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Double domination in rooted product graphs

Abel Cabrera-Martínez [a], Alejandro Estrada-Moreno [b,*]

[a] Departamento de Matematica, Universidad de Cordoba, Campus de Rabanales, Cordoba, 14071, Andalusia, Spain
[b] Departament d'Enginyeria Informatica i Matematiques, Universitat Rovira i Virgili, Av. Paisos Catalans, 26, Tarragona, 43007, Catalonia, Spain

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ABSTRACT

A set D of vertices of a graph G is a double dominating set of G if $|N[v] \cap D| \geq 2$ for every $v \in V(G)$, where $N[v]$ represents the closed neighbourhood of v. The double domination number of G is the minimum cardinality among all double dominating sets of G. In this article, we show that if G and H are graphs with no isolated vertex, then for any vertex $v \in V(H)$ there are six possible expressions, in terms of domination parameters of the factor graphs, for the double domination number of the rooted product graph $G \circ_v H$. Additionally, we characterize the graphs G and H that satisfy each of these expressions.

1. Introduction

Domination in graphs is well studied in graph theory and the literature on this subject has been surveyed and detailed in the books [8,9]. In [7], Harary and Haynes extended the idea of domination in graphs to a more general notion. They introduced the concept of double domination in graphs and, more generally, the concept of k-tuple domination. Given a graph G of minimum degree $\delta(G)$ and a positive integer $k \leq \delta(G) + 1$, a set $D \subseteq V(G)$ is said to be a *k-tuple dominating set* of G if $|N[v] \cap D| \geq k$ for every vertex $v \in V(G)$, where $N[v]$ represents the closed neighbourhood of v. The *k-tuple domination number* of G, denoted by $\gamma_{\times k}(G)$, is the minimum cardinality among all k-tuple dominating sets of G. A $\gamma_{\times k}(G)$-set is a k-tuple dominating set of cardinality $\gamma_{\times k}(G)$. The cases $k = 1$ and $k = 2$ correspond to domination and double domination, respectively. In such a case, $\gamma(G)$ and $\gamma_{\times 2}(G)$ denote the domination number and the double domination number of graph G, respectively.

The double domination in graphs has been extensively studied. In [6], Hansberg and Volkmann put into context all relevant research results on double domination that have been found up to 2020. In addition, we suggest the recent papers [1–5,10]. However, this classic domination parameter has not yet been studied in rooted product graphs. With this work we pretend to solve this gap in the theory.

The aim of this paper is to study this parameter in rooted product graphs. In Section 2 we provide some necessary notation and tools. In Section 3 we obtain closed formulas for the double domination number of rooted product graphs. In particular, we show that this parameter in rooted product graph can be expressed in terms of different domination parameters of the factor graphs involved in the product. Finally, we characterize the graphs that satisfy each of the six expressions obtained.

* Corresponding author.
E-mail address: alejandro.estrada@urv.cat (A. Estrada-Moreno).

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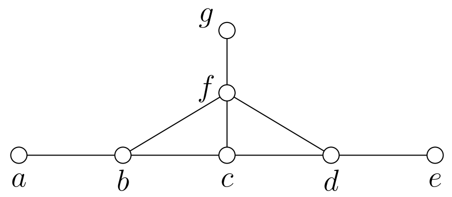

Fig. 1. A graph G with $\gamma_2(G) = 4 < \gamma_{q\times 2}(G) < 6 = \gamma_{\times 2}(G)$.

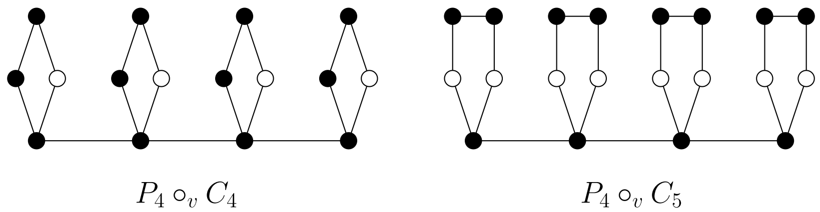

Fig. 2. The set of black-coloured vertices forms a $\gamma_{\times 2}(P_4 \circ_v C_i)$-set, for $i \in \{4, 5\}$.

2. Some necessary notation and tools

Given a graph G, a vertex of degree one is referred as a *leaf* and its unique neighbour is called a *support vertex*. A *strong leaf* is a leaf which is at distance two from another leaf. We use the notation $\mathcal{L}(G)$, $\mathcal{L}_s(G)$ and $\mathcal{S}(G)$ for the sets of leaves, strong leaves and support vertices, respectively. Also $n(G)$ will denote the order of G. Given a vertex $x \in V(G)$, (for simplicity) the subgraph induced by $V(G) \setminus \{x\}$ will be denoted by $G - x$. By analogy, we define the subgraph $G - S$ for an arbitrary set $S \subseteq V(G)$.

For any graph G with no isolated vertex, let $\mathcal{D}_{\times 2}(G)$ the set of double dominating sets (DDSs) of G, and let $\mathcal{D}_2(G)$ be the set of 2-dominating sets of G. Recall that a set D is a 2-*dominating set* of G if $|N(v) \cap D| \geq 2$ for every vertex $v \in V(G) \setminus D$. The 2-*domination number* of G, denoted by $\gamma_2(G)$, is the minimum cardinality among all 2-dominating sets of G.

We now proceed to introduce a new domination parameter which plays an important role in one of the possible values that the double domination number of the rooted product graphs takes.

Let $A, B \subseteq V(G)$. We say that an ordered pair (A, B) of disjoint sets A and B is a *quasi-double dominating pair* of G if $A \cup B \in \mathcal{D}_2(G)$ and $B \in \mathcal{D}_{\times 2}(G - A)$. The *quasi-double domination number* of G, denoted by $\gamma_{q\times 2}(G)$, is defined to be

$$\gamma_{q\times 2}(G) = \min\{|A| + |B| : A \cup B \in \mathcal{D}_2(G) \text{ and } B \in \mathcal{D}_{\times 2}(G - A)\}.$$

A $\gamma_{q\times 2}(G)$-pair is a quasi-double dominating pair (A, B) which satisfies the condition $\gamma_{q\times 2}(G) = |A| + |B|$.

Remark 1. For any graph G with no isolated vertex,

$$\gamma_2(G) \leq \gamma_{q\times 2}(G) \leq \gamma_{\times 2}(G).$$

Proof. Since $A \cup B \in \mathcal{D}_2(G)$ for any $\gamma_{q\times 2}(G)$-pair (A, B), the lower bound is straightforward. Moreover, if D is a $\gamma_{\times 2}(G)$-set, then the ordered pair $(\varnothing, D)$ is a quasi-double dominating pair of G. Therefore, $\gamma_{q\times 2}(G) \leq |D| = \gamma_{\times 2}(G)$, which completes the proof. □

Fig. 1 shows an example of a graph G where the value of $\gamma_{q\times 2}(G)$ does not reach the extreme values of the bounds given in *Remark* 1. For instance, $\{a, c, e, g\}$ is a $\gamma_2(G)$-set, $\{a, b, d, e, f, g\}$ is a $\gamma_{\times 2}(G)$-set and $(\{g\}, \{a, b, d, e\})$ is a $\gamma_{q\times 2}(G)$-pair.

We assume that the reader is familiar with the basic concepts and notation. If this is not the case, we suggest the textbooks [8,9]. For the remainder of the paper, definitions will be introduced whenever a concept is needed.

3. Double domination in rooted product graphs

Let G and H be two graphs with no isolated vertex and v be a vertex of H, which we called *root vertex*. The rooted product graph $G \circ_v H$ is defined as the graph obtained from G and H by taking one copy of G and $n(G)$ copies of H and identifying the ith vertex of G with the root vertex v in the ith copy of H for every $i \in \{1, 2, \ldots, n(G)\}$. Fig. 2 shows the rooted product graphs $P_4 \circ_v C_4$ and $P_4 \circ_v C_5$.

