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ON THE BASIN OF ATTRACTION OF A CRITICAL THREE-CYCLE OF A MODEL FOR THE SECANT MAP

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Abstract. We consider the secant method S_p applied to a real polynomial p of degree $d+1$ as a discrete dynamical system on $\mathbb{R}^2$. If the polynomial p has a local extremum at a point α then the discrete dynamical system generated by the iterates of the secant map exhibits a critical periodic orbit of period 3 or three-cycle at the point (α, α). We propose a simple model map $T_{a,d}$ having a unique fixed point at the origin which encodes the dynamical behaviour of S_p^3 at the critical three-cycle. The main goal of the paper is to describe the geometry and topology of the basin of attraction of the origin of $T_{a,d}$ as well as its boundary. Our results concern global, rather than local, dynamical behaviour. They include that the boundary of the basin of attraction is the stable manifold of a fixed point or contains the stable manifold of a two-cycle, depending on the values of the parameters of d (even or odd) and $a \in \mathbb{R}$ (positive or negative).

1. **Introduction.** A major goal in applied and theoretical mathematical modelling is to find stable equilibria which determines the expected behaviour of the phenomenon we are analyzing. Those equilibria are given by real (or complex) numbers, real (or complex) finite dimensional vectors, or functions belonging to an infinite dimensional space, depending on the nature of the model under consideration.

In the majority of cases, the stable equilibria determining the evolutionary steady states of any model turn out to be solutions of non-linear equations. In general, we cannot solve these equations explicitly. Accordingly, there is a long history of research studying different *algorithms* which efficiently find their solutions.

Among these algorithms the ones given by the special kind of discrete dynamical systems, known as root-finding algorithms, have been shown to be the most useful. Roughly speaking, a *root-finding algorithm* is a system such that for most of the initial conditions the asymptotic behaviour of the corresponding iterative process tends to one of the solutions of the non-linear equation determining the equilibria of the model. We observe that the condition of convergence *for most of the initial conditions* is a global phenomenon rather than (only) a local one. Indeed, this is the

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reason why the *global dynamics* of root-finding algorithms has been an important subject of study for dynamicists.

Moreover, when the model has more than one steady state the phase portrait of the root-finding algorithm splits into regions where the iterates of the seeds converge to different equilibria. Consequently, two natural questions arise. On the one hand, about the boundaries of these regions: do they have easy geometry and topology? what about the restricted dynamics over these boundaries? do they have positive measure? On the other hand, about the stable steady states: are there other stable behaviour of the algorithm unrelated to the steady states of the model? If they exist, there would be open sets of the dynamical plane where the root-finding algorithm is full of *bad* initial conditions. The answers to all these questions have had a great influence on the study of theoretical as well as applied discrete dynamical systems.

There is no discussion that the most famous root-finding algorithm is the well-known Newton's method. More concretely, assume the equation we want to solve is $p(x) = 0$, where, to simplify the exposition, we assume $p(x)$ to be a polynomial with $x \in \mathbb{R}$ or $x \in \mathbb{C}$ but the method extends to higher dimensional problems. *Newton's method* is the study of the dynamical system

$$x_{n+1} = N_p\left(x_n\right) := x_n - \frac{p(x_n)}{p'(x_n)}, \qquad x_0 \in \mathbb{R} \quad \text{or} \quad x_0 \in \mathbb{C}. \tag{1}$$

It is easy to see that $N_p(\alpha) = \alpha$ if and only if $p(\alpha) = 0$ (if $p^{(\ell)}(\alpha) = 0,\ 1 \leq \ell \leq k$, one can still use (1) after some modifications) and moreover if $x_0 \approx \alpha$ then $\{x_n := N_p^n(x_0)\} \to \alpha$ as $n \to \infty$. Consequently, Newton's method is a (one dimensional) dynamical systems whose fixed points correspond to the roots of p and they are local attractors. In fact, it is somehow *the canonical* roof-finding algorithm. The natural dynamical space for one dimensional Newton's method applied to degree d polynomials is $\mathbb{C}$ (rather than $\mathbb{R}$) due to the fundamental theorem of algebra, and it defines one of the most studied family of rational (holomorphic) maps on the Riemann sphere [5, 4, 16, 14]. See also [3, 1] for Newton's method applied to transcendental entire maps.

Despite its fundamental role, Newton's method has some limitations and the literature has explored other root-finding algorithms trying to avoid these weakness (for instance avoiding to compute the derivatives if their computational cost is too high) or to improve the efficiency of the method under certain hypothesis (for instance improving the local speed of convergence of the method to the root(s) of p).

Another basic root-finding algorithm is the *secant method* given by the dynamical system generated by the iterates of the 2-dimensional map

$$S(x,y) = S_p(x,y) := \left(y, y - p(y)\frac{x-y}{p(x)-p(y)}\right). \tag{2}$$

However, in contrast to Newton's method, secant's method is a two dimensional system and it does not require to compute any derivative of p. Nonetheless, as before, we have that $S(\alpha,\alpha) = (\alpha,\alpha)$ if and only if $p(\alpha) = 0$. Moreover if $(x_0, y_0) \approx (\alpha, \alpha)$ then

$$\{(x_n, y_n) := S^n(x_0, y_0)\} \to (\alpha, \alpha)$$

at least for simple roots of p (see [12] for multiple roots). We refer to [11, 2], and references therein, for a detailed discussion of the phase plane ($\mathbb{R}^2$ and $\mathbb{C}^2$) of the secant method.

One (unexpected) fact of the real secant map is that there are no finite periodic points of period two or three in $\mathbb{R}^2$. However, it has *finite* periodic points of period four and some of them determine the geometry and topology of the boundaries of the immediate basin of attraction of its fixed points (see [10]). Moreover, if we extend the domain of the secant method to *infinity* (that is, if we extend the secant map to $\mathbb{P}^1 \times \mathbb{P}^1$) a new three-cycle phenomenon arises. Indeed, in [2] (see also [11]) the authors showed that if $c \in \mathbb{R}$ satisfies that $p'(c) = 0$ (critical point) and $p(c)p''(c) \neq 0$ the secant method exhibits a *critical* three-cycle at (c, c) given by

$$(c,c) \overset{\mathrm{S}}{\mapsto} (c,\infty) \overset{\mathrm{S}}{\mapsto} (\infty,c) \overset{\mathrm{S}}{\mapsto} (c,c).$$

Moreover, the three-cycle has a basin of attraction whose geometry varies depending on the degree of the polynomial. However, its geometry and topology is quite similar among polynomials of the same degree. These basins, and their disparate geometry can be visualized in red in Figure 1 for concrete polynomials of degree of different parity.

The main goal of this work is to go deeper into the understanding of the geometry of the basin of the critical three-cycle by means of a model which captures the relevant information and allows us to give an accurate description.

