.

3.1.3. An alternative approach to Black-Scholes-Merton fuzzy-random differential equation

The literature provides a slightly different enhanced solution to the fuzzyfied version of (3) that allows a more general framework for alternative option types to plain vanilla such as digital options (Li et al., 2018). In that paper, the authors only allow fuzzy $\widetilde{r}$ and $\widetilde{\sigma}$ and apply an analogous arbitrage argument to that proposed for confidence interval valued volatilities in (Avellaneda, Levy, & Paras, 1995). The α-cuts of the fuzzy price of a call option, $C_{0\alpha} = \left[C_{0\alpha-}, \overline{C_{0\alpha}}\right]$ are interpreted as the combined perception of the asked price by potential sellers ($\overline{C_{0\alpha}}$) and bid price by holders ($C_{0\alpha-}$). The dynamic of the asked price of the option is:

$$\frac{\partial \overline{C_{t\alpha}}}{\partial t} + \frac{1}{2}\frac{\partial^2 \overline{C_{t\alpha}}}{\partial {S_t}^2}\sigma^2 {S_t}^2 dt + r\left(\frac{\partial \overline{C_{t\alpha}}}{\partial S_t}S_t - \overline{C_{t\alpha}}\right) = 0,$$

where $\sigma_{\alpha-} \le \sigma \le \overline{\sigma_\alpha}$ and $r_{\alpha-} \le r \le \overline{r_\alpha}$. The result of that equation comes by taking $\sigma = \overline{\sigma_\alpha}$ if $\frac{\partial^2 \overline{C_{t\alpha}}}{\partial {S_t}^2} \ge 0$, and $\sigma = \sigma_{\alpha-}$ otherwise (Avellaneda et al., 1995). Likewise, $r = r_{\alpha-}$ if $\frac{\partial \overline{C_{t\alpha}}}{\partial S_t} S_t \ge C_t$ and $r = \overline{r_\alpha}$ otherwise.

Similarly, to obtain $C_{0\alpha-}$, an arbitrage argument is applied from the holders' perspective in such a way:

$$\frac{\partial C_{t\alpha-}}{\partial t} + \frac{1}{2}\frac{\partial^2 C_{t\alpha-}}{\partial {S_t}^2}\sigma^2 {S_t}^2 dt + r\left(\frac{\partial C_{t\alpha-}}{\partial S_t}S_t - C_{t\alpha-}\right) = 0.$$

Thus, if $\frac{\partial^2 \overline{C_{t\alpha}}}{\partial {S_t}^2} \ge 0$ then $\sigma = \sigma_{\alpha-}$, and $\sigma = \overline{\sigma_\alpha}$ otherwise. With regard to interest rates, $r = \overline{r_\alpha}$ if $\frac{\partial \overline{C_{t\alpha}}}{\partial S_t} S_t \ge C_t$, and $r = r_{\alpha-}$ otherwise.

3.2. Fuzzy-random option pricing in discrete time

3.2.1. Fuzzy binomial option pricing

Mainstream fuzzy random option pricing in discrete time adopts the binomial option model (Cox et al., 1979). This approach has been used to price financial European options (Buckley & Eslami, 2007; Buckley & Eslami, 2008b; Muzzioli & Reynaerts, 2007; Meenakshi & Kennedy, 2021a); to evaluate American options (Yoshida, 2003a; Muzzioli & Reynaerts, 2008; Zmeskal, 2010; Meenakshi & Felbin, 2019; Meenakshi, & Kennedy, 2021b); in a life insurance setting (Anzilli et al., 2018) and to value real options (Ho & Liao, 2011; Anzilli & Facchinetti, 2017; D'Amato et al., 2019; Shang et al., 2020; Chrysafis & Papadopoulos, 2021; Zhang & Yin, 2021; Zmeskal et al., 2022).

The binomial model supposes that a share can only vary in a multiplicative way due to two possible jumps: up (rate $u > 1$) and down (rate $0 < d < 1$). Therefore, if we symbolize the risk-neutral probability for the up jump, u, p_u and to attain a declining rate, d, $p_d = 1 - p_u$, we obtain:

$$p_u = \frac{e^{r\frac{\tau}{n}} - d}{u - d} \text{and} p_d = \frac{u - e^{r\frac{\tau}{n}}}{u - d},$$

where τ is the maturity of the option, which is divided into n periods of extension τ/n years. In this case, the price of a call option at the initial time $j = 0$, C_0 is:

$$C_0(S_0, K, r, u, d) = e^{-r\bullet\tau}\sum_{j=0}^{n}\binom{n}{j}{p_u}^j(1 - p_u)^{n-j}\max\{u^j d^{n-j}S_0 - K, 0\} \tag{8}$$

and for a put option:

$$P_0(S_0, K, r, u, d) = e^{-r\bullet\tau}\sum_{j=0}^{n}\binom{n}{j}{p_u}^j(1 - p_u)^{n-j}\max\{K - u^j d^{n-j}S_0, 0\}. \tag{9}$$

