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Proportional clearing mechanisms in financial systems: An axiomatic approach

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ABSTRACT

We address the problem of clearing mutual obligations among agents when a financial network collapses. To do so, we adopt an axiomatic approach and provide the first comprehensive characterization of the rules based on the principle of proportionality, covering the entire domain of financial systems. While a previous attempt by Csóka and Herings (2021) tackled this issue in a context where agents have strictly positive initial endowments, we show that their properties do not fully capture the set of proportional rules when extended to the full financial systems' domain. To overcome this limitation, we introduce new properties that emphasize the value of equity of the firms in the network. We show that a clearing mechanism satisfies compatibility, limited liability, absolute priority, equity continuity, and non-manipulability by clones if and only if each agent receives a payment proportional to the value of their claims. This characterization holds in the framework studied by Csóka and Herings (2021).

1. Introduction

A financial system comprises a group of diverse agents or firms such as banks, hedge funds, and individual investors, each distinguished by their endowments and obligations towards other agents. In contrast to the traditional bankruptcy problem, where a single firm defaults, in this scenario, agents can act as both debtors and creditors, and the bankruptcy of one firm can trigger a domino effect of insolvencies, endangering the stability of the entire system. The collapse of Lehman Brothers in 2008 and the resulting financial market crisis provide a telling example of such contagion effects. Since then, the study of financial contagion has garnered increasing attention, with Eisenberg and Noe's (2001) work serving as a reference for further studies. For detailed reviews of this subject, we refer readers to Glasserman and Young (2016), Caccioli et al. (2018), and Jackson and Pernoud (2021).

When a financial network collapses, a central question is how to settle the mutual obligations between firms. This mutual liability problem[1] is tackled by means of financial rules that recommend, for each financial network, a set of clearing payment matrices, suggesting the monetary transfer from each node in the network to any other node. To address this problem, in this paper we adopt the axiomatic approach. Taking a normative standpoint, this approach provides valuable insights into selecting suitable mechanisms for resolving unstable financial networks, and more importantly, it allows for an easy justification of these mechanisms. In this regard, and in line with the evidence that the principle of proportionality is significant in practice,[2] it is worth looking into what normative foundations distinguish proportional financial rules. That is, the family of clearing mechanisms satisfying standard conditions in most insolvency laws such as payments bounded by liabilities (*claim boundedness*), limited liability of equity (*limited liability*), priority of creditors over stockholders (*absolute priority*), and proportional repayments of liabilities. Kıbrıs and Kıbrıs (2013), based on an argument supported by egalitarian and utilitarian social welfare considerations, offer a compelling argument why proportionality is preferred in current bankruptcy laws over the principles of equal awards or equal losses.

To our knowledge, Csóka and Herings (2021) is the only attempt at investigating what properties identify the set of proportional payment matrices, but focusing on those financial systems where all agents have strictly positive initial endowments or cash flows. From an economic perspective, this assumption excludes certain real-scenarios in which

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[1] This terminology was introduced by Groote Schaarsberg et al. (2018).
[2] See, for instance, Regulation (EU) 2015/848 of the European Parliament and of the Council of 20 May 2015 on Insolvency Proceedings.
[3] Even in this scenario, unlike the case of the proportional rule, the uniqueness of compatible financial rules is not guaranteed. An illustrative example is the financial rule based on the constrained equal awards bankruptcy rule.

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initial endowments might become negative due to operating costs. As argued by Eisenberg and Noe (2001, p. 238) these negative operating costs can be interpreted as the sum of liabilities to outside factors as workers, suppliers or other debt holders. One approach to address this concern could be to introduce a fictitious agent with no operating cash flow of its own and no obligations to other agents, while negative initial endowments would be liabilities to such fictitious node. Therefore, any financial systems, regardless of its initial endowments (whether negative or positive), can be translated into a system where all agents have zero or positive initial endowments, by internalizing external liabilities. On the other hand, in Csóka and Herings' domain of financial systems, a unique clearing matrix allows to guarantee *claim boundedness*, *limited liability*, *absolute priority*, and proportional repayments. As a result, and for the proportional financial rule, the axiomatic analysis could be restricted to single-valued solutions.[3] Although this route simplifies the analysis to some extent, if the endowments of some agents are allowed to be zero, then several clearing payment matrices can be supported by the principle of proportionality. An implication of this fact is that the characterization by Csóka and Herings (2021) no longer distinguishes all rules relying on this principle. Indeed, as we will show, the accommodation of their properties to multi-valued solution concepts does not characterize all proportional financial rules.

In this paper, we provide an axiomatic ground for the family of proportional rules in the whole domain of financial systems. Along with the basic requirements of *limited liability* and *absolute priority*, we also require *compatibility*. As mutual liability issues often lead to complex insolvency problems spanning across multiple legal principles, it is pertinent to consider financial rules that are in accordance with bankruptcy regulations. To capture, to a certain degree, these law-related prerequisites, the axiom of *compatibility* imposes that monetary transfers should be supported by an inventory of bankruptcy rules that form the basis for the clearing process of each defaulting firm. From a formal viewpoint, the combination of *compatibility*, *limited liability*, and *absolute priority* implies, for multi-valued financial rules, that the set of clearing payment matrices forms a complete lattice.[4] This ensures that they all result in identical equity values for each company, thereby enhancing the transparency and predictability of the insolvency process. Furthermore, when using a multi-valued solution, to understand strategic properties or policy implications, agents must evaluate two sets of allocations under alternative scenarios. In this sense, an advantage of employing compatible financial rules is the ability to uniquely evaluate these sets by means of utility or net worth. In fact, as decisions are derived by utility maximization, we express the properties in terms of equity values.

Additionally, we introduce two new axioms built upon well-established principles in the literature: continuity and non-manipulability. In a broad sense, continuity, a standard requirement in axiomatic studies (e.g., Young, 1987; Moulin, 2000; Kaminski, 2006; Stovall, 2020; Calleja et al., 2020), requires that small changes in the problem's data do not lead to significant changes in the final output. For multi-valued solution concepts, there are two different generalizations of the classical notion of continuity: *lower hemicontinuity* and *upper hemicontinuity*. In words, *lower (upper) hemicontinuity* requires that small changes in a financial system do not make the set of recommended payment matrices suddenly implode (explode). As we will demonstrate, in the whole domain of financial systems, some rules based on the principle of proportionality do not satisfy either *lower hemicontinuity* or *upper hemicontinuity*. To deal with this discontinuity issue, we shift the focus from payment matrices to equity values. Specifically, we introduce *equity-continuity*, which imposes that a small impact on both the initial endowments and the liabilities of the agents does not imply substantial variations in their final equity value. Interestingly, *equity-continuity* weakens *lower hemicontinuity*.

Non-manipulability formalizes the idea that agents cannot take advantage by misrepresenting their characteristics. In the context of classical bankruptcy problems, a rule is non manipulable if it is immune to the strategic behavior of the agents by merging or splitting their claims. This principle has been widely used to characterize the proportional bankruptcy rule (O'Neill, 1982; Chun, 1988; de Frutos, 1999; Ju et al., 2007; Calleja and Llerena, 2022). In financial networks, non-manipulability can be generalized requiring such immunity when agents merge (split) endowments and liabilities, by means of their equity value. However, Calleja et al. (2021) show that, under *claim boundedness*, *limited liability*, and *absolute priority*, the principle of non-manipulability does not endorse any financial rule. To overcome this incompatibility, we introduce *non-manipulability by clones* which requires that the division of a firm into a number of identical firms or clones, that is, with the same endowments, claims, and liabilities, or the merger of a group of identical firms, should have no effect on their utilities. In real-life scenarios, procedures in which an insolvent firm transfers all its liabilities to a spin-off while retaining the endowments and claims for itself are disapproved, indeed.

Our main result concludes that *compatibility*, *limited liability*, *absolute priority*, *equity-continuity*, and *non-manipulability by clones* characterize any selection from the set of clearing payment matrices that adheres to the principle of proportionality. It is worth stressing that our characterization also holds within the framework considered by Csóka and Herings (2021).

The remainder of the paper is organized as follows. In the last part of this introduction, we address some related literature. Section 2 introduces the model. Section 3 connects financial rules and bankruptcy rules. Section 4 contains the characterization result. Section 5 examines the logical implications between our axioms and the accommodation of those used by Csóka and Herings (2021) to a multi-valued setup. Section 6 concludes. The proofs for the results in each section, except for some in Section 4, can be found in the corresponding appendix.

1.1. Related literature

Besides the work of Csóka and Herings (2021), there are other papers that are connected to our research. From an axiomatic perspective, Groote Schaarsberg et al. (2018) study the extension of the Talmudic rule (Aumann and Maschler, 1985) from claims problems to mutual liability problems, whereas Ketelaars and Borm (2021) adapt the joint axiomatization of the proportional, constrained equal awards, and constrained equal losses rules for classical bankruptcy problems proposed by Moulin (2000) to the context of financial systems. These two papers, however, take a different approach by considering financial rules as recommendations on allocations of equity values rather than focusing on clearing payment matrices. Taking a different approach, Stutzer (2018) shows that the strategic justification (coming from bargaining theory) of the proportional rule and the constrained equal awards rule for a standard claims problem cannot be extended to financial networks. Regarding non compatible financial rules, Csóka and Herings (2023b) recently axiomatize the pairwise netting proportional rule. This rule is implemented by applying the proportional rule to the net matrix of liabilities derived from a previous round of pairwise netting. There is also an emerging literature concerning the extension of the classical model of claims problems to network problems (e.g., Bjørndal and Jörnsten, 2010; Moulin and Sethuraman, 2013). For an account of recent contributions dealing with further generalizations, we refer to Thomson (2019).

A related property of *non-manipulability by clones* is *invariance to mitosis* (Csóka and Herings, 2021), which extends the notion of *additivity of claims* (Curiel et al., 1987) or *strong non-manipulability* (Moreno-Ternero, 2006) from the context of bankruptcy problems to the financial systems environment. The idea of imposing restrictions on coalition

[4] To be precise, in order to guarantee that the set of clearing payment matrices has a complete lattice structure, it is necessary to require *resource monotonicity* on the underlying bankruptcy rules (see Section 3).

formation when merges or spin-offs occur appears in the literature when characterizing extensions of the proportional rule to bankruptcy problems with multiple types of assets (Ju et al., 2007; Ju, 2013) or in axiomatizing priority rules in the context of standard insolvencies (Flores-Szwagrzak et al., 2019), to mention some instances. Calleja and Llerena (2022) restrict mergers and splits to agents that are or become symmetric and convey new axiomatizations of the proportional rule for classical bankruptcy problems.

Other authors as Glasserman and Young (2016), Koster (2019), and Csóka and Herings (2023a) have investigated conditions on the network that are sufficient for the uniqueness of clearing payment matrices for compatible financial rules.

2. The model

Before describing the model of financial systems, we first provide some basic definitions and introduce well-known insights from the bankruptcy or claims problems literature.

2.1. Preliminaries

Let $\mathbb{N} = \{1, 2, \dots\}$ (the set of natural numbers) represent the set of all potential agents and let $\mathcal{N}$ be all non-empty finite subsets of $\mathbb{N}$. An element $N \in \mathcal{N}$ describes a finite set of agents. For each $x \in \mathbb{R}^N$ and $T \subseteq N$, x_T denotes the restriction of x to T: $x_T = (x_i)_{i\in T} \in \mathbb{R}^T$. For $N \in \mathcal{N}$, we denote by $\mathcal{M}(N)$ the set of all non-negative real $N \times N$ matrices $M = (M_{ij})_{i,j\in N}$ with a zero diagonal, and $\mathcal{M} = \bigcup_{N\in\mathcal{N}} \mathcal{M}(N)$. For $M \in \mathcal{M}(N)$ and $i \in N$, $M_i = (M_{ij})_{j\in N} \in \mathbb{R}^N_+$ denotes the row i of M being $\bar{M}_i = \sum_{j\in N} M_{ij}$. By $\mathbb{Q}_+ = \{a/b \mid a, b \in \mathbb{N}\}$ we denote the set of positive rational numbers.

An important tool in our analysis is Tarski's fixed-point theorem on lattices (Tarski, 1955). Roughly speaking, a lattice is a partially ordered set A in which any two elements $x, y \in A$ have a supremum (a minimum upper bound) and an infimum (a maximum lower bound) in A. A lattice A is complete if every nonempty subset of A has a supremum and an infimum in A. The Tarski's theorem says that the non-empty set of fixed-points of a monotone function f on a complete lattice A (i.e., the set of elements $x \in A$ such that $x = f(x)$) is a complete lattice. In order not to overload the reading of the paper, Appendix A contains the formal statement of this result.