Following the approach in [2] we assume, without lost of generality, that $c = 0$ and $p(0) = 1$. Thus, assuming also that $\deg(p) = d + 1$, the polynomial p writes as

$$p(x) = 1 + a_2x^2 + \ldots + a_{d+1}x^{d+1}, \tag{3}$$

where $d \geq 2$ and $a_2a_{d+1} \neq 0$. Using the natural extension of S at infinity, via the charts $\varphi_1(x, y) = (1/x, y)$ and $\varphi_2(x, y) = (x, 1/y)$, and some computations (explicit in [2]) the expression of S^3 near the origin is given by

$$S^3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y - \frac{(-a_2)^d}{a_{d+1}}(x+y)^d \\ y - 2\frac{(-a_2)^d}{a_{d+1}}(x+y)^d \end{pmatrix} + \mathcal{O}_{d+1}, \tag{4}$$

where $\mathcal{O}_{d+1}$ indicates terms bounded by $(|x| + |y|)^{d+1}$. The expression (4) motivates the introduction of the following model map $T_{a,d}$ which encodes the dominant terms of S^3 near the origin. Concretely,

$$T_{a,d} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y - a(x+y)^d \\ y - 2a(x+y)^d \end{pmatrix}, \tag{5}$$

where $a \neq 0$ is a parameter. We now are ready to state the main results of this paper about *the basin of attraction* of the origin of (5), defined as

$$\mathcal{A}_{a,d}(0) = \{(x,y) \in \mathbb{R}^2 \mid T^n_{a,d}(x,y) \to (0,0) \text{ as } n \to \infty\}, \tag{6}$$

depending on the parameters a and d. Sometimes the set $\mathcal{A}_{a,d}(0)$ is called *the stable set (of the origin)* but here we prefer to use the name basin of attraction.

Obviously, the origin is the unique fixed point of $T_{a,d}$ and $DT_d(0, 0)$ has eigenvalues 0 and 1. The 0 eigenvalue guarantees that $\mathcal{A}_{a,d}(0) \neq \emptyset$ but a complete topological and geometric description depends on the motion over the center manifold. The main theorem describes $\mathcal{A}_{a,d}(0)$ as well as its boundary $\partial\mathcal{A}_{a,d}(0)$ depending on a and d.

Main Theorem. *Let $\mathcal{A}_{a,d}(0)$ be the basin of attraction of the origin for the map $T_{a,d}$.*

(a) If d is even and $a \neq 0$ then $\mathcal{A}_{a,d}(0)$ is a compact set which is homeomorphic to a closed topological disk and the boundary of $\mathcal{A}_{a,d}(0)$ is the stable manifold of the origin. See Figure 2(a).

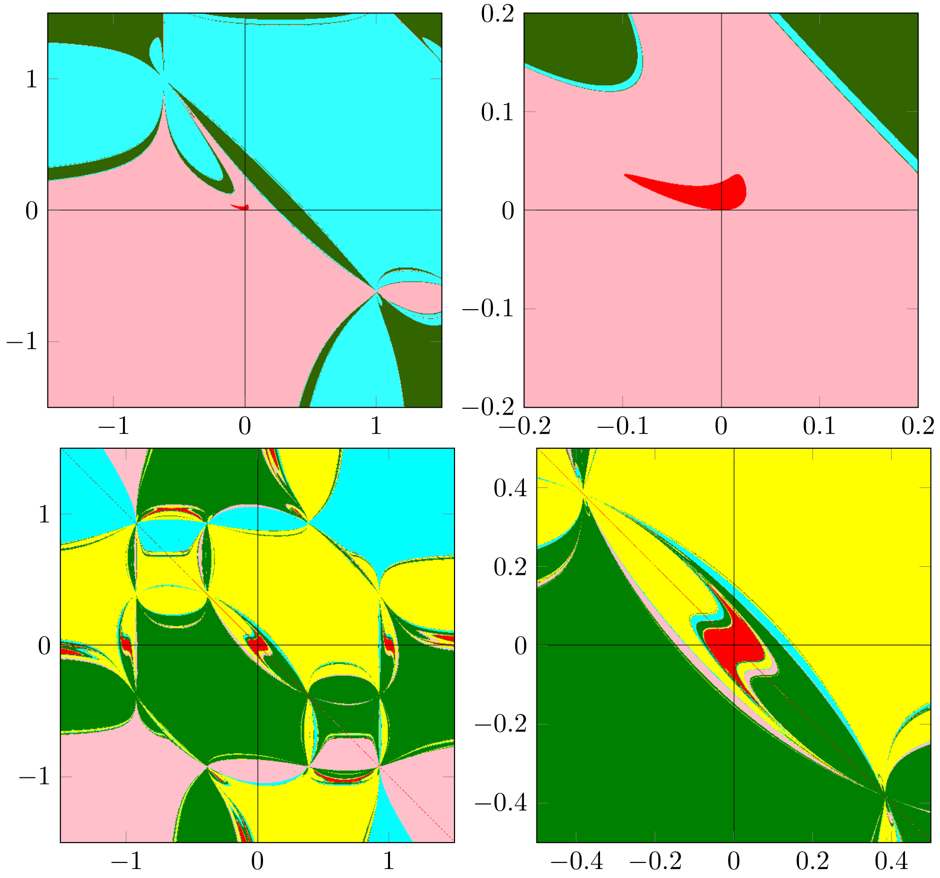
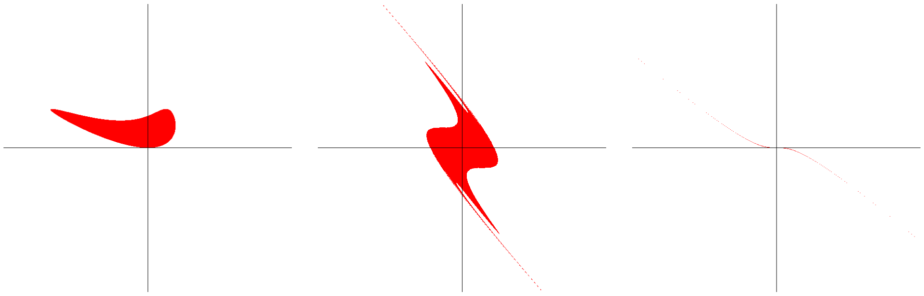


FIGURE 1. Phase planes of the secant map applied to the polynomials $p(x) = 1 - 2x^2 + x^3$ (first row) and $p(x) = 1 - 8x^2 + 8x^4$ (second row). In all the pictures we show in red the set of points converging towards the critical three-cycle $\{(0,0),(0,\infty),(\infty,0)\}$. The second column is a zoom near the origin of the first one.