The fuzzy random approach allows multiplicative jumps of the stock price to be the fuzzy numbers $\widetilde{u}$ and $\widetilde{d}$. Like BSM, we also allow introducing the uncertainty on the actual price of the subjacent asset S_0 by quantifying it with the fuzzy number $\widetilde{S_0}$ and, analogously, a fuzzy risk-free interest rate $\widetilde{r}$. Under our general case, we have to evaluate (8) and (9) with fuzzy parameters. Muzzioli & Torricelli (2004) provide closed expressions of the call prices of European options in a one-period model with triangular up and down factors. The two-period model (Buckley & Eslami, 2007) can be generalized from Eqs. (8) and (9) to n periods. The α-cuts of $\widetilde{C}_0$ and $C_{0\alpha} = \left[C_{0\alpha-}, \overline{C_{0\alpha}}\right]$ are solved by means of two programming models that, due to $\frac{\partial C_0}{\partial S_0} \ge 0, \frac{\partial C_0}{\partial K} \le 0$and $\frac{\partial C_0}{\partial r} \ge 0$, can be expressed as:

$$C_{0\alpha-} = \text{minimize} C_0\left(S_{0\alpha-}, \overline{K_\alpha}, r_{\alpha-}, d, u\right), \text{subject to} d_{\alpha-} \le d \le \overline{d_\alpha}, u_{\alpha-} \le u \le \overline{u_\alpha},$$

$$\overline{C_{0\alpha}} = \text{maximize} C_0\left(\overline{S_{0\alpha}}, K_{\alpha-}, \overline{r_\alpha}, d, u\right), \text{subject to } d_{\alpha-} \le d \le \overline{d_\alpha}, u_{\alpha-} \le u \le \overline{u_\alpha},$$

where $u_\alpha = \left[u_{\alpha-}, \overline{u_\alpha}\right]$ stands for the α-levels of $\widetilde{u}$ and $d_\alpha = \left[d_{\alpha-}, \overline{d_\alpha}\right]$ correspond to those of $\widetilde{d}$. Likewise, we suppose that $\overline{u_\alpha} \ge e^{\overline{r_\alpha}\bullet\tau} \ge \overline{d_\alpha}$ and $u_{\alpha-} \ge e^{r_{\alpha-}\bullet\tau} \ge d_{\alpha-}$.

Of course, we proceed analogously to obtain the fuzzy price of calls $\widetilde{P}_0, P_{0\alpha} = \left[P_{0\alpha-}, \overline{P_{0\alpha}}\right]$ by taking into account that $\frac{\partial C_0}{\partial S_0} \le 0, \frac{\partial C_0}{\partial K} \ge 0, \frac{\partial C_0}{\partial r} \le 0$ and Eq. (9).

Let us free arbitrage up and down factors defined from asset price volatility σ, as $u = e^{\sigma\sqrt{\tau/n}}$ and $d = e^{-\sigma\sqrt{\tau/n}}$ (Cox et al., 1979). This is a common approach in FOP (Muzzioli & Torricelli, 2004; Lee et al., 2005; Zmeskal, 2010; Yu et al., 2011b; Chrysafis & Papadopoulos, 2021). Therefore, the price of a put option (8) turns into a function $C_0(S_0, K, r, \sigma)$ since $p_u = \frac{e^{r\bullet\tau/n} - e^{-\sigma\sqrt{\tau/n}}}{e^{\sigma\sqrt{\tau/n}} - e^{-\sigma\sqrt{\tau/n}}}$ and $p_d = \frac{e^{\sigma\sqrt{\tau/n}} - e^{r\bullet\tau/n}}{e^{\sigma\sqrt{\tau/n}} - e^{-\sigma\sqrt{\tau/n}}}$. Note that again $\frac{\partial C_0}{\partial \sigma} \ge 0$ and thus, the α-cuts of the call option, $C_{0\alpha} = \left[C_{0\alpha-}, \overline{C_{0\alpha}}\right]$ are:

$$C_{0\alpha} = \left[C_{0\alpha-}, \overline{C_{0\alpha}}\right] = \left[C_0\left(S_{0\alpha-}, \overline{K_\alpha}, r_{\alpha-}, \sigma_{\alpha-}\right), C_0\left(\overline{S_{0\alpha}}, K_{\alpha-}, \overline{r_\alpha}, \overline{\sigma_\alpha}\right)\right] \tag{10}$$

By following a similar argument, for the put price, we find:

$$P_{0\alpha} = \left[P_{0\alpha^-}, \overline{P_{0\alpha}}\right] = \left[P_0\left(\overline{S_{0\alpha}}, K_{\alpha^-}, \overline{r_\alpha}, \sigma_{\alpha^-}\right), P_0\left(S_{0\alpha^-}, \overline{K_\alpha}, r_{\alpha^-}, \overline{\sigma_\alpha}\right)\right] \tag{11}$$

where $e^{\overline{\sigma_\alpha}\sqrt{\tau/n}} \geq e^{\overline{r_\alpha}\bullet\frac{\tau}{n}} \geq e^{-\sigma_{\alpha^-}\sqrt{\tau/n}}$ and $e^{\sigma_{\alpha^-}\sqrt{\tau/n}} \geq e^{r_{\alpha^-}\bullet\frac{\tau}{n}} \geq e^{-\overline{\sigma_\alpha}\sqrt{\tau/n}}$.

3.2.2. Numerical application

Let us suppose that the price of a subjacent asset is $\widetilde{S_0} = $ (103, 104, 105), whose annual volatility is $\widetilde{\sigma}$=(0.35, 0.4, 0.45). We price European call and put options on that asset with a maturity $\tau = $ 1 year, a strike option, $\widetilde{K}$=(108, 109, 110) and an annual risk-free rate, $\widetilde{r}$=(0.005, 0.006, 0.007). To fit the price of the options on that asset, we use the fuzzy binomial model that is implemented by supposing quarterly jumps. Therefore, we apply (8) and (10) for the price of call options and (9) and (11) for the value of the put price. The results of our application are displayed in Table 7. Note that the parameters used to price the options are the same as in the case of Table 6. Therefore, prices by the binomial method must be close to those obtained with the fuzzy BSM in Table 6 since it is a discrete-time approximation to the BSM. Binomial pricing is very popular because its flexibility allows pricing options that do not admit the direct application of BSM. This is the case for American options (Yoshida, 2003a; Zmeskal, 2010; Meenakshi, & Kennedy, 2021) or some types of real options (Ho & Liao, 2011; Anzilli & Facchinetti, 2017; Shang et al., 2020; Chrysafis & Papadopoulos, 2021; Zhang & Yin, 2021; Zmeskal et al., 2022). Of course, for $n \longrightarrow \infty$ the price by the binomial method must converge to the fuzzy price by the BSM.