A *bankruptcy problem* (O'Neill, 1982) is a triple (N, E, c) where $N \in \mathcal{N}$ represents the set of creditors of the firm going bankrupt; $c \in \mathbb{R}^N_+$ is the vector of claims, being c_i the claim or the liability of the firm to creditor $i \in N$; and $E \geq 0$ is the net worth or estate of the firm to satisfy its obligations. Additionally, we assume that $\sum_{i\in N} c_i \geq E$. By $\mathcal{B}$ we denote the set of all bankruptcy problems. A *bankruptcy rule* is a function $\beta : \mathcal{B} \longrightarrow \bigcup_{N\in\mathcal{N}} \mathbb{R}^N_+$ that provides for every $(N, E, c) \in \mathcal{B}$ a unique vector or recommendation $\beta(N, E, c) \in \mathbb{R}^N_+$ satisfying $\sum_{i\in N} \beta_i(N, E, c) = E$ (*budget balance*) and $\beta_i(N, E, c) \leq c_i$ for all $i \in N$ (*claim boundedness* (CB)). Instances of well studied bankruptcy rules are the *proportional* (*PR*), the *constrained equal awards* (*CEA*), and the *constrained equal losses* (*CEL*) rules. The *PR* rule makes awards proportional to the claims. Formally, for all $(N, E, c) \in \mathcal{B}$ and all $i \in N$, $PR_i(N, E, c) = \lambda c_i$ where $\lambda \in \mathbb{R}_+$ is such that $\sum_{j\in N} \lambda c_j = E$. The *CEA* rule rewards all claimants equally subject to no one receiving more than her claim. Formally, for all $(N, E, c) \in \mathcal{B}$ and all $i \in N$, $CEA_i(N, E, c) = \min\{c_i, \lambda\}$ where $\lambda \in \mathbb{R}_+$ is such that $\sum_{j\in N} \min\{c_j, \lambda\} = E$. In contrast, the *CEL* rule equalizes the losses of claimants subject to no one receiving a negative amount. That is, for all $(N, E, c) \in \mathcal{B}$ and all $i \in N$, $CEL_i(N, E, c) = \max\{c_i - \lambda, 0\}$ where $\lambda \in \mathbb{R}_+$ is such that $\sum_{j\in N} \max\{c_j - \lambda, 0\} = E$.[5]

Next, we introduce a number of properties for bankruptcy rules that will play a central role in the paper. A bankruptcy rule β satisfies

- *resource monotonicity* (RM) if for all $(N, E, c), (N, E', c) \in \mathcal{B}$ with $E' > E$, $\beta_i(N, E', c) \geq \beta_i(N, E, c)$ for all $i \in N$;
- *equal treatment of equals* (ETE) if for all $(N, E, c) \in \mathcal{B}$ and all $i, j \in N$, if $c_i = c_j$ then $\beta_i(N, E, c) = \beta_j(N, E, c)$;
- *weak continuity* (WCONT) if for all $(N, E, c) \in \mathcal{B}$ and all sequence of bankruptcy problems $\left\{(N, E^n, c^n)\right\}_{n\in\mathbb{N}}$ converging to (N, E, c), there exists a subsequence $\left\{(N, E^{n_k}, c^{n_k})\right\}_{n_k\in\mathbb{N}}$ such that $\left\{\beta(N, E^{n_k}, c^{n_k})\right\}_{n_k\in\mathbb{N}}$ converges to $\beta(N, E, c)$;
- *non-manipulability by clones* (NMC) if for all $(N, E, c), (N', E, c') \in \mathcal{B}$, if $N' \subset N$ and there is $m \in N'$ such that $c_i = \frac{c'_m}{|N\setminus N'|+1}$ for all $i \in N \setminus N' \cup \{m\}$ and $c'_i = c_i$ for all $i \in N' \setminus \{m\}$, then $\beta_i(N', E, c') = \beta_i(N, E, c)$ for all $i \in N' \setminus \{m\}$.

Resource monotonicity says that no one should be worse off when the firm's assets increase. *Equal treatment of equals* is a weak impartiality requirement meaning that symmetric agents (i.e., with the same claim) have to be rewarded equally. *Weak continuity* relaxes continuity, which imposes that small variations in both, the estate and the claims, imply small variations in the resulting allocation vector. *Non-manipulability by clones* specifies that symmetric agents have no incentives to merge, neither an agent incentives to split into equal copies.[6]

The following new characterization of the proportional rule will be important later on in our axiomatic analysis. The proof is contained in Appendix B.

Proposition 1. *A bankruptcy rule satisfies weak continuity and non-manipulability by clones if and only if it is the proportional rule.*

2.2. Financial systems

A financial system (or a mutual liability problem) is a non trivial generalization of a bankruptcy problem where agents are connected to each other in a network of contracts that entail mutual obligations. This implies that the default of an agent may provoke the default of others, leading to some systemic risk.[7] Following Eisenberg and Noe (2001), a financial system is described by a triple $\varepsilon = (N, L, e)$ being $N \in \mathcal{N}$ the set of economic entities in the system; the matrix $L \in \mathcal{M}(N)$ represents the structure of *liabilities*, where L_{ij} stands for the liability of firm $i \in N$ to firm $j \in N$ or, equivalently, the claim of firm j against firm i; and the vector $e \in \mathbb{R}^N_+$ indicates the *initial operating cash flows* (or *endowments*) of the agents, that is, its exogenous funds obtained from sources outside the financial system. At this point, it is important to emphasize that the requirement for non-negative operating cash flows is not really restrictive. Indeed, as we outlined in the introduction, an approach to accommodate the possibility of negative initial endowments is to internalize external liabilities introducing a fictitious agent with zero endowments and no obligations to other agents (see Eisenberg and Noe (2001, page 238). The vector of total obligations in the system is denoted by $\bar{L} = (\bar{L}_i)_{i\in N} \in \mathbb{R}^N_+$. By $\mathcal{F}$ we represent the set of all financial systems. From a bankruptcy perspective, agents play the role of firms and claimants simultaneously.

A bankruptcy problem $(N, E, c) \in \mathcal{B}$ can be translated into a financial system $(\bar{N}, L, e)$ being $\bar{N} = N \cup \{i\}$ the set of agents, where $i \in \mathbb{N} \setminus N$ represents the firm going bankrupt; the matrix of liabilities L is given by $L_{jk} = 0$ for all $j, k \in N$, $L_{ij} = c_j$, and $L_{ji} = 0$ for all $j \in N$; and the initial endowments $e \in \mathbb{R}^N_+$ by $e_i = E$ and $e_j = 0$ for all $j \in N$.

For each financial system (N, L, e), a *payment matrix* $P \in \mathcal{M}(N)$ specifies a recommendation on what monetary transfer P_{ij} should be paid by any agent $i \in N$ to any other agent $j \in N$. Associated to a

[5] For a detailed analysis of bankruptcy rules we refer to Thomson (2019).

[6] *Non-manipulability by clones* (Calleja and Llerena, 2022) weakens the classical non-manipulability property for bankruptcy rules (Curiel et al., 1987; de Frutos, 1999) since only symmetric agents are allowed to split and merge.

[7] This point is addressed, among others, in Chen et al. (2013) and Demange (2018) that focus on measuring the systemic risk of a financial network.

payment matrix P and an endowment vector e, the *asset value* of agent $i \in N$ is determined endogenously as the amount of resources of i to clear its debts, that is, by the sum of its endowment and the payments received from other agents,

$$a_i(P,e) = e_i + \sum_{k \in N} P_{ki}. \tag{1}$$

The entities participating in the system will make evaluations on different payment matrices depending on their associated *value of equity*, or *utility*. Given a payment matrix P and an endowment vector e, the equity value of agent $i \in N$ is defined by

$$E_i(P,e) = a_i(P,e) - \bar{P}_i, \tag{2}$$

where $\bar{P}_i$ is the total payment of agent i according to P. By $E(P,e) \in \mathbb{R}^N$ we denote the vector of equity values of the agents. Observe that, indeed, $\sum_{i \in N} E_i(P,e) = \sum_{i \in N} e_i$. Hence, the choice of a particular payment matrix is, in terms of utility or net worth, a recommendation on the distribution of the total initial endowments in the system.

A financial rule associates to each financial system a non-empty set of payment matrices.

Definition 1. A *financial rule* σ is a correspondence that assigns a non-empty subset $\sigma(N,L,e)$ of $\mathcal{M}(N)$ to each $(N,L,e) \in \mathcal{F}$.

If a financial rule σ always recommends a unique matrix, then we say that σ is single-valued; in a formal manner, if for all $(N,L,e) \in \mathcal{F}$, $|\sigma(N,L,e)| = 1$.

In line with Eisenberg and Noe (2001), we are interested in financial rules fulfilling three basic criteria: *claim boundedness*, which imposes that the payment of a firm to any other firm is bounded from above by the liability to it; *limited liability* of equity, requiring that the payments of the firm to others are limited to its asset value; and *absolute priority* of debt over equity, demanding that stockholders of each firm cannot receive a positive value unless all obligations have been completely paid. Formally, a financial rule σ satisfies

- *claims boundedness* (**CB**) if, for all $(N,L,e) \in \mathcal{F}$, all $P \in \sigma(N,L,e)$, and all $i,j \in N$, $P_{ij} \le L_{ij}$;
- *limited liability* (**LL**) if, for all $(N,L,e) \in \mathcal{F}$, all $P \in \sigma(N,L,e)$, and all $i \in N$, $E_i(P,e) \ge 0$;
- *absolute priority* (**AP**) if, for all $(N,L,e) \in \mathcal{F}$, all $P \in \sigma(N,L,e)$, and all $i \in N$, if $\bar{P}_i < \bar{L}_i$ then $E_i(P,e) = 0$.

In fact, these three basic conditions ensure that the financial rule recommendations clear the debts in the system in a feasible way. The next lemma expresses that, in the presence of *claim boundedness*, the combination of *limited liability* and *absolute priority* is equivalent to require that every firm pays the minimum between its asset value and its total debt obligations. The proof is relegated to Appendix B.

Lemma 1. *Let σ be a financial rule satisfying claim boundedness. Then, the following statements are equivalent:*

1. *σ satisfies limited liability and absolute priority.*
2. *For all $(N,L,e) \in \mathcal{F}$, all $P \in \sigma(N,L,e)$, and all $i \in N$,*

$$\bar{P}_i = \min\left\{ e_i + \sum_{k \in N} P_{ki}, \bar{L}_i \right\}. \tag{3}$$

3. Financial rules compatible with bankruptcy rules

Since the entities in the system may have different tax addresses, one may ask whether the recommendation proposed by a financial rule is compatible with the recommendations of the insolvency laws of each court or administration taking part. Intuitively, clearing payment matrices should be consistent with the legal rules (bankruptcy solution concepts) allocating the value of the estate of a defaulting firm among its debt holders. Obviously, these principles or rules may vary from one court to another, which makes the compatibility issue relevant. Formally,

Definition 2. A financial rule σ is *compatible* with an inventory of bankruptcy rules $\boldsymbol{\beta} = (\beta^i)_{i \in \mathbb{N}}$ if for all $(N,L,e) \in \mathcal{F}$, all $P \in \sigma(N,L,e)$, and all $j \in N$, $P_{jk} = \beta^j_k(N \setminus \{j\}, E, c)$ for all $k \in N \setminus \{j\}$, where $(N \setminus \{j\}, E, c)$ is the bankruptcy problem faced by agent $j \in N$ being $E = \bar{P}_j$ and $c \in \mathbb{R}^{N \setminus \{j\}}_+$ with $c_k = L_{jk}$ for all $k \in N \setminus \{j\}$.

By an inventory of bankruptcy rules, $\boldsymbol{\beta} = (\beta^i)_{i \in \mathbb{N}}$, we identify a bankruptcy rule for each potential agent in the system. In particular, we will denote by $\mathbb{PR} \equiv (PR^i)_{i \in \mathbb{N}}$, $\mathbb{CEA} \equiv (CEA^i)_{i \in \mathbb{N}}$, and $\mathbb{CEL} \equiv (CEL^i)_{i \in \mathbb{N}}$ the set of bankruptcy rules consisting of all agents applying the *PR*, *CEA*, and *CEL* bankruptcy rule, respectively. The next axiom describes financial rules supported by bankruptcy rules. A financial rule σ satisfies

- *compatibility* (**C**) if there exists an inventory of bankruptcy rules $\boldsymbol{\beta} = (\beta^i)_{i \in \mathbb{N}}$ such that σ is compatible with $\boldsymbol{\beta}$.[8]

Any compatible financial rule accomplishes *claim boundedness* as a consequence of the definition of a bankruptcy rule. Moreover, for any given payment matrix P, the value of the estate of any firm $i \in N$ is endogenously determined and defined to be exactly the amount payed to debt holders by the firm according to P, which ensures that the bankruptcy problem faced by $i \in N$ is well defined (independently if the firm defaults or not). Actually,

$$E = \bar{P}_i \underset{\text{BB}}{=} \sum_{k \in N \setminus \{i\}} \beta^i_k \left(N \setminus \{i\}, \bar{P}_i, (L_{ij})_{j \in N \setminus \{i\}} \right) \underset{\text{CB}}{\le} \sum_{k \in N \setminus \{i\}} L_{ik} = \sum_{k \in N \setminus \{i\}} c_k.$$

Given an inventory of bankruptcy rules $\boldsymbol{\beta} = (\beta^i)_{i \in \mathbb{N}}$, and regarding the existence of non-empty financial rules compatible with $\boldsymbol{\beta}$ that additionally meet *limited liability* and *absolute priority*, the approach in Groote Schaarsberg et al. (2018) to show existence when all bankruptcy rules are the same can be extended to the general setting in which different bankruptcy rules apply (see Csóka and Herings (2018). In our analysis, we adopt the methodology of Eisenberg and Noe (2001) that uses Tarski's fixed-point theorem (see Appendix A) to prove non-emptiness for the case of all agents applying the proportional rule, exclusively. To do it, let us introduce the following instrumental function.