(b) *If d is odd and $a > 0$ then $\mathcal{A}_{a,d}(0)$ is an open, simply connected, unbounded set. Moreover, $\partial\mathcal{A}_{a,d}(0)$ contains the stable manifold of a hyperbolic two-cycle $\{p_0, p_1\}$ lying on $\partial\mathcal{A}_{a,d}(0)$. See Figure 2(b).*

(c) *If d is odd and $a < 0$ then $\mathcal{A}_{a,d}(0)$ is the stable manifold of the origin. Moreover, $\mathcal{A}_{a,d}(0)$ is unbounded. See Figure 2(c).*

We finish with an important remark, somehow complementary, on the previous result to calibrate their value. On the one hand, from construction, system (5) encodes the information of system (4) as long as $(x, y) \approx (0, 0)$. But if one reads carefully the Main Theorem, we see that it does not refer to the dynamics in a given small neighbourhood of the origin, as for instance statement (b) of the Main Theorem is showing that $\mathcal{A}_{a,d}$, $a > 0$, is unbounded. Hence, *a priory*, there is no reason to argue that the Main Theorem can be *transported* to explain the red regions in Figure 1. However, comparing the top right picture in Figure 1 with Figure 2(a) (d even), or comparing the bottom right picture in Figure 1 with Figure 2(b) (d



(a) $d = 2$ and $a = 1$. (b) $d = 3$ and $a = 1$. (c) $d = 3$ and $a = -1$.

FIGURE 2. Phase plane of the map $T_{a,d}$ for different values of a and d. The basin of attraction $\mathcal{A}_{a,d}$ is shown in red.

odd) one can immediately see that (5) and (4) share more than expected. A better explanation for this connection, somehow global, will require future work.

In the companion paper [9] we study in more depth the boundary of $\mathcal{A}_{a,d}(0)$ when d is odd and a is positive and we show there is a (topologically transversally) homoclinic intersection between the stable and the unstable manifolds of the hyperbolic two-cycle $\{p_0, p_1\}$ and there are infinitely many periodic points (somehow chaotic dynamics) in $\partial\mathcal{A}_{a,d}(0)$.

The paper is organized as follows. In Section 2 we show that $T_{a,d}$ reduces to three cases: d even with $a = 1$, and d odd with $a = \pm 1$. In Section 3 we study the series expansions of the stable and center invariant manifolds of the origin. The statements (a) and (b) of the Main Theorem are proven in Sections 4 and 5, respectively. Finally, in Section 6 we prove statement (c) of the Main Theorem.

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2. Preliminaries and local dynamics near the origin.

A preliminary simple step is to show that, given $d \geq 2$ fixed, for most values of $a \neq 0$ the maps of the family $T_{a,d}$ in (5) are conjugate to each other, so that we only need to deal with one or two particular values of a. See Corollary 2.2 below.

Lemma 2.1. *We have the following statements.*

(a) If d is even and a_1 and a_2 are such that $a_1 a_2 \neq 0$ then $T_{a_1,d}$ is conjugate to $T_{a_2,d}$.

(b) If d is odd and a_1 and a_2 are such that $a_1 a_2 > 0$ then $T_{a_1,d}$ is conjugate to $T_{a_2,d}$.

Proof. The conjugation will be a rescaling. Given any $\mu \in \mathbb{R}$ we have that

$$T_{a,d}(\mu x, \mu y) = \mu\left(y - a\mu^{d-1}(x+y)^d, y - 2a\mu^{d-1}(x+y)^d\right) = \mu T_{a\mu^{d-1},d}(x,y).$$

Given $a_1, a_2 \neq 0$ we take

$$\mu := (a_2/a_1)^{1/(d-1)}.$$

If d is even and a_1 and a_2 are two parameters with $a_1 a_2 \neq 0$ we immediately have

$$T_{a_1,d}(\mu x, \mu y) = \mu T_{a_2,d}(x,y).$$

If d is odd the same is true but the existence of the $(d-1)$-root requires the condition $a_1 a_2 > 0$. □

Corollary 2.2. *To study the dynamics of the family of maps given by* (5) *it is enough to consider the cases* $\{a = 1,\ d \geq 2\}$ *and* $\{a = -1,\ d \geq 3,\ d \text{ odd}\}$.

To avoid heavy notation (depending on the parameter $a = \pm 1$) in what follows we assume $a = 1$. We will deal with the case $a = -1$ (for d odd) in Section 3, Remark 3.2, and in Section 6. In particular, when $a = 1$, we will use the simplified notation

$$T_d(x, y) := T_{1,d}(x, y) = \begin{pmatrix} y - (x + y)^d \\ y - 2(x + y)^d \end{pmatrix}. \tag{7}$$

Lemma 2.3. *We have*

(a) If d is even, T_d sends $\mathbb{R}^2$ onto $\{x \geq y\}$. The map T_d has two inverses

$$T^{-1}_{\pm,d}(x, y) = \left(-2x + y \pm (x - y)^{1/d},\, 2x - y\right) \tag{8}$$

which determine two one to one maps $T^{-1}_{+,d} : \{x \geq y\} \to \{x \geq -y\}$ and $T^{-1}_{-,d} : \{x \geq y\} \to \{x \leq -y\}$.

Moreover, for any $x_0 \in \mathbb{R}$, T_d maps the line $y = -x + x_0$ onto the line $y = x - x_0^d$ in a one-to-one way.

(b) If d is odd, the map $T_d : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism onto $\mathbb{R}^2$ and its inverse map is real analytic in $\mathbb{R}^2 \setminus \{x = y\}$, but not differentiable on $\{x = y\}$. Its inverse is given by

$$T_d^{-1}(x, y) = \left(-2x + y + (x - y)^{1/d},\, 2x - y\right). \tag{9}$$

Moreover, for any $x_0 \in \mathbb{R}$, the map T_d maps bijectively the line $y = -x + x_0$ onto the line $y = x - x_0^d$.

Proof. All statements come from direct computations. □

3. Local dynamics around the origin: The stable and the center manifolds.

The origin is the only fixed point of the map T_d in (7). In this section we obtain information on the local dynamics near the origin from the analytic expressions (in series expansion) of the (local) invariant manifolds. The derivative of T_d at (x, y) is given by

$$DT_d(x, y) = \begin{pmatrix} -d(x + y)^{d-1} & 1 - d(x + y)^{d-1} \\ -2d(x + y)^{d-1} & 1 - 2d(x + y)^{d-1} \end{pmatrix}, \tag{10}$$

and therefore

$$DT_d(0, 0) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

The matrix $DT_d(0, 0)$ is independent of the parameter d. Its eigenvalues are 0 and 1 with associated eigenvectors $v_1 = (1, 0)$ and $v_2 = (1, 1)$, respectively. In other words the direction v_1 is super-attracting while the direction v_2 is neutral. It follows from the general theory of invariant manifolds of fixed points of maps that there is a stable invariant manifold of $(0, 0)$ being tangent to v_1 and a (non-unique) center invariant manifold of $(0, 0)$ being tangent to v_2. We denote them $W^s_d(0)$ and $W^c_d(0)$, respectively. According to the general theory, $W^s_d(0)$ is analytic and $W^c_d(0)$ is C^k for all $k \geq 1$. Even if $W^c_d(0)$ may not be unique, all its Taylor coefficients are uniquely determined.

