


Review

Fuzzy Random Option Pricing in Continuous Time: A Systematic Review and an Extension of Vasicek's Equilibrium Model of the Term Structure

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Abstract: Fuzzy random option pricing in continuous time (FROPCT) has emerged as an active research field over the past two decades; thus, there is a need for a comprehensive review that provides a broad perspective on the literature and identifies research gaps. In this regard, we conducted a structure review of the literature by using the WoS and SCOPUS databases while following the PRISMA criteria. With this review, we outline the primary research streams, publication outlets, and notable authors in this domain. Furthermore, the literature review revealed a lack of advancements for the equilibrium models of the yield curve. This finding serves as a primary motivation for the second contribution of this paper, which involves an extension of Vasicek's yield curve equilibrium model. Specifically, we introduce the existence of fuzzy uncertainty in the parameters governing interest rate movements, including the speed of reversion, equilibrium short-term interest rate, and volatility. By incorporating fuzzy uncertainty, we enhance the model's ability to capture the complexities of real-world interest rate dynamics. Moreover, this paper presents an empirical application of the proposed extension to the term structure of fixed-income public bonds in European Union. The empirical analysis suggests the suitability of the proposed extension of Vasicek's model for practical applications.

Keywords: option pricing; fuzzy random variables; fuzzy numbers; fuzzy random option pricing; Vasicek's model of term structure

MSC: 62A88; 91G20; 91G30



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1. Introduction

Option pricing mathematics started its development at the end of the 19th century [1] and has been an active research field since the second half of the 20th century [2]. The Black–Scholes–Merton option pricing model [3,4] (the BSM model hereinafter) is commonly considered a milestone, not only in the constrained field of option pricing, but also in the financial arena [2]. The value of the model [3,4] is twofold. From a strict perspective of the options on financial asset pricing, the BSM model provides a very operational formula, as the parameters required for its implementation are easy to obtain and do not depend on subjective risk attitudes. From a more general setting of pricing any asset, valuation through the so-called “arbitrage argument”, which is generalized by the concepts of “risk-neutral valuation” and “equivalent martingale measure”, allows for any asset to be valued according to the mathematical expectation of the present value of its future cash flows [5].

Thus, the BSM model can be applied to options on stocks but also to any asset that includes some type of optionality. Therefore, it can be used to determine the price of assets such as convertible bonds and develop formulas for new types of options such as exotic options or novel derivative assets such as catastrophe bonds [2]. Additionally, the model has also been applied to traditional assets such as mortgage-backed loans or life insurance policies with guaranteed returns [2] and real assets by using real option theory [6]. In

fact, Black and Scholes [3] explicitly state in their work that their formula can be used to value companies.

The analytical framework of the BSM model has been particularly productive at modeling the term structure of interest rates with arbitrage-based models, and it has provided a solid mathematical foundation for pricing interest rate derivative instruments such as bond options, caps, floors, or other options, and also to assess the dynamics of the yield curve. Following [7], the BSM model can be differentiated into models whose parameters do not depend on time [8–10] and those with time-varying parameters [11]. The first type of development, while providing a valuable analytical framework to conduct economic analyses on the yield curve, does not capture its empirical shapes especially well. Thus, due to its parsimony, it is very useful to conduct a great deal of financial and economic analyses, but when valuing interest rate derivatives, their usefulness is limited [7]. On the other hand, time-dependent parameter models are calibrated based on the actual observed prices of zero-coupon instruments, which results in perfect adherence to the observed temporal structure forms, but they suffer from a lack of parsimony [7].

Option valuation models, since their early stages in the late 19th century, have been grounded in the analytical framework of statistics and stochastic calculus. Therefore, they assume a strict risk environment where probabilities of alternative event realizations can be established [12]. Undoubtedly, these models provide a valuable analytical framework. However, financial activities are conducted with information that involves different degrees of knowledge, and these models can combine risk with other sources of uncertainty, such as imprecision or vagueness. Fuzzy set theory (FST) has provided tools for option pricing when modeling nonprobabilistic uncertainty, such as fuzzy measures, fuzzy numbers, or fuzzy regression, since the early 2000s [13]. While stochastic models provide rigorous analytical groundwork, the addition of fuzzy tools can improve their results by allowing for the introduction of additional sources of uncertainty to risk [12,13].

Among the various ways in which FST has been applied to option valuation (FOP), the most fruitful is what we can label as fuzzy random option pricing (FROP) [13]. This set of contributions extends conventional option valuation models to the hypothesis that knowledge of the parameters governing financial asset price movements is not crisp, as they could be affected by issues such as fuzziness or imprecision [12]. The uncertainty existing in these parameters, such as volatility, observed price of the underlying asset, or discount rates, is modeled by using fuzzy numbers [12]. Thus, FROP can be applied to continuous time models by using the BSM framework or to discrete-time models [12]. The first approach, which is the scope of this paper, will be referred to as FROP in continuous time (FROPCT).

This work has two research objectives within the field of FROPCT:

1. The first objective is to present the results of a systematic literature review on FROP that covers the period from the first work up to March 2023 by using Web of Knowledge (WoS) and SCOPUS. We will focus on contributions related to FROPCT. This systematic review aims to offer a valuable perspective on the main contributions and developments in FROPCT and to identify research gaps. Among these research gaps, we have observed that FROPCT has achieved significant development in the context of equity [14–18] and real options [19–21]. However, extensions to the context of fixed-income markets and interest-rate-sensitive instruments are relatively scarce.
2. The aforementioned research gap motivated the second research objective of this study, which involves developing a fuzzy random extension of Vasicek's yield curve model [8]. In this regard, we will assume that the uncertainty about the parameters governing interest rates (mean reverting rate, long-term mean, and volatility) is modeled by using fuzzy numbers. Our extension will be applied to the zero-coupon curve published by the European Central Bank for bonds with the highest credit rating in Europe.

We believe that this paper provides three contributions to FROPCT. The structured literature review serves as a foundation for subsequent advancements in this field by

identifying several research gaps. The introduction of a FROP focus to model uncertainty in parameters that describe the yield curve is a novelty. Additionally, the proposed methodology for estimating these parameters by using objective information from fixed-income markets is an original approach that combines autoregressive time series with a coherent probability–possibility transformation criterion [22] instead of a fuzzy regression method such as those proposed in [23] in a fuzzy option pricing setting.

The rest of the paper is structured as follows: In Section 2, we present the results of a comprehensive review of related works framed within FROPCT, following the PRISMA criteria. The findings of this section motivated Section 3, which proposes an extension of Vasicek’s equilibrium model for the term structure [8]. This extension will be applied to the curve fitting of coupon bonds in the risk-free bond market of European Union issuers. In Section 4, we discuss the main results obtained in this study. We conclude by highlighting the main limitations of the presented work and suggesting future lines of research.

2. Bibliographical Analysis

2.1. Methodology

The bibliographic review was conducted following the PRISMA guidelines [24]. Its implementation required us to specify the methodology used to search for the documents to be reviewed, the databases where these documents should be found, the types of documents that will be considered, etc. Subsequently, how the obtained documents have been compiled must be determined. The next step is to classify the papers identifying the sources, authors, and main areas of application of FROP in continuous time.

Regarding the eligible studies, we only considered articles written in English and published as journal articles or book chapters until 31 March 2023. We did not consider other types of documents that are typically categorized as “grey literature,” such as conference proceedings or documents in digital repositories. The reason is that normally, after a peer review process, these types of contributions are usually published as articles or book chapters. Additionally, we only analyzed works written in English. We chose to combine two databases, SCOPUS and WoS, in this review, as they are commonly used in bibliographic reviews, and it is recommended to combine more than one bibliographic source when the topic is not very broad [13].