Definition 3. Given an inventory of bankruptcy rules $\boldsymbol{\beta} = (\beta^i)_{i \in \mathbb{N}}$ and a financial system $\varepsilon = (N,L,e)$, define the function $\boldsymbol{\Phi}^{\varepsilon,\boldsymbol{\beta}} : [\mathbf{0}, \bar{L}] \longrightarrow [\mathbf{0}, \bar{L}]$ as follows:

$$\Phi^{\varepsilon,\boldsymbol{\beta}}_i(\mathbf{t}) = \min\left\{ e_i + \sum_{k \in N \setminus \{i\}} \beta^k_i \left(N \setminus \{k\}, \mathbf{t}_k, (L_{kj})_{j \in N \setminus \{k\}} \right), \bar{L}_i \right\},$$

for all $i \in N$ and all $\mathbf{t} = (\mathbf{t}_1, \ldots, \mathbf{t}_n) \in [\mathbf{0}, \bar{L}]$, being $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^N$.

Under *limited liability* and *absolute priority*, an interpretation of $\boldsymbol{\Phi}^{\varepsilon,\boldsymbol{\beta}}$ is that, for each firm $i \in N$, $\Phi^{\varepsilon,\boldsymbol{\beta}}_i(\mathbf{t})$ represents the total funds it will employ to satisfy obligations assuming that such a firm will receive, from the other firms in the system, inflows specified by the rules in $\boldsymbol{\beta}$ applied over the vector of payments $\mathbf{t} = (\mathbf{t}_1, \ldots, \mathbf{t}_n)$. If $FIX(\boldsymbol{\Phi}^{\varepsilon,\boldsymbol{\beta}})$ denotes the set of fixed-points of $\boldsymbol{\Phi}^{\varepsilon,\boldsymbol{\beta}}$, a direct implication of Lemma 1 is the following corollary.

Corollary 1. *Let σ be a financial rule compatible with an inventory of bankruptcy rules $\boldsymbol{\beta} = (\beta^i)_{i \in \mathbb{N}}$. Then, the following statements are equivalent:*

[8] Csóka and Herings (2018, 2023a) incorporate *compatibility* into the definition of clearing payment matrices, referring to it as *feasibility*. Groote Schaarsberg et al. (2018) use the term ϕ-transfer scheme to describe compatible financial rules, where ϕ denotes the underlying bankruptcy rule. Most research in this area focuses on financial rules that fulfill *compatibility*, including those developed by Eisenberg and Noe (2001), Stutzer (2018), Koster (2019), Ketelaars et al. (2020), and Ketelaars and Borm (2021), among others.

1. *σ satisfies limited liability and absolute priority.*
2. *For all $\varepsilon = (N, L, e) \in \mathcal{F}$ and all $P \in \sigma(\varepsilon)$, $\bar{P} = (\bar{P}_i)_{i\in N} \in FIX(\Phi^{\varepsilon,\beta})$.*

Note that, indeed, to obtain a financial rule compatible with $\boldsymbol{\beta}$ that additionally fulfills *limited liability* and *absolute priority* is enough to select, for each financial system ε, a vector of payments $\mathbf{t} = (\mathbf{t}_1, \dots, \mathbf{t}_n) \in FIX(\Phi^{\varepsilon,\beta})$ and later apply for each agent i the corresponding bankruptcy rule β^i on $\mathbf{t}_i$ to produce a payment matrix. Remark 1 formally contains this observation.

Remark 1. Given an inventory of bankruptcy rules $\boldsymbol{\beta} = (\beta^i)_{i\in\mathbb{N}}$ and an arbitrary non-empty subset of fixed-points $\mathcal{V}_\varepsilon \subseteq FIX(\Phi^{\varepsilon,\beta})$ for every $\varepsilon = (N, L, e) \in \mathcal{F}$, we can define a financial rule σ compatible with $\boldsymbol{\beta}$ and satisfying *limited liability* and *absolute priority* as follows: for each $\varepsilon = (N, L, e)$ and each $\mathbf{t} \in \mathcal{V}_\varepsilon$, define the matrix $P^{\mathbf{t}}$ as $P^{\mathbf{t}}_{ij} = \beta^i_j(N \setminus \{i\}, \mathbf{t}_i, (L_{ij})_{j\in N\setminus\{i\}})$, for all $i, j \in N$, and then set $\sigma(\varepsilon) = \{P^{\mathbf{t}} \mid \mathbf{t} \in \mathcal{V}_\varepsilon\}$. Note that, for all $\mathbf{t} \in \mathcal{V}_\varepsilon$ and all $i \in N$, by *budget balance* of β^i, we have that $\bar{P}^{\mathbf{t}}_i = \mathbf{t}_i$ and thus $\bar{P}^{\mathbf{t}} \in FIX(\Phi^{\varepsilon,\beta})$.

Hence, the problem of combining *compatibility*, *limited liability*, and *absolute priority* reduces to the existence of fixed-points of $\Phi^{\varepsilon,\beta}$. A way to guarantee that the set of fixed-points is non-empty is requiring *resource monotonicity* on the bankruptcy rules contained in $\boldsymbol{\beta}$, which implies the monotonicity of the function $\Phi^{\varepsilon,\beta}$. These statements are summarized in Remark 2.

Remark 2. Given an inventory of bankruptcy rules $\boldsymbol{\beta} = (\beta^i)_{i\in\mathbb{N}}$ satisfying *resource monotonicity*, there exist financial rules compatible with $\boldsymbol{\beta}$ satisfying *limited liability*, *absolute priority*, and *claim boundedness*. This is a direct consequence of the application of Tarski's fixed-point theorem to the non decreasing function $\Phi^{\varepsilon,\beta}$ for each $\varepsilon \in \mathcal{F}$, which ensures that the set of fixed-points $FIX(\Phi^{\varepsilon,\beta})$ is non-empty and forms a complete lattice.

In view of Remarks 1 and 2, the rich structure of the set of fixed-points of the instrumental function Φ allows us to introduce three very special financial rules associated to any inventory of resource monotonic bankruptcy rules.

Definition 4. Let $\boldsymbol{\beta}$ be an inventory of bankruptcy rules satisfying *resource monotonicity* and let $\mathbf{t}^+_\varepsilon$, $\mathbf{t}^-_\varepsilon$ denote the supremum and the infimum of the set of fixed-points $FIX(\Phi^{\varepsilon,\beta})$ for all $\varepsilon \in \mathcal{F}$, respectively. Define the greatest, σ^{β}_+, the least, σ^{β}_-, and the maximal, σ^{β}_{max}, financial rules compatible with $\boldsymbol{\beta}$ by setting:

1. $\sigma^{\beta}_+(\varepsilon) = \{P^{\mathbf{t}^+_\varepsilon}\}$ for all $\varepsilon \in \mathcal{F}$;
2. $\sigma^{\beta}_-(\varepsilon) = \{P^{\mathbf{t}^-_\varepsilon}\}$ for all $\varepsilon \in \mathcal{F}$;
3. $\sigma^{\beta}_{max}(\varepsilon) = \{P^{\mathbf{t}} \mid \mathbf{t} \in FIX(\Phi^{\varepsilon,\beta})\}$ for all $\varepsilon \in \mathcal{F}$.

Observe that while σ^{β}_+ and σ^{β}_- are single-valued, σ^{β}_{max} is multi-valued.

Next, we introduce the family of financial rules based on the principles of proportionality, equal awards, and equal losses, respectively.

Definition 5. A financial rule σ is a

1. proportional rule if for all $(N, L, e) \in \mathcal{F}$, all $P \in \sigma(N, L, e)$, and all $i, j \in N$, $P_{ij} = \lambda_i L_{ij}$ where $\lambda_i \in \mathbb{R}_+$ satisfies Eq. (3), that is, $\bar{P}_i = \min\{e_i + \sum_{k\in N} \lambda_k L_{ki}, \bar{L}_i\}$;
2. constrained equal awards rule if for all $(N, L, e) \in \mathcal{F}$, all $P \in \sigma(N, L, e)$, and all $i, j \in N$, $P_{ij} = \min\{L_{ij}, \lambda_i\}$ where $\lambda_i \in \mathbb{R}_+$ satisfies Eq. (3), that is, $\bar{P}_i = \min\{e_i + \sum_{k\in N} \min\{L_{ki}, \lambda_k\}, \bar{L}_i\}$;
3. constrained equal losses rule if for all $(N, L, e) \in \mathcal{F}$, all $P \in \sigma(N, L, e)$, and all $i, j \in N$, $P_{ij} = \max\{0, L_{ij} - \lambda_i\}$ where $\lambda_i \in \mathbb{R}_+$ satisfies Eq. (3), that is, $\bar{P}_i = \min\{e_i + \sum_{k\in N} \max\{0, L_{ki} - \lambda_k\}, \bar{L}_i\}$.

Note that, from Corollary 1 and Remark 2, proportional, constrained equal awards or constrained equal losses financial rules are compatible with all agents applying their counterpart bankruptcy rule (all of them being resource monotonic) satisfying, additionally, *limited liability* and *absolute priority*, and vice versa.

Remark 3. It is worth noting that there exist financial rules compatible with an arbitrary inventory of resource monotonic bankruptcy rules $\boldsymbol{\beta}$ that do not fulfill the requirements of *limited liability* and *absolute priority*. Think, for instance, in the following financial rules: $\sigma_1(N, L, e) = \{\mathbf{0}\}$ where $\mathbf{0} \in \mathcal{M}(N)$ denotes the zero matrix and $\sigma_2(N, L, e) = \{L\}$ for all $(N, L, e) \in \mathcal{F}$. Clearly, σ_1 and σ_2 are compatible with $\boldsymbol{\beta}$ since any bankruptcy rule distributing an estate of value zero equals the zero vector, and any other allocating exactly the total debts obligations equal the vector of claims (or liabilities). However, neither σ_1 satisfies *absolute priority*, nor σ_2 satisfies *limited liability*.

The following example illustrates that proportional, constrained equal awards, and constrained equal losses financial rules need not be single-valued.

Example 1 (Eisenberg and Noe, 2001)**.** Let $\varepsilon = (N, L, e) \in \mathcal{F}$ with set of players $N = \{1, 2\}$, initial operating cash flows $e = (0, 0)$, and matrix of liabilities

$$L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now, let σ be an arbitrary financial rule satisfying *claim boundedness*, *limited liability*, *absolute priority*, and $P \in \sigma(\varepsilon)$. By *claim boundedness*, $0 \le P_{12} \le 1$ and $0 \le P_{21} \le 1$. If $E_1(P, e) = P_{21} - P_{12} > 0$ then $E_2(P, e) = P_{12} - P_{21} < 0$, in contradiction with *limited liability*. Thus, $E_1(P, e) = 0$ which implies that $P_{12} = P_{21}$ and

$$P = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix},$$

where $\lambda \in [0, 1]$. Hence,

$$\sigma(\varepsilon) \subseteq \left\{ \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \mid \lambda \in [0, 1] \right\}. \quad (4)$$

Clearly, σ is compatible with any inventory of bankruptcy rules $\boldsymbol{\beta}$. Thus, in this particular case, the family of proportional, constrained equal awards, and constrained equal losses rules coincide and contain multi-valued solutions.

The next example emphasizes that financial rules not supported by bankruptcy rules can lead to different clearing payment matrices with varying equity values.

Example 2. For all $\varepsilon = (N, L, e) \in \mathcal{F}$, define the financial rule $\sigma(\varepsilon) = \{P_1, P_2\}$ where $P_1 = \sigma^{\mathbb{PR}}_+(\varepsilon)$ and $P_2 = \sigma^{\mathbb{CEA}}_-(\varepsilon)$. Since $\bar{P}_1 \in FIX(\Phi^{\varepsilon,\mathbb{PR}})$ and $\bar{P}_2 \in FIX(\Phi^{\varepsilon,\mathbb{CEA}})$, clearly σ satisfies *claim boundedness* and, by Lemma 1, also *limited liability* and *absolute priority*. Observe, however, that it does not meet *compatibility* since P_1 and P_2 are generated by different bankruptcy rules.

To see that P_1 and P_2 may induce different equity values for the firms, take $\varepsilon \in \mathcal{F}$ with set of players $N = \{1, 2, 3\}$, initial operating cash flows $e = (1, 0, 0)$, and matrix of liabilities

$$L = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Easy calculations lead to

$$P_1 = \begin{pmatrix} 0 & 1/3 & 2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, $E_2(P_1, e) = 1/3 \ne 1/2 = E_2(P_2, e)$ and $E_3(P_1, e) = 2/3 \ne 1/2 = E_3(P_2, e)$.

In contrast to Example 2, the following result states that, under *compatibility* and in terms of equity values, agents in the system are indifferent on the chosen clearing payment matrix. This invariance property is a consequence of the lattice structure of the set of fixed-points of the instrumental function Φ (see Definition 3). The proof is relegated to Appendix C.

