Computing the k-metric dimension of graphs

Ismael G. Yero[(1)], Alejandro Estrada-Moreno[(2)]
and Juan A. Rodríguez-Velázquez[(2)]

[(1)]Departamento de Matemáticas, Escuela Politécnica Superior de Algeciras
Universidad de Cádiz, Av. Ramón Puyol s/n, 11202 Algeciras, Spain.
ismael.gonzalez@uca.es

[(2)]Departament d'Enginyeria Informàtica i Matemàtiques,
Universitat Rovira i Virgili, Av. Països Catalans 26, 43007 Tarragona, Spain.
alejandro.estrada@urv.cat, juanalberto.rodriguez@urv.cat

Abstract

Given a connected graph $G = (V, E)$, a set $S \subseteq V$ is a k-metric generator for G if for any two different vertices $u, v \in V$, there exist at least k vertices $w_1, \ldots, w_k \in S$ such that $d_G(u, w_i) \neq d_G(v, w_i)$ for every $i \in \{1, \ldots, k\}$. A metric generator of minimum cardinality is called a k-metric basis and its cardinality the k-metric dimension of G. We make a study concerning the complexity of some k-metric dimension problems. For instance, we show that the problem of computing the k-metric dimension of graphs is NP-hard. However, the problem is solved in linear time for the particular case of trees.

Keywords: k-metric dimension; k-metric dimensional graph; metric dimension; NP-complete problem; NP-hard problem; graph algorithms.

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1 Introduction

Let $\mathbb{R}_{\geq 0}$ denote the set of nonnegative real numbers. A *metric space* is a pair (X, d), where X is a set of points and $d : X \times X \to \mathbb{R}_{\geq 0}$ satisfies $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$ for all $x, y \in X$ and $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$. A *generator* of a metric space (X, d) is a set S of points in the space with the property that every point of the space is uniquely determined by the distances from the elements of S. A point $v \in X$ is said to *distinguish* two points x and y of X if $d(v, x) \neq d(v, y)$. Hence, S is a generator if and only if any pair of points of X is distinguished by some element of S.

Let $\mathbb{N}$ denote the set of nonnegative integers. Given a connected graph $G = (V, E)$, we consider the function $d_G : V \times V \to \mathbb{N}$, where $d_G(u, v)$ is the length of a shortest path between u and v. Clearly, (V, d_G) is a metric space. The diameter of a graph is defined with this metric.

A vertex set $S \subseteq V$ is said to be a *metric generator* for G if it is a generator of the metric space (V, d_G). A minimum metric generator is called a *metric basis*, and its cardinality the *metric dimension* of G, denoted by $\dim(G)$. Motivated by some problems regarding unique location of intruders in a network, the concept of metric dimension of a graph was introduced by Slater [28], where the metric generators were called *locating sets*. The concept of metric dimension of a graph was also introduced by Harary and Melter [17], where metric generators were called *resolving sets*. Applications of this invariant to the navigation of robots in networks are discussed in [21] and applications to chemistry in [19, 20]. This graph parameter was studied further in a number of other papers including, for instance [3, 6, 7, 22, 23, 30]. Several variations of metric generators including resolving dominating sets [5], locating dominating sets [25], independent resolving sets [8], local metric sets [24], strong resolving sets [27], etc. have been introduced and studied.

On the other hand, complexity studies concerning the metric dimension of graphs have recently attracted the attention of several researchers. This is mainly based on the fact that finding the metric dimension of graphs is NP-hard [21], even when restricted to planar graphs [9]. However, there exist a linear-time and a polynomial-time algorithm for determining the metric dimension for trees [21] and outerplanar graphs [9], respectively. For these reasons, many efforts have been made to computationally solve the problem of finding a metric generator of a graph in the last few years. For instance, an increasing interest into algorithmic questions on this topic has been raised (see [10, 11, 15, 18] as some examples).

From now on we consider an extension of the concept of metric generators introduced in [12] by the authors of this paper, and also independently by Adar and Epstein in [1]. Given a simple and connected graph $G = (V, E)$, a set $S \subseteq V$ is said to be a *k-metric generator* for G if and only if any pair of vertices of G is distinguished by at least k elements of S, *i.e.*, for any pair of different vertices $u, v \in V$, there exist at least k vertices $w_1, w_2, \ldots, w_k \in S$ such that

$$d_G(u, w_i) \neq d_G(v, w_i), \text{ for every } i \in \{1, \ldots, k\}. \tag{1}$$

A k-metric generator of minimum cardinality in G is called a *k-metric basis* and its cardinality the *k-metric dimension* of G, which is denoted by $\dim_k(G)$, [12]. Note that every k-metric generator S satisfies that $|S| \geq k$ and, if $k > 1$, then S is also a $(k-1)$-metric generator. Moreover, 1-metric generators are the standard metric generators (resolving sets or locating sets as defined in [17] or [28], respectively). Notice that if $k = 1$, then the problem of checking if a set S is a metric generator reduces to check condition (1) only for those vertices $u, v \in V \setminus S$, as every vertex in S is distinguished at least by itself. Also, if $k = 2$, then condition (1) must be checked only for those pairs having at most one vertex in S, since two vertices of S are distinguished at least by themselves. Nevertheless, if $k \geq 3$, then condition (1) must be checked for every pair of different vertices of the graph. The k-metric dimension of connected graphs has been also studied in [2, 4, 13, 14]. Among them, a remarkable study concerns an interesting application of the k-metric generators (whether $k \geq 3$) to the theory of error correcting codes which was presented in [4].

In this article we show the NP-hardness of the problem of computing the k-metric dimension of graphs. To do so, we first prove that the decision problem regarding whether $\dim_k(G) \leq r$ for some graph G and some integer $r \geq k+1$ is NP-complete. The particular case of trees is separately addressed, based on the fact that for trees, the problem mentioned above becomes polynomial. We must remark that, for the tree graphs, a similar approach was also dealt with in [2] for the case $k = 2$. We say that a connected graph G is *k-metric dimensional* if k is the largest integer such that there exists a k-metric basis for G. We also show that the problem of finding the integer k such that a graph G is k-metric dimensional can be solved in polynomial time. The reader is referred to [12] for combinatorial results on the k-metric dimension, including tight bounds and some closed formulae. The article is organized as follows. In Section 2 we analyze the problem of computing the largest integer k such that there exists a k-metric basis. In Section 3 we show that the decision problem regarding whether the k-metric dimension of a graph does not exceed a positive integer is NP-complete, which gives also the NP-hardness of computing $\dim_k(G)$ for any graph G. The procedure of such proof is done by using some similar techniques like those ones already presented in [21] while studying the computational complexity problems related to the standard metric dimension of graphs. Finally, in Section 4 we give an algorithm for determining the value of k such that a tree is k-metric dimensional and present two algorithms for computing the k-metric dimension and obtaining a k-metric basis of any tree. We also show that all algorithms presented in this section run in linear time.