For every vertex $x \in V(G)$, $H_x \cong H$ will denote the copy of H in $G \circ_v H$ containing x. The restriction of any set $D \subseteq V(G \circ_v H)$ to $V(H_x)$ will be denoted by D_x, i.e., $D_x = D \cap V(H_x)$. Since $V(G \circ_v H) = \cup_{x\in V(G)} V(H_x)$, we have that $|D| = \sum_{x\in V(G)} |D_x|$ for every set $D \subseteq V(G \circ_v H)$.

The following result gives a first approximation for the double domination number of a rooted product graph.

Proposition 1. *The following statements hold for any graphs G and H with no isolated vertex and any vertex $v \in V(H)$.*

(i) $\gamma_{\times 2}(G \circ_v H) \leq n(G)\gamma_{\times 2}(H)$.
(ii) *If* $v \in \mathcal{S}(H)$, *then* $\gamma_{\times 2}(G \circ_v H) = n(G)\gamma_{\times 2}(H)$.
(iii) *If* $v \in V(H) \setminus \mathcal{S}(H)$, *then* $\gamma_{\times 2}(G \circ_v H) \leq \gamma_{\times 2}(G) + n(G)\gamma_{\times 2}(H - v)$.

Proof. Let S be a $\gamma_{\times 2}(H)$-set and $D \subseteq V(G \circ_v H)$ such that D_x is the subset of $V(H_x)$ induced by S for every $x \in V(G)$. Observe that D is a DDS of $G \circ_v H$, which implies that $\gamma_{\times 2}(G \circ_v H) \leq |D| = \sum_{x \in V(G)} |D_x| = n(G)|S| = n(G)\gamma_{\times 2}(H)$. Therefore, (i) follows.

Now, we proceed to prove (ii) and (iii). Let W be a $\gamma_{\times 2}(G \circ_v H)$-set. First, we assume that $v \in \mathcal{S}(H)$. Let $x \in V(G)$. Notice that W_x is a DDS of H_x as $(N(x) \cap \mathcal{L}(H_x)) \cup \{x\} \subseteq W_x$, which implies that $|W_x| \geq \gamma_{\times 2}(H_x) = \gamma_{\times 2}(H)$. Hence, $\gamma_{\times 2}(G \circ_v H) = |W| = \sum_{x \in V(G)} |W_x| \geq n(G)\gamma_{\times 2}(H)$. The equality follows by (i), which completes the proof of (ii).

Finally, assume that $v \in V(H) \setminus \mathcal{S}(H)$. Let S' be a $\gamma_{\times 2}(H - v)$-set and $D' \subseteq V(G \circ_v H) \setminus V(G)$ such that D'_x is the subset of $V(H_x - x)$ induced by S' for every $x \in V(G)$. For any $\gamma_{\times 2}(G)$-set X, it follows that $X \cup D'$ is a DDS of $G \circ_v H$. Hence, $\gamma_{\times 2}(G \circ_v H) \leq |X \cup D'| = \gamma_{\times 2}(G) + n(G)\gamma_{\times 2}(H - v)$, which completes the proof of (iii). □

As a consequence of Proposition 1, we will focus our study on rooted product graphs where the root vertex is not a support vertex. Furthermore, we will assume that the two vertices of the path graph P_2 are support vertices.

Following Proposition 1-(iii) we have an upper bound on $\gamma_{\times 2}(G \circ_v H)$ in terms of $\gamma_{\times 2}(H - v)$. For that reason, in the next lemma we analyse the relationship between $\gamma_{\times 2}(H - v)$ and $\gamma_{\times 2}(H)$.

Lemma 1. *Let H be a graph with no isolated vertex. If $v \in V(H) \setminus \mathcal{S}(H)$, then*

$$\gamma_{\times 2}(H - v) \geq \gamma_{\times 2}(H) - 2.$$

Furthermore, if $\gamma_{\times 2}(H - v) = \gamma_{\times 2}(H) - 2$, then $N(v) \cap S = \varnothing$ for every $\gamma_{\times 2}(H - v)$-set S.

Proof. Let $v \in V(H) \setminus \mathcal{S}(H)$ and S be a $\gamma_{\times 2}(H - v)$-set. For every $u \in N(v)$, the set $S \cup \{u, v\}$ is a DDS of H. Hence, $\gamma_{\times 2}(H) - 2 \leq |S \cup \{u, v\}| - 2 \leq |S| = \gamma_{\times 2}(H - v)$, as desired.

We now assume that $|S| = \gamma_{\times 2}(H) - 2$. If there exists a vertex $u \in N(v) \cap S$, then $S \cup \{v\}$ is a DDS of H, which is a contradiction as $|S \cup \{v\}| = \gamma_{\times 2}(H) - 1$. Therefore, $N(v) \cap S = \varnothing$, which completes the proof. □

Lemma 2. *For every $\gamma_{\times 2}(G \circ_v H)$-set D and $x \in V(G)$,*

$$|D_x| \geq \gamma_{\times 2}(H) - 2.$$

Furthermore, if $|D_x| = \gamma_{\times 2}(H) - 2$, then $N[x] \cap D_x = \varnothing$.

Proof. Let D be a $\gamma_{\times 2}(G \circ_v H)$-set and $x \in V(G)$. Notice that for every $y \in N(x) \cap V(H_x)$, the set $D'_x = D_x \cup \{x, y\}$ is a DDS of H_x. Hence, $\gamma_{\times 2}(H) - 2 = \gamma_{\times 2}(H_x) - 2 \leq |D'_x| - 2 \leq |D_x|$, as desired.

Now, assume that $|D_x| = \gamma_{\times 2}(H) - 2$. Suppose there exists a vertex $y \in N[x] \cap D_x$ and let $z \in N(x) \cap V(H_x)$. Next, we define a set $D'_x \subseteq V(H_x)$ as follows. If $y = x$ then $D'_x = D_x \cup \{z\}$, otherwise $D'_x = D_x \cup \{x\}$. Notice that D'_x is a DDS of H_x, which is a contradiction as $|D'_x| = \gamma_{\times 2}(H) - 1$. Hence, $N[x] \cap D_x = \varnothing$, which completes the proof. □

From Lemma 2 we deduce that any $\gamma_{\times 2}(G \circ_v H)$-set D generates three subsets $\mathcal{A}_D$, $\mathcal{B}_D$, $\mathcal{C}_D$ of $V(G)$ as follows.

$$\mathcal{A}_D = \{x \in V(G) : |D_x| \geq \gamma_{\times 2}(H)\}$$
$$\mathcal{B}_D = \{x \in V(G) : |D_x| = \gamma_{\times 2}(H) - 1\}$$
$$\mathcal{C}_D = \{x \in V(G) : |D_x| = \gamma_{\times 2}(H) - 2\}.$$

In order to determine all possible values of $\gamma_{\times 2}(G \circ_v H)$, the following lemma is a tool that allows us to bound and give a value for $\gamma_{\times 2}(G \circ_v H)$ in the case that $\mathcal{B}_D \neq \varnothing$ and $\mathcal{C}_D = \varnothing$ for a $\gamma_{\times 2}(G \circ_v H)$-set D.

Lemma 3. *Given a $\gamma_{\times 2}(G \circ_v H)$-set D with $\mathcal{B}_D \neq \varnothing$ and $\mathcal{C}_D = \varnothing$, the following statements hold.*

(i) *If* $\mathcal{B}_D \cap D \neq \varnothing$, *then* $\gamma_{\times 2}(G \circ_v H) = n(G)(\gamma_{\times 2}(H) - 1)$.
(ii) *If* $\mathcal{B}_D \cap D = \varnothing$, *then* $\gamma_{\times 2}(H - v) = \gamma_{\times 2}(H) - 1$, *and as a consequence,*

$$\gamma(G) + n(G)(\gamma_{\times 2}(H) - 1) \leq \gamma_{\times 2}(G \circ_v H) \leq \gamma_{q \times 2}(G) + n(G)(\gamma_{\times 2}(H) - 1).$$

Proof. First, we proceed to prove (i). Let $z \in \mathcal{B}_D \cap D$ and $S \subseteq V(G \circ_v H)$, where S_x is the subset of $V(H_x)$ induced by D_z for every $x \in V(G)$. Notice that S is a DDS of $G \circ_v H$, which implies that $\gamma_{\times 2}(G \circ_v H) \leq |S| = \sum_{x \in V(G)} |S_x| = n(G)|D_z| = n(G)(\gamma_{\times 2}(H) - 1)$. Moreover, since $\mathcal{C}_D = \varnothing$, we deduce that $\gamma_{\times 2}(G \circ_v H) = |D| \geq n(G)(\gamma_{\times 2}(H) - 1)$ and so, (i) follows.