Lemma 2. *Let $\boldsymbol{\beta} = (\beta^i)_{i\in\mathbb{N}}$ be an inventory of resource monotonic bankruptcy rules and $\varepsilon = (N, L, e) \in \mathcal{F}$. Then,*

1. *if $\mathbf{t}, \mathbf{t}' \in FIX(\Phi^{\varepsilon,\beta})$, then $E(P^t, e) = E(P^{t'}, e)$ where the payment matrices P^t and $P^{t'}$ are defined as in Remark 1;*
2. *if σ is a financial rule compatible with $\boldsymbol{\beta}$ satisfying limited liability and absolute priority and $P, P' \in \sigma(\varepsilon)$, then $E(P, e) = E(P', e)$;*
3. *if σ and σ' are two different financial rules compatible with $\boldsymbol{\beta}$ satisfying limited liability and absolute priority, $P \in \sigma(\varepsilon)$, and $P' \in \sigma'(\varepsilon)$, then $E(P, e) = E(P', e)$.*

4. Axiomatic characterization

In this section, we provide an axiomatic foundation for the family of proportional financial rules, considering the whole domain of financial systems. In addition to *compatibility*, *limited liability*, and *absolute priority*, we will introduce two new axioms. To emphasize the importance of utility maximization in the decision-making process of agents, we express these axioms in relation to equity values. The proofs omitted in this section are collected in Appendix D. The first axiom is based on the standard principle of continuity. In our context, this axiom concerns how a marginal impact in the endowments and liabilities of the agents affects their utility. Formally, a financial rule σ satisfies

- *equity-continuity* (**E-CONT**) if for all $(N, L, e) \in \mathcal{F}$, all sequence of financial systems $\left\{(N, L^n, e^n)\right\}_{n\in\mathbb{N}}$ converging to (N, L, e) and all clearing payment matrix $P \in \sigma(N, L, e)$, there exists a sequence $\left\{P^n \in \sigma(N, L^n, e^n)\right\}_{n\in\mathbb{N}}$ with a subsequence of clearing payment matrices $\left\{P^{n_k} \in \sigma(N, L^{n_k}, e^{n_k})\right\}_{n_k\in\mathbb{N}}$ such that the associated sequence of equity values $\{E(P^{n_k}, e^{n_k})\}_{n_k\in\mathbb{N}}$ converges to $E(P, e)$.

Equity-continuity implies that small changes in the structure of liabilities and in the initial endowments should not lead to large changes in the value of equity. That is, as long as we approach to a financial system (N, L, e), and for any clearing payment matrix P in the solution, there exists a path to approach to the equity values of the agents according to P.

The following lemma states that any proportional rule meets *equity-continuity*.

Lemma 3. *Let σ be a proportional financial rule. Then, σ satisfies equity-continuity.*

The second axiom focuses on the strategic behavior of agents to avoid them from taking advantage of mergers or spin-offs. In contrast to bankruptcy problems, in financial systems no rule is immune to these types of manipulations when combined with the basic requirements of *claim boundedness*, *limited liability*, and *absolute priority*.[9] Therefore, we weaken non-manipulability by restricting splits and mergers to symmetric agents or clones, that is, those with the same initial operating cash flow, claims, and debts to the rest of agents. Formally, a financial rule σ satisfies

- *non-manipulability by clones* (**NMC**) if for all (N, L, e), $(N', L', e') \in \mathcal{F}$, if $N' \subset N$ and there is $m \in N'$ such that

$$\begin{array}{lcl} e_i & = & \frac{e'_m}{|N\setminus N'|+1} \text{ for all } i \in N \setminus N' \cup \{m\} \\ e_i & = & e'_i \text{ for all } i \in N' \setminus \{m\} \\ L_{kl} & = & 0 \text{ for all } k, l \in N \setminus N' \cup \{m\} \\ L_{ij} & = & L'_{ij} \text{ for all } i, j \in N' \setminus \{m\} \\ L_{ki} & = & \frac{L'_{mi}}{|N\setminus N'|+1} \text{ for all } k \in N \setminus N' \cup \{m\}, i \in N' \setminus \{m\} \\ L_{ik} & = & \frac{L'_{im}}{|N\setminus N'|+1} \text{ for all } k \in N \setminus N' \cup \{m\}, i \in N' \setminus \{m\} \end{array} \tag{5}$$

then,

(a) for each $P \in \sigma(N, L, e)$ there exists $P' \in \sigma(N', L', e')$ and
(b) for each $P' \in \sigma(N', L', e')$ there exists $P \in \sigma(N, L, e)$

such that, for all $i \in N' \setminus \{m\}$,

$$E_i(P', e') = E_i(P, e). \tag{6}$$

Non-manipulability by clones asserts that the split of an agent into symmetric agents or the merger of a group of symmetric agents should not affect the utility of the remaining agents, and consequently, neither the utility of the agents merging nor splitting. Observe that,

$$\sum_{i\in N} E_i(P, e) = \sum_{i\in N} e_i = \sum_{i\in N'} e'_i = \sum_{i\in N'} E_i(P', e'),$$

which, by (6), automatically implies that,

$$E_m(P', e') = E_m(P, e) + \sum_{k\in N\setminus N'} E_k(P, e).$$

The way in which *non-manipulability by clones* compares the set of payment matrices proposed by the rule before and after the merging is by imposing that, regardless of the chosen payment matrix in the initial financial system (N, L, e), the policy maker can select a payment matrix in the new financial system (N', L', e') that does not provide incentives to merge. This condition is stated in (a). On the other hand, condition (b) says that, independently of the chosen matrix in (N', L', e'), the policy maker has some room to prevent firms from splitting. This property could be stated in its stronger version, meaning that irrespective of the chosen matrices in both scenarios, agents should have no incentives to either merge or split. Furthermore, the combination of *non-manipulability by clones* with *compatibility*, *limited liability*, and *absolute priority* allows for the unique evaluation of the two sets of matrices in terms of the utility or net worth of the firms, making it easier for policy makers to assess the precise implications of a particular decision. Choosing either the weak or the strong version of the property makes no difference regarding our characterization result in Theorem 1 below.

The following lemma shows that any proportional financial rule is non-manipulable by identical agents.

Lemma 4. *Let σ be a proportional financial rule. Then, σ satisfies non-manipulability by clones.*

Now, we are in the position to prove our characterization result.

Theorem 1. *A financial rule satisfies compatibility, limited liability, absolute priority, equity-continuity, and non-manipulability by clones if and only if it is a proportional financial rule.*

Proof. The definition of a proportional financial rule ensures *compatibility*, *limited liability*, and *absolute priority*; Lemmas 3 and 4 guarantee *equity-continuity* and *non-manipulability by clones*, respectively.

To prove the reverse implication, let σ be a financial rule compatible with an inventory of bankruptcy rules $\boldsymbol{\beta} = (\beta^i)_{i\in\mathbb{N}}$ and satisfying *limited liability*, *absolute priority*, *equity-continuity*, and *non-manipulability by clones*.

[9] See Theorem 3 in Calleja et al. (2021).

- **Claim 1**: For all $i \in \mathbb{N}$, β^i satisfies *weak continuity*.
To show Claim 1, take $i \in \mathbb{N}$. Let $(N, E, c) \in \mathcal{B}$ and $\{(N, E^n, c^n)\}_{n\in\mathbb{N}}$ be a sequence of bankruptcy problems converging to (N, E, c) with $i \notin N$. Let $\varepsilon = (\bar{N}, L, e) \in \mathcal{F}$ and $\varepsilon^n = (\bar{N}, L^n, e^n) \in \mathcal{F}$ be the corresponding associated financial systems with $\bar{N} = N \cup \{i\}$ as defined in Section 2.2. Next, we see that $\sigma(\varepsilon)$ and $\sigma(\varepsilon^n)$ select a unique payment matrix. Indeed, let $P \in \sigma(\varepsilon)$. As σ satisfies *claim boundedness* (received from *compatibility*), $P_{jl} = 0$ for all $j \in N, l \in \bar{N}$. Moreover, since σ is compatible with $\boldsymbol{\beta}$, $P_{ij} = \beta^i_j(N, E, c)$ for all $j \in N$, which is unique by definition of β^i. Hence, $\sigma(\varepsilon) = \{P\}$. In a similar way, we obtain $\sigma(\varepsilon^n) = \{P^n\}$ for all $n \in \mathbb{N}$, being $P^n_{jl} = 0$ and $P^n_{ij} = \beta^i_j(N, E^n, c^n)$ for all $j \in N$, $l \in \bar{N}$. Note that $E = \bar{P}_i$ and $E^n = \bar{P}^n_i$.
Clearly, the sequence of financial systems $\{\varepsilon^n\}_{n\in\mathbb{N}}$ converges to ε and, by *equity-continuity*, there exists a subsequence of clearing payment matrices $\{P^{n_k}\}_{n_k\in\mathbb{N}}$ such that the associated sequence of equity values $\{E(P^{n_k}, e^{n_k})\}_{n_k\in\mathbb{N}}$ converges to $E(P, e)$. Let $j \in N$ and $n_k \in \mathbb{N}$. Then, we have
$$E_j(P^{n_k}, e^{n_k}) = e^{n_k}_j + \sum_{l\in\bar{N}} P^{n_k}_{lj} - \sum_{l\in\bar{N}} P^{n_k}_{jl} = P^{n_k}_{ij} \underset{\mathbf{C}}{=} \beta^i_j(N, E^{n_k}, c^{n_k})$$
and
$$E_j(P, e) = e_j + \sum_{l\in\bar{N}} P_{lj} - \sum_{l\in\bar{N}} P_{jl} = P_{ij} \underset{\mathbf{C}}{=} \beta^i_j(N, E, c).$$
Hence, for all $j \in N$, the sequence $\{\beta^i_j(N, E^{n_k}, c^{n_k})\}_{n_k\in\mathbb{N}}$ converges to $\beta^i_j(N, E, c)$, which means that β^i meets *weak continuity*.
- **Claim 2**: For all $i \in \mathbb{N}$, β^i satisfies *non-manipulability by clones*.
To show Claim 2, take $i \in \mathbb{N}$. Let $(N, E, c), (N' E, c') \in \mathcal{B}$ where $N' \subset N$, $i \notin N$, and there is $m \in N'$ such that $c_j = \frac{c'_m}{|N\setminus N'|+1}$ for all $j \in N \setminus N' \cup \{m\}$ and $c'_j = c_j$ for all $j \in N' \setminus \{m\}$. Let $\varepsilon = (\bar{N}, L, e)$ and $\varepsilon' = (\bar{N}', L', e')$ be the associated financial systems being $\bar{N} = N \cup \{i\}$ and $\bar{N}' = N' \cup \{i\}$ as defined in Section 2.2. It can easily checked that ε and ε' satisfy all the conditions in (5). Moreover, following the same arguments as in the proof of Claim 1 we have that $\sigma(\varepsilon) = \{P\}$ where $P_{ij} = \beta^i_j(N, E, c)$ and $P_{jk} = 0$ for all $j \in N, k \in \bar{N}$; and $\sigma(\varepsilon') = \{P'\}$ where $P'_{ij} = \beta^i_j(N', E, c')$ and $P'_{jk} = 0$ for all $j \in N', k \in \bar{N}'$. By *non-manipulability by clones*, for all $j \in \bar{N}' \setminus \{m\}$, we have
$$E_j(P', e') = E_j(P, e).$$
In particular, if $j \neq i$, we obtain
$$E_j(P', e') = e'_j + \sum_{l\in\bar{N}'} P'_{lj} - \sum_{l\in\bar{N}'} P'_{jl} = P'_{ij} \underset{\mathbf{C}}{=} \beta^i_j(N', E, c')$$
and
$$E_j(P, e) = e_j + \sum_{l\in\bar{N}} P_{lj} - \sum_{l\in\bar{N}} P_{jl} = P_{ij} \underset{\mathbf{C}}{=} \beta^i_j(N, E, c),$$
which implies that β^i satisfies *non-manipulability by clones*.

Hence, from claims 1 and 2, all bankruptcy rules in $\boldsymbol{\beta}$ satisfy *weak continuity* and *non-manipulability by clones* which imply, by Proposition 1, that $\beta^i = PR$, for all $i \in \mathbb{N}$. Finally, since σ is compatible with $\mathbb{PR}$ and, additionally, meets *limited liability* and *absolute priority*, we conclude that it is a proportional financial rule. □

The independence of the axioms in Theorem 1 can be found in Appendix D.

5. Comparative axiomatic analysis

A previous axiomatization of the (unique) proportional financial rule was provided by Csóka and Herings (2021) within a subdomain of regular financial systems,[10] where all agents possess a strictly positive initial endowment. In this section, we demonstrate that accommodating the axioms used by these authors to multi-valued solution concepts is no longer suitable for characterizing the family of proportional rules across the entire domain. The proofs for this section are included in Appendix E.

5.1. Continuity

Since Csóka and Herings (2021) implicitly demand single-valuedness, which defines a financial rule as a function that associates a unique clearing payment matrix to each financial system, they use the classical notion of continuity for functions. However, when dealing with multi-valued solutions, there are two possible generalizations of continuity: *lower hemicontinuity* and *upper hemicontinuity*. Formally, a financial rule σ satisfies

- *lower hemicontinuity* (**LHC**) if for all $(N, L, e) \in \mathcal{F}$, all sequence of financial systems $\{(N, L^n, e^n)\}_{n\in\mathbb{N}}$ converging to (N, L, e), and all clearing payment matrix $P \in \sigma(N, L, e)$, there exists a sequence $\{P_n \in \sigma(N, L^n, e^n)\}_{n\in\mathbb{N}}$ converging to P;
- *upper hemicontinuity* (**UHC**) if for all $(N, L, e) \in \mathcal{F}$, all sequence of financial systems $\{(N, L^n, e^n)\}_{n\in\mathbb{N}}$ converging to (N, L, e), and all sequence of clearing payment matrices $\{P_n \in \sigma(N, L^n, e^n)\}_{n\in\mathbb{N}}$ converging to the matrix P it holds that $P \in \sigma(N, L, e)$;
- *continuity* (**CONT**) if it satisfies simultaneously *lower hemicontinuity* and *upper hemicontinuity*.