Let us outline whether initial data are given by TFNs, a TFN approximation with the same support and core works as well as in fuzzy-random BSM. Table 7 again displays the triangular approximation to $\widetilde{C_0} \approx \widetilde{c_0} = (c_{01}, c_{02}, c_{03})$ and $\widetilde{P_0} \approx \widetilde{p_0} = (p_{01}, p_{02}, p_{03})$. In both cases, errors in the extremes of α-cuts rarely exceed 1%.

4. Other approaches to fuzzy option pricing

4.1. Option pricing with fuzzy measures and Choquet's integral

4.1.1. Implementing Black-Scholes-Merton model with Choquet's integral

The application of fuzzy measures has been a productive research line in fuzzy option pricing. This approach does not fuzzify the parameters of the option pricing formula (price, volatility, etc.) but the risk-neutral probability measure, Q, which is additive. In this regard, we can outline the seminal contribution (Cherubini, 1997) and within the reviewed papers (Cherubini & Della, 2001a; Cherubini & Della, 2001b; Han & Zheng, 2002; Han & Zheng, 2005; Han & Chen, 2006; Han & Zhou, 2007; Kaino & Hirota, 2007; Liu, 2009; Liu & Jiang, 2010; Jiang et al., 2012; Cherubini & Mulinacci, 2021). The most common fuzzy measure used is Sugeno's measure, which is built on a parameter λ ∈ [−1,∞). Then, λ <0 stands for a superadditive measure, and λ >0 stands for a subadditive measure (Sugeno, Narukawa, & Murofushi, 1998). Thus, λ = 0 implies an additive measure, which in our setting is the original probability risk-neutral distribution Q. Let be a subadditive λ-measure, λ >0, and its dual superadditive measure λ^* is obtained by doing the following:

$$\lambda^* = -\frac{\lambda}{1+\lambda} \tag{12}$$

Table 7

Alpha-cuts of call and put fuzzy prices based on binomial formula (Cox et al., 1979) and its triangular approximates with $\widetilde{S_0} = (103, 104, 105)$, $\widetilde{K}$=(108, 109, 110), $\widetilde{\sigma}$=(0.35, 0.4, 0.45), $\widetilde{r}$=(0.005, 0.006, 0.007) and $\tau = $ 1.

	$\widetilde{C_0}$		$\widetilde{c_0} \approx (9.75, 14.74, 20.67)$		Error in $C_{0\alpha^-}$ and $\overline{C_{0\alpha}}$	
	$C_{0\alpha^-}$	$\overline{C_{0\alpha}}$	$c_{0\alpha^-}$	$\overline{c_{0\alpha}}$	Error in $C_{0\alpha^-}$	Error in $\overline{C_{0\alpha}}$
1	15.64	15.64	15.64	15.64	0.00%	0.00%
0.75	15.03	16.25	15.03	16.24	0.03%	0.03%
0.5	14.42	16.85	14.41	16.85	0.05%	0.03%
0.25	13.80	17.45	13.80	17.45	0.04%	0.02%
0	13.19	18.05	13.19	18.05	0.00%	0.00%
	$\widetilde{P_0}$		$\widetilde{p_0} = (15.19, 19.09, 23.17)$		Error in $P_{0\alpha^-}$ and $\overline{P_{0\alpha}}$	
	$P_{0\alpha^-}$	$\overline{P_{0\alpha}}$	$p_{0\alpha^-}$	$\overline{p_{0\alpha}}$	Error in $P_{0\alpha^-}$	Error in $\overline{P_{0\alpha}}$
1	18.11	18.11	18.11	18.11	0.00%	0.00%
0.75	17.62	18.99	17.71	18.90	0.52%	0.50%
0.5	17.11	19.87	17.30	19.68	1.14%	0.95%
0.25	16.59	20.75	16.90	20.46	1.88%	1.37%
0	16.49	21.25	16.49	21.25	0.00%	0.00%

Note: The errors are fitted as the relative difference between the actual extremes of α-cuts and those provided by the triangular approximate.

This approach to FOP separates risky behaviour of assets, which are modelled stochastically, and uncertainty about the information needed to implement analytical models due to the existence of ambiguity, vagueness and/or fuzziness in the available information. In practice, that uncertainty is because of market frictions, asymmetric information, or incompleteness, whose manifestations are transaction costs, the existence of bid-asked prices or liquidity risk. These issues explain why in a given moment, asset prices are not unique but multiple. For example, in stock exchange settings, bid and ask prices coexist (Cherubini & Della, 2001a; Cherubini & Della, 2001b). In this regard, λ-measures quantify the degree of uncertainty in the market and thus the vagueness of the information about traded prices of assets such as bid-asked spreads. Similar bid-asked prices denote frictionless and high liquidity, which is the usual hypothesis of financial models. In contrast, higher spreads imply that the market is further to attain perfection. If the information has no ambiguity, λ=0, which implies that $\lambda^* = 0$ and thus the ideal of a perfect market is reached. In contrast, if $\lambda \longrightarrow \infty$, $\lambda^* \rightarrow -1$, the situation is of complete uncertainty and full market incompleteness (Cherubini & Della, 2001b; Han & Zhou, 2007).