The search was executed on titles, abstracts, and keywords using the following search: (“option pricing” OR “arbitrage model” OR “risk neutral pricing” OR “equivalent martingale measure”) AND (“fuzzy sets” OR “fuzzy mathematics” OR “fuzzy numbers”). Figure 1 graphically shows how we proceeded.

There were a total of 117 documents reported by SCOPUS and 144 by WoS. We eliminated duplicate works and examined the title, abstract, keywords, and, if necessary, the full document to ensure that the papers were effectively related to FROP.

Finally, we identified 104 documents related to FROP. We found 83 documents common to both databases; 7 were exclusively provided by WoS, and SCOPUS provided 15. At this stage, the Meyer’s index, which quantifies the level of coverage attributable to each database, recorded a rate of 46.15% for WoS and 53.85% for SCOPUS. The degree of overlap, i.e., the redundancy rate of a database, was 92.13% for WoS and 84.54% for SCOPUS.

Our bibliographic exploration showed that FROPCT was developed in 77 articles. Within these documents, 2 came exclusively from the WoS database and 11 from SCOPUS. Therefore, 64 documents were common to both SCOPUS and WoS. At this step, Meyer’s index was 55.84% for SCOPUS and 44.16% for WoS. We verified an overlap level of 85.33% for SCOPUS and 96.97% for WoS.

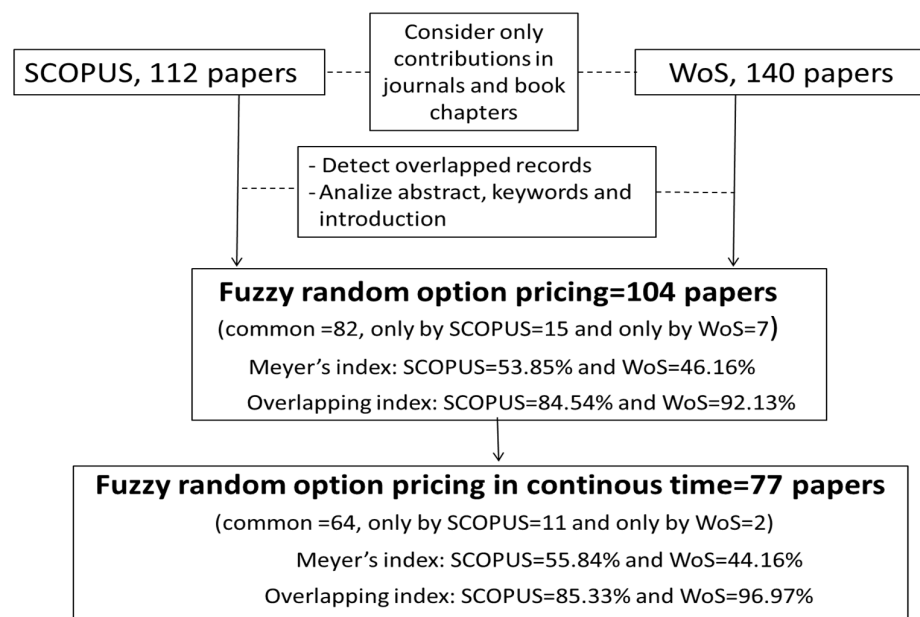


Figure 1. Protocol used to select articles on FROP and FROPCT for review. Source: own elaboration by following PRISMA guidelines [22].

2.2. Classification

Figure 2 shows the evolution of the production of FROPCT by year of publication. The first works were published in the years 2001–2003, which means that the introduction of fuzzy numbers in option valuation started in the beginning of the 21st century. We could observe a trend until 2013 that, although not monotonic, was clearly increasing. In that year, the maximum number of published works (8) was reached. From that year, works on FROPCT remained within a fluctuation range of 3 to 8 annual contributions. Therefore, although FROP is not a mainstream topic in fuzzy mathematics, it can be considered a consolidated topic within the applications of fuzzy logic.

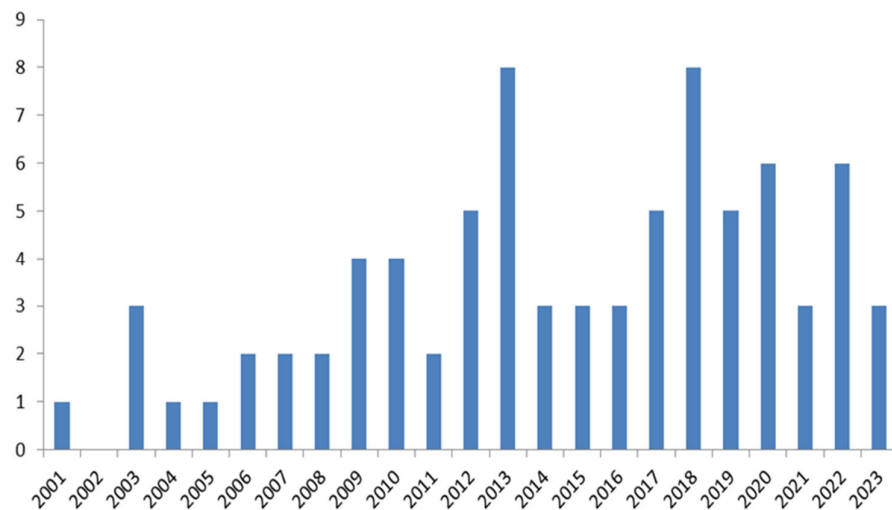


Figure 2. Evolution of the papers on fuzzy random option pricing in continuous time published during 2003–2023. Source: own elaboration based on the WoS and Scopus databases.

Table 1 displays the works reviewed in this article that were published in 2015 or received more than 25 citations in the WoS or SCOPUS databases. These were classified in columns according to the stochastic process that serves as the basis for fuzzy extensions.

Table 1. Revised papers on fuzzy stochastic option pricing in continuous time from 2015 or at least with 25 citations in the WoS or Scopus databases.

	Geometric Brownian Process (BSM)	Geometric Brownian Process (More than One Asset)		Other Brownian Processes
All papers in the table	[14,15,21,25–50]	[19,20,51–53]		[48,54]
Fuzzy numbers of higher degree	[42,47]			[48]
Non-European options	[38,49]	[19,20,51–53]		[48,54]
Hedging	[14,28,31,33,35]			[54]
Application to financial markets	[23,25,26,36,37]			
Real Options	[21,30,42]	[19,20,51–53]		[54]
Other financial applications	[45,47,50]			[48]
	Jump diffusion	Heston	Fractional	Levy
All papers in the table	[55,56]	[57]	[18,58–63]	[16,17,55,64–71]
Fuzzy numbers of higher degree			[61]	
Non-European options			[60,61]	[17,70]
Hedging				
Application to financial markets		[57]	[18]	[17,71]
Real Options				
Other financial applications				[67,69]

Source: own elaboration based on WoS and Scopus databases.

Table 1 shows that the majority of works (29) assumed a single underlying asset and geometric Brownian motion. Thus, the majority were based on the application of the Black–Scholes–Merton (BSM) model. However, there are several nuances to consider in this regard. For example, ref. [49] extended the Asian option valuation formula of Kemma and Vorst [72]. Additionally, within the framework of geometric Brownian motion, there were 5 works that assumed that various parameters of the Margrabe exchange valuation model [73] and Geske’s compound option model pricing formula [74] are given by fuzzy numbers.

In the framework of more complex stochastic models, fractional Brownian motion (7 works) and the more general Levy modeling (11 works) have been extensively addressed. It is also worth mentioning that Merton’s jump-diffusion model [75] and the Heston formula [76] have been objects of attention in the FROPCT literature.