Next, we proceed to prove (ii). Assume that $\mathcal{B}_D \cap D = \varnothing$, and let $z \in \mathcal{B}_D$. Since $z \notin D$, we deduce that $z \notin \mathcal{S}(H_z)$, which implies that D_z is a DDS of $H_z - z$. Hence, $\gamma_{\times 2}(H - v) = \gamma_{\times 2}(H_z - z) \leq |D_z| = \gamma_{\times 2}(H) - 1$.

By inequality above and Lemma 1 we have that $\gamma_{\times 2}(H-v) \in \{\gamma_{\times 2}(H)-2, \gamma_{\times 2}(H)-1\}$. If $\gamma_{\times 2}(H-v) = \gamma_{\times 2}(H)-2$, then by Proposition 1-(iii) and the condition $\mathcal{C}_D = \varnothing$, we obtain that $n(G)(\gamma_{\times 2}(H)-1) \leq \gamma_{\times 2}(G \circ_v H) \leq \gamma_{\times 2}(G)+n(G)(\gamma_{\times 2}(H)-2)$, which implies that $\gamma_{\times 2}(G \circ_v H) = n(G)(\gamma_{\times 2}(H)-1)$, and as a consequence, $V(G) = \mathcal{B}_D$, which contradicts the fact that D is a DDS of $G \circ_v H$ because $\mathcal{B}_D \cap D = \varnothing$. Hence, $\gamma_{\times 2}(H-v) = \gamma_{\times 2}(H)-1$, as desired.

Now, we proceed to prove the upper bound. Let (A, B) be a $\gamma_{q\times 2}(G)$-pair, S' a $\gamma_{\times 2}(H-v)$-set and S'' a $\gamma_{\times 2}(H)$-set. We construct a set $W \subseteq V(G \circ_v H)$ as follows. If $x \in A$, then W_x is the subset of $V(H_x)$ induced by S''; while if $x \in V(G) \setminus A$, then $W_x = W'_x$, where W'_x is the subset of $V(H_x - x)$ induced by S'. Notice that $W \cup B$ is a DDS of $G \circ_v H$, which implies that $\gamma_{\times 2}(G \circ_v H) \leq |W \cup B| = \gamma_{q\times 2}(G) + n(G)(\gamma_{\times 2}(H)-1)$.

In order to prove the lower bound, notice that $|N(x) \cap D_x| \leq 1$ for every $x \in \mathcal{B}_D$, which implies that $\mathcal{A}_D$ is a dominating set of G. Hence,

$$\begin{aligned}\gamma_{\times 2}(G \circ_v H) &= \sum_{x\in\mathcal{A}_D} |D_x| + \sum_{x\in\mathcal{B}_D} |D_x| \\ &\geq |\mathcal{A}_D|\gamma_{\times 2}(H) + |\mathcal{B}_D|(\gamma_{\times 2}(H)-1) \\ &= |\mathcal{A}_D| + n(G)(\gamma_{\times 2}(H)-1) \\ &\geq \gamma(G) + n(G)(\gamma_{\times 2}(H)-1),\end{aligned}$$

which completes the proof. □

Now, we proceed to give the six possible expressions that can take $\gamma_{\times 2}(G \circ_v H)$ in terms of domination parameters of its factor graphs. It may be noted that we have ordered the possible values reached by $\gamma_{\times 2}(G \circ_v H)$ in non-increasing order for convenience.

Theorem 1. *Let G and H be two graphs with no isolated vertex. If $v \in V(H)$, then*

$$\gamma_{\times 2}(G \circ_v H) \in \left\{\begin{array}{l} n(G)\gamma_{\times 2}(H), \\ \gamma_{q\times 2}(G) + n(G)(\gamma_{\times 2}(H)-1), \\ \gamma_2(G) + n(G)(\gamma_{\times 2}(H)-1), \\ \gamma(G) + n(G)(\gamma_{\times 2}(H)-1), \\ n(G)(\gamma_{\times 2}(H)-1), \\ \gamma_{\times 2}(G) + n(G)(\gamma_{\times 2}(H)-2) \end{array}\right\}.$$

Proof. Let D be a $\gamma_{\times 2}(G \circ_v H)$-set and consider the three subsets $\mathcal{A}_D$, $\mathcal{B}_D$, $\mathcal{C}_D$ of $V(G)$ defined above. Observe that $\mathcal{A}_D \cup \mathcal{B}_D \cup \mathcal{C}_D = V(G)$. Now, we distinguish the following cases.

Case 1. $\mathcal{B}_D \cup \mathcal{C}_D = \varnothing$. In this case, we have that $V(G) = \mathcal{A}_D$, i.e, every vertex of G satisfies that $|D_x| \geq \gamma_{\times 2}(H)$ and, as a consequence, $\gamma_{\times 2}(G \circ_v H) \geq n(G)\gamma_{\times 2}(H)$. Therefore, by Proposition 1-(i) we have that $\gamma_{\times 2}(G \circ_v H) = n(G)\gamma_{\times 2}(H)$.

Case 2. $\mathcal{C}_D = \varnothing$ and $\mathcal{B}_D \neq \varnothing$. If $\mathcal{B}_D \cap D \neq \varnothing$, then by Lemma 3-(i) we obtain that $\gamma_{\times 2}(G \circ_v H) = n(G)(\gamma_{\times 2}(H)-1)$. From now on, we assume that $\mathcal{B}_D \cap D = \varnothing$. So, by Lemma 3-(ii) we have the following inequality chain.

$$\gamma(G) + n(G)(\gamma_{\times 2}(H)-1) \leq \gamma_{\times 2}(G \circ_v H) \leq \gamma_{q\times 2}(G) + n(G)(\gamma_{\times 2}(H)-1). \tag{1}$$

Now, observe that if $x \in \mathcal{B}_D$, then $|N(x) \cap D_x| \leq 1$. Hence, we introduce the following subsets of $\mathcal{B}_{\mathcal{D}}$. Let $\mathcal{B}'_D = \{x \in \mathcal{B}_D : |N(x) \cap D_x| = 1\}$ and $\mathcal{B}''_D = \mathcal{B}_D \setminus \mathcal{B}'_D$, i.e., $\mathcal{B}''_D = \{x \in \mathcal{B}_D : |N(x) \cap D_x| = 0\}$. Also, let $\mathcal{A}'_D = \{x \in \mathcal{A}_D : |D_x| = \gamma_{\times 2}(H)\}$ and $\mathcal{A}''_D = \mathcal{A}_D \setminus \mathcal{A}'_D$. Notice that $\mathcal{A}''_D \subseteq D$.

Next, we distinguish the following three subcases.

Subcase 2.1. $\mathcal{B}'_D \neq \varnothing$. Let $z \in \mathcal{B}'_D$ and $W \subseteq V(G \circ_v H) \setminus V(G)$ such that W_x is the subset of $V(H_x - x)$ induced by D_z for every $x \in V(G)$. Notice that for any $\gamma(G)$-set X, the set $X \cup W$ is a DDS of $G \circ_v H$. Since $|W| = n(G)|D_z| = n(G)(\gamma_{\times 2}(H)-1)$, it follows that $\gamma_{\times 2}(G \circ_v H) \leq |X \cup W| = |X| + |W| = \gamma(G) + n(G)(\gamma_{\times 2}(H)-1)$. Therefore, we conclude that $\gamma_{\times 2}(G \circ_v H) = \gamma(G) + n(G)(\gamma_{\times 2}(H)-1)$ by inequality chain (1).