Let be a European call option of an asset. Its price in conventional financial models C_0 is the expected value of its payoff under the probability measure Q, i.e., $C_0 = e^{-r\bullet\tau}E_Q[\max\{S_\tau - K, 0\}]$.

In a BSM framework, for a strike price $K = k$, the probability measure is $\Phi(d_2)$, $d_2 = \frac{ln\left(\frac{S_0}{k}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$, and so the probability that the option will be exercised is:

$$C_0 = e^{-r\tau}\int_K^\infty \Phi(d_2)dk.$$

The transformation of Q into a λ-fuzzy measure is done by means of a distortion function $\theta(\lambda, Q)$ in such a way that the generic price of the call function (4) is now the expectation under a Choquet integral of the positive payoff $\max\{S_\tau - K, 0\}$:

$$C_{0\lambda} = e^{-r\bullet\tau}CE_{\theta(\lambda,Q)}[\max\{S_\tau - K, 0\}]$$

where $CE_{\theta(\lambda,Q)}$, we denote the Choquet-expected value under the λ-fuzzy measure induced by $\theta(\lambda, Q)$. Given that we are dealing with two λ-fuzzy measures, λ and λ^*, informational uncertainty provides us with two prices, $C_{0\lambda} \leq C_{0\lambda^*}$. After stating λ and, by using (12),λ^*, the key issue is choosing the function $\theta(\lambda, Q)$. In this regard, whereas it can be used $\theta(\lambda, Q) = \frac{Q}{1+\lambda(1-Q)}$, (Cherubini & Della, 2001b), it is also used in the distortion function in Han and Zhou (2007):

$$\theta(\lambda, Q) = \begin{cases} \frac{1}{\lambda}\left[(1+\lambda)^Q - 1\right], & \lambda \neq 0, \\ Q, & \lambda = 0. \end{cases}$$

Therefore, in a BSM framework and by using $\theta(\lambda, Q) = \frac{Q}{1+\lambda(1-Q)}$, we obtain:

$$C_{0\lambda} = e^{-r\bullet\tau}\int_K^{\infty}\frac{\Phi(d_2)}{1+\lambda(1-\Phi(d_2))}dk$$

$$C_{0\lambda^*} = e^{-r\bullet\tau}\int_K^{\infty}\frac{\Phi(d_2)}{1+\lambda^*(1-\Phi(d_2))}dk$$

On the other hand, with the distortion function (Han & Zhou, 2007):

$$C_{0\lambda} = \begin{cases} e^{-r\bullet\tau}\displaystyle\int_K^{\infty}\frac{1}{\lambda}\left[(1+\lambda)^{\Phi(d_2)}-1\right]dk, & \lambda \neq 0, \\ e^{-r\bullet\tau}\displaystyle\int_K^{\infty}\Phi(d_2)dk, & \lambda = 0. \end{cases}$$

$$C_{0\lambda^*} = \begin{cases} e^{-r\bullet\tau}\displaystyle\int_K^{\infty}\frac{1}{\lambda^*}\left[(1+\lambda^*)^{\Phi(d_2)}-1\right]dk, & \lambda^* \neq 0, \\ e^{-r\bullet\tau}\displaystyle\int_K^{\infty}\Phi(d_2)dk, & \lambda^* = 0. \end{cases}$$

We proceed similarly to price put options. Therefore, by applying the distortion function used in (Cherubini & Della, 2001b):

$$P_{0\lambda} = e^{-r\bullet\tau}\int_0^{k}\frac{\Phi(-d_2)}{1+\lambda(1-\Phi(-d_2))}dk$$

$$P_{0\lambda^*} = e^{-r\bullet\tau}\int_0^{K}\frac{\Phi(-d_2)}{1+\lambda^*(1-\Phi(-d_2))}dk$$

On the other hand, with the distortion function (Han & Zhou, 2007):

$$P_{0\lambda} = \begin{cases} e^{-r\bullet\tau}\displaystyle\int_0^{K}\frac{1}{\lambda}\left[(1+\lambda)^{\Phi(-d_2)}-1\right]dk, & \lambda \neq 0, \\ e^{-r\bullet\tau}\displaystyle\int_0^{K}\Phi(-d_2)dk, & \lambda = 0. \end{cases}$$

$$P_{0\lambda^*} = \begin{cases} e^{-r\bullet\tau}\displaystyle\int_0^{K}\frac{1}{\lambda^*}\left[(1+\lambda^*)^{\Phi(-d_2)}-1\right]dk, & \lambda^* \neq 0, \\ e^{-r\bullet\tau}\displaystyle\int_0^{K}\Phi(-d_2)dk, & \lambda^* = 0. \end{cases}$$

4.1.2. Numerical application

Let us price European call and put options with a strike price $K = 109$, time to expiration 1 year over an asset with price $S_0 = 104$ and annual volatility $\sigma = 0.4$. The annual risk-free rate is 0.6%. While Table 8 shows the prices of the call option for several λ-fuzzy measures and their dual, Table 9 does so for put options. In both cases, we use the distortion functions described above. We can check that the election of such functions is a key step, especially when the market presents a great uncertainty level. For example, in the case of λ = 2, the use of the Cherubini and Della Lunga (2001b) distortion function leads to obtaining the price of the put option within 8.91 and 57.28. On the other hand, by using the distortion function in Han and Zhou (2007), we obtain a price between 13.30 and 21.90, which is a notably narrower range.