In most cases, the analyzed topics are technical aspects that emerge from the juxtaposition of stochastic calculus with fuzzy mathematics. The most common mathematical issue is the application of fuzzy arithmetic, which quantifies certain parameters of the generalized option formula. Without being exhaustive, we can indicate that [43,44] quantified these parameters in the context of the BSM model, refs. [20,51] quantified them by using a multiple underlying asset options framework that was governed by Brownian motion, ref. [55] quantified them by using the framework of a diffusion and jump model, ref. [57] quantified them via the fuzzification of Heston’s model, and ref. [58] quantified them via fuzzy fractional models or the fuzzy extension to Levy stochastic processes [16,71]. In some cases, especially in articles that were based on the standard BSM model, issues associated with fuzzy analysis were refined. These issues may embed in areas such as defuzzification [39,40,48] or the construction of membership functions for the inputs of the pricing formula or the final price of the asset [26,27,36,37].

The first row of Table 1 indicates that the modeling of uncertainty in the parameters of the valuation formula is usually conducted by using type-1 fuzzy numbers (i.e., conventional fuzzy numbers). Exceptions are [42,61], which model uncertain parameters with type-2 fuzzy numbers, and [47], which use intuitionistic fuzzy numbers. In most cases, the assumed form of the fuzzy magnitude is linear, i.e., triangular or trapezoidal. However, within type-1 fuzzy numbers, the literature has also used other shapes, such as adaptive fuzzy numbers [14,39,40], Gaussian fuzzy numbers [28], or parabolic fuzzy numbers [17]. The parameters that are considered crisp and those that are considered fuzzy are estab-

lished ad hoc depending on the problem being addressed. In options on financial assets traded in financial markets, the volatility (always), risk-free interest rate, and underlying asset value (in most cases) are assumed to be estimated with fuzzy numbers. However, the strike price and expiration are crisp parameters because they are clearly defined in the contract. However, in terms of real options, the strike price [21] or even the expiration [13] may not be known with precision and, therefore, are susceptible to being quantified with fuzzy numbers.

Starting from the second row (included), relevant topics on financial pricing addressed by concrete papers are indicated. A great proportion of papers price European options. However, other types of options, such as American [17], binary [60,66], exchange [19,20,51], or compound [53,63,70] options, were analyzed. A sensitivity analysis of the option prices from the perspective of the BSM model has been the subject of attention of various authors [28,31]. Likewise, while fuzzy Malliavin calculus has been applied in [33,54], ref. [26] showed that the use of Greeks can be useful in the linear approximation of the membership functions of fuzzy option prices.

We must acknowledge that there is a relative scarcity of empirical applications of FROPCT [12,13]. Among these works, notable contributions include [27], which proposes the utilization of congruent probability–possibility transformations [27] to model option volatility based on empirical data, and [23,36,37], which employ fuzzy regression models to estimate the volatility, kurtosis, and skewness of the asset returns. The papers [25,26] demonstrated the good adherence of the fuzzy version of the standard BSM formula to the traded prices on the Spanish stock index IBEX-35. Additionally, ref. [57], which is an extension of [76]; ref. [18], which explored the fuzzy extension of the geometric fractional Brownian motion; and [17,71], who conducted fuzzy Levy modeling, present comprehensive empirical applications for options regarding indexes such as the SSE 50 or Standard and Poor’s indexes.

Beyond the options for stocks or indexes traded on stock exchanges, a very fruitful field of FROPCT has been real options. In a sense, it is logical since for this type of option, the underlying asset is usually an investment project, and the data on it are often given by subjective estimates from experts that can be well represented through trapezoidal or triangular numbers [19,21]. While the simplest real options can be valued by using a fuzzy version of the BSM formula [21,30,41,42], other works [19,20,51–53,63,70] extend more complex option valuation frameworks to real asset-related optionality.

Other residual applications of FROPCT to asset valuation include assessing the firms’ value [50], as suggested by the seminal work of Black and Scholes [3]; credit default swaps [47,48]; bank deposit insurance [45]; catastrophe bonds [69]; and forward contracts in energy markets [67].

Table 2 shows the main outlets where the contributions of FROPCT have been published. We only included journals with two or more contributions. Undoubtedly, the main journal was Fuzzy Sets and Systems (10 contributions), which was one of the principal academic references in fuzzy mathematics. Other journals whose scope was fuzzy logic and where FROP had a significant impact were the International Journal of Fuzzy Systems (4 contributions), IEEE Transactions on Fuzzy Systems, Fuzzy Optimization and Decision Making, and Journal of Intelligent and Fuzzy Systems (2 contributions). Other types of journals that publish a large proportion of studies of the contributions of FROPCT are more generalist journals dedicated to computing and/or soft computing (for example, Journal of Computational and Applied Mathematics, 4 contributions; Soft Computing, 3 contributions; or International Journal of Intelligent Systems, 2 contributions). Likewise, generalist journals of operational research have also been a vehicle for relevant productions of FROPCT (e.g., European Journal of Operational Research with 4 contributions; International Journal of Information Technology and Decision Making and Journal of Applied Mathematics and Statistics with 2 contributions).

Table 2. Principal outlets of fuzzy random option pricing in continuous time.

Journal	Number of Items
Fuzzy Sets and Systems	10
Journal of Computational and Applied Mathematics	4
International Journal of Fuzzy Systems	4
European Journal of Operational Research	4
Soft Computing	3
IEEE Transactions on Fuzzy Systems	2
Fuzzy Optimization and Decision Making	2
Journal of Intelligent and Fuzzy Systems	2
Computers and Mathematics with Applications	2
Discrete Dynamics in Nature and Society	2
International Journal of Information Technology and Decision Making	2
International Journal of Intelligent Systems	2
International Journal of Applied Mathematics and Statistics	2
Journal of Applied Mathematics	2

Source: own elaboration based on WoS and Scopus databases.

Table 3a,b list the most relevant works according to the WoS (Table 3a) and SCOPUS (Table 3b) databases. We determined the relevance based on the number of citations, and we included works referenced at least 25 times. We observed that both databases essentially included the same works, and the ordering, although not identical, was very similar. It can also be noted that works were usually more cited in SCOPUS than in WoS, which was expected since the SCOPUS database is more comprehensive. The most cited works were usually the oldest works that fell between 2001 and 2010 and were all within the framework of the BSM formula.

Table 3. (a) Papers with at least 25 citations in the WoS database. (b) Papers with at least 25 citations in the Scopus database.

(a)				
Year	Authors	Article Title	Source Title	Citations
2003	Carlsson, C; Fuller, R. [21]	A fuzzy approach to real option valuation	Fuzzy Sets and Systems	168
2003	Yoshida, Y. [15]	The valuation of European options in uncertain environment	European Journal of Operational Research	119
2004	Wu, H.C. [43]	Pricing European options based on the fuzzy pattern of Black-Scholes formula	Computers and Operations Research	105
2001	Zmeskal, Z. [50]	Application of the fuzzy-stochastic methodology to appraising the firm value as an European call option	European Journal of Operational Research	81
2007	Wu, H.C. [44]	Using fuzzy sets theory and Black-Scholes formula to generate pricing boundaries of European options	Applied Mathematics and Computation	80
2009	Chrysafis, K.A.; Papadopoulos, BK. [14]	On theoretical pricing of options with fuzzy estimators	Journal of Computational and Applied Mathematics	50