More concretely, the local invariant manifolds can be parametrized as graphs

$$W^s_{d,\text{loc}}(0) = \{(x, \varphi^s_d(x)) \mid |x| < \varepsilon_0\} \quad \text{and} \quad W^c_{d,\text{loc}}(0) = \{(x, \varphi^c_d(x)) \mid |x| < \varepsilon_0\}, \tag{11}$$

for some $\varepsilon_0 > 0$, where

$$\varphi_d^s(x) = \sum_{n=2}^{\infty} \alpha_n(d) x^n \qquad \text{and} \qquad \varphi_d^c(x) = x + \sum_{n=2}^{\infty} \beta_n(d) x^n. \tag{12}$$

We also denote by R_d^s and R_d^c the maps which encode the induced dynamics on the invariant manifolds. Thus, locally, we have

$$T_d\left(x, \varphi_d^s(x)\right) = \left(R_d^s(x), \varphi_d^s(R_d^s(x))\right) \ \text{ and } \ T_d\left(x, \varphi_d^c(x)\right) = \left(R_d^c(x), \varphi_d^c(R_d^c(x))\right), \tag{13}$$

respectively.

See [13, 6, 7] for a general discussion on the theory of local invariant manifolds. In the next lemma we provide the structure of the Taylor expansion of φ_d^s and φ_d^c. The lower order terms will determine the local dynamics near the origin.

Lemma 3.1. *Let $d \geq 2$. The Taylor series of φ_d^s and φ_d^c have the following structure*

$$\varphi_d^s(x) = x^d \sum_{k=0}^{\infty} \alpha_{d+k(d-1)}(d) x^{k(d-1)} \quad \textit{and} \quad \varphi_d^c(x) = x + x^d \sum_{k=0}^{\infty} \beta_{d+k(d-1)}(d) x^{k(d-1)}. \tag{14}$$

Moreover, $\alpha_d(d) = 2$, $\alpha_{2d-1}(d) = 4d$, $\beta_d(d) = -2^d$ and $\beta_{2d-1}(d) = -3d2^{2d-1}$ and thus we have that

$$\varphi_d^s(x) = 2x^d + \mathcal{O}(x^{2d-1}) \qquad \textit{and} \qquad \varphi_d^c(x) = x - 2^d x^d + \mathcal{O}(x^{2d-1})$$

and the one-dimensional dynamics induced by T_d on the stable and center manifolds are governed by

$$R_d^s : x \mapsto x^d + \mathcal{O}(x^{2d-1}) \qquad \textit{and} \qquad R_d^c : x \mapsto x - 4^4 x^d + \mathcal{O}(x^{2d-1}), \tag{15}$$

respectively.

See Figure 3(a) and 3(b) for the induced dynamics of the map T_d on the invariant manifolds.

Proof. To simplify the notation below we introduce the symbol $\{\cdot\}_n$ so that if Φ is a formal series around the origin, we write

$$\Phi(x) = \sum_{n \geq 0} \{\Phi\}_n \ x^n.$$

We prove (14) for the case of the stable manifold $W_d^s(0)$ (see (12)). Using that the stable manifold is an invariant graph for T_d we obtain that if $W_d^s(0) = \operatorname{graph} \varphi_d^s$ then

$$\varphi_d^s(x) - 2\left[x + \varphi_d^s(x)\right]^d = \varphi_d^s\left(\varphi_d^s(x) - \left[x + \varphi_d^s(x)\right]^d\right). \tag{16}$$

From the above equation, some computations show that, on the one hand $\alpha_2(2) = 2$ and $\alpha_2(d) = 0$ for all $d \geq 3$, and on the other hand, for all $n \geq 3$ we have that $\alpha_n(d)$ in (12) can be written recursively as

$$\alpha_n(d) = 2\left\{\left(x + \sum_{j=2}^{n-1} \alpha_j(d) x^j\right)^d\right\}_n + \sum_{i=2}^{n-1} \alpha_i(d) \left\{\left(\sum_{j=2}^{n-1} \alpha_j(d) x^j - \left(x + \sum_{j=2}^{n-1} \alpha_j(d) x^j\right)^d\right)^i\right\}_n. \tag{17}$$

Proving (14) for the stable manifold is equivalent to see that in (12) the coefficient $\alpha_n(d) = 0$ for all $n \geq 2$ such that $n - d$ is not a multiple of $d - 1$, or equivalently, not of the form $n = d + k(d-1)$ for $k \geq 0$. We argue by induction. We claim that for any $N \geq 1$, up to order

$$n = d + (N-1)(d-1)$$

the stable manifold writes as

$$x^d \sum_{k=0}^{N-1} \alpha_{d+k(d-1)}(d) x^{k(d-1)} =: x^d \Psi(x^{d-1}). \tag{18}$$

When $N = 1$ the result is true since $\alpha_n(d) = 0$ for $2 \leq n \leq d-1$ and $\alpha_d(d) = 2$. Indeed, from (17), $a_j(d) = 0$ implies $\alpha_{j+1}(d) = 0$ for all $j = 1, \ldots, d-2$. Also, $\alpha_d(d) = 2$ since we have a unique term of degree d with coefficient 2 associated to the first $\{\Phi\}_n$ term in the right hand side of (17).

Assuming the claim is true for N, we are going to prove that in the right hand side of (17) the coefficients $\alpha_{n+j}(d)x^{n+j}$, $j \geq 1$, are involved in terms of order $n+d$ or higher. This is easy to check for $j = 1$. For $j > 1$ the coefficients appear in terms of order bigger or equal than $n + d + 1$. In the right hand side of (17) the first term is

$$2(x + x^d \Psi(x^{d-1}) + \alpha_{n+1}(d) x^{n+1} + \ldots)^d$$

and the lower term in which $\alpha_{n+1}(d)$ appears is

$$2dx^{d-1}\alpha_{n+1}(d)x^{n+1} = 2d\alpha_{n+1}(d)x^{n+d}.$$

The second term of the right hand side of (17) can be written as

$$\begin{aligned} \alpha_d(d)[x^d\Psi(x^{d-1}) + \alpha_{n+1}(d)x^{n+1} + \ldots \\ - (x + x^d\Psi(x^{d-1}) + \alpha_{n+1}(d)x^{n+1} + \ldots)^d]^d + \ldots \end{aligned}$$

and the lower term in which $\alpha_{n+1}(d)$ appears is

$$2d(2x^d)^{d-1}\alpha_{n+1}(d)x^{n+1} = \mathcal{O}(x^{n+d(d-1)+1}).$$

This finishes the induction.