Subcase 2.2. $\mathcal{B}_D = \mathcal{B}''_D$ and there exists $z \in \mathcal{A}'_D$ such that D_z is a $\gamma_{\times 2}(H_z)$-set containing z. From a fixed vertex $y \in \mathcal{B}_D$ and any $\gamma_2(G)$-set X, we can construct a set $W \subseteq V(G \circ_v H)$ as follows. If $x \in X$, then W_x is induced by D_z, while if $x \in V(G) \setminus X$, then W_x is induced by D_y. Notice that W is a DDS of $G \circ_v H$, which implies that $\gamma_{\times 2}(G \circ_v H) \leq |W| = \gamma_2(G) + n(G)(\gamma_{\times 2}(H)-1)$.

Moreover, since $\mathcal{B}_D = \mathcal{B}''_D$, it follows that $\mathcal{B}_D \cap D = \varnothing$. This previous condition and the fact that $\mathcal{C}_D = \varnothing$ lead to $D \cap V(G) \subseteq \mathcal{A}_D$ and $D \cap V(G) \in \mathcal{D}_2(G)$, which implies that $\gamma_2(G) \leq |D \cap V(G)| \leq |\mathcal{A}_D|$. Thus,

$$\begin{aligned}\gamma_{\times 2}(G \circ_v H) &= \sum_{x\in\mathcal{A}_D} |D_x| + \sum_{x\in\mathcal{B}_D} |D_x| \\ &\geq |\mathcal{A}_D|\gamma_{\times 2}(H) + |\mathcal{B}_D|(\gamma_{\times 2}(H)-1) \\ &= |\mathcal{A}_D| + n(G)(\gamma_{\times 2}(H)-1) \\ &\geq \gamma_2(G) + n(G)(\gamma_{\times 2}(H)-1).\end{aligned}$$

Therefore, $\gamma_{\times 2}(G \circ_v H) = \gamma_2(G) + n(G)(\gamma_{\times 2}(H)-1)$.

Subcase 2.3. $\mathcal{B}_D = \mathcal{B}''_D$ and also, $\mathcal{A}'_D = \varnothing$ or for any $x \in \mathcal{A}'_D$, either D_x is not a $\gamma_{\times 2}(H_x)$-set or $x \notin D_x$. Notice that if $\mathcal{A}'_D \neq \varnothing$, then every vertex $x \in \mathcal{A}'_D$ satisfies one of the following conditions.

(a) D_x is a $\gamma_{\times 2}(H_x)$-set such that $x \notin D_x$.
(b) D_x is not a DDS of H_x and $x \in D_x$.
(c) D_x is not a DDS of H_x and $x \notin D_x$.

From now on we do not consider condition (c). Note that if this condition holds, then we can always obtain, from (a) and (b), a $\gamma_{\times 2}(G \circ_v H)$-set that satisfies only these first two conditions.

Let us construct a set $X \subset V(G)$ of minimum cardinality as follows.

- $\mathcal{A}_D \subseteq X$.
- $N(x) \cap X \neq \varnothing$ for any $x \in \mathcal{A}''_D$ with $N(x) \cap D \cap V(G) = \varnothing$.

Since $\mathcal{B}_D = \mathcal{B}''_D$, $\mathcal{B}_D \cap D = \varnothing$ and $\mathcal{C}_D = \varnothing$, it follows that $X \in \mathcal{D}_2(G)$. Let $I'_D = \{x \in \mathcal{A}'_D : x$ satisfies the condition (a)$\}$. Observe that $X \setminus I'_D \in \mathcal{D}_{\times 2}(G - I'_D)$. This implies that $(I'_D, X \setminus I'_D)$ is a quasi-double dominating pair of G. Therefore, $\gamma_{q\times 2}(G) \leq |I'_D| + |X \setminus I'_D| = |X| \leq 2|\mathcal{A}''_D| + |\mathcal{A}'_D|$. Thus,

$$\begin{aligned}\gamma_{\times 2}(G \circ_v H) &= \sum_{x \in \mathcal{A}''_D} |D_x| + \sum_{x \in \mathcal{A}'_D} |D_x| + \sum_{x \in \mathcal{B}_D} |D_x| \\ &\geq |\mathcal{A}''_D|(\gamma_{\times 2}(H) + 1) + |\mathcal{A}'_D|\gamma_{\times 2}(H) + |\mathcal{B}_D|(\gamma_{\times 2}(H) - 1) \\ &= (2|\mathcal{A}''_D| + |\mathcal{A}'_D|) + n(G)(\gamma_{\times 2}(H) - 1) \\ &\geq \gamma_{q\times 2}(G) + n(G)(\gamma_{\times 2}(H) - 1),\end{aligned}$$

Therefore, we conclude that $\gamma_{\times 2}(G \circ_v H) = \gamma_{q\times 2}(G) + n(G)(\gamma_{\times 2}(H) - 1)$ by inequality chain (1).

Case 3. $\mathcal{C}_D \neq \varnothing$. Let $x \in \mathcal{C}_D$. By Lemma 2 we have that $N[x] \cap D_x = \varnothing$. This condition implies that the root vertex v is not a support vertex of H, i.e., $v \in V(H) \setminus \mathcal{S}(H)$, and as a consequence, D_x is a DDS of $H_x - x$. Hence, $\gamma_{\times 2}(H - v) = \gamma_{\times 2}(H_x - x) \leq |D_x| = \gamma_{\times 2}(H) - 2$. Therefore, Lemma 1 leads to $\gamma_{\times 2}(H - v) = \gamma_{\times 2}(H) - 2$.

In order to prove that $\gamma_{\times 2}(G \circ_v H) = \gamma_{\times 2}(G) + n(G)(\gamma_{\times 2}(H) - 2)$, by Proposition 1-(iii) and the fact that $\gamma_{\times 2}(H - v) = \gamma_{\times 2}(H) - 2$ we only need to prove that $|D| \geq \gamma_{\times 2}(G) + n(G)(\gamma_{\times 2}(H) - 2)$. For this, we define a set $S \subseteq V(G)$ of minimum cardinality as follows.

- $\mathcal{B}_D \cup \mathcal{A}_D \subseteq S$.
- $N(x) \cap S \neq \varnothing$ for any $x \in \mathcal{A}_D$ such that $N(x) \subseteq \mathcal{C}_D$.

We claim that S is a DDS of G. If $x \in \mathcal{C}_D$, then $N[x] \cap D_x = \varnothing$. Hence, $|N(x) \cap V(G) \cap D| \geq 2$, which implies that $|N(x) \cap S| \geq 2$ because $V(G) \cap D \subseteq S$. If $x \in \mathcal{A}_D$, then $|N(x) \cap S| \geq 1$ by definition of S. Finally, we assume that $x \in \mathcal{B}_D$. Since D_x is not a DDS of H_x, then either $N(x) \cap D_x = \varnothing$ or $|N(x) \cap D_x| = 1$ and $x \notin D$. In both cases it follows that $|N(x) \cap V(G) \cap D| \geq 1$, and so, $|N(x) \cap S| \geq 1$. Therefore, S is a DDS of G, as desired. Thus, $\gamma_{\times 2}(G) \leq |S| \leq 2|\mathcal{A}_D| + |\mathcal{B}_D|$, and so,

$$\begin{aligned}\gamma_{\times 2}(G \circ_v H) &= \sum_{x \in \mathcal{A}_D} |D_x| + \sum_{x \in \mathcal{B}_D} |D_x| + \sum_{x \in \mathcal{C}_D} |D_x| \\ &\geq |\mathcal{A}_D|\gamma_{\times 2}(H) + |\mathcal{B}_D|(\gamma_{\times 2}(H) - 1) + |\mathcal{C}_D|(\gamma_{\times 2}(H) - 2) \\ &= (2|\mathcal{A}_D| + |\mathcal{B}_D|) + n(G)(\gamma_{\times 2}(H) - 2) \\ &\geq \gamma_{\times 2}(G) + n(G)(\gamma_{\times 2}(H) - 2),\end{aligned}$$

which completes the proof. □

Now, we proceed to show some simple examples in which we can observe that the expressions of $\gamma_{\times 2}(G \circ_v H)$ given in Theorem 1 are realizable.