For functions, both *lower hemicontinuity* and *upper hemicontinuity* are equivalent to *continuity*. Informally speaking, these continuity properties require that small changes in the financial system imply small changes in the payment matrices. In the full domain of financial systems, and given the potential multiplicity of payment matrices, it is not surprising that there are financial rules that do not satisfy either *lower hemicontinuity* or *upper hemicontinuity*. To illustrate this point, consider the following example.

Example 3 (*Eisenberg and Noe, 2001*). Let $\varepsilon = (N, L, e) \in \mathcal{F}$ be the financial system described in Example 1. Now, consider the sequence of financial systems $\{\varepsilon^n = (N, L^n, e^n)\}_{n\in\mathbb{N}}$ with set of players $N = \{1, 2\}$, initial operating cash flows $e^n = (0, 0)$, and matrices of liabilities
$$L^n = \begin{pmatrix} 0 & 1+\frac{1}{n} \\ 1+\frac{1}{n} & 0 \end{pmatrix}, \text{ for all } n \in \mathbb{N}.$$
Let $\sigma^{\mathbb{PR}}_+$ be the greatest, $\sigma^{\mathbb{PR}}_-$ the least, and $\sigma^{\mathbb{PR}}_{max}$ the maximal proportional financial rules (see Definition 4). From Example 1 it follows that
$$\sigma^{\mathbb{PR}}_{max}(\varepsilon) = \left\{ \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \mid \lambda \in [0, 1] \right\}.$$
Consequently,
$$\sigma^{\mathbb{PR}}_+(\varepsilon) = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \text{ and } \sigma^{\mathbb{PR}}_-(\varepsilon) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$
Following similar arguments, we can additionally set $\sigma^{\mathbb{PR}}_-(\varepsilon^n) = \{\mathbf{0}\}$ and $\sigma^{\mathbb{PR}}_+(\varepsilon^n) = \{L^n\}$ for all $n \in \mathbb{N}$. Now, define the proportional financial rule σ as follows: for all $\varepsilon' \in \mathcal{F}$,
$$\sigma(\varepsilon') = \begin{cases} \sigma^{\mathbb{PR}}_+(\varepsilon) & \text{if} \quad \varepsilon' = \varepsilon, \\ \sigma^{\mathbb{PR}}_-(\varepsilon') & \text{if} \quad \varepsilon' \neq \varepsilon. \end{cases} \tag{7}$$
So, while the sequence of financial systems $\{\varepsilon^n\}_{n\in\mathbb{N}}$ converges to ε when $n \to \infty$, there is not a sequence of matrices $\{P_n \in \sigma(\varepsilon^n)\}_{n\in\mathbb{N}}$ converging to $P \in \sigma(\varepsilon) = \sigma^{\mathbb{PR}}_+(\varepsilon)$, showing that σ fails to satisfy *lower hemicontinuity*. To see that it neither satisfies *upper hemicontinuity*, it is enough to observe that the sequence $\{P_n \in \sigma(\varepsilon^n) = \sigma^{\mathbb{PR}}_-(\varepsilon^n)\}_{n\in\mathbb{N}}$ converges to the zero matrix that is not contained in $\sigma(\varepsilon) = \sigma^{\mathbb{PR}}_+(\varepsilon)$. Furthermore, σ is *single-valued*, pointing out that even though we restrict

[10] See Eisenberg and Noe (2001) for a formal definition of regular financial systems. In this domain, the proportional financial rule is single-valued.

ourselves to financial rules that are functions, in the richer domain of all financial systems continuity of proportional financial rules may fail.[11]

The arguments used in Example 3 hold if we replace the proportional bankruptcy rule by any arbitrary inventory of resource monotonic bankruptcy rules, such as the constrained equal awards or the constrained equal losses. These bankruptcy rules satisfy *continuity*; however, they do not necessarily produce continuous financial rules. Thus, there are financial rules compatible with continuous bankruptcy rules, satisfying additionally *limited liability* and *absolute priority*, that do not fulfill either *lower hemicontinuity* or *upper hemicontinuity*. Nevertheless, it is worth highlighting that all of them satisfy *equity-continuity*, which allows to overcome the lack of continuity of compatible financial rules in a multi-valued setting.[12] Next, we show that *equity-continuity* is weaker than *lower hemicontinuity*.

Lemma 5. *Lower hemicontinuity implies equity-continuity.*

5.2. Impartiality

A further axiom imposed in Csóka and Herings (2021) is *impartiality*, which requires that two agents j and k with the same claim on the asset of agent i should receive the same payment from i. We now extend *impartiality* for multi-valued financial rules. A financial rule σ satisfies

- *impartiality* (**I**) if, for all $(N, L, e) \in \mathcal{F}$ and all $i, j, k \in N$ such that $L_{ij} = L_{ik}$ then, for all $P \in \sigma(N, L, e)$, it holds that $P_{ij} = P_{ik}$.

Impartiality applies only to payments made by agent i to agents j and k, but the repayment capacity of these two agents is not taken into account. Even though *impartiality* appears to be a mild condition, it applies to pairs of agents that need not be symmetric or identical. We interpret that symmetric agents should be treated equally, meaning they should end up with the same utility. Formally, a financial rule σ satisfies

- *equal treatment of equals* (**ETE**) if for all $(N, L, e) \in \mathcal{F}$ and all $i, j \in N$ such that $e_i = e_j$, $L_{ij} = L_{ji}$, $L_{ik} = L_{jk}$, and $L_{ki} = L_{kj}$ for all $k \in N \setminus \{i, j\}$ then, for all $P \in \sigma(N, L, e)$ it holds that $E_i(P, e) = E_j(P, e)$.

Equal treatment of equals ensures that symmetric agents should get the same value of equity. Under the basic requirements of *claim boundedness*, *limited liability*, and *absolute priority*, the next result establishes that *equal treatment of equals* is weaker than *impartiality*.

Lemma 6. *Under claim boundedness, limited liability, and absolute priority; impartiality implies equal treatment of equals.*

Remarkably, when we impose the stronger condition of *compatibility* instead of *claim boundedness*, we find that *impartiality* and *equal treatment of equals* become equivalent. This connection stems from the fact that *equal treatment of equals* of the bankruptcy rules underlying a financial rule connects both properties.

Proposition 2. *Let σ be a financial rule compatible with an inventory of bankruptcy rules $\boldsymbol{\beta} = (\beta^i)_{i\in\mathbb{N}}$, satisfying limited liability and absolute priority. Then, the following statements are equivalent:*

1. *σ satisfies impartiality.*
2. *σ satisfies equal treatment of equals.*
3. *β^i satisfies equal treatment of equals for all $i \in \mathbb{N}$.*

A straightforward consequence of Proposition 2 is the following.

Corollary 2. *Let σ be a proportional financial rule. Then, σ satisfies impartiality and equal treatment of equals.*

However, as showed in the next example, *impartiality* and *equal treatment of equals* are not generally equivalent.

Example 4. We first show that *equal treatment of equals* does not imply *impartiality*, neither under *claim boundedness*, *limited liability*, and *absolute priority*. Consider the financial system $\varepsilon' = (N', L', e')$ being $N' = \{1, 2, 3\}$, initial operating cash flows $e' = (0, 0, 0)$, and matrix of liabilities

$$L' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix}.$$

Now define the financial rule σ_1 as follows:

$$\sigma_1(\varepsilon) = \begin{cases} \sigma_+^{\mathbb{PR}}(\varepsilon) & \text{if} \quad \varepsilon \neq \varepsilon', \\ \left\{ P' = \begin{pmatrix} 0 & 1/2 & 1 \\ 0 & 0 & 1 \\ 3/2 & 1/2 & 0 \end{pmatrix} \right\} & \text{if} \quad \varepsilon = \varepsilon'. \end{cases}$$

To see that σ_1 satisfies *claim boundedness*, *limited liability*, and *absolute priority*, it is enough to observe that, by definition, $\sigma_+^{\mathbb{PR}}$ fulfills the properties and, in the financial system ε', we have $E_1(P', e') = E_2(P', e') = E_3(P', e') = 0$. To check that it also satisfies *equal treatment of equals*, we distinguish two cases. If agents i and j are symmetric in $\varepsilon \neq \varepsilon'$, then *equal treatment of equals* follows since proportional financial rules satisfy the property. Otherwise, the only symmetric players in ε' are 1 and 2, which receive the same equity value according to P'. However, σ_1 does not meet *impartiality* since $L'_{12} = L'_{13}$ but $P'_{12} \neq P'_{13}$.

To see that *impartiality* does not imply *equal treatment of equals*, define the financial rule σ_2 as follows:

$$\sigma_2(\varepsilon) = \begin{cases} \sigma_+^{\mathbb{PR}}(\varepsilon) & \text{if} \quad \varepsilon \neq \varepsilon', \\ \left\{ P'' = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 2 & 0 \end{pmatrix} \right\} & \text{if} \quad \varepsilon = \varepsilon'. \end{cases}$$

Clearly, σ_2 satisfies *impartiality* as, in case that $\varepsilon \neq \varepsilon'$, $\sigma_+^{\mathbb{PR}}$ meets the property. Otherwise, if $\varepsilon = \varepsilon'$, $L'_{12} = L'_{13}$ and $P''_{12} = P''_{13} = 1$; $L'_{21} = L'_{23}$ and $P''_{21} = P''_{23} = 0$; $L'_{31} = L'_{32}$ and $P''_{31} = P''_{32} = 2$. However, σ_2 fails to satisfy *equal treatment of equals* because players 1 and 2 are symmetric in ε' but they obtain a different equity value according to P''. Indeed, $E_1(P'', e') = 0$ and $E_2(P'', e') = 3$.

5.3. Invariance to mitosis

In the setting of financial networks, Csóka and Herings (2021) interpret non-manipulability as some invariance conditions on the clearing payment matrices, enforcing invariance not only on payments made by and received from the merging or splitting agents but also on payments between agents that are not involved in the merger or split, in the spirit of *additivity of claims* (Curiel et al., 1987) or *strong non-manipulability* for bankruptcy problems (Moreno-Ternero, 2006). Moreover, only mergers or spin-offs involving identical agents are allowed. A natural extension of this property to multi-valued solutions can be derived as follows. A financial rule σ satisfies

- *invariance to mitosis* (**IM**) if for all $(N, L, e), (N', L', e') \in \mathcal{F}$, if $N' \subset N$ and there is $m \in N'$ such that all conditions listed in (5) hold, then

[11] The alternate application of either the least or greatest proportional rule in financial systems could be justified in terms of its adaptability to different contexts.

[12] Indeed, Lemma 3 in Section 4, which states that the proportional financial rule satisfies *equity-continuity*, can be readily extended to any finance rule that complies with *limited liability* and *absolute priority* and is compatible with a set of continuous bankruptcy rules.

(a) for each $P \in \sigma(N, L, e)$ there exists $P' \in \sigma(N', L', e')$ and
(b) for each $P' \in \sigma(N', L', e')$ there exists $P \in \sigma(N, L, e)$

such that

$$\begin{array}{rcl} P'_{mi} & = & P_{mi} + \sum_{k \in N \setminus N'} P_{ki} \text{ for all } i \in N' \setminus \{m\}; \\ P'_{im} & = & P_{im} + \sum_{k \in N \setminus N'} P_{ik} \text{ for all } i \in N' \setminus \{m\}; \\ P'_{kl} & = & P_{kl} \text{ for all } k, l \in N' \setminus \{m\}. \end{array} \quad (8)$$

Since financial rules may select a number of payment matrices, we demand that for every payment recommendation made by the rule $P \in \sigma(N, L, e)$ there exists $P' \in \sigma(N', L', e')$ satisfying all conditions in (8). This ensure that for any payment matrix in the initial financial system (N, L, e) there is another one in the new financial system (N', L', e') that does not provide incentives to merge since payoffs remain invariant. On the other hand, imposing that any $P' \in \sigma(N', L', e')$ can be assigned to a $P \in \sigma(N, L, e)$ for which all equalities in (8) hold guarantees that the rule does not provide incentives to split neither. Observe that, indeed, *invariance to mitosis* requires payments made by and received from agents not involved in the split or the merge to remain constant as well.

Unexpectedly, as the next example points out, there are proportional financial rules that do not meet *invariance to mitosis*.

Example 5. Let σ be the proportional financial rule defined as follows: for all $\varepsilon = (N, L, e) \in \mathcal{F}$,

$$\sigma(\varepsilon) = \begin{cases} \sigma_+^{\mathbb{PR}}(\varepsilon) & \text{if} \quad |N| \le 2, \\ \sigma_-^{\mathbb{PR}}(\varepsilon) & \text{if} \quad |N| > 2, \end{cases} \quad (9)$$

where $|N|$ denotes the cardinality of N.

Let $\varepsilon_1 = (N, L, e)$ be the financial system defined in Example 1, that is, $N = \{1, 2\}$, $e = (0, 0)$, and matrix of liabilities

$$L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now consider $\varepsilon_2 = (N', L', e')$ where agent 2 splits into agents 2 and 3, being $N' = \{1, 2, 3\}$, $e' = (0, 0, 0)$, and

$$L' = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}.$$

Recall that,

$$\sigma_+^{\mathbb{PR}}(\varepsilon_1) = \left\{ P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Some calculations lead to

$$\sigma(\varepsilon_2) = \sigma_-^{\mathbb{PR}}(\varepsilon_2) = \left\{ P' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Observe that, $1 = P_{21} \neq P'_{21} + P'_{31} = 0$, which proves that σ does not meet *invariance to mitosis*. However, $E_2(P, e) = 0 = E_2(P', e) + E_3(P', e)$.