Once the expression of φ_d^s given in (14) is proved and the first terms of the expansion have been calculated we only need to justify the expression in (15). For this we compute the image of a point on the stable invariant manifold only using the lowest term of the series expansion

$$\begin{aligned} T_d\left(x, \varphi_d^s(x)\right) &= \left(2x^d - \left(x + 2x^d\right)^d, 2x^d - 2\left(x + 2x^d\right)^d\right) \\ &= \left(x^d + \mathcal{O}(x^{2d-1}), -4dx^{2d-1} + \mathcal{O}(x^{3d-2})\right). \end{aligned}$$

Therefore, the one-dimensional dynamics is given by

$$x \mapsto x^d + \mathcal{O}(x^{2d-1}).$$

Similar computations provide the result for φ_d^c. □

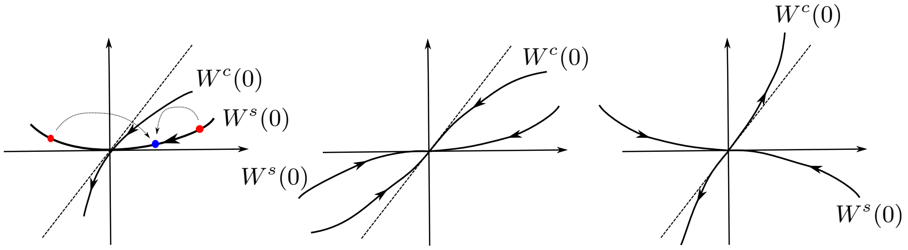
Remark 3.2. Using the same arguments as the ones in Lemma 3.1 one can get similar results for the case d odd and $a = -1$. The difference is the sign of some leading coefficients. More precisely if d is odd and $a = -1$ in the definition of T_d we have that $\alpha_d(d) = -2$, $\alpha_{2d-1}(d) = 4d$, $\beta_d(d) = 2^d$ and $\beta_{2d-1}(d) = -3d2^{2d-1}$ and hence we have that

$$\varphi_d^s(x) = -2x^d + \mathcal{O}(x^{2d-1}) \qquad \text{and} \qquad \varphi_d^c(x) = x + 2^d x^d + \mathcal{O}(x^{2d-1})$$

and the one-dimensional dynamics induced by T_d over the stable and center manifold are governed by

$$x \mapsto -x^d + \mathcal{O}(x^{2d-1}) \qquad \text{and} \qquad x \mapsto x + 4^4 x^d + \mathcal{O}(x^{2d-1}),$$

respectively. See Figure 3(c) for the induced dynamics of the map T_d over the invariant manifolds in this case.



(a) d even and $a = 1$ (b) d odd and $a = 1$ (c) d odd and $a = -1$

FIGURE 3. Local dynamics of $T_{a,d}$ near the origin.

We close this section by completing the discussion above, for the case d even. We have shown in Lemma 3.1 that many coefficients of the series expansion of the stable and center manifolds are zero (no mater the parity of d). Next we prove that, for d even, all non-zero coefficients of φ_d^s are positive.

Lemma 3.3. *Let d be even. Then, $\alpha_\ell(d) \geq 0$ for all $\ell \geq 0$.*

Proof. In Lemma 3.1 we proved that the coefficients of the series expansion of the analytic expression of the local stable manifold at the origin satisfy certain properties. In particular we proved that all coefficients $\alpha_\ell(d)$, $\ell \geq 0$, of the monomials x^ℓ with $\ell \neq d + k(d-1)$ for some $k \geq 0$ are zero. Moreover we also proved that $\alpha_d(d) = 2$ for all $d \geq 2$.

We fix $d \geq 2$ even. To simplify the notation we remove the dependence of the coefficients with respect to d; that is, we write $\alpha_k := \alpha_k(d)$. Let $\gamma(x)$ be the auxiliary analytic function given by the series expansion

$$\gamma(x) = \sum_{k=d}^{\infty} \gamma_k x^k := \left(x + \sum_{k=d}^{\infty} \alpha_k x^k \right)^d .$$

Note that γ_{k+1} depends on α_j, $d \leq j \leq k$. The lemma follows from the following claim.

Claim. *If $n \geq d$ then, for all $d \leq k \leq n$, we have $\alpha_k \geq 2\gamma_k \geq 0$.*

We prove the claim by induction. For $n = d$ it is obviously true because $\alpha_d = 2$ and $\gamma_d = 1$. Assuming the claim is true for n, from (17) we can write

$$\alpha_{n+1} = 2\left\{ \left(x + \sum_{k=d}^{n} \alpha_k x^k \right)^d \right\}_{n+1} + \left\{ \sum_{i=d}^{n} \alpha_i \left[\sum_{k=d}^{n} \alpha_k x^k - \left(x + \sum_{k=d}^{n} \alpha_k x^k \right)^d \right]^i \right\}_{n+1} . \tag{19}$$

The induction assumption implies

$$\alpha_k \geq 2\left\{ \left(x + \sum_{j=d}^{k} \alpha_j x^j \right)^d \right\}_k \geq \left\{ \left(x + \sum_{j=d}^{k} \alpha_j x^j \right)^d \right\}_k , \qquad d \leq k \leq n.$$

This implies that all coefficients of the terms of order $n+1$ of the second term of the right hand side of (19) are non-negative (because $i \geq d \geq 2$). Then, we conclude from (19) that $\alpha_{n+1} \geq 2\gamma_{n+1} \geq 0$, and the claim follows. □

4. **Proof of statement (a) of the main Theorem: The case d even and $a=1$.** Let $d \geq 2$ be even. From Corollary 2.2 we can take $a=1$ to cover all cases ($a \neq 0$). We simplify the notation writing $T_d := T_{1,d}$ and $\mathcal{A}_d(0) := \mathcal{A}_{1,d}(0)$. We will show that the origin belongs to the boundary of the basin, that the basin is contained in the upper half plane and that its boundary is the stable manifold of the origin.

Let us introduce some notation. Given $(x_0, y_0) \in \mathbb{R}^2$ we will write $(x_k, y_k) = T_d^k(x_0, y_0)$ for $k \geq 0$. Set

$$R_d := \big(1 - \frac{1}{d}\big)(2d)^{\frac{-1}{d-1}}.$$

Note that $R_2 = 1/8$ and, in general, $R_d < 1$. Finally let

$$\mathcal{T} = \{(x,y) \mid \ y \leq x\} \qquad \text{and} \qquad \mathcal{T}_{R_d} = \{(x,y) \in \mathcal{T} \mid \ y \geq 0,\ 0 \leq x \leq R_d\}.$$

Since the proof of statement (a) of the Main Theorem is quite long we split the arguments into several lemmas. The first one is just an observation.

Lemma 4.1. *We have that $(x_k, y_k) \in \mathcal{T}$ for $k \geq 1$ and then the sequences $\{x_k\}_{k\geq 1}$ and $\{y_k\}_{k \geq 1}$ are monotonically decreasing. Moreover, while $x_k > -y_k$ the sequences are strictly decreasing.*

Proof. The first assertion follows from Lemma 2.3(a). The second one follows directly from the inequalities:

$$x_{k+1} = y_k - (x_k + y_k)^d \leq y_k \leq x_k, \qquad k \geq 1 \quad (k \geq 0 \text{ if } (x_0, y_0) \in \mathcal{T}), \tag{20}$$

$$y_{k+1} = y_k - 2(x_k + y_k)^d \leq y_k, \qquad k \geq 0. \tag{21}$$

□

Next lemma shows that $\mathcal{A}_d(0)$ is a bounded set.