Table 3. *Cont.*

(a)				
Year	Authors	Article Title	Source Title	Citations
2009	Thavaneswaran, A.; Appadoo, S.S.; Paseka, A. [39]	Weighted possibilistic moments of fuzzy numbers with applications to GARCH modeling and option pricing	Mathematical and Computer Modeling	50
2007	Thiagarajah, K.; Appadoo, S.S.; Thavaneswaran, A. [40]	Option valuation model with adaptive fuzzy numbers	Computers and Mathematics with Applications	49
2005	Wu, H.C. [53]	European option pricing under fuzzy environments	International Journal of Intelligent Systems	36
2010	Nowak, P.; Romaniuk, M. [16]	Computing option price for Levy process with fuzzy parameters	European Journal of Operational Research	35
2013	Thavaneswaran, A.; Appadoo, S.S.; Frank, J. [38]	Binary option pricing using fuzzy numbers	Applied Mathematics Letters	33
2015	Muzzioli, S.; Ruggieri, A.; De Baets, B. [23]	A comparison of fuzzy regression methods for the estimation of the implied volatility smile function	Fuzzy Sets and Systems	31
2009	Xu, W.; Wu, C.; Xu, W.; Li, H. [55]	A jump-diffusion model for option pricing under fuzzy environments	Insurance Mathematics and Economics	31
2014	Nowak, P.; Romaniuk, M. [68]	Application of Levy processes and Esscher transformed martingale measures for option pricing in fuzzy framework	Journal of Computational and Applied Mathematics	29
2012	Zhang, L.H.; Zhang, W.G.; Xu, W.J.; Xiao, W.J. [56]	The double exponential jump diffusion model for pricing European options under fuzzy environments	Economic Modeling	29
2017	Muzzioli, S.; De Baets, B. [12]	Fuzzy Approaches to Option Price Modeling	IEEE Transactions on Fuzzy Systems	28
(b)				
Year	Author	Title	Source Title	Citations
2003	Carlsson, C., Fullér, R. [21]	A fuzzy approach to real option tvaluation	Fuzzy Sets and Systems	195
2003	Yoshida, Y. [15]	The valuation of European options in uncertain environment	European Journal of Operational Research	122
2004	Wu, H.C. [43]	Pricing European options based on the fuzzy pattern of Black-Scholes formula	Computers and Operations Research	112
2007	Wu, H.C. [44]	Using fuzzy sets theory and Black-Scholes formula to generate pricing boundaries of European options	Applied Mathematics and Computation	76
2001	Zmeškal, Z. [50]	Application of the fuzzy-stochastic methodology to appraising the firm value as an European call option	European Journal of Operational Research	73

Table 3. Cont.

(b)				
Year	Author	Title	Source Title	Citations
2009	Thavaneswaran, A., Appadoo, S.S., Paseka, A. [39]	Weighted possibilistic moments of fuzzy numbers with applications to GARCH modeling and option pricing	Mathematical and Computer Modeling	55
2009	Chrysafis, K.A., Papadopoulou, B.K. [14]	On theoretical pricing of options with fuzzy estimators	Journal of Computational and Applied Mathematics	51
2007	Thiagarajah, K., Appadoo, S.S., Thavaneswaran, A. [40]	Option valuation model with adaptive fuzzy numbers	Computers and Mathematics with Applications	51
2005	Wu, H.C., [53]	European option pricing under fuzzy environments	International Journal of Intelligent Systems	46
2015	Muzzioli, S.; Ruggieri, A.; De Baets, B. [23]	A comparison of fuzzy regression methods for the estimation of the implied volatility smile function	Fuzzy Sets and Systems	31
2010	Nowak, P., Romaniuk, M., [16]	Computing option price for Levy process with fuzzy parameters	European Journal of Operational Research	36
2017	Muzzioli, S., De Baets, B. [12]	Fuzzy Approaches to Option Price Modeling	IEEE Transactions on Fuzzy Systems	32
2009	Xu, W., Wu, C., Xu, W., Li, H. [55]	A jump-diffusion model for option pricing under fuzzy environments	Insurance: Mathematics and Economics	32
2013	Thavaneswaran, A., Appadoo, S.S., Frank, J. [38]	Binary option pricing using fuzzy numbers	Applied Mathematics Letters	31
2014	Nowak, P., Romaniuk, M. [68]	Application of Levy processes and Esscher transformed martingale measures for option pricing in fuzzy framework	Journal of Computational and Applied Mathematics	29
2012	Zhang, L.-H., Zhang, W.-G., Xu, W.-J., Xiao, W.-L. [56]	The double exponential jump diffusion model for pricing European options under fuzzy environments	Economic Modeling	29
2011	Guerra, M.L., Sorini, L., Stefanini, L. [31]	Option price sensitivities through fuzzy numbers	Computers and Mathematics with Applications	27

(a) Source: own elaboration based on WoS database. (b) Source: own elaboration based on the Scopus database.

Within the WoS database (Table 3a), the most cited paper was [21], which applied a fuzzy version of the BSM model to real options, followed by [15,43], which developed the valuation of European-style options with the BSM model on stocks. The papers [14,15,39,40,43,44,50] also fell within the scope of the BSM model and value European-style options, but some of them introduced new nuances related to nonlinear fuzzy numbers [14,40], defuzzification [39], sensitivity analysis of prices [14], or valuation of companies [50]. It was not until the tenth work [16] that we observed an analytical framework different from that provided by the geometric Brownian motion; specifically, it was a more general Levy stochastic process. In the eleventh cited contribution, ref. [38], a different type of option from the European binary options was evaluated. In later positions, there were several contributions whereby more alternative fuzzy stochastic modeling was used to model stock prices movements such as in [55,56], whereby the authors addressed the

jump-diffusion processes, and in [68], whereby the authors used fuzzy Levy processes. Additionally, noteworthy are the contributions of [23], who applied fuzzy regression in the adjustment of implied volatility, and [12], who provided a review of fuzzy random option pricing in both continuous and discrete time. The patterns observed in the SCOPUS database (Table 3b) were very similar to those in WoS, although there may have been small changes. Changes in the ranking were not very pronounced in any case. The top three most cited works continued to be [15,21,43]. However, starting from the fourth work, the order underwent subtle modifications. For example, in the SCOPUS database, ref. [50] was the fifth most referenced work instead of the fourth. On the other hand, ref. [44] went from being the fifth in SCOPUS to the fourth in WoS.

Table 4 shows that the authors who, as of March 2023, had the highest indexed production in WoS and SCOPUS in the field of FROPCT were Nowak (7 contributions), followed by Muzzioli, Romaniuk, and Guerra (4 contributions). These 4 authors were followed by 11 authors with 3 papers.

Table 4. Authors with at least 3 items.

Author	Country	Items	Author	Country	Items
Nowak, P.	Poland	7	Liu, S.	China	3
Muzzioli, S.	Italy	4	Pawlowski, M.	Poland	3
Romaniuk, M.	Poland	4	Sorini, L.	Italy	3
Guerra, M.L.	Italy	4	Stefanini, L.	Italy	3
Andres-Sanchez, J.	Spain	3	Thavaneswaran, A.	Canada	3
Appadoo, S.S.	Canada	3	Vilani, G.	Italy	3
de Baets, B.	Belgium	3	Wu, H.C.	Taiwan	3
Figa-Talamanca, G.	Italy	3			

Source: own elaboration based on WoS and Scopus databases.

3. Fuzzy Random Extension to Vasicek’s Equilibrium Term Model

3.1. Preliminary Questions

Table 1 shows that the main focus of fuzzy random option pricing in continuous time is on options on stocks, primarily European-style options and real options. However, we did not find any fuzzy random approaches to equilibrium models of yield rates or option pricing formulas derived from these models. We can only mention [48], which assumes a crisp model of the term structure by using the Cox–Ingersoll and Ross model [10] to determine the risk-free interest rate to evaluate credit default swaps. However, the observed short rate is considered a fuzzy number. This observation motivates the extension developed in this section, which applies the one-factor model for the yield curve by Vasicek [8] to incorporate fuzziness in the parameters. This extension is based on the following hypotheses:

1. The evolution of short rates can be described by using a mean-reverting process with a constant mean. This hypothesis is commonly employed in practical applications, such as in [77], whereby it was employed in the context of life insurance.
2. Uncertainty regarding the parameters governing interest rate movements can be represented by using fuzzy numbers. Following the concept commonly adopted in FROPCT, we consider fuzzy parameters as epistemic fuzzy numbers, assuming they are disjunctive sets [78].
3. A fuzzy number can represent an ill-defined deterministic value and provide a rough approximation of information about the population being observed [79]. This can be achieved through an appropriate transformation of a probability distribution function into a possibility distribution [22].
4. In the empirical application presented in this paper, the fuzzy parameters are constructed based on objective information obtained from bond markets.