The next lemma states that *non-manipulability by clones* is weaker than *invariance to mitosis*.

Lemma 7. *Invariance to mitosis implies non-manipulability by clones.*

To conclude this section, we point out that the axioms used by Csóka and Herings (2021) do not distinguish the set of proportional rules in the whole domain of financial system. On the contrary, our axiomatic characterization (Theorem 1) is also valid in the subdomain considered by these authors. We impose weaker non-manipulability and continuity axioms and get rid of single-valuedness, *claim boundedness*, and *impartiality* at the price of imposing *compatibility*. Although *claim boundedness* is clearly implied by *compatibility*, the absence of *impartiality* is not direct. Indeed, Claim 2 in the proof of Theorem 1 shows that *non-manipulability by clones* of a compatible financial rule implies the parallel property for the underlying bankruptcy rules, which, by Lemma 1 in Calleja and Llerena (2022), also meet *equal treatment of equals*. So, by Corollary 2, the financial rule inherits *impartiality*. However, *impartiality* and *non-manipulability by clones* are logically independent. Indeed, the financial rule σ^4 defined to show the independence of the properties in Theorem 1 (see Appendix D) satisfies *impartiality* but not *non-manipulability by clones*. On the other hand, Csóka and Herings (2023b) introduce the pairwise netting proportional rule for financial systems. This rule makes pairwise bilateral payments first, and the remaining liabilities are settled employing the proportional rule. In a general framework, this rule is multi-valued. Nevertheless, taking the greatest proportional rule (see Definition 4) in the second step produces a single-valued rule that can be proved to satisfy *invariance to mitosis* (see Csóka and Herings (2023b) and consequently, by Lemma 7, also *non-manipulability by clones*, but not *impartiality* (see Example 3.7 in (Csóka and Herings, 2023b).

6. Final comments

In this paper, we provide an axiomatic ground for the family of proportional financial rules in the whole domain of financial systems. Assuming that all agents dispose of a strictly positive operating cash flow, Csóka and Herings (2021) identify a set of axioms that characterizes the unique proportional clearing mechanism. However, allowing some agents to initially have zero cash flow may result in a multiplicity of proportional payoff matrices, which requires a different set of axioms, putting the focus on the equity value of the entities rather than on the clearing matrices themselves. Remarkably, Theorem 1 not only refines the characterization provided by the aforementioned authors but also opens the door for multi-valued rules. This result establishes a parallelism with the axiomatization of the proportional bankruptcy rule by means of *weak continuity* and *non-manipulability by clones* (Proposition 1). Recently, Calleja and Llerena (2022) show that, in the context of claims problems, *claims monotonicity* can replace *weak continuity*. In this sense, and given that monotonicity principles are widely accepted, an interesting open question is whether or not a suitable monotonicity requirement on liabilities could be used to provide new characterizations of proportional financial rules.

An important result in the literature of bankruptcy problems is owed to Young (1987), who characterizes the so-called parametric rules by means of *symmetry* (or *equal treatment of equals*), *resource continuity*, and *consistency*, a classical invariant principle with respect to variations of population (see Thomson (2012)). A possibility for future research could be to introduce parametric rules in the context of financial networks and extend Young's result to this setup. The main issue in applying the principle of consistency is that the reduced problem may be outside the original domain. A natural way to address this drawback is using conditional consistency, a weak form of consistency imposing that the initial payments must be reconfirmed in the reduced problem only when it is a financial network. Another direction worth exploring is to study the implications of considering axioms formalizing reasonable lower bounds, either on a firm's payments to its debtors or on its equity value, within the framework of mutual liability problems. This approach might shift the attention to alternative rules, as the proportional rule in conventional bankruptcy problems violates such type of properties. Neither Groote Schaarsberg et al. (2018) nor Ketelaars and Borm (2021) have undertaken an analysis of the talmudic financial rule or the constrained equal awards and the constrained equal losses financial rules, respectively, taking this axiomatic approach into account.

A closely related result to Young's can be found in Ju (2003), who characterizes the set of parametric rules that are not manipulable via (pairwise) merging or splitting. Although non-manipulability via splitting is incompatible with the basic requirements of *claims boundedness*, *limited liability*, and *absolute priority*, Calleja et al. (2021) identify a broad class of financial rules immune to manipulations via merging and compatible with these requirements. Therefore, an interesting line of research could be the identification of the class of financial rules that fulfill non-manipulability via (pairwise) merging.

The model of Eisenberg and Noe (2001) implicitly assumes uniform priority among all liabilities in the financial system during the settlement process. On the contrary, according to many bankruptcy regulations, some claims (or agents) in the system may take priority over others, for instance, by forcing payments to workers or the government first. Thus, an interesting avenue of research is to investigate generalizations of the priority rules (relative to an order on the agents) or the random arrival rule for classical bankruptcy problems. Recently, Flores-Szwagrzak et al. (2019) considered a more general setting in which the order on the agents need not be strict, and the proportional rule is applied to each class of indifferent agents, taking into consideration different levels of repayment priorities. In this setting, analyzing the adaptability of the properties described by Flores-Szwagrzak et al. (2019) to characterize the set of priority-augmented proportional rules in the domain of standard claims problems could serve as a starting point.

CRediT authorship contribution statement

Pedro Calleja: Conceptualization, Formal analysis, Investigation, Methodology. **Francesc Llerena:** Conceptualization, Formal analysis, Investigation, Methodology, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare the have no known competing financial interests or personal relationships that could have appeared to influence the work of this paper.

Data availability

No data was used for the research described in the article.

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Appendix A. Tarski's fixed-point theorem

A *lattice* is a pair $(A, \leq)$ formed by a non-empty set A and a transitive and antisymmetric binary relation $\leq$ on A that determines a partial order on A such that, for any two elements $x, y \in A$, there is a *supremum (join)*, denoted by $x \vee y$, and an *infimum (meet)*, denoted by $x \wedge y$. We write $x < y$ if $x \leq y$ but $x \neq y$. The *supremum* $x \vee y$ is the unique element of A such that $x, y \leq x \vee y$ and if $z \in A$ is such that $z \geq x, y$, then $z \geq x \vee y$. The *infimum* $x \wedge y$ is the unique element of A such that $x, y \geq x \wedge y$ and if $z \in A$ is such that $z \leq x, y$, then $z \leq x \wedge y$. The lattice $(A, \leq)$ is called *complete* if every non-empty subset $B \subseteq A$ has a *supremum* and an *infimum*. Given two elements $x, y \in A$ with $x \leq y$, we denote by $[x, y]$ the interval with the endpoints x and y, i.e., $[x, y] = \{z \in A \mid x \leq z \leq y\}$. Clearly, $([x, y], \leq)$ is a lattice, and it is a complete lattice if $(A, \leq)$ is complete. We shall consider functions $f : B \to C$, where $B, C \subseteq A$. Such a function f is called *non-decreasing* if, for any pair of elements $x, y \in B$, $x \leq y$ implies $f(x) \leq f(y)$. A *fixed point* of f is an element x of B such that $x = f(x)$. Let $FIX(f)$ denote the set of fixed-points of f. The Tarski's fixed-point theorem states that if $(A, \leq)$ is a complete lattice and $f : B \to C$ is a non-decreasing function, then $(\text{FIX}(f), \leq)$ is a complete lattice.

Appendix B. Proofs of Section 2

Proof (*Proposition* 1)**.** Let β be a bankruptcy rule satisfying WCONT and NMC, and $(N, E, c) \in \mathcal{B}$. If $c_i \in \mathbb{Q}_+$ for all $i \in N$ then, by Theorem 1 in Calleja and Llerena (2022), $\beta(N, E, c) = PR(N, E, c)$. If not, there exists a sequence of bankruptcy problems $\{(N, E, c^n)\}_{n \in \mathbb{N}}$ converging to (N, E, c) with $c_i^n \in \mathbb{Q}_+$ for all $i \in N$ and all $n \in \mathbb{N}$. Again by Theorem 1 in Calleja and Llerena (2022), $PR(N, E, c^n) = \beta(N, E, c^n)$ for all $n \in \mathbb{N}$. By WCONT, there exists a subsequence $\{(N, E, c^{n_k})\}_{n_k \in \mathbb{N}}$ converging to (N, E, c) such that $PR(N, E, c) = \lim_{n_k \to \infty} PR(N, E, c^{n_k}) = \lim_{n_k \to \infty} \beta(N, E, c^{n_k}) = \beta(N, E, c)$.

Proof (*Lemma* 1)**.** Let σ be a financial rule satisfying **CB**, **LL**, and **AP**, $(N, L, e) \in \mathcal{F}$, and $P \in \sigma(N, L, e)$. By **LL**, $E_i(P, e) \geq 0$ for all $i \in N$. If $E_i(P, e) = 0$, then $e_i + \sum_{k \in N} P_{ki} = \bar{P}_i \leq \bar{L}_i$, where the inequality comes from **CB**. If $E_i(P, e) > 0$, by **AP** and **CB**, $\bar{P}_i = \bar{L}_i$ and thus $e_i + \sum_{k \in N} P_{ki} > \bar{P}_i = \bar{L}_i$. Hence, $\bar{P}_i = \min\{e_i + \sum_{k \in N} P_{ki}, \bar{L}_i\}$. To see the reverse implication, let σ be a financial rule fulfilling **CB**, $(N, L, e) \in \mathcal{F}$, and $P \in \sigma(N, L, e)$. If $\bar{P}_i = \min\{e_i + \sum_{k \in N} P_{ki}, \bar{L}_i\}$, for all $i \in N$, then $E_i(P, e) = e_i + \sum_{k \in N} P_{ki} - \bar{P}_i \geq 0$, which proves **LL**. To check **AP**, select $i \in N$ and suppose that $E_i(P, e) > 0$. Then, $e_i + \sum_{k \in N} P_{ki} > \bar{P}_i$ and thus $\bar{P}_i = \bar{L}_i$. $\square$

Appendix C. Proofs of Section 3

Proof (*Lemma* 2)**.**

1. Let $\boldsymbol{\beta} = (\beta^i)_{i \in \mathbb{N}}$ be an inventory of resource monotonic bankruptcy rules and $\varepsilon = (N, L, e) \in \mathcal{F}$. Since, for all $i \in N$, β^i satisfies RM, by Tarski's theorem the set of fixed-points $FIX(\Phi^{\varepsilon, \boldsymbol{\beta}})$ is non-empty and forms a complete lattice. Let $\mathbf{t} \in [\mathbf{0}, \bar{L}]$ be an arbitrary element of $FIX(\Phi^{\varepsilon, \boldsymbol{\beta}})$ and $P^{\mathbf{t}} \in \mathcal{M}(N)$ defined by $P^{\mathbf{t}}_{ij} = \beta^i_j \left(N \setminus \{i\}, \mathbf{t}_i, (L_{ij})_{j \in N \setminus \{i\}}\right)$ for all $i, j \in N$. As $\bar{P}^{\mathbf{t}} = \mathbf{t}$, for all $i \in N$ we have that
$$\begin{aligned} E_i(P^{\mathbf{t}}, e) &= e_i + \sum_{k \in N} P^{\mathbf{t}}_{ki} - \bar{P}^{\mathbf{t}}{}_i \\ &= e_i + \sum_{k \in N} P^{\mathbf{t}}_{ki} - \min\left\{e_i + \sum_{k \in N} P^{\mathbf{t}}_{ki}, \bar{L}_i\right\} \\ &= \max\left\{0, e_i + \sum_{k \in N} P^{\mathbf{t}}_{ki} - \bar{L}_i\right\}. \end{aligned} \tag{10}$$
Let $\mathbf{t}^+$ be the supremum of $FIX(\Phi^{\varepsilon, \boldsymbol{\beta}})$ and $P^{\mathbf{t}^+} \in \mathcal{M}(N)$ the corresponding matrix. Since $\mathbf{t}^+ \geq \mathbf{t}$, by RM of β^i for all $i \in N$, we have that $P^{\mathbf{t}^+} \geq P^{\mathbf{t}}$ and thus, from (10), $E_i(P^{\mathbf{t}^+}, e) \geq E_i(P^{\mathbf{t}}, e)$. If there is $i \in N$ such that $E_i(P^{\mathbf{t}^+}, e) > E_i(P^{\mathbf{t}}, e)$, then $\sum_{i \in N} e_i = \sum_{i \in N} E_i(P^{\mathbf{t}^+}, e) > \sum_{i \in N} E_i(P^{\mathbf{t}}, e) = \sum_{i \in N} e_i$ getting a contradiction. Thus, $E(P^{\mathbf{t}^+}, e) = E(P^{\mathbf{t}}, e)$, which finishes the proof.
2. It is a direct consequence of item 1.
3. It is a direct consequence of item 1. $\square$

Appendix D. Proofs of Section 4

Proof (*Lemma* 3)**.** Let σ be a proportional financial rule. Hence, σ satisfies **CB**, **LL**, and **AP**. Let $\{\varepsilon^n = (N, L^n, e^n)\}_{n \in \mathbb{N}}$ be a sequence of financial systems converging to $\varepsilon = (N, L, e)$, $P \in \sigma(\varepsilon)$, and $\{P^n \in \sigma(\varepsilon^n)\}_{n \in \mathbb{N}}$ be a sequence of clearing payment matrices. By **CB**, $0 \leq P^n \leq L^n$ for all $n \in \mathbb{N}$. Therefore, by the Bolzano–Weierstrass theorem,[13] we can suppose, w.l.o.g., that the sequence $\{P^n \in \sigma(\varepsilon^n)\}_{n \in \mathbb{N}}$ converges to

[13] In real analysis, this result states that every bounded sequence in the finite-dimensional Euclidean space $\mathbb{R}^n$ has a convergent subsequence.