Lemma 4.2. *$\mathcal{A}_d(0)$ is a bounded set. More concretely,*

$$\mathcal{A}_d(0) \subset \mathcal{K} := (-5R_d, R_d) \times (0, 2R_d) \cup \{(0,0)\}.$$

Proof. We decompose

$$\mathbb{R}^2 \setminus \mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3 \cup \mathcal{K}_4,$$

where

$$\begin{aligned}
\mathcal{K}_1 &= \{(x,y) \mid \ y \leq 0\} \setminus \{(0,0)\},\\
\mathcal{K}_2 &= \{(x,y) \mid \ x \geq R_d\},\\
\mathcal{K}_3 &= \{(x,y) \mid \ y \geq 2R_d\},\\
\mathcal{K}_4 &= \{(x,y) \mid \ x \leq -5R_d,\ 0 < y < 2R_d\}
\end{aligned}$$

and we argue that $\mathcal{A}_d(0) \cap \mathcal{K}_j = \emptyset$, $1 \leq j \leq 4$, so that $\mathcal{A}_d(0) \subset \mathcal{K}$. We will use the following property: by the invariance of $\mathcal{A}_d(0)$ by T_d and its inverses we have that $T^{-1}_{\pm,d}$, $(x_0, y_0) \in \mathcal{A}_d(0)$ if and only if $(x_k, y_k) \in \mathcal{A}_d(0)$ for some $k \geq 0$.

Let $(x_0, y_0) \in \mathcal{K}_1$. Since $(x_0, y_0) \neq (0,0)$, if $y_0 = 0$ then $y_1 = -2x_0^d < 0$ and if $y_0 < 0$ then $y_1 \leq y_0 < 0$. Hence, in both cases $y_1 < 0$ and by (21) the sequence of iterates cannot converge to $(0,0)$.

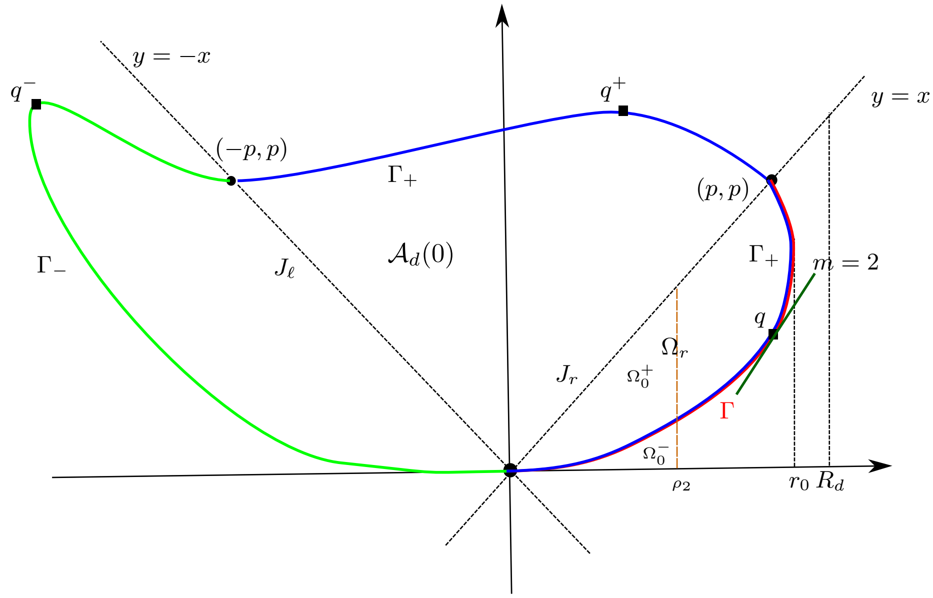


FIGURE 4. Sketch of the construction of $\mathcal{A}_d(0)$.

Next, we claim that if $(x_0,y_0)\in\mathcal{K}_2$ then $(x_1,y_1)\in\mathcal{K}_1$. Indeed, we consider the line $\{x=x_0\}$ with $x_0\ge R_d$ and we look at the second component of its image $\Psi_1(y):=\pi_y T_d(x_0,y)=y-2(x_0+y)^d$. Since $d\ge 2$ is even, $\lim_{t\to\pm\infty}\Psi(y)=-\infty$ and therefore Ψ_1 has a global maximum. Actually, it has a unique maximum whose location is obtained from the condition $\Psi_1'(y)=0$ and is $y^{(m)}:=1/(2d)^{1/(d-1)}-x_0$. Then $\Psi_1^{(m)}:=\Psi_1(y^{(m)})=1/(2d)^{1/(d-1)}-2/(2d)^{d/(d-1)}-x_0=R_d-x_0$ and therefore $y_1\le 0$. Moreover, $(x_1,y_1)\ne(0,0)$ because the only preimage of $(0,0)$ is $(0,0)\notin\mathcal{K}_2$.

Next we take $(x_0,y_0)\in\mathcal{K}_3$ and we claim that $(x_1,y_1)\in\mathcal{K}_1\cup\mathcal{K}_2$. Indeed, consider the line $\{y=y_0\}$ with $y_0\ge 2R$. Its image is contained in the line $\{(u,v)\mid\ v=2u-y_0\}$ and it is contained in $\mathcal{K}_1\cup\mathcal{K}_2$ because we have that either $u\ge R_d$ and the claim is true or $u<R_d$ and then $v=2u-y_0<2R_d-y_0\le 0$ and $(x_1,y_1)\ne(0,0)$.

Finally, if $(x_0,y_0)\in\mathcal{K}_4$ we claim that $(x_1,y_1)\in\mathcal{K}_1\cup\mathcal{K}_2$. Indeed, notice that $x_0+y_0<-5R_d+2R_d=-3R_d$ and then $(x_0+y_0)^d>(3R_d)^d$. If $x_1\ge R_d$ the claim is true. If $x_1<R_d$ then $y_1=y_0-2(x_0+y_0)^d=x_1-(x_0+y_0)^d<R_d-(3R_d)^d=(1-\frac{3^d}{2d}(1-\frac{1}{d})^{d-1})R_d<0$. □

Lemma 4.3. *We have that* $(x_0,y_0)\in\mathcal{A}_d(0)$ *if and only if* $(x_k,y_k)\in\mathcal{T}_{R_d}$ *for all* $k\ge 1$.

Proof. Assume $(x_0,y_0)\in\mathcal{A}_d(0)$. By Lemma 2.3(a), $x_k\ge y_k$ for all $k\ge 1$ and by Lemma 4.2, $x_k<R_d$ for all $k\ge 0$. Since the sequences $\{x_k\}$ and $\{y_k\}$ are decreasing for $k\ge 1$, if there exists $m>0$ such that $y_m<0$, then $y_k\le y_m<0$ for all $k\ge m$ and (x_k,y_k) cannot converge to $(0,0)$. Then, $y_k\ge 0$ for all k and the limit $y^\star=\lim_{k\to\infty}y_k$ exists, $y^\star\ge 0$ and then $x_k\ge y_k\ge 0$. As a consequence $(x_k,y_k)\in\mathcal{T}_{R_d}$ for all $k\ge 1$.