The fuzzy extension of [8] presented in this section was applied to the prices of default-free zero-coupon bonds in European Union fixed income markets, which serve as the main

reference for investors of euro-denominated fixed-income assets and for the European Central Bank when assessing the evolution of long-term interest rates.

3.2. The Equilibrium Model of the Yield Curve by Vasicek

As with any one-factor model, in ref. [8] it is assumed that the stochastic variability comes from short-term interest rate (r) fluctuations. The ultimate goal of equilibrium models is to obtain the price that would be agreed upon at t for a risk-free zero-coupon bond that pays one monetary unit (u.m) at $T \geq t$, $P(t, T)$. In term structure models, stochastic variation does not directly apply to $P(t, T)$, as in the case of stocks, but rather to the interest rate (r), which is directly connected to $P(t, T)$. In general, one-factor models suppose that r follows an Ito process such as [8]:

$$dr = m(r, t)dt + \sigma(r, t)dz, \tag{1}$$

where $m(r, t)$ and $\sigma(r, t)$ are the instantaneous drift and variance, respectively, and dz is a Wiener process with a standard deviation of dt . The price of any asset affected by r (bonds, derivatives on fixed-income assets, etc.), P , must accomplish:

$$\frac{\partial P}{\partial t} + m(r, t) \frac{\partial P}{\partial r} dt + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} \sigma^2(r, t) = rP, \tag{2}$$

and thus, to obtain $P(t, T)$, we have to consider the condition $P(T, T) = 1$.

Among the many models proposed for dr (see Hull [80]), we extend the classical Vasicek's model [8]. It supposes that the fluctuations in the short interest rate follow a mean-reverting process:

$$dr = a(b - r)dt + \sigma dz, \tag{3}$$

where $a \geq 0$ is the reversion rate, i.e., the speed at which the interest rates return to equilibrium, and the b is the equilibrium short rate. By naming r_T the short interest rate in year T and r_t to that rate in $t \leq T$, the expectation of r_T in t $E_t(r_T)$ is

$$E_t(r_T) = b + (r_t - b)e^{-a(T-t)}, \tag{4a}$$

and the variance $V_t(r_T)$ is

$$V_t(r_T) = \frac{\sigma^2}{2a} \left(1 - e^{-2a(T-t)}\right). \tag{4b}$$

Note that for $T \rightarrow \infty$, $E_t(r_T) = b$, i.e., r_T is increasing with respect to b . Likewise, (4b) suggests that the variance in the short-term rate is affected by an exponential decay at rate a . That decreasing behavior can be easily checked by the limit because if $T \rightarrow \infty$, $V_t(r_T) = \frac{\sigma^2}{2a}$, and the long-term variance is decreasing with respect to a and of course increasing with respect to σ .

Therefore, in the mean-reverting groundwork (3), the general Equation (2) becomes

$$\frac{\partial P}{\partial t} + a(b - r) \frac{\partial P}{\partial r} dt + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} \sigma^2 = rC. \tag{5}$$

Thus, for the zero-coupon bond, we find

$$P(t, T, a, b, \sigma, r_t) = A(t, T, a, b, \sigma) e^{-B(t, T, a) \cdot r_t}, \tag{6a}$$

where

$$B(t, T, a) = \frac{1 - e^{-a(T-t)}}{a}, \tag{6b}$$

$$A(t, T, a, b, \sigma) = \exp \left[\frac{(B(t, T, a) - T + t) \left(a^2 b - \frac{\sigma^2}{2} \right)}{a^2} - \frac{\sigma^2 B^2(t, T, a)}{4a} \right]. \tag{6c}$$

Note that the fact that b and r_t are the discount rates of the zero-coupon bond implies that the partial derivatives must be $\frac{\partial P}{\partial b} < 0$ and $\frac{\partial P}{\partial r_t} < 0$. Likewise, the price of a bond is a decreasing function of the interest rate volatility because its profit is positively linked with volatility [81]. Therefore, $\frac{\partial P}{\partial \sigma} < 0$ and $\frac{\partial P}{\partial a} > 0$ since a basically negatively affects the volatility of the short-term interest rate (see (4b)).

3.3. An Extension of Vasicek’s Yield Curve with Fuzzy Parameters

In the following, we will suppose that with the exception of the maturities t and T , all the parameters of (6a)–(6c) are imprecise and are given by means of fuzzy numbers (FNs). A FN is a fuzzy set \tilde{f} defined on the reference set \mathbb{R} and is normal (i.e., $\max_{x \in \mathbb{R}} \mu_f(x) = 1$, where $\mu_f(x)$ is its membership function) and convex, i.e., all its α -cuts are convex and compact sets. Therefore, the fuzzy set can be represented as confidence intervals (so-called α -cuts or α -level sets) $f_\alpha = [\underline{f}_\alpha, \overline{f}_\alpha]$, where \underline{f}_α (\overline{f}_α) are continuously increasing (decreasing) functions of α . A FN can be interpreted as a fuzzy quantity that is approximately equal to the value $x \in \mathbb{R}$ whose membership function is one, f_1 .

Therefore, the parameters a, b, σ , and r_t are now the imprecise quantities $\tilde{a}, \tilde{b}, \tilde{\sigma}$, and \tilde{r}_t whose α -cuts are denoted as $a_\alpha = [\underline{a}_\alpha, \overline{a}_\alpha]$, $b_\alpha = [\underline{b}_\alpha, \overline{b}_\alpha]$, $\sigma_\alpha = [\underline{\sigma}_\alpha, \overline{\sigma}_\alpha]$ and $r_{t\alpha} = [\underline{r}_{t\alpha}, \overline{r}_{t\alpha}]$.

Therefore, under our hypothesis, the mean-reverting process (3) turns into a fuzzy random process where the parameters that rule the interest rate movements are fuzzy numbers. In the fuzzy random approach setting of the FOP, the differential Equation (5) has fuzzy parameters but a crisp boundary condition $P(T, T) = 1$.