Throughout the paper, we use the notation $K_{1,n-1}$, C_n and P_n for star graphs, cycle graphs and path graphs of order n, respectively. For a vertex v of a graph G, $N_G(v)$ denotes the set of neighbors or *open neighborhood* of v in G. The *closed neighborhood*, denoted by $N_G[v]$, equals $N_G(v) \cup \{v\}$. If there is no ambiguity, we simple write $N(v)$ or $N[v]$. We also refer to the degree of v as $\delta(v) = |N(v)|$. The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

2 k-metric dimensional graphs.

Notice that if a graph G is a k-metric dimensional graph, then for every positive integer $k' \leq k$, G has at least a k'-metric basis. Since for every pair of vertices x, y of a graph G we have that they are distinguished at least by themselves, it follows that the whole vertex set $V(G)$ is a 2-metric generator for G and, as a consequence it follows that every graph G is k-metric dimensional for some $k \geq 2$. On the other hand, for any connected graph G of order $n > 2$ there exists at least one vertex $v \in V(G)$ such that $\delta(v) \geq 2$. Since v does not distinguish any pair $x, y \in N_G(v)$, there is no n-metric dimensional graph of order $n > 2$.

Remark 1. [12] *Let G be a k-metric dimensional graph of order n. If $n \geq 3$ then, $2 \leq k \leq n-1$. Moreover, G is n-metric dimensional if and only if $G \cong K_2$.*

Next we present a characterization of k-metric dimensional graphs already known from [12]. To this end, we need some additional terminology. Given two vertices $x, y \in V(G)$, we say that the set of *distinctive vertices* of x, y is

$$\mathcal{D}_G(x, y) = \{z \in V(G) : d_G(x, z) \neq d_G(y, z)\}.$$

Theorem 2. [12] *A graph G is k-metric dimensional if and only if $k = \min_{x,y \in V(G)} |\mathcal{D}_G(x, y)|$.*

We now consider the problem of finding the integer k for which a given graph G of order n is k-metric dimensional.

k-DIMENSIONAL GRAPH PROBLEM
INSTANCE: A connected graph G of order $n \geq 3$
PROBLEM: Find the integer k, $2 \leq k \leq n-1$, such that G is k-metric dimensional

Remark 3. *Let G be a connected graph of order $n \geq 3$. The time complexity of computing the value k for which G is k-metric dimensional is $O(n^3)$.*

Proof. We can initially compute the distance matrix DistM_G, by using the well-known Floyd-Warshall algorithm [26, 29], which has time complexity $O(n^3)$. The distance matrix DistM_G is symmetric of order $n \times n$ whose rows and columns are labeled by vertices, with entries between 0 and $n-1$. Now observe that $z \in \mathcal{D}_G(x, y)$ if and only if $\mathrm{DistM}_G(x, z) \neq \mathrm{DistM}_G(y, z)$.

Given the matrix DistM_G, the process of computing how many vertices belong to $\mathcal{D}_G(x, y)$ for each of the $\binom{n}{2}$ pairs $x, y \in V(G)$ can be checked in linear time. Therefore, since the overall running time of such a process is dominated by the cubic time of the Floyd-Warshall algorithm, the proof is completed. □

3 The k-metric dimension problem

Since the problem of computing the value k' for which a given graph is k'-metric dimensional can be solved in polynomial time, we can study the problem of deciding whether the k-metric dimension, $k \leq k'$, of G is less than or equal to r, for some $r \geq k+1$, *i.e.*, the following decision problem.

k-METRIC DIMENSION PROBLEM
INSTANCE: A k'-metric dimensional graph G of order $n \geq 3$ and integers k, r such that $1 \leq k \leq k'$ and $k+1 \leq r \leq n$.
QUESTION: Is $\dim_k(G) \leq r$?

Next we prove that the k-METRIC DIMENSION PROBLEM is NP-complete. We must remark that for $k = 1$ the problem above was proved to be NP-complete by Khuller *et al.* in [21], although a previous claim about it was first presented in [16]. Moreover, the NP-completeness of this problem (when $k = 1$) restricted to the case of planar graphs was settled in [9]. As a kind of generalization of the technique used in [21] for $k = 1$, we also use a reduction from 3-SAT in order to prove the NP-completeness of the k-METRIC DIMENSION PROBLEM.

Our problem is clearly in NP, since verifying that a given subset $S \subseteq V(G)$ with $k+1 \leq |S| \leq r$ is a k-metric generator for a graph G, can be done in polynomial time by using some similar procedure like that described in the proof of Remark 3. In order to present the reduction from 3-SAT, we need some terminology and notation. From now on, we assume $x_1, \ldots, x_n$ are variables; $Q_1, \ldots, Q_s$ are clauses; and $x_1, \overline{x_1}, x_2, \overline{x_2}, \ldots, x_n, \overline{x_n}$ are literals, where x_i represents a positive literal of the variable, while $\overline{x_i}$ represents a negative literal.

We consider an arbitrary input to 3-SAT, that is, a boolean formula F with n variables and s clauses. In this reduction, without loss of generality, we assume that the formula F has $n \geq 4$ variables. Let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of variables and let $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_s\}$ be the set of clauses. Now we construct a graph G_F in the following way.

- For every $x_i \in X$, we take an even cycle C^i of order $4\left\lceil \frac{k}{2} \right\rceil + 2$ and we denote by F_i (the false node) and by T_i (the true node) two diametral vertices of C^i. Then we denote by $f_i^1, f_i^2, \ldots, f_i^{2\left\lceil \frac{k}{2} \right\rceil}$ the half vertices of $C^i \setminus \{T_i, F_i\}$ closest to F_i and we denote by $t_i^1, t_i^2, \ldots, t_i^{2\left\lceil \frac{k}{2} \right\rceil}$ the half vertices of $C^i \setminus \{T_i, F_i\}$ closest to T_i (see Figure 1).

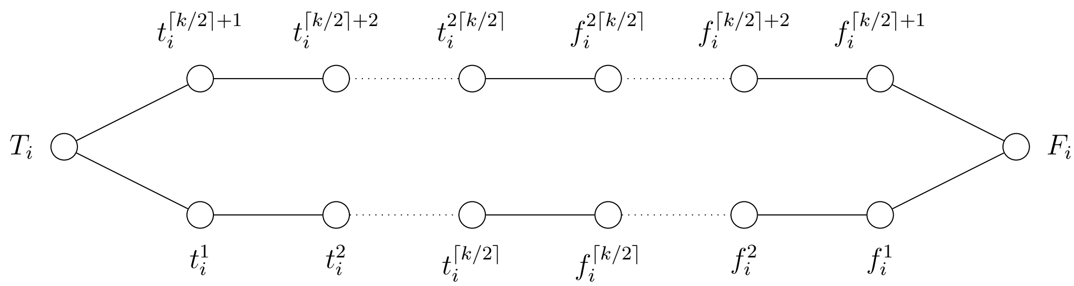

Figure 1: The cycle C^i associated to the variable x_i.