Table 8
Call prices with BSM and λ-measures for $S_0 = 104$, $K = 109$, $\sigma = 0.4$, $r = 0.006$ and $\tau = 1$ year.

λ-measure		Distortion function (Cherubini & Della, 2001b)		Distortion function (Han & Zhou, 2007)	
λ	λ*	$C_{0\lambda}$	$C_{0\lambda^*}$	$C_{0\lambda}$	$C_{0\lambda^*}$
0	0	19.09	19.09	14.74	14.74
0.001	−0.001	14.82	14.84	14.83	14.75
0.010	−0.010	14.71	14.95	14.77	14.80
0.100	−0.091	13.73	16.00	14.27	15.31
0.25	−0.2	12.29	17.58	13.45	16.10
0.5	−0.333	10.60	20.31	12.53	17.24
1	−0.5	8.26	25.01	11.03	19.13
2	−0.667	5.73	32.79	9.12	21.90

Table 9
Put prices with BSM and λ-measures for $S_0 = 104$, $K = 109$, $\sigma = 0.4$, $r = 0.006$ and $\tau = 1$ year.

λ-measure		Distortion function (Cherubini & Della, 2001b)		Distortion function (Han & Zhou, 2007)	
λ	λ*	$P_{0\lambda}$	$P_{0\lambda^*}$	$P_{0\lambda}$	$P_{0\lambda^*}$
0	0	19.09	19.09	19.09	19.09
0.001	−0.001	19.08	19.11	19.09	19.10
0.010	−0.010	18.98	19.28	19.04	19.15
0.100	−0.091	18.03	21.00	18.56	19.63
0.25	−0.2	16.65	23.87	17.84	20.37
0.5	−0.333	14.78	28.64	16.85	21.42
1	−0.5	12.10	38.18	15.32	23.10
2	−0.667	8.91	57.28	13.30	25.48

4.2. *Pricing real options with the fuzzy payoff method*

Option pricing methods such as the BSM or binomial method suppose that the subjacent asset is traded in a market that is not far from efficient. It does not follow in real investments since every investment project is unique and cannot be traded in any market or they may rely on assets that do not already exist, such as those linked to R&D (Trigeorgis & Reuer 2017). Thus, the fuzzy payoff method (Collan et al., 2009) prices real options with the sole use of fuzzy numbers. Therefore, it is not based on conventional option pricing models, and therefore, the hypothesis of market completeness is avoided.

This method supposes that the net present value (NPV) of an investment project can be represented by means of a trapezoidal or a triangular fuzzy number (Collan et al., 2009), which in practice could be provided by managers (Carlsson & Fuller, 2003). For simplicity, we consider that the NPV is a TFN $\widetilde{v} = (v_1, v_2, v_3)$ such as that depicted in Fig. 3. If the variables needed to calculate NPV (inflows, outflows and interest rate) were estimated by TFNs, even though the NPV does not conserve the linear shape of input data, a TFN with the same support and core usually provides a good approximation (Terceño et al., 2003).

Let us suppose that a firm has the option to defer an investment project until there is evidence enough that its NPV is positive, i.e., until the corporation is certain that NPV in Fig. 3 will lie between 0 and v_3. In this case, the possibility to defer the project can be understood as a call option on that project whose strike price is $K = 0$. To price the real option, two variables that are deduced from the shape of $\widetilde{v}$ are used. The first is the expected value of the positive side of $\widetilde{v}$, which is denoted as $E\left(\widetilde{v}_+\right)$. That value plays a similar role to the intrinsic value of a financial call option. The second variable is the ratio of the area delimited by the membership function of negative values of $\widetilde{v}$ with respect to the area of the whole membership function of $\widetilde{v}$, which we symbolize as $m_v(x)$. This ratio plays a similar role to the risk-neutral probability of attaining an NPV (Collan et al., 2009). Therefore, the real option value of the fuzzy NPV, $R(\widetilde{v})$ is:

$$R(\widetilde{v}) = \frac{\int_0^{\infty} m_v(x)dx}{\int_{-\infty}^{\infty} m_v(x)dx}E\left(\widetilde{V}_+\right)$$
$$= \left[\frac{{v_1}^2}{(v_3-v_1)(-v_1)}\right]\left[\frac{-{v_1}^3}{6(v_2-v_1)^2}+v_2+\frac{v_3-v_1}{6}\right]$$

We are aware that Fig. 3 and the analytical results presented in this paper only display case $v_1 \leq 0 \leq v_2$. A deeper study of all possible situations and under the more general framework in which $\widetilde{v}$ is a trapezoidal fuzzy number of course are possible (Collan et al., 2009). This method has been the object of several applications (Hassanzadeh et al., 2012; Luukka & Collan, 2015; Collan et al., 2017; Mintah et al., 2017; Servati et al., 2017; Stoklasa et al., 2021).