Studies in the FROPCT literature obtain fuzzy option prices by evaluating the crisp pricing formula that comes from the assumed stochastic process (e.g., the BSM model after accepting that geometrical Brownian motion explains the price movements of subjacent assets) with fuzzy numbers by using the rules in [82]. This procedure is theoretically supported by the concept of the solution of differential equations with fuzzy coefficients [83]. Thus, in our case, the assumption of fuzzy parameters in (3) leads to the need to solve (6a)–(6c) with FNs. Therefore, the price of a zero-coupon bond turns into a fuzzy number $\tilde{P}(t, T) = P(t, T, \tilde{a}, \tilde{b}, \tilde{\sigma}, \tilde{r}_t)$ whose α -levels $P(t, T)_\alpha = [\underline{P}(t, T)_\alpha, \overline{P}(t, T)_\alpha]$ can be obtained by evaluating (6a)–(6c) in $a_\alpha, b_\alpha, \sigma_\alpha$, and $r_{t\alpha}$:

$$P(t, T)_\alpha = \left\{ x \mid x = P(t, T, a, b, \sigma, r_t) = A(t, T, a, b, \sigma) e^{-B(t, T, a) \cdot r_t}, a \in a_\alpha, b \in b_\alpha, \sigma \in \sigma_\alpha, r_t \in r_{t\alpha} \right\}, \tag{7a}$$

and given that $\frac{\partial P}{\partial b} < 0$, $\frac{\partial P}{\partial r_t} < 0$, $\frac{\partial P}{\partial \sigma} < 0$, and $\frac{\partial P}{\partial a} > 0$, we can obtain the extremes of the α -cuts by evaluating (6a)–(6c) in the extremes of $a_\alpha, b_\alpha, \sigma_\alpha$, and $r_{t\alpha}$ as

$$\underline{P}(t, T)_\alpha = P(t, T, \underline{a}_\alpha, \overline{b}_\alpha, \overline{\sigma}_\alpha, \overline{r}_{t\alpha}) = A(t, T, \underline{a}_\alpha, \overline{b}_\alpha, \overline{\sigma}_\alpha) e^{-B(t, T, \underline{a}_\alpha) \cdot \overline{r}_{t\alpha}}, \tag{7b}$$

and

$$\overline{P}(t, T)_\alpha = P(t, T, \overline{a}_\alpha, \underline{b}_\alpha, \underline{\sigma}_\alpha, \underline{r}_{t\alpha}) = A(t, T, \overline{a}_\alpha, \underline{b}_\alpha, \underline{\sigma}_\alpha) e^{-B(t, T, \overline{a}_\alpha) \cdot \underline{r}_{t\alpha}}. \tag{7c}$$

3.4. Empirical Application of Fuzzy Vasicek’s Model in the Public Debt Bond Market of Europe

The empirical application developed in this work estimates the theoretical yield curve by using the Vasicek model [8] for European public bonds with the highest rating (AAA) on April 18, 2023. The data we worked with were obtained from the website of the European Central Bank (<https://www.ecb.europa.eu/home/html/index.en.html> accessed on 18 April 2023). The short-term interest rate considered was the 3-month interest

rate, which was the lowest term published in the public bond market by the European Central Bank.

We will use the parameters in (3) as the fuzzy numbers $\tilde{a}, \tilde{b}, \tilde{\sigma}$ by using the probability–possibility criterion exposed in [22], in which these FN will be modeled by means of symmetrical fuzzy numbers, which we will also suppose are triangular. A triangular symmetrical fuzzy number (TSFN) can be represented as $\tilde{f} = (f_C, f_R)$, where f_C is the center of the FN and f_R is the spread. Its membership function is $\mu_f(x) = \max\left\{0, \frac{|x-f_C|}{f_R}\right\}$, and its α -cut representation $f_\alpha = [\underline{f}_\alpha, \overline{f}_\alpha] = [f_C - f_R(1 - \alpha), f_C + f_R(1 - \alpha)]$. Therefore, its support is $f_0 = [\underline{f}_0, \overline{f}_0] = [f_C - f_R, f_C + f_R]$. For example, for the equilibrium short-term rate $\tilde{b} = (3\%, 0.5\%)$, it can be interpreted that the most reliable value for that rate is 3%, but deviations of approximately 0.5% are viewed as possible.

To fit the parameters, we used the ground of the conventional AR(1) model that serves as the basis to empirically estimate (3) [7]. Therefore, the time series model to fit is [7]

$$r_{t+1} = \gamma + \beta r_t + \varepsilon_{t+1}, \quad t = 1, 2, \dots, n. \tag{8}$$

where γ is the intercept, β is the slope, and ε_{t+1} is the normal white noise and i.i.d. with mean 0 and standard deviation σ_ε . A commonly used methodology to adjust (8) is ordinary least squares. In this regard, it is easy to check that the relations between the parameters of (3), (4a), and (8) are $a = -\ln \beta \cdot h, b = \frac{\gamma}{1-\beta}$, and $\sigma = \sigma_\varepsilon \sqrt{h}$, where h is the frequency of the data. Thus, for daily data, $h = 252$.

To fit the temporal structure in the calendar date m , we adjusted (8) in the moments $i = 1, 2, \dots, m$ in all cases by using n observations. Therefore, for the i th adjustment of (8), we obtain point estimates in (8) $\hat{\gamma}^{(i)}, \hat{\beta}^{(i)}, \hat{\sigma}_\varepsilon^{(i)}, i = 1, 2, \dots, m$, and consequently, $\hat{a}^{(i)}, \hat{b}^{(i)}, \hat{\sigma}^{(i)}, i = 1, 2, \dots, m$.

After fitting the m AR(1) models (8), we can fit an empirical probability distribution for every parameter; for the parameter f , we symbolize it as \hat{f} , and its outcomes are $\hat{f}^{(i)}, i = 1, 2, \dots, m$. Therefore, we can calculate the mean and the standard deviation of \hat{f} simply as

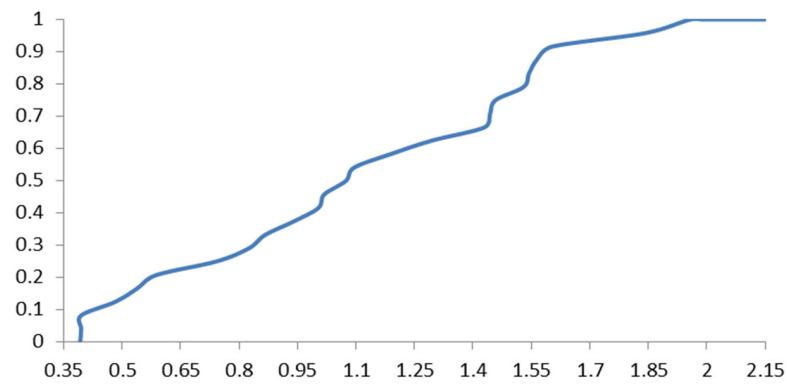
$$\hat{f}_{mean} = \frac{\sum_i \hat{f}^{(i)}}{m} \quad \text{and} \quad \hat{f}_{sd} = \sqrt{\frac{\sum_i (\hat{f}^{(i)} - \hat{f}_{mean})^2}{m}}. \tag{9}$$

A natural way to adjust a fuzzy number to a unimodal probability distribution f with mean \hat{f}_{mean} and standard deviation \hat{f}_{sd} is to fit an STFM $\tilde{f} = (f_C, f_R)$ with a center $f_C = \hat{f}_{mean}$ and fit the spread f_R by considering the Chebyshev inequality [22]. Specifically, given that

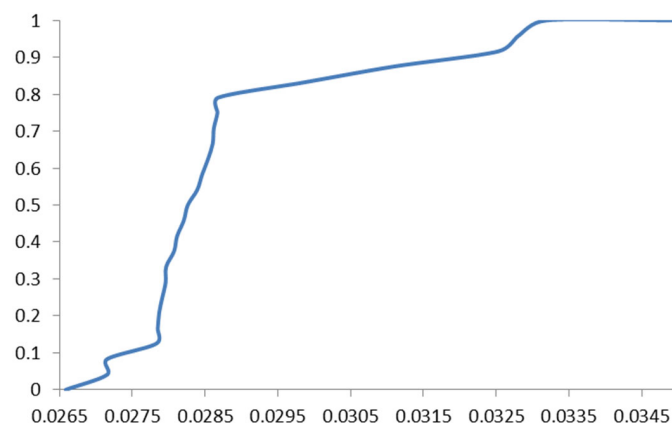
$$P\left(x \in [\hat{f}_{mean} - k \cdot \hat{f}_{sd}, \hat{f}_{mean} + k \cdot \hat{f}_{sd}]\right) \geq 1 - \frac{1}{2.25 \cdot k^2}, \quad k \geq 1, \tag{10}$$

where $P(\cdot)$ is a probability measure, and we choose f_R such that $P(x \in [\hat{f}_{mean} - f_R, \hat{f}_{mean} + f_R]) \geq 1 - \frac{1}{2.25 \cdot k^2}$, after fixing $k, f_R = k \cdot \hat{f}_{sd}$.