P^*. Let $\{E(P^n, e^n)\}_{n\in\mathbb{N}}$ be the associated sequence of equity values. We claim that $\{E(P^n, e^n)\}_{n\in\mathbb{N}}$ converges to $E(P,e)$.

To prove it, we first see that

$$E(P^*, e) = E(P, e). \tag{11}$$

By **LL**, **AP**, and Corollary 1, $\bar{P}^n \in FIX(\Phi^{\varepsilon^n,\mathbb{PR}})$ for all $n \in \mathbb{N}$. Taking the limit when $n \to \infty$ we have that

$$\begin{aligned}\bar{P}^*_i &= \lim_{n\to\infty} \bar{P}^n_i \\ &= \lim_{n\to\infty} \min\left\{ e^n_i + \sum_{k\in N} PR^k_i\left(N\setminus\{k\}, \bar{P}^n_k, \left(L^n_{kj}\right)_{j\in N\setminus\{k\}}\right), \bar{L}^n_i \right\} \\ &\underset{\text{CONT of } PR}{=} \min\left\{ e_i + \sum_{k\in N} PR^k_i\left(N\setminus\{k\}, \bar{P}^*_k, \left(L_{kj}\right)_{j\in N\setminus\{k\}}\right), \bar{L}_i \right\},\end{aligned}$$

for all $i \in N$, where the last equality follows from the continuity of the proportional bankruptcy rule. Thus, $\bar{P}^* \in FIX(\Phi^{\varepsilon,\mathbb{PR}})$ and, consequently, $E(P^*, e) = E(P, e)$, which follows from Lemma 2 and the observation that since $P \in \sigma(\varepsilon)$, by **LL**, **AP**, and Corollary 1, $\bar{P} \in FIX(\Phi^{\varepsilon,\mathbb{PR}})$.

Finally, for all $i \in N$, we obtain

$$\begin{aligned}\lim_{n\to\infty} E_i(P^n, e^n) &= \lim_{n\to\infty}\Bigg(e^n_i + \sum_{k\in N} PR^k_i\left(N\setminus\{k\}, \bar{P}^n_k, \left(L^n_{kj}\right)_{j\in N\setminus\{k\}}\right) \\ &\quad - \sum_{k\in N} PR^i_k\left(N\setminus\{i\}, \bar{P}^n_i, \left(L^n_{ij}\right)_{j\in N\setminus\{i\}}\right)\Bigg) \\ &\underset{\text{CONT of } PR}{=} e_i + \sum_{k\in N} PR^k_i\left(N\setminus\{k\}, \bar{P}^*_k, \left(L_{kj}\right)_{j\in N\setminus\{k\}}\right) \\ &\quad - \sum_{k\in N} PR^i_k\left(N\setminus\{i\}, \bar{P}^*_i, \left(L_{ij}\right)_{j\in N\setminus\{i\}}\right) \\ &= E_i(P^*, e) \\ &\underset{(11)}{=} E_i(P, e),\end{aligned}$$

which concludes the proof. □

Proof (*Lemma* 4).

Let σ be a proportional financial rule and $\varepsilon = (N, L, e)$, $\varepsilon' = (N', L', e')$ two financial systems as described in (5).

First, we prove item (a). Let $P \in \sigma(\varepsilon)$ and define $P' \in \mathcal{M}(N')$ as in (8) from P. Observe that P' is well defined and unique. We are going to prove that $P' \in \sigma^{\mathbb{PR}}_{max}(\varepsilon')$. Suppose, w.l.o.g., $\bar{L'}_i \neq 0$ for all $i \in N'$. Then, for all $i, j \in N'$, we claim that

$$P'_{ij} = \frac{L'_{ij}}{\bar{L'}_i}\bar{P'}_i. \tag{12}$$

First, let us note that if $i \in N' \setminus \{m\}$, then

$$\begin{aligned}\bar{P'}_i &= \textstyle\sum_{j\in N'} P'_{ij} \\ &= \textstyle\sum_{j\in N'\setminus\{m\}} P'_{ij} + P'_{im} \\ &= \textstyle\sum_{j\in N'\setminus\{m\}} P_{ij} + P_{im} + \sum_{k\in N\setminus N'} P_{ik} \\ &= \bar{P}_i.\end{aligned} \tag{13}$$

Otherwise, if $i = m$ we have that

$$\begin{aligned}\bar{P'}_m &= \textstyle\sum_{j\in N'} P'_{mj} \\ &= \textstyle\sum_{j\in N'}\left(P_{mj} + \sum_{k\in N\setminus N'} P_{kj}\right) \\ &\underset{P_{kj}=P_{mj}}{=} \textstyle\sum_{j\in N'}\left(P_{mj} + \sum_{k\in N\setminus N'} P_{mj}\right) \\ &= \textstyle\sum_{j\in N'}\left(P_{mj} + (|N\setminus N'|)P_{mj}\right) \\ &= (|N\setminus N'|+1)\textstyle\sum_{j\in N'} P_{mj} \\ &= (|N\setminus N'|+1)\bar{P}_m,\end{aligned} \tag{14}$$

where the last equality comes from the fact that $P_{mj} = 0$ for all $j \in N \setminus N'$.

Now, to prove (12) we distinguish three cases:

Case 1: $i, j \in N' \setminus \{m\}$. In this situation,

$$P'_{ij} = P_{ij} = \frac{L_{ij}}{\bar{L}_i}\bar{P}_i \underset{(13)}{=} \frac{L'_{ij}}{\bar{L'}_i}\bar{P'}_i.$$

Case 2: $i \in N' \setminus \{m\}$ and $j = m$. In this situation,

$$\begin{aligned}P'_{im} &= P_{im} + \textstyle\sum_{k\in N\setminus N'} P_{ik} \\ &= \left(L_{im} + \textstyle\sum_{k\in N\setminus N'} L_{ik}\right)\frac{\bar{P}_i}{\bar{L}_i} \\ &= L'_{im}\frac{\bar{P}_i}{\bar{L'}_i} \\ &\underset{(13)}{=} \frac{L'_{im}}{\bar{L'}_i}\bar{P'}_i.\end{aligned}$$

Case 3: $i = m$ and $j \in N' \setminus \{m\}$. In this situation,

$$\begin{aligned}P'_{mj} &= P_{mj} + \textstyle\sum_{k\in N\setminus N'} P_{kj} \\ &\underset{P_{kj}=P_{mj}}{=} P_{mj} + \textstyle\sum_{k\in N\setminus N'} P_{mj} \\ &= \frac{L_{mj}}{\bar{L}_m}\bar{P}_m(|N\setminus N'|+1) \\ &\underset{(14)}{=} \frac{L_{mj}}{\bar{L}_m}\bar{P'}_m \\ &= \frac{L'_{mj}/(|N\setminus N'|+1)}{(\sum_{k\in N'\setminus\{m\}} L'_{mk})/(|N\setminus N'|+1)}\bar{P'}_m \\ &= \frac{L'_{mj}}{\bar{L'}_m}\bar{P'}_m.\end{aligned}$$

Thus, (12) holds.

Next, we show that $\bar{P'} \in FIX(\Phi^{\varepsilon',\mathbb{PR}})$. Indeed, if $i \in N' \setminus \{m\}$, then

$$\begin{aligned}\bar{P'}_i &= \bar{P}_i \\ &= \min\left\{ e_i + \sum_{k\in N'\setminus\{m\}} P_{ki} + P_{mi} + \sum_{k\in N\setminus N'} P_{ki}, \bar{L}_i\right\} \\ &= \min\left\{ e'_i + \sum_{k\in N'\setminus\{m\}} P'_{ki} + P'_{mi}, \bar{L'}_i\right\} \\ &= \min\left\{ e'_i + \sum_{k\in N'} P'_{ki}, \bar{L'}_i\right\} \\ &\underset{(12)}{=} \min\left\{ e'_i + \sum_{k\in N'} \frac{L'_{ki}}{\bar{L'}_k}\bar{P'}_k, \bar{L'}_i\right\} \\ &= \min\left\{ e'_i + \sum_{k\in N'} PR^k_i\left(N'\setminus\{k\}, \bar{P'}_k, \left(L'_{kj}\right)_{j\in N'\setminus\{k\}}\right), \bar{L'}_i\right\}.\end{aligned}$$

In a similar way, and taking into account that $\bar{P'}_m = (|N \setminus N'| + 1)\bar{P}_m$, we obtain

$$\bar{P'}_m = \min\left\{ e'_m + \sum_{k\in N'} PR^k_m\left(N'\setminus\{k\}, \bar{P'}_k, \left(L'_{kj}\right)_{j\in N'\setminus\{k\}}\right), \bar{L'}_m\right\}.$$

Hence, $\bar{P'} \in FIX(\Phi^{\varepsilon',\mathbb{PR}})$. Moreover, since σ satisfies **CB**, **LL**, and **AP**, for all $P'' \in \sigma(\varepsilon')$ it holds that $\bar{P''} \in FIX(\Phi^{\varepsilon',\mathbb{PR}})$. Finally, making use of Lemma 2, we have that $E(P', e') = E(P'', e')$. But then, for all $i \in N' \setminus \{m\}$, we obtain

$$\begin{aligned}E_i(P'', e') &= E_i(P', e') \\ &= e'_i + \sum_{k\in N'} P'_{ki} - \sum_{k\in N'} P'_{ik} \\ &= e_i + \sum_{k\in N'\setminus\{m\}} P'_{ki} + P'_{mi} - \sum_{k\in N'\setminus\{m\}} P'_{ik} - P'_{im} \\ &= e_i + \sum_{k\in N'\setminus\{m\}} P_{ki} + P_{mi} \\ &\quad + \textstyle\sum_{k\in N\setminus N'} P_{ki} - \sum_{k\in N'\setminus\{m\}} P_{ik} - P_{im} - \sum_{k\in N\setminus N'} P_{ik} \\ &= e_i + \sum_{k\in N} P_{ki} - \sum_{k\in N} P_{ik} \\ &= E_i(P, e).\end{aligned}$$

To conclude, we prove item (b). Let $P' \in \sigma(\varepsilon')$ and define $P \in \mathcal{M}(N)$ as follows: for all $i, j \in N' \setminus \{m\}$, $P_{ij} = P'_{ij}$; for all $i \in N' \setminus \{m\}$ and all

$j \in N \setminus N' \cup \{m\}, P_{ij} = P'_{im}/(|N \setminus N'|+1)$ and $P_{ji} = P'_{mi}/(|N \setminus N'|+1)$; and for all $i, j \in N \setminus N' \cup \{m\}, P_{ij} = 0$. Note that $\bar{P}_i = \bar{P}'_i$ for all $i \in N' \setminus \{m\}$; $\bar{P}_j = \bar{P}'_m/(|N \setminus N'|+1)$ for all $j \in N \setminus N' \cup \{m\}$. From this point, the same arguments as before lead to $\bar{P} \in FIX(\Phi^{\varepsilon,\mathbb{PR}})$ and that $E_i(P,e) = E_i(P',e')$ for all $i \in N' \setminus \{m\}$. This concludes the proof. □

Next, we show that the axioms in Theorem 1 are logically independent.