Example 1. Let G be a graph with no isolated vertex. If H is C_4, C_5 or one of the graphs shown in Fig. 3, then the resulting values of $\gamma_{\times 2}(G \circ_v H)$ for some specific roots are described below.

(i) $\gamma_{\times 2}(G \circ_v C_4) = 3n(G) = n(G)\gamma_{\times 2}(C_4)$ (See Fig. 2).
(ii) $\gamma_{\times 2}(G \circ_v C_5) = 3n(G) = n(G)(\gamma_{\times 2}(C_5) - 1)$ (See Fig. 2).
(iii) $\gamma_{\times 2}(G \circ_v H_1) = \gamma_{\times 2}(G) + 2n(G) = \gamma_{\times 2}(G) + n(G)(\gamma_{\times 2}(H_1) - 2)$.
(iv) $\gamma_{\times 2}(G \circ_v H_2) = \gamma(G) + 2n(G) = \gamma(G) + n(G)(\gamma_{\times 2}(H_2) - 1)$.
(v) $\gamma_{\times 2}(G \circ_v H_3) = \gamma_2(G) + 2n(G) = \gamma_2(G) + n(G)(\gamma_{\times 2}(H_3) - 1)$.
(vi) $\gamma_{\times 2}(G \circ_v H_4) = \gamma_{q\times 2}(G) + 2n(G) = \gamma_{q\times 2}(G) + n(G)(\gamma_{\times 2}(H_4) - 1)$.

Now, we characterize the graphs with $\gamma_{\times 2}(G \circ_v H) = \gamma_{\times 2}(G) + n(G)(\gamma_{\times 2}(H) - 2)$. Before, we need the following lemma.

Lemma 4. *Let D be a $\gamma_{\times 2}(G \circ_v H)$-set and assume that $\gamma_{\times 2}(G) < n(G)$. Then $\mathcal{C}_D \neq \varnothing$ if and only if $\gamma_{\times 2}(H - v) = \gamma_{\times 2}(H) - 2$.*

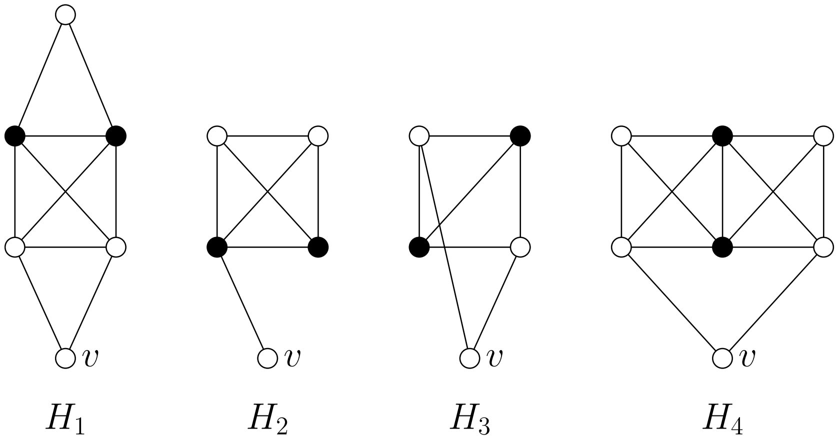

Fig. 3. The set of black-coloured vertices forms a $\gamma_{\times 2}(H_i - v)$-set, for $i \in \{1, 2, 3, 4\}$.

Proof. By Lemma 2, if $x \in \mathcal{C}_D$, then $N[x] \cap D_x = \varnothing$, which implies that D_x is a DDS of $H_x - x$ since $x \in V(H_x) \setminus \mathcal{S}(H_x)$. Hence, $\gamma_{\times 2}(H - v) = \gamma_{\times 2}(H_x - x) \le |D_x| = \gamma_{\times 2}(H) - 2$, and the equality follows by Lemma 1.

On the other side, if $\gamma_{\times 2}(H-v) = \gamma_{\times 2}(H)-2$, then by Proposition 1-(iii) we obtain that $|D| \le \gamma_{\times 2}(G)+n(G)(\gamma_{\times 2}(H)-2)$. Therefore, as $\gamma_{\times 2}(G) < n(G)$, by Lemma 2 we have $\mathcal{C}_D \neq \varnothing$. □

Theorem 2. *If $\gamma_{\times 2}(G) < n(G)$, then the following statements are equivalent.*

(i) $\gamma_{\times 2}(G \circ_v H) = \gamma_{\times 2}(G) + n(G)(\gamma_{\times 2}(H) - 2)$.
(ii) $\gamma_{\times 2}(H - v) = \gamma_{\times 2}(H) - 2$.

Proof. Let D be a $\gamma_{\times 2}(G \circ_v H)$-set. By Lemma 4, we deduce that (ii) is equivalent to $\mathcal{C}_D \neq \varnothing$, and by Case 3 of the proof of Theorem 1 we obtain that (i) holds. On the other side, if (i) holds then $|D| = \gamma_{\times 2}(G) + n(G)(\gamma_{\times 2}(H) - 2) < n(G)+n(G)(\gamma_{\times 2}(H)-2) = n(G)(\gamma_{\times 2}(H)-1)$, which leads to $\mathcal{C}_D \neq \varnothing$. Therefore, by Lemma 4 we deduce that (ii) holds and the proof is complete. □

Next, we characterize the graphs with $\gamma_{\times 2}(G \circ_v H) = n(G)(\gamma_{\times 2}(H) - 1)$. Before, we need to introduce the following well-known definition. A subset $D \subseteq V(G)$ is a *total dominating set* (TDS) of a graph G if $N(v) \cap D \neq \varnothing$ for every vertex $v \in V(G)$.

Theorem 3. *Let G and H be two graphs with no isolated vertex and $v \in V(H)$. Then $\gamma_{\times 2}(G \circ_v H) = n(G)(\gamma_{\times 2}(H) - 1)$ if and only if one of the following conditions holds.*

(i) $\gamma_{\times 2}(G) = n(G)$ *and* $\gamma_{\times 2}(H - v) = \gamma_{\times 2}(H) - 2$.
(ii) $\gamma_{\times 2}(H - v) \ge \gamma_{\times 2}(H) - 1$ *and there exists a set* $W \subseteq V(H) \setminus N[v]$ *of cardinality* $\gamma_{\times 2}(H) - 2$ *which is both a DDS of* $H - N[v]$ *and a TDS of* $H - v$.

Proof. We first assume that $\gamma_{\times 2}(G \circ_v H) = n(G)(\gamma_{\times 2}(H) - 1)$. Let D be a $\gamma_{\times 2}(G \circ_v H)$-set. By Proposition 1-(ii) we deduce that $v \in V(H) \setminus \mathcal{S}(H)$. Also, Lemma 1 leads to $\gamma_{\times 2}(H - v) \ge \gamma_{\times 2}(H) - 2$. We consider the next two cases.

Case 1. $\gamma_{\times 2}(H - v) = \gamma_{\times 2}(H) - 2$. In this case, by Proposition 1-(iii) we obtain that $n(G)(\gamma_{\times 2}(H) - 1) = |D| \le \gamma_{\times 2}(G) + n(G)(\gamma_{\times 2}(H) - 2)$, which implies that $\gamma_{\times 2}(G) = n(G)$. Therefore, (i) follows.

Case 2. $\gamma_{\times 2}(H-v) \ge \gamma_{\times 2}(H)-1$. If $\mathcal{C}_D \neq \varnothing$, then applying a procedure analogous to that of proof of Lemma 4, we deduce that $\gamma_{\times 2}(H - v) = \gamma_{\times 2}(H) - 2$, which is a contradiction. Hence, $\mathcal{C}_D = \varnothing$, which implies that $\mathcal{B}_D = V(G)$. So, by Lemma 3 there exists a vertex $x \in V(G) \cap D$. If $N(x) \cap D_x \neq \varnothing$, then D_x is a DDS of H_x, which is a contradiction as $|D_x| = \gamma_{\times 2}(H) - 1$. Thus, $N(x) \cap D_x = \varnothing$. From all the above, it follows that $D_x \setminus \{x\} \subseteq V(H_x) \setminus N[x]$ has cardinality $\gamma_{\times 2}(H) - 2$, and it is both a DDS of $H_x - N[x]$ and a TDS of $H_x - x$. Hence, (ii) holds as $H \cong H_x$.