From Fig. 3, we can deduce $R(\widetilde{v})$ in an intuitive way. It is obtained by

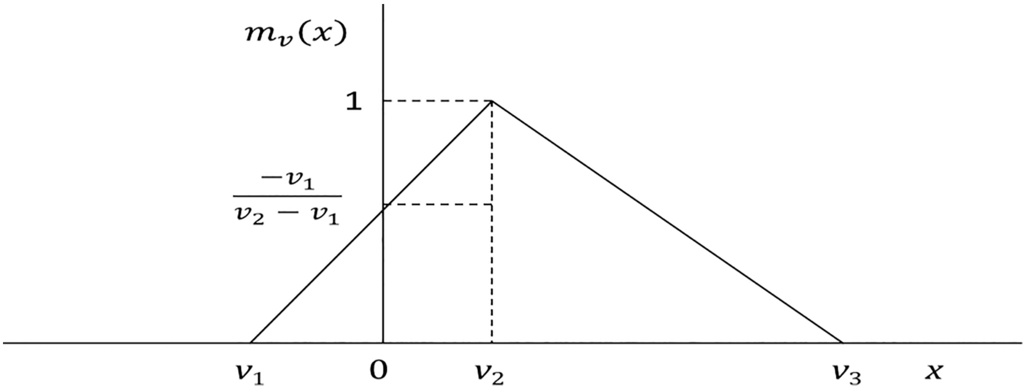


Fig. 3. Triangular fuzzy net present value.

weighting $E\left(\widetilde{v}_+\right)$, with an artificial probability that is determined as the rate of the area delimited by the membership function in the positive part of $\widetilde{v}$, whose height in $x = 0$ is $m_{\widetilde{v}}(0) = \frac{-v_1}{v_2 - v_1}$; with respect to the area of the whole membership function of $\widetilde{v}$, which is a triangle with base $v_3 - v_1$ and height $m_{\widetilde{v}}(v_2) = 1$.

Numerical application

Let us suppose that an investment project that could be initiated in a few months has a $\widetilde{v} = (-300, 2500, 4000)$. The expected value of the positive part of $\widetilde{v}$, $E\left(\widetilde{v}_+\right)$, is:

$$E\left(\widetilde{v}_+\right) = \frac{(-300)^3}{6(2500 + 300)^2} + 2500 + \frac{4000 + 300}{6} = 3217.24$$

The area delimited under the whole $m_v(x)$ is $\frac{4000-(-300)}{2} = 2150$. Likewise, $m_{\widetilde{v}}(0) = \frac{300}{2800}$ in such a way that the area of $m_v(x)$ under its positive part of $\widetilde{v}$ is $2150 - \frac{300 \bullet \frac{300}{2800}}{2} = 2133.93$. Therefore, the artificial probability of a positive net present value is $\frac{\int_0^\infty m_v(x)dx}{\int_{-\infty}^\infty m_v(x)dx} = \frac{2133.93}{2150} =$ 0.9925. Therefore:

$$R(\widetilde{v}) = \frac{\int_0^\infty m_v(x)dx}{\int_{-\infty}^\infty m_v(x)dx} E\left(\widetilde{v}_+\right) = 3193.19$$

4.3. *Nonparametric option pricing*

The mainstream of this set of FOP methods consists of using neural fuzzy systems, which are a combination of fuzzy controllers and neural networks, to price options (Wang, 2007; Kakati 2008; Gradojevic & Kukolj, 2011; Tung & Quek, 2011; Maciel et al., 2012; Sharma et al., 2014; Abdollahi, 2020; Hajizadeh, 2020). Within this group of approaches, we have also considered the combination of fuzzy time series with neural networks or support vector regression (Leu et al., 2011; Lee et al., 2014; Yang et al., 2014), fuzzy semantic networks (Tung & Quek, 2011), cloud computing with linguistic variables (Sharma et al., 2014) and fuzzy expert systems of Madamni's type (Magni et al., 2004). The objective of this focus of FOP is to take advantage of the capability of techniques such as fuzzy expert systems and neural networks to provide good numerical approximations for any function (Hornik, Stinchcombe, & White, 1989; Castro, 1995). On the other hand, these approaches do not provide closed formulas to compute option prices.

To give an approximate idea about how the fuzzy controller works to price options, we design a simple fuzzy expert system of the Takagi-Sugeno type (Takagi & Sugeno, 1985) based on the contract of European call options on futures of the Spanish stock index IBEX-35. As a reference framework, we will consider the pricing model for options on futures (Black, 1976), which is similar to the BSM with the slight difference that the riskless interest rate is waived:

$$C_0(F_0, K, \sigma, \tau) = F_0\Phi\left(\frac{ln\left(\frac{F_0}{K}\right) + \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}\right) - K\Phi\left(\frac{ln\left(\frac{F_0}{K}\right) - \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}\right)$$