- For every clause $Q_j \in \mathcal{Q}$, we take a star graph $K_{1,4}$ with central vertex u_j and leaves $u_j^1, u_j^2, u_j^3, u_j^4$. If $k \geq 3$, then we subdivide the edge $u_j u_j^2$ until we obtain a shortest $u_j - u_j^2$ path of order $\left\lceil \frac{k}{2} \right\rceil + 1$, as well as, we subdivide the edge $u_j u_j^3$ until we obtain a shortest $u_j - u_j^3$ path of order $\left\lfloor \frac{k}{2} \right\rfloor + 1$ (see Figure 2). We denote by $P(u_j^2, u_j^3)$ the shortest $u_j^2 - u_j^3$ path of length k obtained after subdivision. The star graph remains unchanged for $k \in \{1, 2\}$.

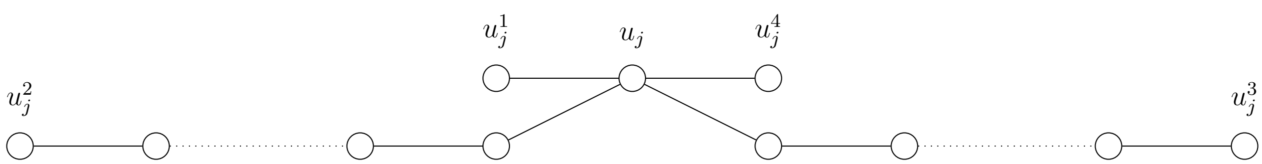

Figure 2: The subgraph associated to the clause Q_j.

- If a variable x_i occurs as a positive literal in a clause Q_j, then we add the edges $T_i u_j^1$, $F_i u_j^1$ and $F_i u_j^4$ (see Figure 3).

- If a variable x_i occurs as a negative literal in a clause Q_j, then we add the edges $T_i u_j^1$, $F_i u_j^1$ and $T_i u_j^4$ (see Figure 3).

- Finally, for every $l \in \{1, \ldots, n\}$ such that x_l and $\overline{x_l}$ do not occur in a clause Q_j, we add the edges $T_l u_j^1$, $T_l u_j^4$, $F_l u_j^1$ and $F_l u_j^4$.

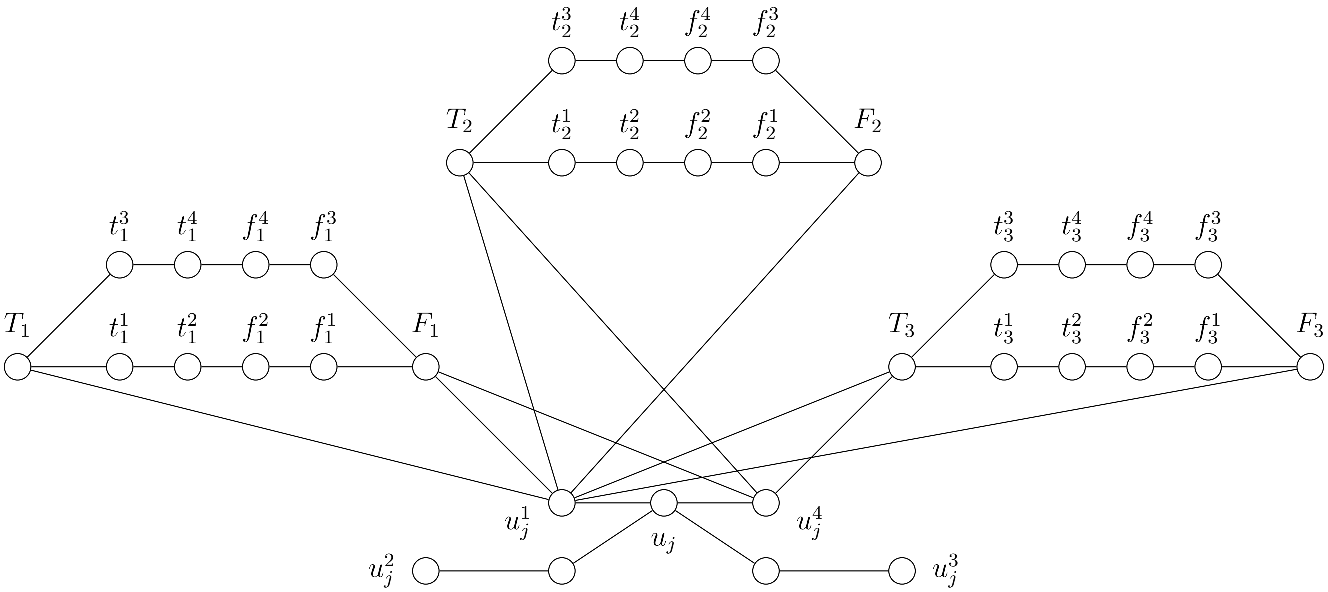

Figure 3: The subgraph associated to the clause $Q_j = (x_1 \vee \overline{x_2} \vee \overline{x_3})$ (taking $k = 4$).

Notice that the graph G_F obtained from the procedure above has order $n\left(4\left\lceil \frac{k}{2} \right\rceil + 2\right) + s(k+3)$. Also, we observe that given the formula F, the graph G_F can be constructed in polynomial time. Next we prove that F is satisfiable if and only if $\dim_k(G_F) = k(n+s)$. To do so, we first notice some properties of G_F.

Remark 4. *Let $x_i \in X$. Then there exist two different vertices $a, b \in V(C^i)$ such that they are distinguished only by vertices of the cycle C^i and, as a consequence, for any k-metric basis S of G_F, we have that $|S \cap V(C^i)| \geq k$.*

Proof. To observe that it is only necessary to take the two vertices of C^i adjacent to T_i or adjacent to F_i. □

Remark 5. *Let $Q_j \in \mathcal{Q}$. Then there exist two different vertices x, y in the shortest $u_j^2 - u_j^3$ path such that they are distinguished only by vertices of the itself shortest $u_j^2 - u_j^3$ path and, as consequence, for any k-metric basis S of G_F, we have that $|S \cap V(P(u_j^2, u_j^3))| \geq k$.*

Proof. To observe that it is only necessary to take the two vertices of $P(u_j^2, u_j^3)$ adjacent to u_j. □

Proposition 6. *Let F be an arbitrary input to the 3-SAT problem. Then the graph G_F associated to F satisfies that $\dim_k(G_F) \geq k(n+s)$.*

Proof. As a consequence of Remarks 4 and 5 we obtain that for every variable $x_i \in X$ and for every clause $Q_j \in \mathcal{Q}$ the set of vertices of G_F associated to each variable or clause, contains at least k vertices of every k-metric basis for G_F. Thus, the result follows. □

Theorem 7. *The k-METRIC DIMENSION PROBLEM is NP-complete.*

Proof. Let F be an arbitrary input to the 3-SAT problem having more than three variables and let G_F be the graph associated to F. We shall show that F is satisfiable if and only if $\dim_k(G_F) = k(n+s)$.

We first assume that F is satisfiable. From Proposition 6 we have that $\dim_k(G_F) \geq k(n+s)$. Now, based on a satisfying assignment of F, we shall give a set S of vertices of G_F, of cardinality $|S| = k(n+s)$, which is a k-metric generator.