- (All except *compatibility*): for all $(N,L,e) \in \mathcal{F}$, let $\sigma^1(N,L,e) = \sigma_+^{\mathbb{PR}}(N,2L,e)$. Clearly, σ^1 does not meet *claim boundedness* and thus neither *compatibility*. Since $\sigma_+^{\mathbb{PR}}$ satisfies *limited liability*, *absolute priority*, *equity-continuity*, and *non-manipulability by clones*, it follows that σ^1 inherits these properties.
- (All except *limited liability*): for all $(N,L,e) \in \mathcal{F}$, let $\sigma^2(N,L,e) = \{L\}$. Obviously, σ^2 satisfies *compatibility*, since any bankruptcy rule distributing an estate equal to its total liabilities equals, by *claim boundedness* and *budget balance*, the vector of its liabilities. Clearly, σ^2 satisfies *absolute priority*. *Equity-continuity* comes from σ^2 satisfying *lower hemicontinuity* and Lemma 5, while *non-manipulability by clones* holds from σ^2 satisfying *invariance to mitosis* and Lemma 7. However, σ^2 does not meet *limited liability* since there might exist firms with insufficient resources to cover all its liabilities and, consequently, ending up with a negative equity value.
- (All except *absolute priority*): for all $(N,L,e) \in \mathcal{F}$, let $\sigma^3(N,L,e) = \{\mathbf{0}\}$ where $\mathbf{0} \in \mathcal{M}(N)$ denotes the zero matrix. Note that σ^3 satisfies *compatibility* since any bankruptcy rule distributing an estate of zero equals the zero vector. Clearly, σ^3 satisfies *limited liability*. *Equity-continuity* and *non-manipulability by clones* come from *lower hemicontinuity* (Lemma 5) and *invariance to mitosis* (Lemma 7), respectively. However, it does not meet *absolute priority* since the equity value of each firm coincides with its initial endowment but it could be positive.
- (All except *non-manipulability by clones*): for all $(N,L,e) \in \mathcal{F}$, let $\sigma^4(N,L,e) = \sigma_-^{\mathbb{CEA}}(N,L,e)$. Clearly, σ^4 meets *compatibility*, *limited liability*, and *absolute priority*. Obviously, σ^4 is a constrained equal awards financial rule. Moreover, the proof of Lemma 3 can be followed almost step by step, if we take a constrained equal awards rule, instead. Thus, σ^4 satisfies *equity-continuity*. Finally, to see that it fails to satisfy *non-manipulability by clones*, consider the financial system ε as defined in Example 2. Then,

$$\sigma^4(\varepsilon) = \left\{ P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Suppose now that agent 3 splits into clones 3 and 4, defining the corresponding four agents financial system ε'. Some easy algebra yields to

$$\sigma^4(\varepsilon') = \left\{ P' = \begin{pmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

Observe that $E_3(P,e) = 1/2 < 2/3 = E_3(P',e') + E_4(P',e')$, showing that constrained equal awards financial rules may provide incentives to split.
- (All except *equity-continuity*): define first the bankruptcy rule β^* that gives priority to non-rational claims, i.e, belonging to the set $\mathbb{R} \setminus \mathbb{Q}_+$, over claims in $\mathbb{Q}_+$, and distributing any amount proportionally in each group. Formally, let $(N,E,c) \in \mathcal{B}$ and $N_{\mathbb{Q}_+}$ be the set of agents with a positive rational claim:
  - If $\sum_{k \in N \setminus N_{\mathbb{Q}_+}} c_k \geq E$, then

    $\beta_i^*(N,E,c) = PR_i(N \setminus N_{\mathbb{Q}_+}, E, c_{N \setminus N_{\mathbb{Q}_+}})$ for all $i \in N \setminus N_{\mathbb{Q}_+}$

    and

    $\beta_i^*(N,E,c) = 0$ for all $i \in N_{\mathbb{Q}_+}$.
  - If $\sum_{k \in N \setminus N_{\mathbb{Q}_+}} c_k < E$, then

    $\beta_i^*(N,E,c) = c_i$ for all $i \in N \setminus N_{\mathbb{Q}_+}$

    and

    $$\beta_i^*(N,E,c) = PR_i\left(N_{\mathbb{Q}_+}, E - \sum_{k \in N \setminus N_{\mathbb{Q}_+}} c_k, c_{N_{\mathbb{Q}_+}}\right) \text{ for all } i \in N_{\mathbb{Q}_+}.$$

  Note that β^* is *resource monotonicity* but it does not satisfies *weak continuity*. So, the financial rule $\sigma^5(N,L,e) = \sigma_+^{\beta^*}(N,L,e)$ is well defined. Obviously, σ^5 satisfies *compatibility* (with respect to β^*). This rule was first introduced in Csóka and Herings (2021) and clearly satisfies *limited liability* and *absolute priority*. As they argue, it meets *invariance to mitosis* and, thus, from Lemma 7, also *non-manipulability by clones*. To see that it does not satisfy *equity-continuity*, consider the financial system $\varepsilon = (N,L,e) \in \mathcal{F}$ with set of players $N = \{1,2,3\}$, initial operating cash flows $e = (1,0,0)$, and matrix of liabilities

$$L = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

  Now, consider the sequence of financial systems $\{\varepsilon^n = (N, L^n, e^n)\}_{n \in \mathbb{N}}$ with set of players $N = \{1,2,3\}$, initial operating cash flows $e^n = (1,0,0)$, and matrices of liabilities

$$L^n = \begin{pmatrix} 0 & 1 + \frac{\sqrt{2}}{n} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ for all } n \in \mathbb{N}.$$

  Clearly, $\{\varepsilon^n\}_{n \in \mathbb{N}}$ converges to ε. It is not difficult to check that

$$\sigma^5(\varepsilon) = \left\{ P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \text{ and}$$

$$\sigma^5(\varepsilon^n) = \left\{ P^n = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \text{ for all } n \in \mathbb{N}.$$

  Note that $\{E(P^n,e^n)\}_{n \in \mathbb{N}}$ converges to $(0,1,0)$ while $E(P,e) = (0,1/2,1/2)$. Hence, σ^5 does not meet *equity-continuity*.

Table 1
Solutions and properties.

	σ^1	σ^2	σ^3	σ^4	σ^5
Compatibility	No	Yes	Yes	Yes	Yes
Limited liability	Yes	No	Yes	Yes	Yes
Absolute priority	Yes	Yes	No	Yes	Yes
Non-manipulability by clones	Yes	Yes	Yes	No	Yes
Equity-continuity	Yes	Yes	Yes	Yes	No

Table 1 collects the financial rules and the axioms they satisfy.

Appendix E. Proofs of Section 5

Proof (*Lemma* 5**).**
Let $\varepsilon = (N,L,e) \in \mathcal{F}$ and $\{\varepsilon^n = (N,L^n,e^n)\}_{n \in \mathbb{N}}$ be a sequence of financial systems converging to ε. Let σ be a financial rule satisfying *lower hemicontinuity* and $P \in \sigma(\varepsilon)$. By **LHC**, there exists a sequence of clearing payment matrices $\{P^n \in \sigma(\varepsilon^n)\}_{n \in \mathbb{N}}$ converging to P. Then, for

the associated sequence of equity values $\{E(P^n, e^n)\}_{n\in\mathbb{N}}$ we have, for all $i \in N$,

$$\begin{aligned}\lim_{n\to\infty} E_i(P^n, e^n) &= \lim_{n\to\infty}\Big(e_i^n + \sum_{k\in N} P_{ki}^n - \sum_{k\in N} P_{ik}^n\Big)\\ &= e_i + \sum_{k\in N} P_{ki} - \sum_{k\in N} P_{ik}\\ &= E_i(P,e)\end{aligned}$$

which proves **E-CONT** of σ. □

Proof (*Lemma 6*).

Let σ be a financial rule satisfying **CB**, **LL**, **AP**, and **I**. Let $(N, L, e) \in \mathcal{F}$, $P \in \sigma(N, L, e)$, and $i, j \in N$ such that $e_i = e_j$, $L_{ij} = L_{ji}$, $L_{ik} = L_{jk}$, and $L_{ki} = L_{kj}$ for all $k \in N \setminus \{i, j\}$. For all $k \in N \setminus \{ij\}$, since $L_{ki} = L_{kj}$, by **I** we have that

$$P_{ki} = P_{kj}. \quad (15)$$

By **LL**, $E_i(P,e) \geq 0$ and $E_j(P,e) \geq 0$. If $E_i(P,e) = E_j(P,e)$ we are done. If not, it is sufficient to consider two cases: **(a)** $E_i(P,e) > 0$ and $E_j(P,e) > 0$; **(b)** $E_i(P,e) > 0$ and $E_j(P,e) = 0$.

In case **(a)**, by **AP** and **CB**, $P_{ik} = L_{ik}$ and $P_{jk} = L_{jk}$ for all $k \in N$. In particular, $P_{ij} = L_{ij} = L_{ji} = P_{ji}$. Hence, since $L_{ik} = L_{jk}$ for all $k \in N \setminus \{ij\}$, we have $\bar{P}_i = \bar{L}_i = \bar{L}_j = \bar{P}_j$. Moreover, by (15) and $P_{ij} = P_{ji}$, we obtain

$$E_i(P,e) = e_i + \sum_{k\in N\setminus\{i\}} P_{ki} - \bar{L}_i = e_j + \sum_{k\in N\setminus\{j\}} P_{kj} - \bar{L}_j = E_j(P,e).$$

In case **(b)**, by **AP** and **CB**, $P_{ik} = L_{ik}$ for all $k \in N$ and thus

$$\bar{P}_i = \bar{L}_i \quad (16)$$

Then,

$$\begin{aligned}E_j(P,e) &= e_j + \sum_{k\in N\setminus\{i,j\}} P_{kj} + P_{ij} - \bar{P}_j\\ &= e_j + \sum_{k\in N\setminus\{i,j\}} P_{kj} + L_{ij} - \bar{P}_j\\ &\underset{(15)}{=} e_i + \sum_{k\in N\setminus\{i,j\}} P_{ki} + L_{ji} - \bar{P}_j\\ &\underset{CB}{\geq} e_i + \sum_{k\in N} P_{ki} - \bar{L}_j\\ &= e_i + \sum_{k\in N} P_{ki} - \bar{L}_i\\ &\underset{(16)}{=} e_i + \sum_{k\in N} P_{ki} - \bar{P}_i\\ &= E_i(P,e),\end{aligned}$$

in contradiction with $E_i(P,e) > E_j(P,e) = 0$.

Hence, in both cases, $E_i(P,e) = E_j(P,e)$, which implies **ETE**. □

Proof (*Proposition 2*).

First, we show that [1] $\Longrightarrow$ [2]. Let σ be a financial rule compatible with $\boldsymbol{\beta} = (\beta^i)_{i\in\mathbb{N}}$ fulfilling **LL** and **AP**. Then, from CB of all β^i, σ satisfies **CB**. Hence, by Lemma 6, if σ satisfies **I** then also **ETE**.

Secondly, we show that [2] $\Longrightarrow$ [3]. Suppose that σ satisfies **ETE**, we prove that each bankruptcy rule in $\boldsymbol{\beta}$ fulfills ETE. Indeed, select an arbitrary $i \in \mathbb{N}$ and let $(N, E, c) \in \mathcal{B}$ with $i \in \mathbb{N} \setminus N$ and $j, k \in N$ such that $c_j = c_k$. Define the associated financial system $(\bar{N}, L, e) \in \mathcal{F}$, as in Section 2.2, being $\bar{N} = N \cup \{i\}$; $L_{lh} = 0$ for all $l, h \in N$, $L_{il} = c_l$ and $L_{li} = 0$ for all $l \in N$; $e_i = E$ and $e_l = 0$ for all $l \in N$. Next, we see that $\sigma(\bar{N}, L, e)$ selects a unique payment matrix P. As σ satisfies **CB** (received from CB of all β^i), $P_{lh} = 0$ for all $l, h \in N$ and $P_{li} = 0$ for all $l \in N$. Moreover, since σ is compatible with $\boldsymbol{\beta}$, $P_{il} = \beta^i_l(N, E, c)$ for all $l \in N$, which is unique by definition of β^i. Thus, $E_j(P,e) = e_j + \sum_{l\in\bar{N}} P_{lj} - \sum_{l\in\bar{N}} P_{jl} = P_{ij} = \beta^i_j(N, E, c)$ and, analogously, $E_k(P,e) = P_{ik} = \beta^i_k(N, E, c)$. To finish, observe that since $e_j = e_k = 0$, $L_{jl} = L_{kl} = 0$ for all $l \in \bar{N}$, $L_{lj} = L_{lk} = 0$ for all $l \in N$ and $L_{ij} = c_j = c_k = L_{ik}$, players j and k are symmetric in $(\bar{N}, L, e)$ and then, by **ETE**, $\beta^i_j(N, E, c) = E_j(P,e) = E_k(P,e) = \beta^i_k(N, E, c)$, which proves ETE of β^i.

Finally, we show that [3] $\Longrightarrow$ [1]. Let (N, L, e) be a financial system with $i, j, k \in N$ such that $L_{ij} = L_{ik}$, and let $P \in \sigma(N, L, e)$. Then, as σ is compatible with $\boldsymbol{\beta}$, by ETE of β^i it holds that $P_{ij} = \beta^i_j(N \setminus \{i\}, \bar{P}_i, (L_{il})_{l\in N\setminus\{i\}}) = \beta^i_k(N \setminus \{i\}, \bar{P}_i, (L_{il})_{l\in N\setminus\{i\}}) = P_{ik}$, which shows that σ satisfies **I**. □

Proof (*Lemma 7*).

Let σ be a financial rule satisfying **IM**. Let $(N, L, e), (N', L', e')$ be two related financial systems as described in (5). Let $P \in \sigma(N, L, e)$ then, by **IM**, there exist $P' \in \sigma(N', L', e')$ satisfying the conditions in (8). From the relation between P and P', for all $i \in N' \setminus \{m\}$, it follows that

$$\begin{aligned}E_i(P', e') &= e'_i + \sum_{k\in N'} P'_{ki} - \sum_{k\in N'} P'_{ik}\\ &= e'_i + \sum_{k\in N'\setminus\{m\}} P'_{ki} + P'_{mi} - \sum_{k\in N'\setminus\{m\}} P'_{ik} - P'_{im}\\ &= e_i + \sum_{k\in N'\setminus\{m\}} P_{ki} + P_{mi}\\ &\quad + \textstyle\sum_{k\in N\setminus N'} P_{ki} - \sum_{k\in N'\setminus\{m\}} P_{ik} - P_{im} - \sum_{k\in N\setminus N'} P_{ik}\\ &= e_i + \sum_{k\in N} P_{ki} - \sum_{k\in N} P_{ik}\\ &= E_i(P,e).\end{aligned}$$

Following parallel arguments, by **IM**, for all $P' \in \sigma(N', L', e')$ there exist $P \in \sigma(N, L, e)$ satisfying the conditions in (8) and, consequently, for all $i \in N' \setminus \{m\}$ we also obtain $E_i(P', e') = E_i(P, e)$. Thus, σ satisfies **NMC**. □

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