On the other side, we assume that one of the conditions (i) and (ii) holds. First, we suppose that (i) holds. By Proposition 1-(iii) and the equality $\gamma_{\times 2}(H - v) = \gamma_{\times 2}(H) - 2$ we obtain that $\gamma_{\times 2}(G \circ_v H) \le \gamma_{\times 2}(G) + n(G)(\gamma_{\times 2}(H) - 2)$. Moreover, Theorem 1 and the equality $\gamma_{\times 2}(G) = n(G)$ lead to $\gamma_{\times 2}(G \circ_v H) = \gamma_{\times 2}(G)+n(G)(\gamma_{\times 2}(H)-2) = n(G)(\gamma_{\times 2}(H)-1)$, as desired. Finally, we suppose that (ii) holds. So, there exists a set $W \subseteq V(H) \setminus N[v]$ of cardinality $\gamma_{\times 2}(H) - 2$ which is both a DDS of $H - N[v]$ and a TDS of $H - v$. Let $S \subseteq V(G \circ_v H)$ such that for every $x \in V(G)$ the set S_x is induced by $W \cup \{x\}$. Notice that S is a DDS of $G \circ_v H$, which implies that $\gamma_{\times 2}(G \circ_v H) \le |S| = n(G)(|W| + 1) = n(G)(\gamma_{\times 2}(H) - 1)$. Now, by Theorem 1 we deduce that $\gamma_{\times 2}(G \circ_v H) \in \{n(G)(\gamma_{\times 2}(H) - 1), \gamma_{\times 2}(G) + n(G)(\gamma_{\times 2}(H) - 2)\}$. If $\gamma_{\times 2}(G) = n(G)$, then we are done. Moreover, if $\gamma_{\times 2}(G) < n(G)$, then Theorem 2 and the inequality $\gamma_{\times 2}(H-v) \geq \gamma_{\times 2}(H) - 1$ lead to $\gamma_{\times 2}(G \circ_v H) \neq \gamma_{\times 2}(G) + n(G)(\gamma_{\times 2}(H) - 2)$. Therefore, $\gamma_{\times 2}(G \circ_v H) = n(G)(\gamma_{\times 2}(H) - 1)$, which completes the proof. □

We now discuss the case in which the graph H and its root v satisfy the condition $\gamma_{\times 2}(H-v) \geq \gamma_{\times 2}(H)$.

Proposition 2. *Let G be a graph such that $\gamma_{\times 2}(G) < n(G)$. Let H be a graph with no isolated vertex and $v \in V(H) \setminus \mathcal{S}(H)$. If $\gamma_{\times 2}(H-v) \geq \gamma_{\times 2}(H)$, then*

$$\gamma_{\times 2}(G \circ_v H) \in \{n(G)\gamma_{\times 2}(H), n(G)(\gamma_{\times 2}(H) - 1)\}.$$

Proof. Let D be a $\gamma_{\times 2}(G \circ_v H)$-set. Theorem 1 leads to $|D| \leq n(G)\gamma_{\times 2}(H)$. If $|D| = n(G)\gamma_{\times 2}(H)$, then we are done. Now, let us suppose that $|D| < n(G)\gamma_{\times 2}(H)$. By Lemma 4 it follows that $\mathcal{C}_D = \varnothing$. If $\mathcal{B}_D = \varnothing$, then $V(G) = \mathcal{A}_D$ which contradicts the fact that $|D| < n(G)\gamma_{\times 2}(H)$. Hence, $\mathcal{B}_D \neq \varnothing$. By Lemma 3 and the fact that $\gamma_{\times 2}(H-v) \geq \gamma_{\times 2}(H)$, it follows that $\mathcal{B}_D \cap D \neq \varnothing$, and as a consequence, $|D| = n(G)(\gamma_{\times 2}(H) - 1)$, which completes the proof. □

The following result shows an example where $\gamma_{\times 2}(G \circ_v H) = n(G)\gamma_{\times 2}(H)$.

Proposition 3. *Let G be a graph such that $\gamma_{\times 2}(G) < n(G)$. Let H be a graph with no isolated vertex and $v \in V(H) \setminus \mathcal{S}(H)$. If $\gamma_{\times 2}(H-v) \geq \gamma_{\times 2}(H)$ and v does not belong to any $\gamma_{\times 2}(H)$-set, then*

$$\gamma_{\times 2}(G \circ_v H) = n(G)\gamma_{\times 2}(H).$$

Proof. Since $\gamma_{\times 2}(H-v) \geq \gamma_{\times 2}(H)$, then by Proposition 2 we have that $\gamma_{\times 2}(G \circ_v H) = n(G)\gamma_{\times 2}(H)$ or $\gamma_{\times 2}(G \circ_v H) = n(G)(\gamma_{\times 2}(H) - 1)$. Let D be a $\gamma_{\times 2}(G \circ_v H)$-set, and suppose that $|D| = n(G)(\gamma_{\times 2}(H) - 1)$. Since $\gamma_{\times 2}(G) < n(G)$, it is straightforward to see that $\mathcal{C}_D = \varnothing$, which implies that $\mathcal{B}_D \neq \varnothing$. Let $x \in \mathcal{B}_D$. By Lemma 3 and the fact that $\gamma_{\times 2}(H-v) \geq \gamma_{\times 2}(H)$, it follows that $x \in D$. So, from any $u \in N(x) \cap V(H_x)$, the set $D_x \cup \{u\}$ is a $\gamma_{\times 2}(H_x)$-set, which contradicts the fact that v does not belong to any $\gamma_{\times 2}(H)$-set because $H \cong H_x$. Therefore, $|D| \neq n(G)(\gamma_{\times 2}(H) - 1)$, and so $|D| = n(G)\gamma_{\times 2}(H)$, which completes the proof. □

Now, we characterize the graphs $G \circ_v H$ that satisfy $\gamma_{\times 2}(G \circ_v H) = n(G)\gamma_{\times 2}(H)$, under the condition that G satisfies $\gamma_{\times 2}(G) < n(G)$.

Theorem 4. *Let G be a graph such that $\gamma_{\times 2}(G) < n(G)$. Let H be a graph with no isolated vertex and $v \in V(H)$. Then $\gamma_{\times 2}(G \circ_v H) = n(G)\gamma_{\times 2}(H)$ if and only if either $v \in \mathcal{S}(H)$ or the following conditions hold.*

(i) *$\gamma_{\times 2}(H-v) \geq \gamma_{\times 2}(H)$.*
(ii) *If $H - N[v]$ has no isolated vertices, then every subset of $V(H) \setminus N[v]$ of cardinality $\gamma_{\times 2}(H) - 2$ is not a DDS of $H - N[v]$ or it is not a TDS of $H - v$.*

Proof. Let us consider that $\gamma_{\times 2}(G) < n(G)$. We first assume that $\gamma_{\times 2}(G \circ_v H) = n(G)\gamma_{\times 2}(H)$. If $v \in \mathcal{S}(H)$, then we are done. From now on we suppose that $v \in V(H) \setminus \mathcal{S}(H)$. By Lemma 1 and Theorem 2, it follows that $\gamma_{\times 2}(H-v) \geq \gamma_{\times 2}(H) - 1$. Now, if $\gamma_{\times 2}(H-v) = \gamma_{\times 2}(H) - 1$, then by Proposition 1-(iii) we deduce that $\gamma_{\times 2}(G \circ_v H) \leq \gamma_{\times 2}(G) + n(G)(\gamma_{\times 2}(H) - 1) < n(G)\gamma_{\times 2}(H)$, which is a contradiction. Therefore, $\gamma_{\times 2}(H-v) \geq \gamma_{\times 2}(H)$, which completes the proof of (i). Moreover, (ii) follows as a consequence of negating the second condition given in statement (ii) of Theorem 3.