Conversely, let $(x_0,y_0)\in\mathbb{R}^2$ and assume that $(x_k,y_k)\in\mathcal{T}_{R_d}$ for all $k\ge 1$. Since the sequence $\{y_k\}_{k\ge 0}$ is strictly decreasing and bounded from below by 0 there exists the limit $y^\star = \lim_{k\to\infty} y_k \geq 0$. From the recurrence

$$y_{k+1} = y_k - 2(x_k + y_k)^d$$

we obtain that $\lim_{k\to\infty}(x_k + y_k)$ exists and it is 0. This implies that $-y^\star = \lim_{k\to\infty} x_k \geq 0$. Then $y^\star = 0$ and $(x_0, y_0) \in \mathcal{A}_d(0)$. ☐

We now turn the attention to $\partial\mathcal{A}_d(0)$. Our goal is to prove that $\partial\mathcal{A}_d(0)$ coincides with the global stable manifold of the origin, $W_d^s(0)$.

Lemma 4.4. *$W_d^s(0)$ cuts the line $\{x = y\}$ at some point (p, p) with $0 < p < R_d$.*

Proof. In Lemma 3.3 it is proven that the local expression of $W_d^s(0)$ is given by the graph of an analytic function $\varphi_d^s(x) = 2x^d + \dots$ whose series expansion in the x-variable has all its coefficients non-negative, and therefore there exists $\rho_1 > 0$ such that $\gamma = \{(x, \varphi_d^s(x)) \mid \ x \in (0, \rho_1)\}$ is contained in $\mathcal{T}_{R_d}$. We claim that the extension of this local piece γ of $W_d^s(0)$ eventually leaves $\mathcal{T}_{R_d}$. Indeed, assume the contrary. We globalize γ iterating with $T_{+,d}^{-1}$ (see (8)). Let $(x_0, y_0) \in \gamma$ and denote

$$(x_{-k}, y_{-k}) = T_{+,d}^{-k}(x_0, y_0), \qquad k \geq 0.$$

We have

$$y_{-k-1} = y_{-k} + 2(x_{-k} - y_{-k}) > y_{-k}. \tag{22}$$

If all $(x_{-k}, y_{-k}) \in \mathcal{T}_{R_d}$ we have that the sequence $\{y_{-k}\}_{k\geq 0}$ is strictly increasing and bounded, and we conclude that there exists $y^\star > 0$ such that

$$y^\star = \lim_{k\to\infty} y_{-k} > 0. \tag{23}$$

Moreover, from (22) we have that $x_{-k} = (y_{-k} + y_{-k-1})/2 \to y^\star$. Now, using the recurrence

$$x_{-k-1} = -2x_{-k} + y_{-k} + (x_{-k} - y_{-k})^{1/d},$$

we get that $y^\star = 0$, which provides a contradiction with (23). Finally, since we have seen that $\mathcal{A}_d(0)$ does not meet $\{y = 0\} \setminus \{(0,0)\}$ nor $\{x = R_d\}$ the globalization of γ has to cross $\{x = y\}$. ☐

We denote by Γ the piece of the stable manifold $W_d^s(0)$ from $(0, 0)$ to (p, p) contained in $\mathcal{T}_{R_d}$. We plot Γ in red colour in Figure 4. Let φ_d^s be given in (11).

Lemma 4.5. *The following properties for φ_d^s hold.*

- *(a) There exists a unique point $\bar{q} = (\bar{q}_x, \bar{q}_y) \in \Gamma$ whose tangent vector has slope $m = 1/2$.*
- *(b) If we denote by $r_0 > 0$ the radius of convergence of φ_d^s (as a function of a complex variable) then $0 < r_0 < R_d$, φ_d^s is increasing and convex in the interval $(0, r_0)$ and decreasing and convex $(-r_0 - 2\varphi_d^s(r_0), 0)$.*

Proof. We observe that since all coefficients of the series expansion of φ_d^s are non-negative (see Lemma 3.3) we conclude from Vivanti-Pringsheim's Theorem [8] that φ_d^s as a function of a complex variable has a singularity at $x = r_0 > 0$ and

$$\varphi_0 := \varphi_d^s(r_0) = \sum_{k\geq d} \alpha_k r_0^k.$$

In fact we have that $r_0 < R_d < \infty$ and $\varphi_0 < 2R_d$ since graph $\varphi_d^s \subset W_d^s(0) \subset \mathcal{A}_d(0)$ and by Lemma 4.2, $\mathcal{A}_d(0) \subset \mathcal{K}$. In particular $\varphi_d^s|_{(0,r_0)}$ is an increasing and convex function and

$$\lim_{x\to r_0^-} (\varphi_d^s)'(x) = +\infty.$$

Indeed, if $(\varphi_d^s)'(r_0) < \infty$, then $(\varphi_d^s)'$ could be extended in a differentiable way for $x > r_0$. The graph close to $x = r_0$ will be the image by $T_{+,d}^{-1}$ of a piece of the graph of φ_d^s, say γ_2, closer to the origin. The piece γ_2 does not contain the point (p,p) since its image is the point $(-p,p)$ outside $\mathcal{T}_{R_d}$. Therefore $T_{+,d}^{-1}$ is analytic on γ_2 and then φ_d^s would be analytic in a neighborhood of r_0 which provides a contradiction. This proves statement (a) and provides the existence of r_0.

We have the symmetry (d is even)

$$T_d(x,y) = T_d(-2y - x, y). \tag{24}$$

Hence if $(x,y) \in W_d^s(0)$ then also $(-2y-x,y) \in W_d^s(0)$. More concretely, since $(x, \varphi_d^s(x)) \in W_d^s(0)$, $(-x - 2\varphi_d^s(x), \varphi_d^s(x)) \in W_d^s(0)$ and then

$$\varphi_d^s(-x - 2\varphi_d^s(x)) = \varphi_d^s(x). \tag{25}$$

This means that φ_d^s is defined for $x \in (-r_0 - 2\varphi_d^s(r_0), 0)$. Moreover, taking derivatives in (25) we get

$$(\varphi_d^s)'(-x - 2\varphi_d^s(x))(-1 - 2(\varphi_d^s)'(x)) = (\varphi_d^s)'(x), \tag{26}$$

$$(\varphi_d^s)''(-x - 2\varphi_d^s(x))(-1 - 2(\varphi_d^s)'(x))^2 + (\varphi_d^s)'(-x - 2\varphi_d^s(x))(-2(\varphi_d^s)''(x)) = (\varphi_d^s)''(x), \tag{27}$$

and hence we can conclude that φ_d^s is decreasing and convex in $(-r_0 - 2\varphi_d^s(r_0), 0)$. Indeed, substituting $(\varphi_d^s)'(-x - 2\varphi_d^s(x))$ from (26) into (27) we obtain

$$(\varphi_d^s)''(-x - 2\varphi_d^s(x))(-1 - 2(\varphi_d^s)'(x))^2 = (\varphi_d^s)''(x)\left(1 - \frac{2(\varphi_d^s)'(x)}{1 + 2(\varphi_d^s)'(x)}\right) > 0.$$