Suppose we have a satisfying assignment for F. For every clause $Q_j \in \mathcal{Q}$, we add to S all the vertices of the set $V(P(u_j^2, u_j^3)) \setminus \{u_j\}$. For a variable $x_i \in X$ we consider two possibilities according to the parity of k, that is the following.

- k is even. Hence, clearly $2\left\lceil \frac{k}{2} \right\rceil = k$. If the value of x_i is true, then we add to S the vertices $t_i^1, t_i^2, \ldots, t_i^{2\left\lceil \frac{k}{2} \right\rceil}$. On the contrary, if the value of x_i is false, then we add to S the vertices $f_i^1, f_i^2, \ldots, f_i^{2\left\lceil \frac{k}{2} \right\rceil}$.
- k is odd. Hence, $2\left\lceil \frac{k}{2} \right\rceil = k+1$. In this sense, if the value of x_i is true, then we add to S the vertices $t_i^1, t_i^2, \ldots, t_i^{2\left\lceil \frac{k}{2} \right\rceil - 1}$. On the contrary, if the value of x_i is false, then we add to S the vertices $f_i^1, f_i^2, \ldots, f_i^{2\left\lceil \frac{k}{2} \right\rceil - 1}$.

In concordance with the items above, since $k = 2\left\lceil \frac{k}{2} \right\rceil$ for k even and $k = 2\left\lceil \frac{k}{2} \right\rceil - 1$ for k odd, from now on in this proof we simply assume that vertices added to S are $t_i^1, t_i^2, \ldots, t_i^k$ or $f_i^1, f_i^2, \ldots, f_i^k$ accordingly.

We shall show that S is a k-metric generator for G_F. Let a, b be two different vertices of G_F. We consider the following cases.

Case 1. $a, b \in V(C^i)$ for some $i \in \{1, \ldots, n\}$. Hence, there exists at most one vertex $y \in S \cap V(C^i)$ such that $d(a,y) = d(b,y)$. If $d(a,w) \neq d(b,w)$ for every vertex $w \in S \cap V(C^i)$, then since $|S \cap V(C^i)| = k$, we have that $|\mathcal{D}_G(a,b) \cap S| = k$. On the other hand, if there exist one vertex $y \in S \cap V(C^i)$ such that $d(a,y) = d(b,y)$, then $d(a,T_i) \neq d(b,T_i)$ and $d(a,F_i) \neq d(b,F_i)$. Thus, for every $w \in S \setminus V(C^i)$ it follows that $d(a,w) \neq d(b,w)$ and a, b are distinguished by more than k vertices of S.

Case 2. $a, b \in V(P(u_j^2, u_j^3))$. Hence, there exists at most one vertex $y' \in S \cap V(P(u_j^2, u_j^3))$ such that $d(a,y') = d(b,y')$. But, in this case, $d(a,u_j) \neq d(b,u_j)$ and so for every $w \in S \setminus V(P(u_j^2, u_j^3))$ it follows that $d(a,w) \neq d(b,w)$ and a, b are distinguished by at least k vertices of S.

Case 3. $a = u_j^1$ and $b = u_j^4$. Since the clause Q_j is satisfied, there exists $i \in \{1, \ldots, n\}$, *i.e*, a variable x_i occurring in the clause Q_j such that either

- $a \sim T_i$, $b \not\sim T_i$ and $S \cap V(C^i) = \left\{t_i^1, t_i^2, \ldots, t_i^k\right\}$, *i.e*, a variable x_i occurring as a positive literal in Q_j and has the value true in the assignment, or
- $a \sim F_i$, $b \not\sim F_i$ and $S \cap V(C^i) = \left\{f_i^1, f_i^2, \ldots, f_i^k\right\}$, *i.e*, a variable x_i occurring as a negative literal in Q_j and has the value false in the assignment.

Thus, in any case we have that for every $w \in S \cap V(C^i)$ it follows $d(a,w) < d(b,w)$ and a,b are distinguished by at least k vertices of S.

Case 4. $a \in V(C^i)$ and $b \in V(C^l)$ for some $i,l \in \{1,\dots,n\}$, $i \neq l$. In this case, if there is a vertex $z \in S \cap V(C^i)$ such that $d(a,z) = d(b,z)$, then for every vertex $w \in S \cap V(C^l)$ it follows that $d(a,w) \neq d(b,w)$. So a,b are resolved by at least k vertices of S.

Case 5. $a \in V(C^i)$ and $b \in V(P(u_j^2,u_j^3))$. It is similar to the case above.

Case 6. $a \in \{u_j^1,u_j^4\}$ and $b \notin \{u_j^1,u_j^4\}$. If $b \in V(C^i)$, for some $i \in \{1,\dots,n\}$, then all elements of $S \cap V(P(u_j^2,u_j^3))$ distinguish a,b. Similarly, if $b \in V(P(u_l^2,u_l^3)) \cup \{u_l^1,u_l^4\}$ for some $l \in \{1,\dots,m\} \setminus \{j\}$, then all elements of $S \cap V(P(u_j^2,u_j^3))$ distinguish a,b. Now, let w be one of the two vertices adjacent to u_j in $P(u_j^2,u_j^3)$. If $b \in V(P(u_j^2,u_j^3)) \setminus \{w\}$, then all elements of $S \cap V(C^i)$ distinguish a,b. On the other hand, since $n \geq 4$, if $b = w$, then there exists a variable x_l not occurring in the clause Q_j. Thus, the vertex a is adjacent to T_l and to F_l and, as a consequence, the vertices of $S \cap V(C^l)$ distinguish a,b.

As a consequence of the cases above, we have that S is a k-metric generator for G_F. Therefore, $\dim_k(G_F) = k(n+s)$.

Next we prove that, if $\dim_k(G_F) = k(n+s)$, then F is satisfiable. To this end, we show that there exists a k-metric basis S of G_F such that we can set an assignment of the variables, so that F is satisfiable. We take S in the same way as the k-metric generator for G_F described above. Since S is a k-metric generator for G_F of cardinality $k(n+s)$, it is also a k-metric basis. Note that for any cycle C_i either $S \cap V(C_i) = \left\{t_i^1, t_i^2, \dots, t_i^k\right\}$ or $S \cap V(C_i) = \left\{f_i^1, f_i^2, \dots, f_i^k\right\}$.

In this sense, we set an assignment of the variables as follows. Given a variable $x_i \in X$, if $S \cap \left\{t_i^1, t_i^2, \dots, t_i^k\right\} = \emptyset$, then we set x_i to be false. Otherwise we set x_i to be true. We claim that this assignment satisfies F.

Consider any clause $Q_j \in \mathcal{Q}$ and let $x_{j_1}, x_{j_2}, x_{j_3}$ be the variables occurring in Q_j. Recall that for each clause Q_h, we have that $S \cap V(P(u_h^2,u_h^3)) = V(P(u_h^2,u_h^3)) \setminus \{u_h\}$. Besides neither any vertex of $V(C_l)$ associated to a variable x_l, $l \neq j_1, j_2, j_3$, nor any vertex of $S \cap V(P(u_h^2,u_h^3))$ associated to a clause Q_h, distinguishes the vertices u_j^1 and u_j^4. Thus u_j^1 and u_j^4 must be distinguished by at least k vertices belonging to $V(C_{j_1}) \cup V(C_{j_2}) \cup V(C_{j_3})$ associated to the variables $x_{j_1}, x_{j_2}, x_{j_3}$.