Conversely, we assume that either $v \in \mathcal{S}(H)$ or statements (i) and (ii) hold. If $v \in \mathcal{S}(H)$, then by Proposition 1-(ii) we have that $\gamma_{\times 2}(G \circ_v H) = n(G)\gamma_{\times 2}(H)$. From now on we consider that (i) and (ii) hold. Let D be a $\gamma_{\times 2}(G \circ_v H)$-set. Since (ii) contradicts the condition (ii) of Theorem 3, it follows that $|D| \neq n(G)(\gamma_{\times 2}(H) - 1)$. Finally, by (i) and Proposition 2 it follows that $|D| = n(G)\gamma_{\times 2}(H)$, which completes the proof. □

Next, we characterize the graphs satisfying $\gamma_{\times 2}(G \circ_v H) = \gamma(G) + n(G)(\gamma_{\times 2}(H) - 1)$.

Theorem 5. *Let G and H be two graphs with no isolated vertex and $v \in V(H)$. Then $\gamma_{\times 2}(G \circ_v H) = \gamma(G) + n(G)(\gamma_{\times 2}(H) - 1)$ if and only if the following conditions hold.*

(i) *$\gamma_{\times 2}(H-v) = \gamma_{\times 2}(H) - 1$ and one of the following conditions holds.*
    (a) *$\gamma(G) = \gamma_2(G)$ and there exists a $\gamma_{\times 2}(H)$-set containing the vertex v.*
    (b) *$\gamma(G) < \gamma_2(G)$ and there exists a $\gamma_{\times 2}(H-v)$-set W such that $N(v) \cap W \neq \varnothing$.*
(ii) *If $H - N[v]$ has no isolated vertices, then every subset of $V(H) \setminus N[v]$ of cardinality $\gamma_{\times 2}(H) - 2$ is not a DDS of $H - N[v]$ or it is not a TDS of $H - v$.*

Proof. We first assume that $\gamma_{\times 2}(G \circ_v H) = \gamma(G) + n(G)(\gamma_{\times 2}(H) - 1)$. Observe that (ii) follows as a consequence of Theorems 2 and 3. Now, we proceed to prove (i). Let D be a $\gamma_{\times 2}(G \circ_v H)$-set. If $\mathcal{C}_D \neq \varnothing$, then proceeding as in Case 3 of the proof of Theorem 1 we deduce that $\gamma_{\times 2}(G \circ_v H) = \gamma_{\times 2}(G) + n(G)(\gamma_{\times 2}(H) - 2)$, which is a contradiction. Hence, $\mathcal{C}_D = \varnothing$, and as

$\gamma(G) < n(G)$, it follows that $\mathcal{B}_D \neq \varnothing$. So, Lemma 3 leads to $\mathcal{B}_D \cap D = \varnothing$, and as a consequence, $\gamma_{\times 2}(H - v) = \gamma_{\times 2}(H) - 1$. Let $x \in \mathcal{B}_D$ such that $|N(x) \cap D_x|$ is maximum. Now, we analyse the following two cases.

Case 1. $|N(x) \cap D_x| = 0$. By the maximality of $|N(x) \cap D_x|$ we have that $|N(u) \cap D_u| = 0$ for every $u \in \mathcal{B}_D$. Now, we observe that

$$\begin{aligned}\gamma_{\times 2}(G \circ_v H) &= \gamma(G) + n(G)(\gamma_{\times 2}(H) - 1)\\ &= \sum_{x \in \mathcal{A}_D} |D_x| + \sum_{x \in \mathcal{B}_D} |D_x|\\ &\geq |\mathcal{A}_D|\gamma_{\times 2}(H) + |\mathcal{B}_D|(\gamma_{\times 2}(H) - 1)\\ &= |\mathcal{A}_D| + n(G)(\gamma_{\times 2}(H) - 1)\end{aligned}$$

Hence, $|\mathcal{A}_D| \leq \gamma(G) \leq \gamma_2(G)$. Since $\mathcal{B}_D \cap D = \varnothing$ and $N(x) \cap D_x = \varnothing$ for every $x \in \mathcal{B}_D$, it follows that $\mathcal{A}_D \cap D \neq \varnothing$, $\mathcal{A}_D$ is a $\gamma_2(G)$-set of cardinality $\gamma(G)$ and $|D_x| = \gamma_{\times 2}(H)$ for every $x \in \mathcal{A}_D$. So, $\mathcal{A}_D$ is an independent set of G, which implies that for some $x \in \mathcal{A}_D \cap D$, the set D_x is a $\gamma_{\times 2}(H_x)$-set containing the vertex x. Therefore, (a) holds, which implies that (i) follows.

Case 2. $|N(x) \cap D_x| > 0$. In this case, it is straightforward that D_x is a $\gamma_{\times 2}(H_x - x)$-set such that $N(x) \cap D_x \neq \varnothing$ and $D_x \cup \{x\}$ is a $\gamma_{\times 2}(H_x)$-set containing the vertex x. Therefore, either (a) or (b) holds, which implies that (i) follows.

On the other side, we assume that conditions (i) and (ii) hold. By Theorems 1–3 we deduce that $\gamma_{\times 2}(G \circ_v H) \geq \gamma(G) + n(G)(\gamma_{\times 2}(H) - 1)$. Let X be a $\gamma_2(G)$-set, Z be a $\gamma(G)$-set and Y be a $\gamma_{\times 2}(H - v)$-set.

We first suppose that (a) holds, i.e., $\gamma(G) = \gamma_2(G)$ and there exists a $\gamma_{\times 2}(H)$-set W containing the vertex v. From W, X and Y, we define a set $S \subseteq V(G \circ_v H)$ as follows. If $x \in X$, then S_x is induced by W; and if $x \in V(G) \setminus X$, then S_x is induced by Y. Notice that S is a DDS of $G \circ_v H$, which implies that $\gamma_{\times 2}(G \circ_v H) \leq |S| = |X| + n(G)|W| = \gamma(G) + n(G)(\gamma_{\times 2}(H) - 1)$. Therefore, $\gamma_{\times 2}(G \circ_v H) = \gamma(G) + n(G)(\gamma_{\times 2}(H) - 1)$, as required.

Finally, suppose that (b) holds, i.e., $\gamma(G) < \gamma_2(G)$ and there exists a $\gamma_{\times 2}(H - v)$-set W' such that $N(v) \cap W' \neq \varnothing$. Recall that $|W'| = \gamma_{\times 2}(H) - 1$. From W' and Z, we define a set $S \subseteq V(G \circ_v H)$ as follows. If $x \in Z$, then S_x is induced by $W' \cup \{x\}$; and if $x \in V(G) \setminus Z$, then S_x is induced by W'. Notice that S is a DDS of $G \circ_v H$, which implies that $\gamma_{\times 2}(G \circ_v H) \leq |S| = |Z| + n(G)|W'| = \gamma(G) + n(G)(\gamma_{\times 2}(H) - 1)$. Therefore, $\gamma_{\times 2}(G \circ_v H) = \gamma(G) + n(G)(\gamma_{\times 2}(H) - 1)$, which completes the proof. □

The following proposition shows a particular example where $\gamma_{\times 2}(G \circ_v H) = \gamma(G) + n(G)(\gamma_{\times 2}(H) - 1)$.

Proposition 4. *Let G and H be two graphs with no isolated vertex. If $v \in \mathcal{L}_s(H)$, then*

$$\gamma_{\times 2}(G \circ_v H) = \gamma(G) + n(G)(\gamma_{\times 2}(H) - 1).$$

Proof. Let X be a $\gamma_{\times 2}(H)$-set. Since $\mathcal{S}(H) \cup \mathcal{L}(H) \subseteq X$ and $v \in \mathcal{L}_s(H)$, it is easy to deduce that $X \setminus \{v\}$ is a $\gamma_{\times 2}(H - v)$-set, and so $\gamma_{\times 2}(H - v) = |X \setminus \{v\}| = \gamma_{\times 2}(H) - 1$. Hence, statements given in Theorem 5 hold (notice that $H - N[v]$ has at least one isolated vertex), which implies that $\gamma_{\times 2}(G \circ_v H) = \gamma(G) + n(G)(\gamma_{\times 2}(H) - 1)$, as desired. □

Next, we characterize the graphs with $\gamma_{\times 2}(G \circ_v H) = \gamma_2(G) + n(G)(\gamma_{\times 2}(H) - 1)$.