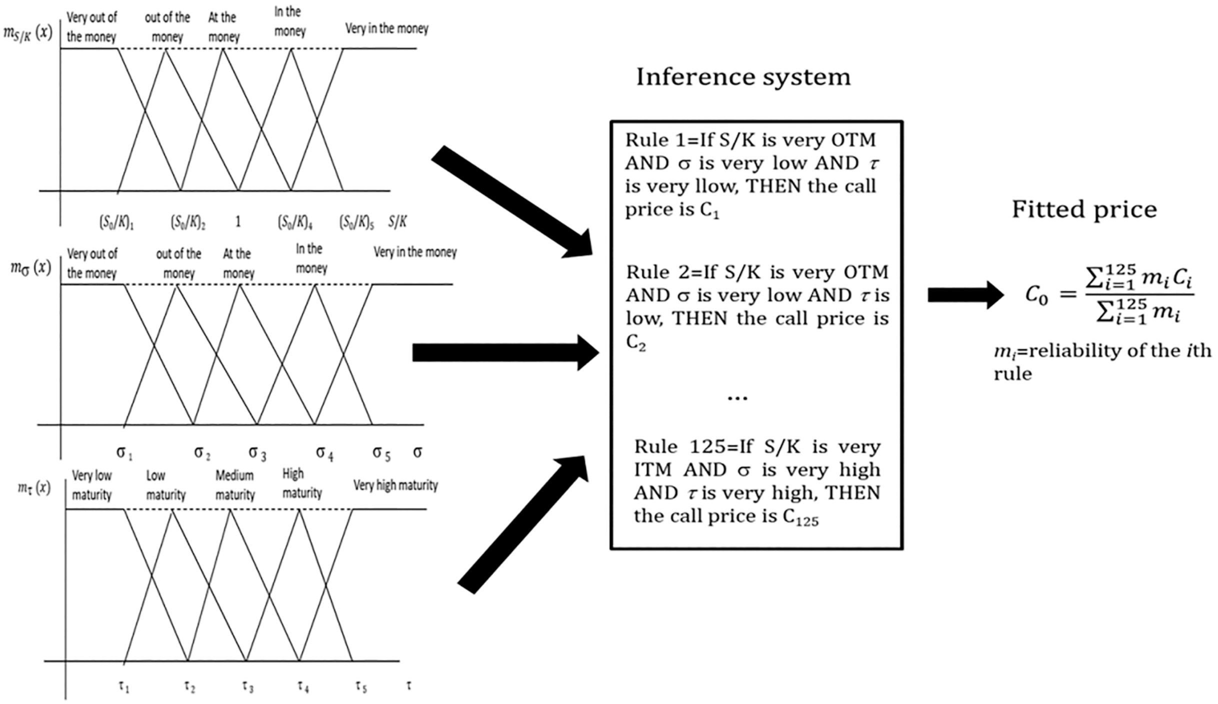
where F_0 stands for the price of the future contract at $t = 0$. In our example of fuzzy expert systems, which is depicted in Fig. 4, there are only three input variables (Black, 1976): the moneyness degree (F_0/K), volatility σ and time to expiration, τ. However, the use of techniques such as fuzzy expert systems allows greater flexibility since it allows the introduction of more input variables than the theoretical models. For example, to price exchange options, two volatilities (implied volatility and historical volatility) and a measure of qualitative risks linked to countries such as those of political and economic type can be introduced (Wang, 2007).

To build up an expert system such as Fig. 4, the first step consists of stating the universe of discourse of input variables. Subsequently, we must granulate these possible values to transform input variables into linguistic variables. It implies stating the number of linguistic labels for each variable and the membership function of each label. For example, the rate F_0/K, which measures the moneyness of the option, has been represented as a linguistic variable with five labels {"very out of the money", "out of the money", "at the money", "in the money", "very in the money"}. Then, a set of rules must be fitted whose consequence is a concrete option price. This is usually performed by means of neural networks (Wang, 2007; Kakati 2008; Gradojevic & Kukolj, 2011; Tung & Quek, 2011; Maciel et al., 2012; Abdollahi, 2020; Hajizadeh, 2020). Finally, the set of possible prices given by the set of rules must be defuzzified with a weighted mean.

5. Conclusions and future research

This paper has developed a comprehensive review of the literature exploring how fuzzy set theory (FST) has contributed to option pricing (OP). We have identified four main approaches to fuzzy option pricing (FOP). The mainstream of papers, based on the so-called fuzzy-random approach, consists of fuzzifying conventional option pricing formulas under the hypothesis that the parameters governing the stochastic models are fuzzy numbers. We are aware that this focus was reviewed in detail previously (Muzzioli and De Baets, 2016). However, this paper updates the literature on fuzzy-random option pricing and identifies three additional approaches to FOP.

The second surveyed approach also fuzzyfies conventional models, but this is done with the help of fuzzy measures and Choquet's integral. The third stream of FOP, known as the fuzzy pay-off method (Collan et al., 2009), is used to price real options by applying solely fuzzy number concepts. The fourth focus uses instruments such as fuzzy neural



Rule 1=If S/K is very OTM AND σ is very low AND τ is very llow, THEN the call price is C_1

Rule 2=If S/K is very OTM AND σ is very low AND τ is low, THEN the call price is C_2

…

Rule 125=If S/K is very ITM AND σ is very high AND τ is very high, THEN the call price is C_{125}

$$C_0 = \frac{\sum_{i=1}^{125} m_i C_i}{\sum_{i=1}^{125} m_i}$$

m_i=reliability of the ith rule

Fig. 4. Fuzzy expert system for the price of a call on the IBEX-35 future.

networks to take advantage of their capacity to be universal function approximators from empirical data to fit option prices.

The principal outlets of FOP are journals devoted to fuzzy mathematics and soft computing. The number of contributions on this topic grew continuously from the early 2000s to 2011. Since that year, the number of papers has reached a constant level in such a way that FOP, despite not being a massive topic, can be considered a consolidated issue of fuzzy mathematics.

The literature on fuzzy random option pricing is probably the most productive topic in FOP. The mainstream in this approach consists of analysing several arithmetical aspects that arise from the introduction of fuzzy numbers in conventional option pricing formulas, especially in the BSM formula and in the binomial model. This implies issues such as defuzzification (Yoshida, 2003; Yoshida et al., 2006; Thavaneswaran et al., 2009), approximation (Guerra et al., 2017; de Andres-Sanchez, 2018), or evaluating the option price when that of the subjacent asset is negative, which is possible in real option pricing (Carlsson & Fuller, 2003).