Now, according to the way in which we have added the edges between the vertices T_{j_1}, T_{j_2}, T_{j_3}, F_{j_1}, F_{j_2}, F_{j_3} and u_j^1, u_j^4, we have that u_j^1 and u_j^4 are distinguished by at least k vertices of S if and only if one of the following statements holds.

- There exists $l \in \{1,2,3\}$ for which the variable x_{j_l} occurs as a negative literal in the clause Q_j and $S \cap \left\{t_j^1, t_j^2, \dots, t_j^k\right\} = \emptyset$ (in such a case x_{j_l} is set to be false).
- There exists $l \in \{1,2,3\}$ for which the variable x_{j_l} occurs as a positive literal in the clause Q_j and $S \cap \left\{t_j^1, t_j^2, \dots, t_j^k\right\} \neq \emptyset$ (in such a case x_{j_l} is set to be true).

Consequently, if at least k vertices of S distinguish u_j^1, u_j^4, then the corresponding setting of x_{j_l}, $l \in \{1,2,3\}$, according to the cases above, satisfies the clause Q_j. Therefore F is satisfiable. $\square$

As a consequence of the theorem above we have the following result.

Corollary 8. *The problem of finding the k-metric dimension of graphs is NP-hard.*

4 The particular case of trees

We must first recall that for the particular case of trees, it is already known from [21] that the problem of computing its 1-metric dimension can be done in linear time. Moreover, it was recently proved in [9] that also for the case of outerplanar graphs, this problem can be solved in polynomial time. We next deal with the problem of computing the k-metric dimension of trees for $k \geq 2$.

In order to continue presenting our results, we need to introduce some definitions. A vertex of degree at least three in a tree T is called a *major vertex* of T. Any leaf u of T is said to be a *terminal vertex* of a major vertex v of T, if $d_T(u, v) < d_T(u, w)$ for every other major vertex w of T. The *terminal degree* $\text{ter}(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of T is an *exterior major vertex* of T if it has positive terminal degree. Let $\mathcal{M}(T)$ be the set of exterior major vertices of T having terminal degree greater than one.

Given $w \in \mathcal{M}(T)$ and a terminal vertex u_j of w, we denote the shortest path that starts at u_j and ends at w by $P(u_j, w)$. Now, given $w \in \mathcal{M}(T)$ and two terminal vertices u_j, u_r of w we denote the shortest path from u_j to u_r containing w by $P(u_j, w, u_r)$, and the length of $P(u_j, w, u_r)$ by $\varsigma(u_j, u_r)$. Notice that, by the definition of exterior major vertex, $P(u_j, w, u_r)$ is obtained by concatenating the paths $P(u_j, w)$ and $P(u_r, w)$, where w is the only vertex of degree greater than two lying on these paths.

Finally, given $w \in \mathcal{M}(T)$ and the set of terminal vertices $U = \{u_1, u_2, \ldots, u_k\}$ of w, for $j \neq r$ we define

$$\varsigma(w) = \min_{u_j, u_r \in U} \{\varsigma(u_j, u_r)\}$$

and

$$l(w) = \min_{u_j \in U} \{d(u_j, w)\}.$$

From the local parameters above we define the following global parameter

$$\varsigma(T) = \min_{w \in \mathcal{M}(T)} \{\varsigma(w)\}.$$

An example of a tree T which helps to better understand the notation above is given in Figure 4. In such a case we have that $\mathcal{M}(T) = \{6, 12, 26\}$, $\{1, 4\}$ is the set of terminal vertices of 6, $\{9, 11\}$ is the set of terminal vertices of 12 and $\{15, 20, 23\}$ is the set of terminal vertices of 26. For instance, for the vertex 26 we have that $l(26) = \min\{d(15, 26), d(20, 26), d(23, 26)\} = \min\{5, 3, 3\} = 3$ and $\varsigma(26) = \min\{\varsigma(15, 20), \varsigma(15, 23), \varsigma(20, 23)\} = \min\{8, 8, 6\} = 6$. Analogously, we deduce that $l(6) = 2$, $\varsigma(6) = 5$, $l(12) = 1$ and $\varsigma(12) = 3$. Therefore, we conclude that $\varsigma(T) = \min\{\varsigma(6), \varsigma(12), \varsigma(26)\} = \min\{5, 3, 6\} = 3$.

4.1 On k-metric dimensional trees different from paths

In this section we focus on finding the positive integer k for which a tree is k-metric dimensional. We now show a result presented in [12] that allows us to consider only those trees that are not paths.

Theorem 9. [12] *A graph G of order $n \geq 3$ is $(n-1)$-metric dimensional if and only if G is a path or G is an odd cycle.*

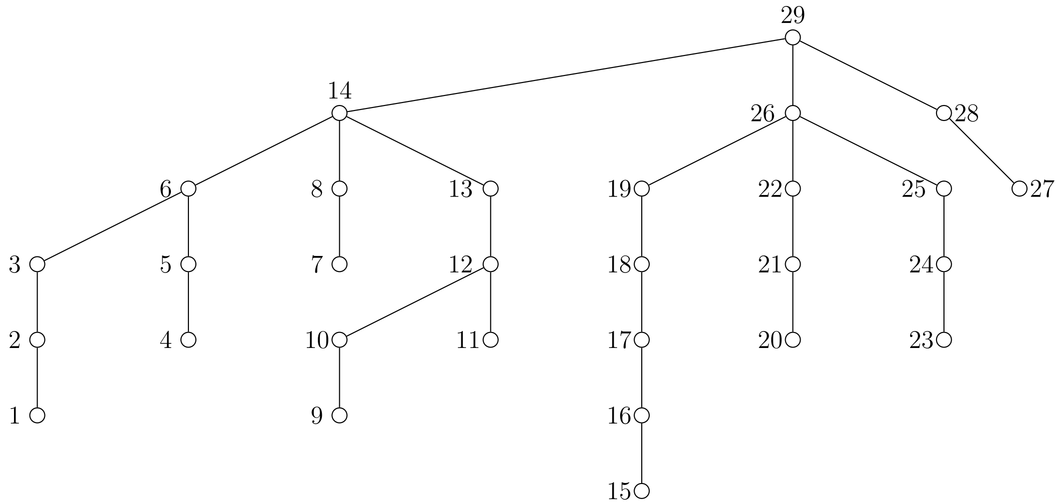

Figure 4: A tree T where $\varsigma(T) = 3$. Note that vertices are labeled through a post-order traversal.

The following theorem presented in [12] is the base of the algorithm presented in this subsection.