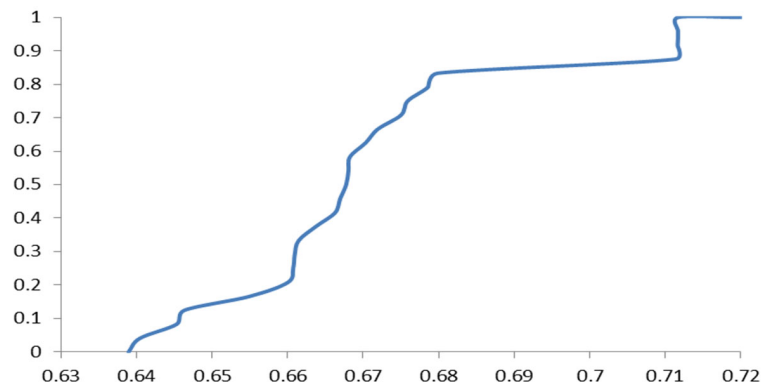
In our empirical application, to implement the set of regressions (8), we considered $n = 50$ and $s = 25$ so that all the observations from the year 2023 until 18 April were used. The data used had a daily periodicity, so $h = 252$. Figure 3a–c show the estimated empirical distribution functions for the parameters ruling the model. The coefficients must be quantified through fuzzy numbers that must be coherent with the empirical distribution function of these parameters. Although there are various criteria for this purpose [22], we used the one described in Equations (9) and (10). We found it interesting to analyze whether the observations were compatible with a unimodal probability distribution, as it was the underlying hypothesis in Equations (9) and (10). In this regard, by using a Chi-square test, we tested whether the observations of parameters a and σ were compatible with a normal probability distribution while parameter b was compatible with a Gamma distribution which, in any case, is also a unimodal distribution.



(a)



(b)



(c)

Figure 3. (a) Empirical distribution function of the speed of adjustment rate a ; (b) empirical distribution function of the equilibrium interest rate b ; (c) empirical distribution function of the parameter σ . Source: own elaboration-based data from the European Central Bank.

Table 5 displays the center of the fuzzy numbers \tilde{a} , \tilde{b} , and $\tilde{\sigma}$, as well as their possible spreads for $k = 2, 3, 4$, considering Equation (10). Note that there is no optimal value for k . On the one hand, a higher value of k results in more uncertain and imprecise estimates of zero-coupon bond prices, but on the other hand, it allows for a better fit of the real yield curve shapes.

Table 5. Estimates of \tilde{a} , \tilde{b} , and $\tilde{\sigma}$ on 18 April 2023.

		\hat{a}	\hat{b}	$\hat{\sigma}$
mean		1.117	0.02882	0.0067075
std. dev.		0.245	0.00174	0.0002121
k		\tilde{a}	\tilde{b}	$\tilde{\sigma}$
	center	1.117	0.02882	0.0067075
2	spread	0.491	0.00349	0.0004242
3	spread	0.736	0.00523	0.0006363
4	spread	0.982	0.00697	0.0008484

Source: own elaboration-based data from the European Central Bank.

Table 6 presents the observed prices of zero-coupon bonds with a face value of 100, ranging from 3 months to 20 years, as well as the fuzzy estimations of these prices using the proposed extension to [8]. At the given date, the 3-month interest rate was $r_0 = 2.877\%$. Despite one of the main criticisms of one-factor models with fixed parameters being their inability to capture all possible shapes of the term structure [7], the proposed fuzzy extension to [8] successfully generated predictions of zero-coupon bond prices that aligned with the observed prices. Therefore, ref. [8] can be considered valuable for explaining term structure equilibria and conducting economic analyses that require a parametric and parsimonious estimation of the yield rates. It can be observed that the 0 cut of Equations (7a) and (7b) adequately captured the prices of the zero-coupon bonds across various maturities, with the exception of the bonds maturing in 3, 4, and 5 years. However, even in these cases, the maximum deviation between the 0-cut predictions and the actual prices was consistently less than 0.5%. For instance, for the 4-year maturity bond, the deviation was $\frac{90.56-90.37}{90.56} = 0.002$.

Table 6. Spot prices of zero-coupon bonds with face value 100 monetary units on 18 April 2023 in the European Union public debt market and α -cuts of the estimates ($\alpha = 0, 0.5, 1$) from fuzzy Vasicek’s model.

T	Estimated by Vasicek’s Model					
	Observed	1-Cut	0.5-Cut		0-Cut	
	$P(0,T)$	$P(0,T)_1$	$\underline{P(0,T)}_{0.5}$	$\overline{P(0,T)}_{0.5}$	$\underline{P(0,T)}_0$	$\overline{P(0,T)}_0$
3 months	99.29	99.28	99.27	99.29	99.25	99.29
1 year	97.04	97.16	97.00	97.26	96.79	97.28
2 years	94.62	94.40	93.96	94.72	93.44	94.81
3 years	92.57	91.72	90.98	92.31	90.16	92.52
4 years	90.56	89.12	88.09	89.98	86.99	90.37
5 years	88.49	86.59	85.29	87.72	83.93	88.32
6 years	86.36	84.13	82.58	85.53	80.98	86.35
7 years	84.20	81.74	79.96	83.39	78.14	84.45
8 years	82.02	79.42	77.42	81.31	75.39	82.61
9 years	79.86	77.17	74.96	79.28	72.74	80.82
10 years	77.73	74.98	72.57	77.30	70.18	79.07
11 years	75.64	72.85	70.27	75.37	67.72	77.37
12 years	73.61	70.78	68.03	73.48	65.34	75.70
13 years	71.64	68.77	65.87	71.65	63.04	74.08
14 years	69.73	66.82	63.78	69.86	60.82	72.49
15 years	67.90	64.92	61.75	68.12	58.69	70.93
16 years	66.13	63.08	59.79	66.41	56.62	69.41
17 years	64.44	61.29	57.89	64.76	54.63	67.93

Table 6. *Cont.*

<i>T</i>	Estimated by Vasicek’s Model					
	Observed	1-Cut	0.5-Cut		0-Cut	
	$P(0,T)$	$P(0,T)_1$	$P(0,T)_{0.5}$	$\overline{P(0,T)}_{0.5}$	$\underline{P(0,T)}_0$	$\overline{P(0,T)}_0$
18 years	62.82	59.55	56.05	63.14	52.71	66.47
19 years	61.27	57.86	54.27	61.56	50.86	65.05
20 years	59.79	56.22	52.54	60.03	49.07	63.65

Note: The spread of parameters used in the estimation of prices was stated as $k = 4$. Likewise, the 3-month rate at that date was 2.877%. Source: own elaboration-based data from the European Central Bank.

4. Discussion and Implications

4.1. Discussion of Findings

This paper presents a double contribution. On the one hand, we systematically reviewed a set of contributions that we can label as fuzzy random option prices in continuous time (FROPCT), which is the dominant approach within fuzzy option pricing [13]. On the other hand, we extended the fuzzy random approach to model the term structure with an equilibrium model. We concretely extended the classical model [12] to the existence of imprecise parameters in the mean-reverting process.

We checked that journals of soft computing and fuzzy mathematics were the most burgeoning outlets of FROPCT. However, journals devoted to the wider fields of computational mathematics and operational research have also actively published papers on FROPCT. We checked that the contributions to FROPCT grew continuously from early 2000 to 2013. In the mid 2010s, the literature on FROPCT stopped having a growing constant production; thus, FROPCT can be labeled as a small but well-established branch of fuzzy mathematics.