The mainstream of papers evaluates fuzzy option pricing formulas by means of the min T-norm. The consequence is obtaining option prices that may be too uncertain. For example, in Table 6, we found that the price of the call option with fuzzy BSM in the 0-level could oscillate between 11.83 and 17.71. In Table 7, we found with a fuzzy binomial option model that the price of the same call option could oscillate between 13.19 and 18.05. This explains why the mainstream literature focuses on higher degrees of α (for example, $\alpha \geq 0.9$), (Guerra et al., 2011; Wang et al., 2014; Wu, 2004; Wu, 2007; Nowak & Romaniuk, 2014; Nowak & Romaniuk, 2010). An alternative approach that needs to be researched is the use of the alternative T-norm to the min-norm to implement arithmetical operations with less uncertain results, such as the weakest T-norm (Hong & Do, 1997).

Although the principal focus of fuzzy-random in continuous time has consisted of fuzzifying the BSM formula, other continuous models have been the object of fuzzification with the same methodology. Of course, further research can be done in continuous time, e.g., extending the results grounded in the Black-Scholes framework for cliquet-style options (Korn, Temoçin, & Wenzel, 2017). Likewise, it could be of interest to develop an FOP for new stochastic processes such as Markov switching jump-diffusion, which has already been applied in portfolio management issues (Azevedo, Pinheiro, & Weber, 2014; Savku & Weber, 2018).

The great number of models in continuous time extended to fuzzy parameters has not been followed in discrete time. In that case, practically all the contributions are grounded on the binomial approximation by Cox et al. (1979). In our opinion, it could be of interest to extend alternative binomial option pricing approaches such as those reviewed in Chance (2015) to the use of fuzzy parameters. For example, although the binomial model (Trigeorgis, 1991) is specifically addressed to price real options, fuzzy-random FOP has been unanimously adopted (Cox et al., 1979). Likewise, other discrete time models such as trinomial (Boyle, 1988) or Monte-Carlo (Boyle, 1977) have been underused in FOP in spite of their adoption being of interest in concrete settings.

The mainstream fuzzy random option pricing models the uncertainty of parameters with fuzzy numbers. More complex modelling of uncertain quantities such as Type-2 fuzzy numbers or intuitionistic fuzzy numbers is rarely used. More research introducing these tools to represent uncertainty must be welcome. In this regard, advances in fuzzy games with these kinds of fuzzy numbers (Bhaumik, Roy, & Li, 2017; Roy & Bhaumik 2018) could be applied to develop fuzzy real option pricing with game theory instruments. This research line must embed solving several arithmetical questions but also investigate how to fit these kinds of fuzzy numbers, which, on the one hand, allows more sophisticated option pricing formulas but, on the other hand, are less parsimonious and require estimating more parameters. Likewise, other nonfuzzy approaches to vagueness and imprecision, such as parallelepiped uncertainty, which has been used in a portfolio selection setting (Kara, Özmen, & Weber, 2019), could be of interest to build up new option pricing formulas.

Similar to Muzzioli and De Baets (2016), we feel that the FOP literature is more theoretically oriented and does not provide much empirical research. Therefore, it seems necessary for more literature to put FOP theoretical papers into work with the data market. This work may cover how to use fuzzy data analysis to fit implied moments such as volatility but also assess the performance of fuzzy approaches, either by comparing fuzzy and crisp pricing formulas or by contrasting goodness of fit between fuzzyfied option pricing methods.

Within these empirical challenges of the FOP approach, we also outline research about how to fit uncertain parameters of OP formulas. Some remarkable approaches have been proposed for volatility. Chrysafis & Papadopoulos (2009), Chrysafis & Papadopoulos (2021) propose to quantify this parameter from a fuzzy number that has been fitted from the statistical interval-confidence representation of the subjacent asset standard deviation of the price (Sfiris & Papadopoulos, 2014). Alternative approaches consist of empirically adjusting the implied volatility by means of fuzzy numbers (Capotorti & Figa-Talamanca 2013; Muzzioli et al., 2015; Muzzioli et al., 2018; Capotorti & Figà-Talamanca 2020; Muzzioli et al., 2020). However, other parameters such as subjacent asset price, free-risk rate or parameters in more complex option pricing models such as by Merton (1976) need to be the object of deeper attention.

The principal application of FOP has been made on options on stock and indexes and in real options setting. However, FOP can be applied to price any asset that has an embedded option. Thus, practically all assets can be valued like an option and thus with FOP. Therefore, FOP has also been applied to evaluate firms (Zmeskal, 2001), deposit insurance (Wu et al., 2020), catastrophe bonds (Nowak & Romaniuk 2017) and minimum guarantee insurance (Anzilli et al., 2018). Therefore, additional financial assets, such as fixed income instruments or mortgages, are susceptible to being covered by FOP analysis.

FOP based on fuzzy measures and Choquet's integral is mainly developed on the BSM formula. Therefore, the use of Choquet's integral may be extended to more complex continuous-time models and to discrete time methods such as the binomial model (Cox et al., 1979). Likewise, how to calibrate the parameter λ (and its dual λ*) when using λ measures to introduce the existence of Knightian uncertainty in option prices is also an open question.

The fuzzy pay-off method (Collan et al., 2009) is mainly devoted to pricing real options. It is very useful due to its intuitiveness and the scarce requirements to estimate its parameters. This method only requires predicting the net present value of the investment project by means of a triangular or a trapezoidal fuzzy number, and thus, basic real options can be easily valued. However, we feel that this methodology needs to be extended to price investment projects with complex and interdependent multioption features or when there is uncertainty in the maturity of the real option. The fuzzy pay-off method in its basic formulation could be hard to apply in complex architectures of embedded real options within investment projects.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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