Theorem 6. *Let G be a graph with no isolated vertex such that $\gamma(G) < \gamma_2(G) < \gamma_{q\times 2}(G)$. Let H be a graph with no isolated vertex and $v \in V(H)$. Then $\gamma_{\times 2}(G \circ_v H) = \gamma_2(G) + n(G)(\gamma_{\times 2}(H) - 1)$ if and only if the following conditions hold.*

(i) *$\gamma_{\times 2}(H - v) = \gamma_{\times 2}(H) - 1$.*
(ii) *$N(v) \cap W = \varnothing$ for every $\gamma_{\times 2}(H - v)$-set W.*
(iii) *There exists a $\gamma_{\times 2}(H)$-set containing the vertex v.*
(iv) *If $H - N[v]$ has no isolated vertices, then every subset of $V(H) \setminus N[v]$ of cardinality $\gamma_{\times 2}(H) - 2$ is not a DDS of $H - N[v]$ or it is not a TDS of $H - v$.*

Proof. We first assume that $\gamma_{\times 2}(G \circ_v H) = \gamma_2(G) + n(G)(\gamma_{\times 2}(H) - 1)$. Observe that (iv) follows as a consequence of Theorems 2 and 3. Moreover, (ii) holds as an immediate consequence of condition (i)–(b) given in Theorem 5. Now, we proceed to prove the conditions (i) and (iii). Let D be a $\gamma_{\times 2}(G \circ_v H)$-set. If $\mathcal{C}_D \neq \varnothing$, then proceeding as in Case 3 of the proof of Theorem 1 we deduce that $\gamma_{\times 2}(G \circ_v H) = \gamma_{\times 2}(G) + n(G)(\gamma_{\times 2}(H) - 2)$, which is a contradiction. Hence, $\mathcal{C}_D = \varnothing$, and as $\gamma_2(G) < n(G)$, it follows that $\mathcal{B}_D \neq \varnothing$. So, Lemma 3 leads to $\mathcal{B}_D \cap D = \varnothing$, and as a consequence, $\gamma_{\times 2}(H - v) = \gamma_{\times 2}(H) - 1$, i.e., (i) follows. Finally, we observe that

$$\begin{aligned}\gamma_{\times 2}(G \circ_v H) &= \gamma_2(G) + n(G)(\gamma_{\times 2}(H) - 1)\\ &= \sum_{x \in \mathcal{A}_D} |D_x| + \sum_{x \in \mathcal{B}_D} |D_x|\\ &\geq |\mathcal{A}_D|\gamma_{\times 2}(H) + |\mathcal{B}_D|(\gamma_{\times 2}(H) - 1)\\ &= |\mathcal{A}_D| + n(G)(\gamma_{\times 2}(H) - 1)\end{aligned}$$

Hence $|\mathcal{A}_D| \leq \gamma_2(G)$. In addition, since $\mathcal{C}_D = \emptyset$, $\mathcal{B}_D \cap D = \emptyset$ and $N(x) \cap D_x = \emptyset$ for every $x \in \mathcal{B}_D$, it follows that $\mathcal{A}_D \cap D \neq \emptyset$, $\mathcal{A}_D$ is a $\gamma_2(G)$-set and $|D_x| = \gamma_{\times 2}(H)$ for every $x \in \mathcal{A}_D$. Now, if D_x is not a $\gamma_{\times 2}(H_x)$-set for every $x \in \mathcal{A}_D \cap D$, then $\mathcal{A}_D \cap D \in \mathcal{D}_{\times 2}(G - \mathcal{A}_D \setminus D)$. This implies that $(\mathcal{A}_D \setminus D, \mathcal{A}_D \cap D)$ is a quasi-double dominating pair of G. Hence, $\gamma_{q\times 2}(G) \leq |\mathcal{A}_D| = \gamma_2(G)$, which contradicts the fact that $\gamma_2(G) < \gamma_{q\times 2}(G)$. Therefore, there exists a vertex $x \in \mathcal{A}_D \cap D$ such that D_x is a $\gamma_{\times 2}(H_x)$-set containing the vertex x, which implies that (iii) follows as $H_x \cong H$.

On the other side, we assume that conditions (i)-(iv) hold. By Theorems 1, 2, 3 and 5 we deduce that $\gamma_{\times 2}(G \circ_v H) \geq \gamma_2(G) + n(G)(\gamma_{\times 2}(H) - 1)$. Now, let X be a $\gamma_2(G)$-set, W be a $\gamma_{\times 2}(H)$-set containing the vertex v, and Y be a $\gamma_{\times 2}(H - v)$-set. From X, Y and W, we define a set $S \subseteq V(G \circ_v H)$ as follows. If $x \in X$, then S_x is induced by W; and if $x \in V(G) \setminus X$, then S_x is induced by Y. Notice that S is a DDS of $G \circ_v H$, which implies that $\gamma_{\times 2}(G \circ_v H) \leq |S| = |X| + n(G)|Y| = \gamma_2(G) + n(G)(\gamma_{\times 2}(H) - 1)$. Therefore, $\gamma_{\times 2}(G \circ_v H) = \gamma_2(G) + n(G)(\gamma_{\times 2}(H) - 1)$, which completes the proof. □

As shown in Theorem 1, there are six possible expressions for $\gamma_{\times 2}(G \circ_v H)$. Moreover, the graphs G and H (and the root $v \in V(H)$) that satisfy five of these expressions were characterized in Theorems 2, 3, 4, 5 and 6. For the case of the equality $\gamma_{\times 2}(G \circ_v H) = \gamma_{q\times 2}(G) + n(G)(\gamma_{\times 2}(H) - 1)$, the corresponding characterization can be derived by eliminating the previous ones from the family of all graphs G and H with no isolated vertices and roots v of H.

Data availability

No data was used for the research described in the article.

References

[1] A. Cabrera-Martínez, New bounds on the double domination number of trees, Discrete Appl. Math. 315 (2022) 97–103, http://dx.doi.org/10.1016/j.dam.2022.03.022.

[2] A. Cabrera Martínez, S. Cabrera García, J.A. Rodríguez-Velázquez, Double domination in lexicographic product graphs, Discrete Appl. Math. 284 (2020) 290–300, http://dx.doi.org/10.1016/j.dam.2020.03.045.

[3] A. Cabrera-Martínez, F.A. Hernández Mira, New bounds on the double total domination number of graphs, Bull. Malays. Math. Sci. Soc. 45 (2022) 443–453, http://dx.doi.org/10.1007/s40840-021-01200-0.

[4] A. Cabrera-Martínez, J.A. Rodríguez-Velázquez, A note on double domination in graphs, Discrete Appl. Math. 300 (2021) 107–111, http://dx.doi.org/10.1016/j.dam.2021.05.011.

[5] M. Hajian, N.J. Rad, A new lower bound on the double domination number of a graph, Discrete Appl. Math. 254 (2019) 280–282, http://dx.doi.org/10.1016/j.dam.2018.06.009.

[6] A. Hansberg, L. Volkmann, Multiple Domination, in: Developments in Mathematics, vol. 64, Springer, Cham, 2020, pp. 151–203.

[7] F. Harary, T.W. Haynes, Double domination in graphs, Ars Combin. 55 (2000) 201–213.

[8] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, in: Chapman and Hall/CRC Pure and Applied Mathematics Series, Marcel Dekker, Inc. New York, 1998.

[9] T. Haynes, S. Hedetniemi, P. Slater, Domination in Graphs: Volume 2: Advanced Topics, in: Chapman & Hall/CRC Pure and Applied Mathematics, Taylor & Francis, 1998.

[10] D.A. Mojdeh, B. Samadi, Total 2-domination number in digraphs and its dual parameter, Discuss. Math. Graph Theory 43 (3) (2023) 587–606, http://dx.doi.org/10.7151/dmgt.2387.