The mainstream FROPCT models have imprecise knowledge of the parameters that govern the subjacent random movements of subjacent assets with type-1 fuzzy numbers that are often triangular and trapezoidal. More complex forms of uncertain quantities such as type-2 fuzzy numbers or intuitionistic fuzzy numbers are rarely applied.

The main applications of FROPCT have focused on the development of the valuation of stocks on stock and stock market indexes. Most developments take the analytical framework of the Black–Scholes–Merton (BSM) formula as a reference [3,4], which is based on the consideration of the geometric Brownian motion of the subjacent asset price. However, the literature has also provided other approaches with more complex stochastic processes such as jump diffusion [55,56], stochastic variance [57], fractional stochastic movements (e.g., [58,59,66]), or Levy processes (e.g., [64,65,71]). The development of real options with fuzziness in the parameters has been another relevant stream of FROPCT. The reviewed developments of fuzzy real options are based on the assumption of conventional geometric Brownian motion. Thus, when it comes to the modeling of the simplest options, the FROPCT literature uses the BSM framework (e.g., [21]), but in the case of options on options or compound options, the analytical frameworks that support contributions such as [19,20,42,51,52] are option-pricing models [73,74]. The use of fuzzy subset theory has been more extensive than the approach through FROPCT, as other tools such as fuzzy measures or fuzzy inference systems have been applied [13]. In our opinion, the combination of game theory and fuzzy inference systems in the assessment of real options can be particularly fruitful, as it has already been applied to similar business problems such as the establishment of hotel pricing policies within the Stackelberg game framework [84].

The no-arbitrage approach to option valuation initiated in [3,4] has been particularly fruitful when modeling the term structure of interest rates [7,80]. Surprisingly, we observed that developments by the FROPCT literature in this setting have been scarce, if not non-existent. In this regard, we highlight [85], which does not use a fuzzy random approach but rather uncertain Liu processes. This lack of the no-arbitrage approach has motivated the second contribution of the paper, in which we extended the equilibrium model of the

yield rate curve by Vasicek [8] to fuzziness in the reverting rate, long-term equilibrium short-term rate, and volatility.

A key aspect in the fuzzy random modeling of asset prices is the calibration of the fuzzy parameters that govern their variation [7]. Naturally, since the parameters that govern the movements of short-term interest rates have a clear economic interpretation, it could be assumed that they can be set intuitively by experts. Alternatively, following [14,17,18,23,25–27,57,71], we can adjust the parameters that are assumed to be fuzzy numbers based on existing evidence in financial markets. Likewise, it should be noted that the existence of studies that empirically apply FROPCT developments is relatively scarce [12,13]. Both considerations motivate the parameter adjustment methodology of the mean reversion process outlined in this study, which is also a novelty. The speed of the return to equilibrium, the asymptotic short-term interest rate, and the volatility are fitted by using symmetric triangular fuzzy numbers that are obtained by combining the conventional autoregressive modeling of the Ornstein–Uhlenbeck process [7] and the application of a consistent probability–possibility transformation criterion [22].

The empirical application developed in the European public debt market suggests that the extension of model [8] can reasonably capture the prices of zero-coupon bonds for all analyzed maturities (up to 20 years); this is performed by using only the fuzzy random modeling of the 3-month interest rate as the input with a fuzzy stochastic process with mean reversion and constant volatility. This finding is in accordance with the reviewed literature. In Spanish option markets, it has been observed that the introduction of fuzzy uncertainty on the parameters governing market prices yields good estimates of observed prices of valued assets [25,26]. Similarly, in the option market of Shanghai [18] and over options on Standard and Poor’s index [18,57,71], similar results have been obtained.

4.2. Practical Implications

In our opinion, this work presents various consequences both from a theoretical perspective and from the perspective of financial practice. From an academic point of view, we provided an overview of the contributions of FROPCT to the field of option theory, which allows for the identification of research gaps. In this regard, we can highlight the additional use of comprehensive fuzzy modeling of uncertainty over type-1 fuzzy numbers, the further development of FROPCT in equilibrium models of the term structure and the valuation of interest-sensitive instruments, or the increased applications of FROPCT with empirical data from financial derivative markets.

The results of this paper have practical implications as well. We observed the viability of a parametric interest rate model such as Vasicek [8], which has wide practical applications, such as modeling the returns obtained by life insurance companies [77] while considering the fact that the uncertainty governing the movement of interest rates is modeled through fuzzy numbers. The proposed extension, which allows for the uncertainty of risk-free interest rates to be captured with fuzzy numbers based on market data, is particularly applicable in fuzzy capital budgeting [86]. Furthermore, the implementation of the presented developments and their interpretation from the perspective of a nonexpert professional in fuzzy logic is straightforward. For the interpretation of the results, while the endpoints of the 0 cut provide an understanding of the values of zero-coupon bonds in the most extreme scenarios regarding equilibrium interest rates, the equilibrium return ratio, and volatility, the 1 cut quantifies the price in the most plausible scenario. In all cases, the calculations were performed by using formulas widely known to practitioners.

We believe that the proposed methodology of adjusting the parameters governing the movement of short-term interest rates also has potential implications. We constructed a method for estimating the fuzzy parameters of the mean-reverting process which, starting from the widely used autoregressive modeling and a coherent probability–possibility transformation criterion, allows us to capture the uncertainty of the parameters governing the mean-reversion process through fuzzy numbers. Of course, this method can be ap-

plied to any variable whose fluctuations may exhibit mean reversion, such as commodity prices [87].

5. Conclusions and Further Research

Given that the majority of the literature in the field of FROPCT has introduced uncertainty when using type-1 fuzzy numbers, the application of such more complex fuzzy numbers in FROPCT may be a natural and fruitful research field. However, it must also be taken into account that the introduction of these more sophisticated forms of imprecise quantities could also be a source of drawbacks. Note that defining their shape requires estimating more parameters than for conventional fuzzy numbers, and their arithmetical handle is less parsimonious in such a way that the developments of FROPCT with this type of uncertainty may be more difficult to put into practice.

The fuzzy random extension of the model [8] has been applied in an empirical application in the European market for public debt bonds with the highest credit rating. We are aware that one of the main criticisms of [8] is that the model allows negative interest rates that, from an economic perspective, have no meaning. However, the empirical evidence in European fixed-income markets, such as that of public debt assets, contradicts this alleged disadvantage because in 2010, the internal rate of returns was consistently negative.

Of course, there are better alternatives to equilibrium term structure models if the only objective is to obtain the best adjustment of the yield curve. For example, econometric models usually provide better results, and some of them have been implemented with fuzzy regression [88]. However, in our empirical application, we were interested in demonstrating that the extension of the FROPCT in [8], which is a parsimonious parametric model, can be useful in further analytical developments, not only because of its good properties and ease of interpretation but also because it is coherent with empirical evidence.

A natural extension of this paper involves extending the possibility of fuzziness in the coefficients that govern the movement of interest rates to other yield curve models based on arbitrage arguments; these models can be either single factor [10] or multifactorial [9], and this extension can occur in continuous time, such as in [10], or in discrete time [11]. It should be noted that while models with fixed parameters, such as those in [8–10], which were built simply on the basis of a no-arbitrage argument, do not necessarily provide a perfect fit for the term structure, they are very useful in a wide variety of economic and financial analyses. Conversely, models that are referred to as consistent with variable parameters such as [11] require a greater number of parameters to be estimated, which allows them to perfectly fit the zero-coupon yield curve existing on a particular date; however, they are less parsimonious, and their application is usually limited to the valuation of interest-rate-sensitive instruments such as swaptions, options on bonds, cap and floor options, etc.

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