Theorem 10. [12] *If T is a k-metric dimensional tree different from a path, then $k = \varsigma(T)$.*

Now we consider the problem of finding the integer k such that a tree T of order n is k-metric dimensional.

k-DIMENSIONAL TREE PROBLEM
INSTANCE: A tree T different from a path of order n
PROBLEM: Find the integer k, $2 \leq k \leq n-1$, such that T is k-metric dimensional

Algorithm 1:
Input: A tree T different from a path rooted in a major vertex v.
Output: The value k for which T is k-metric dimensional.

1. For any vertex $u \in V(T)$ visited by post-order traversal as shown in Figure 4, assign a pair (a_u, b_u) in the following way:

   (a) If u does not have any child (u is a leaf), then $a_u = 1$ and $b_u = \infty$.

   (b) If u has only one child (u has degree 2), then $a_u = a_{u'} + 1$ and $b = b_{u'}$, where the pair $(a_{u'}, b_{u'})$ was assigned to the child vertex of u. Note that $a_{u'}$ can be ∞. Thus, in such case, $a_u = \infty$.

   (c) If u has at least two children (u is a major vertex), then $a_u = \infty$ and $b_u = \min\{a_{u_1} + a_{u_2}, b_{min}\}$, where a_{u_1} and a_{u_2} are the two minimum values among all possible pairs (a_{u_i}, b_{u_i}) assigned to the children of u, and b_{min} is the minimum value among all the b_{u_i}'s.

2. The value k for which T is k-metric dimensional equals b_v (the second element of the pair assigned to the root v).

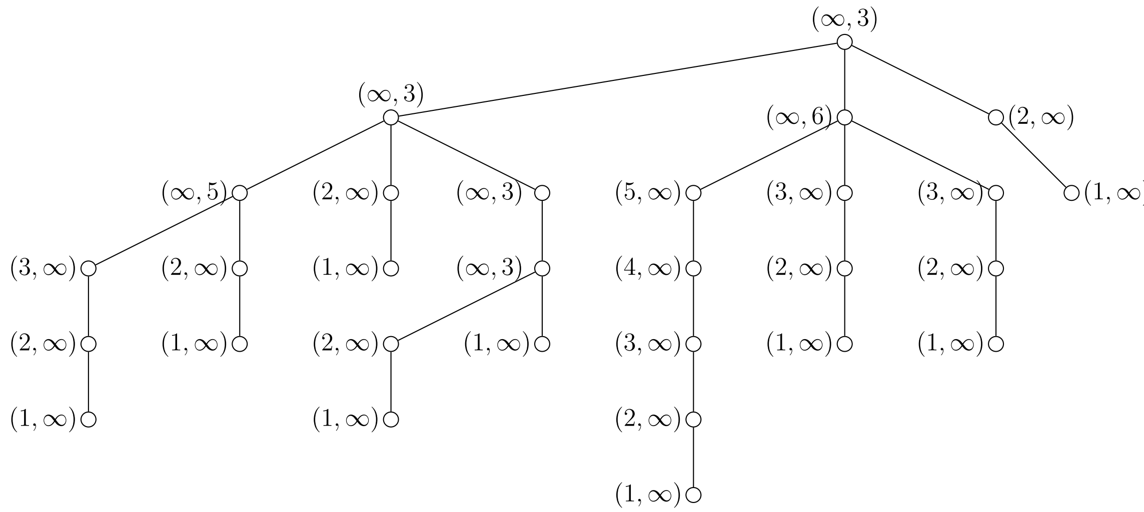

Figure 5: Algorithm 1 yields that this tree is 3-metric dimensional.

Figure 5 shows an example of a run of Algorithm 1 for the tree shown in Figure 4.

Remark 11. *Let T be a tree different from a path of order n. The* Algorithm 1 *finds the integer k such that T is k-metric dimensional and $2 \leq k \leq n-1$.*

Proof. Let v be the major vertex taken as the root of the tree T different from a path, and let (a_v, b_v) be the pair stored in v by Algorithm 1. We show that $b_v = \varsigma(T)$. Since v is a major vertex, it has at least three children. Let $t \geq 3$ be the number of children of v and let $S_1, \dots, S_t$ be the subtrees whose roots are the children $v_1, \dots, v_t$ of v, respectively. We differentiate two cases:

1. There exist at least two subtrees that are paths. In this case $v \in \mathcal{M}(T)$. Let $S_1, \dots, S_{t'}$ be the subtrees that are paths, where $2 \leq t' \leq t$. Hence, after running Algorithm 1, each root v_i of S_i, $1 \leq i \leq t'$, stores the pair (a_{v_i}, ∞), where a_{v_i} is the number of vertices of S_i. Note that $\varsigma(v) = a_{v_1} + a_{v_2}$, where a_{v_1} and a_{v_2} are the two minimum values among all a_{v_i}'s belonging to the pairs (a_{v_i}, b_{v_i}) stored as the children of v such that $1 \leq i \leq t'$. If $t' = t$, then v is the only exterior major vertex of T. Hence, the pair $(a_v, b_v) = (\infty, \varsigma(v)) = (\infty, \varsigma(T))$ is stored in v. Assume now that $t' < t$. Thus, there exists at least one subtree that is not a path. Let $S_{t'+1}, \dots, S_t$ be the subtrees that are not paths. For each root v_i of S_i, $t'+1 \leq i \leq t$, if v_i is a major vertex, then we take the vertex $v'_i = v_i$. Otherwise, v'_i is the first descendant of v_i that is a major vertex. Hence, the pair $(\infty, b_{v'_i})$, where $b_{v'_i} = \min\limits_{v' \in \mathcal{M}(T) \cap V(S_i)} \{\varsigma(v')\}$ is recursively stored in v'_i while running Algorithm 1. In both cases, $b_{v_i} = b_{v'_i}$, where (∞, b_{v_i}) is the pair stored in v_i by Algorithm 1. Therefore, the pair $(a_v, b_v) = (\infty, \min\{\varsigma(v), b_{min}\}) = (\infty, \varsigma(T))$, where $b_{min} = \min\limits_{t'+1 \leq i \leq t} \{b_{v_i}\}$ is stored in the root v by running the Algorithm 1.

2. There exists at most one subtree that is a path. In this case $v \notin \mathcal{M}(T)$. Let $S_1, \dots, 1.00, 0.00, 0.00 S_{t'}$ be the subtrees that are not paths, where $1 \leq t' \leq t$. For each root v_i of S_i, $1 \leq i \leq t'$, if v_i is a major vertex, then we take the vertex $v'_i = v_i$. Otherwise, v'_i is the first descendant of v_i that is a major vertex. In this case, the pair $(\infty, b_{v'_i})$, where $b_{v'_i} = \min\limits_{v' \in \mathcal{M}(T) \cap V(S_i)} \{\varsigma(v')\}$ is

recursively stored in v'_i. In both cases, $b_{v_i} = b_{v'_i}$, where (∞, b_{v_i}) is the pair stored in v_i by Algorithm 1. Note in this case, at least one of two minimum values among all a_{v_i} of pairs (a_{v_i}, b_{v_i}) stored by the children of v is infinity. Therefore, the pair $(a_v, b_v) = (\infty, b_{min}) = (\infty, \varsigma(T))$, where $b_{min} = \min_{1\le i\le t'}\{b_{v_i}\}$ is stored in v, by Algorithm 1.

In any case, $b_v = \varsigma(T)$, and the result follows. □

Corollary 12. *The positive integer k for which a tree different from a path is k-metric dimensional can be computed in linear time with respect to the order of the tree.*

4.2 On the k-metric bases and the k-metric dimensions of trees different from paths

Based on the fact that any tree different from a path is $\varsigma(T)$-metric dimensional, in this section we propose an algorithm to find a k-metric basis for a tree T for any $k \le \varsigma(T)$ and, as a consequence, the k-metric dimension of T. We first present a result with the value of the k-metric dimensions of paths, already presented in [12]. Further on, we center our attention to those trees different from paths.

Proposition 13. [12] *Let $k \ge 3$ be an integer. For any path graph P_n of order $n \ge k+1$,*

$$\dim_k(P_n) = k + 1.$$

We observe that, for instance, if P_n is a path of order n and the two leaves of P_n belong to a set $S \subseteq V(P_n)$ of cardinality $k+1$, then S is a k-metric basis of P_n.

We now present a function for any exterior major vertex $w \in \mathcal{M}(T)$, shown in [12], that allows us to compute the k-metric dimension of any $k \le \varsigma(T)$. Notice that this function uses the concepts already defined at the beginning of the Section 4. Given an integer $k \le \varsigma(T)$,

$$I_k(w) = \begin{cases} (ter(w) - 1)\,(k - l(w)) + l(w), & \text{if } l(w) \le \lfloor \frac{k}{2} \rfloor, \\ \\ (ter(w) - 1)\,\lceil \frac{k}{2} \rceil + \lfloor \frac{k}{2} \rfloor, & \text{otherwise.} \end{cases}$$

The following theorem presented in [12] is the base of the algorithm presented in this subsection.

Theorem 14. [12] *If T is a tree which is not a path, then for any $k \in \{1, \ldots, \varsigma(T)\}$,*

$$\dim_k(T) = \sum_{w \in \mathcal{M}(T)} I_k(w).$$

k-METRIC BASIS TREE PROBLEM
INSTANCE: A tree T of order n different from a path
PROBLEM: Find a k-metric basis of T, for any $k \le \varsigma(T)$

Algorithm 2:
Input: A tree T different from a path rooted in a major vertex v.
Output: A k-metric basis of T for any $k \le \varsigma(T)$.

1. For any vertex $u \in V(T)$ visited by post-order traversal as shown in Figure 4, assign a pair (a_u, b_u) in the following way:

   (a) If u does not have any child (u is a leaf), then $a_u = \{u\}$ and $b_u = \emptyset$.

   (b) If u has only one child (u has degree 2), then $b_u = b_{u'}$, where the pair $(a_{u'}, b_{u'})$ was assigned to the child vertex of u. If $a_{u'} = \emptyset$, then $a_u = \emptyset$. If $a_{u'} \neq \emptyset$, then $a_u = a_{u'} \cup \{u\}$.

   (c) If u has at least two children (u is a major vertex), then $a_u = \emptyset$. Let a_{min} be a set of minimum cardinality among all a_{u_i} belonging to the pairs (a_{u_i}, b_{u_i}) assigned to the children of u, let c_u be the number of a_{u_i} which are different from an empty set, and let d_u be the union of all b_{u_i}. If $c_u \leq 1$, then $b_u = d_u$. If $c_u \geq 2$ and $|a_{min}| \leq \lfloor \frac{k}{2} \rfloor$, then we remove elements of each $a_{u_i} \neq a_{min}$ until its cardinality is $k - |a_{min}|$. If $c_u \geq 2$ and $|a_{min}| > \lfloor \frac{k}{2} \rfloor$, then we remove elements of each $a_{u_i} \neq a_{min}$ until its cardinality is $\lceil \frac{k}{2} \rceil$, and we remove elements of a_{min} until its cardinality is $\lfloor \frac{k}{2} \rfloor$.

   Then $b_u = a_{min} \cup \left(\bigcup_{a_{u_i} \neq a_{min}} a_{u_i} \right) \cup d_u$.

2. A k-metric basis of T is stored in b_v.

Remark 15. *Let T be a tree different from a path.* Algorithm 2 *finds a k-metric basis of T for any $k \leq \varsigma(T)$.*

Proof. Given an exterior major vertex $w \in \mathcal{M}(T)$ such that $u_1, u_2, \ldots, u_t$ are its terminal vertices and $l(w) = d(u_{min}, w)$, we define the vertex set $B_k(w)$ in the following way. If $l(w) \leq \lfloor \frac{k}{2} \rfloor$, then $|B_k(w) \cap (V(P(u_j, w)) \setminus \{w\})| = k - l(w)$, for any $j \neq min$, and $V(P(u_{min}, w)) \setminus \{w\} \subset B_k(w)$. Otherwise, $|B_k(w) \cap (V(P(u_j, w)) \setminus \{w\})| = \lceil \frac{k}{2} \rceil$, for any $j \neq min$, and $|B_k(w) \cap (V(P(u_{min}, w)) \setminus \{w\})| = \lfloor \frac{k}{2} \rfloor$. It was shown in [12], that $\bigcup_{w \in \mathcal{M}(T)} B_k(w)$ is a k-metric basis of T. Let v be the major vertex taken as a root of the tree T different from a path, and let (a_v, b_v) be the pair assigned to v once executed Algorithm 2. We show that the vertex set $b_v = \bigcup_{w \in \mathcal{M}(T)} B_k(w)$. Since v is a major vertex, it has at least three children. Let $t \geq 3$ be the number of children of v and let $S_1, \ldots, S_t$ be the subtrees whose roots are the children $v_1, \ldots, v_t$ of v, respectively. We differentiate two cases:

1. There exist at least two subtrees that are paths. In this case $v \in \mathcal{M}(T)$. Let $S_1, \ldots, S_{t'}$ be the subtrees that are paths, where $2 \leq t' \leq t$. Hence, Algorithm 2 assigns to each root v_i of S_i, $1 \leq i \leq t'$, the pair $(a_{v_i}, \emptyset)$, where $a_{v_i} = V(S_i)$. Note that in that situation $\mathrm{ter}(v) = c_v = t' \geq 2$ and $l(v) = |a_{min}|$. If $t' = t$, then v is the only exterior major vertex and $d_v = \emptyset$. As a consequence, Algorithm 2 assigns to v the pair $(\emptyset, B_k(v))$. Assume now that $t' < t$. Thus, there exists at least one subtree that is not a path. Let $S_{t'+1}, \ldots, S_t$ be the subtrees that are not paths. For each root v_i of S_i, $t'+1 \leq i \leq t$, if v_i is a major vertex, then we take the vertex $v'_i = v_i$. Otherwise, v'_i is the first descendant of v_i that is a major vertex. Hence, the pair $(\emptyset, b_{v'_i})$, where $b_{v'_i} = \bigcup_{v' \in \mathcal{M}(T) \cap V(S_i)} B_k(v')$ is recursively stored in v'_i by running Algorithm 2. In both cases, $b_{v_i} = b_{v'_i}$, where $(\emptyset, b_{v_i})$ is the pair stored in v_i by

Algorithm 2. Hence, $d_v = \bigcup_{v' \in \mathcal{M}(T) \setminus \{v\}} B_k(v')$. Therefore, Algorithm 2 assigns to v the pair

$$(a_v, b_v) = \left(\emptyset, B_k(v) \cup \bigcup_{v' \in \mathcal{M}(T) \setminus \{v\}} B_k(v') \right) = \left(\emptyset, \bigcup_{v' \in \mathcal{M}(T)} B_k(v') \right).$$

2. There exists at most one subtree that is a path. In this case $v \notin \mathcal{M}(T)$ and $c_v \leq 1$. Let $S_1, \ldots, S_t'$ be the subtrees that are not paths, where $1 \leq t' \leq t$. For each root v_i of S_i, $1 \leq i \leq t'$, if v_i is a major vertex, then we take the vertex $v_i' = v_i$. Otherwise, v_i' is the first descendant of v_i that is a major vertex. Hence, the pair $(\emptyset, b_{v_i'})$, where $b_{v_i'} = \bigcup_{v' \in \mathcal{M}(T) \cap V(S_i)} B_k(v')$ is recursively assigned to v_i'. Again, $b_{v_i} = b_{v_i'}$, where $(\emptyset, b_{v_i})$ is the pair stored in v_i by Algorithm 2. Thus, $d_v = \bigcup_{v' \in \mathcal{M}(T)} B_k(v')$. Note in such case, that at most one of all possible a_{v_i}'s belonging to the pairs (a_{v_i}, b_{v_i}) assigned to the children of v, is different from infinity. As a consequence, $c_v \leq 1$. Therefore, the pair $(a_v, b_v) = \left(\emptyset, \bigcup_{v' \in \mathcal{M}(T)} B_k(v') \right)$ is assigned to v by Algorithm 2.

In any case, $b_v = \bigcup_{v' \in \mathcal{M}(T)} B_k(v')$, and the result follows. □

Corollary 16. *A k-metric basis of any tree different from a path, for any $k \leq \varsigma(T)$, can be computed in linear time with respect to the order of T.*

References

[1] R. Adar, L. Epstein, Models for the k-metric dimension, arXiv:1410.4209 [cs.DS].

[2] R. Adar, L. Epstein, The weighted 2-metric dimension of trees in the non-landmarks model, Discrete Optimization 17 (2015) 123–135.

[3] R. F. Bailey, K. Meagher, On the metric dimension of grassmann graphs, Discrete Mathematics & Theoretical Computer Science 13 (4) (2011) 97–104.

[4] R. F. Bailey, I. G. Yero, Error-correcting codes from k-resolving sets, arXiv:1605.03141 [math.CO].

[5] R. C. Brigham, G. Chartrand, R. D. Dutton, P. Zhang, Resolving domination in graphs, Mathematica Bohemica 128 (1) (2003) 25–36.

[6] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, D. R. Wood, On the metric dimension of Cartesian product of graphs, SIAM Journal on Discrete Mathematics 21 (2) (2007) 423–441.

[7] G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, Discrete Applied Mathematics 105 (1-3) (2000) 99–113.

[8] G. Chartrand, V. Saenpholphat, P. Zhang, The independent resolving number of a graph, Mathematica Bohemica 128 (4) (2003) 379–393.

[9] J. Díaz, O. Pottonen, M. Serna, E. J. van Leeuwen, On the Complexity of Metric Dimension, Lecture Notes in Computer Science Volume 7501 (2012) 419–430.

[10] D. Epstein, Metric dimension parameterized by max leaf number, Journal of Graph Algorithms and Applications 19 (1) (2015) 313–323.

[11] L. Epstein, A. Levin, G. J. Woeginger, The (weighted) metric dimension of graphs: Hard and easy cases, Algorithmica 72 (4) (2015) 1130–1171.

[12] A. Estrada-Moreno, J. A. Rodríguez-Velázquez, I. G. Yero, The k-metric dimension of a graph, Applied Mathematics & Information Sciences 9 (6) (2015) 2829–2840.

[13] A. Estrada-Moreno, I. Yero, J. A. Rodríguez-Velázquez, The k-metric dimension of the lexicographic product of graphs, Discrete Mathematics 339 (7) (2016) 1924–1934.

[14] A. Estrada-Moreno, I. G. Yero, J. A. Rodríguez-Velázquez, The k-metric dimension of corona product graphs, Bulletin of the Malaysian Mathematical Sciences Society 39 (1) (2016) 135–156.

[15] H. Fernau, P. Heggernes, P. van 't Hof, D. Meister, R. Saei, Computing the metric dimension for chain graphs, Information Processing Letters 115 (9) (2015) 671–676.

[16] M. R. Garey, D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W. H. Freeman & Co., New York, NY, USA, 1979.

[17] F. Harary, R. A. Melter, On the metric dimension of a graph, Ars Combinatoria 2 (1976) 191–195.

[18] S. Hartung, A. Nichterlein, On the parameterized and approximation hardness of metric dimension, in: Computational Complexity (CCC), 2013 IEEE Conference on, 2013.

[19] M. Johnson, Structure-activity maps for visualizing the graph variables arising in drug design, Journal of Biopharmaceutical Statistics 3 (2) (1993) 203–236.

[20] M. Johnson, Browsable structure-activity datasets, in: R. Carbó-Dorca, P. Mezey (eds.), Advances in Molecular Similarity, chap. 8, JAI Press Inc, Stamford, Connecticut, 1998, pp. 153–170.

[21] S. Khuller, B. Raghavachari, A. Rosenfeld, Landmarks in graphs, Discrete Applied Mathematics 70 (3) (1996) 217–229.

[22] D. Kuziak, I. G. Yero, J. A. Rodríguez-Velázquez, On the strong metric dimension of corona product graphs and join graphs, Discrete Applied Mathematics 161 (7–8) (2013) 1022–1027.

[23] R. A. Melter, I. Tomescu, Metric bases in digital geometry, Computer Vision, Graphics, and Image Processing 25 (1) (1984) 113–121.

[24] F. Okamoto, B. Phinezy, P. Zhang, The local metric dimension of a graph, Mathematica Bohemica 135 (3) (2010) 239–255.

[25] D. F. Rall, P. J. Slater, On location-domination numbers for certain classes of graphs, Congressus Numerantium 45 (1984) 97–106.

[26] B. Roy, Transitivité et connexité, Comptes Rendus de l'Académie des Sciences 249 (1959) 216–218.

[27] A. Sebö, E. Tannier, On metric generators of graphs, Mathematics of Operations Research 29 (2) (2004) 383–393.

[28] P. J. Slater, Leaves of trees, Congressus Numerantium 14 (1975) 549–559.

[29] S. Warshall, A theorem on boolean matrices, Journal of the ACM 9 (1) (1962) 11–12.

[30] I. G. Yero, D. Kuziak, J. A. Rodríguez-Velázquez, On the metric dimension of corona product graphs, Computers & Mathematics with Applications 61 (9) (2011) 2793–2798.