From the previous properties there exists a unique point $\bar{q} = (\bar{q}_x, \bar{q}_y) \in \Gamma$ whose tangent vector has slope $m = 1/2$. $\square$

Let

$$\Omega_0^+ = \{(x,y) \in \mathcal{T}_{R_d} \mid y \geq \varphi_d^s(x),\ 0 \leq x \leq \rho_2\},$$
$$\Omega_0^- = \{(x,y) \in \mathcal{T}_{R_d} \mid 0 < y < \varphi_d^s(x),\ 0 \leq x \leq \rho_2\},$$

with $\rho_2 < \min\left\{\frac{1}{2}\frac{1}{(4d)^{1/(d-1)}}, \bar{q}_x\right\}$. See Figure 4.

Lemma 4.6. *The domain* Ω_0^+ *is invariant by* T_d *and* $\Omega_0^+ \subset \mathcal{A}_d(0)$. *Moreover,* $\Omega_0^- \cap \mathcal{A}_d(0) = \emptyset$.

Proof. Let $(x,y) := (x_0,y_0) \in \Omega_0^+$. The sequence $\{x_k\}_{k\geq 0}$ is decreasing by Lemma 4.1 . So, it is enough to show that $y_1 - \varphi_d^s(x_1) \geq 0$. Indeed

$$\begin{aligned} y_1 - \varphi_d^s(x_1) &= y - 2(x+y)^d - \varphi_d^s(y - (x+y)^d) \\ &= y - \varphi_d^s(x) + \varphi_d^s(x) - \varphi_d^s(y - (x+y)^d) - 2(x+y)^d \\ &= y - \varphi_d^s(x) + H(x,y), \end{aligned} \tag{28}$$

where

$$H(x,y) = \varphi_d^s(x) - \varphi_d^s(y - (x+y)^d) - 2(x+y)^d. \tag{29}$$

Taking into account that φ_d^s satisfies the invariance equation

$$\varphi_d^s(x) - 2(x + \varphi_d^s(x))^d = \varphi_d^s(\varphi_d^s(x) - (x + \varphi_d^s(x))^d), \tag{30}$$

H can be rewritten as

$$\begin{aligned}H(x,y) &= \varphi^s_d(\varphi^s_d(x)-(x+\varphi^s_d(x))^d)-\varphi^s_d(y-(x+y)^d)+2(x+\varphi^s_d(x))^d-2(x+y)^d\\ &= \int_0^1 \Big[\frac{d}{dt}\varphi^s_d(\xi_t-(x+\xi_t)^d)(1-d(x+\xi_t)^{d-1})+2d(x+\xi_t)^{d-1}\Big](\varphi^s_d(x)-y)\,dt\\ &=: (\varphi^s_d(x)-y)\widehat{H}(x,y),\end{aligned} \tag{31}$$

where $\xi_t = y+t(\varphi^s_d(x)-y)$ and hence since $(x,y)\in\Omega_0^+$ we have $0<\varphi^s_d(x)\le\xi_t\le y\le x\le\rho_2$.

From (28) and (31) we have

$$y_1-\varphi^s_d(x_1)=(y-\varphi^s_d(x))\left(1-\widehat{H}(x,y)\right),$$

so that it is enough to see that $\widehat{H}(x,y)<1$.

Given $x\in(0,\rho_2)$, we introduce $\Psi_2(\xi)=\xi-(x+\xi)^d$ for $\xi\in(0,x)$. The function $\Psi_2(x)$ is concave and we have that $\Psi_2(0)=-x^d<0$, with $-x^d>-\rho_2^d$ and $\Psi_2(x)=x-(2x)^d=x(1-2^dx^{d-1})>x(1-1/(2d))>0$ by one of the conditions in the definition of ρ_2. Hence, $\xi_t-(x+\xi_t)^d\ge-\rho_2^d$ for all $t\in[0,1]$. Note that, by the fact that the coefficients of the expansion of φ^s_d are non-negative (Lemma 3.3), at the symmetric point the absolute value of the derivative is smaller, i.e. for $x\in[0,r_0)$, $|(\varphi^s_d)'(-x)|\le(\varphi^s_d)'(x)$ so that, for $-x^d<\zeta<0$, $|(\varphi^s_d)'(\zeta)|\le(\varphi^s_d)'(-\zeta)\le(\varphi^s_d)'(x^d)\le(\varphi^s_d)'(\rho_2)\le 1/2$. Then

$$|(\varphi^s_d)'(\xi_t-(x+\xi_t)^d)|<1/2. \tag{32}$$

Moreover,

$$2d(x+\xi_t)^{d-1}<2d(2x)^{d-1}\le 1/2. \tag{33}$$

By (32) and (33) we obtain $\widehat{H}(x,y)<1$ and therefore we obtain that the iterates stay in the same side of graph φ^s_d.

Now we deal with Ω_0^-. To prove that $\Omega_0^-\cap\mathcal{A}_d(0)=\emptyset$ we will see that if $(x_0,y_0)\in\Omega_0^-$ then not all its iterates can remain in Ω_0^-. Assume the contrary. To simplify the estimates we do a (x-depending) translation to put the (local) stable manifold at $\{y=0\}$. Actually, we make the change $C(x,y)=(x,y+\varphi^s_d(x))$. The transformed map is

$$\widehat{T}_d\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}F(x,y)\\ G(x,y)\end{pmatrix}:=\begin{pmatrix}y+\varphi^s_d(x)-(x+y+\varphi^s_d(x))^d\\ y+\varphi^s_d(x)-2(x+y+\varphi^s_d(x))^d-\varphi^s_d(F(x,y))\end{pmatrix}.$$

The domain Ω_0^- is transformed into

$$\widehat{\Omega}_0^-=\{(x,y)\mid\ 0<x<\rho_2,\ -\varphi^s_d(x)<y<0\}.$$

Let $(x_0,y_0)\in\widehat{\Omega}_0^-$. We use again the notation $(x_k,y_k)=\widehat{T}_d^k(x_0,y_0)$ for $k\ge0$.

Let $\rho_3\in(0,\rho_2]$ be such that

$$0<\varphi^s_d(x)<3x^d \qquad \text{for} \qquad x\in(0,\rho_3).$$

Assume that $(x_k,y_k)\in\widehat{\Omega}_0^-$ for all $k\ge0$. Since d is even, we also have $0<x_{k+1}\le y_k+\varphi^s_d(x_k)<\varphi^s_d(x_k)\le x_k$. Then $\{x_k\}$ is also decreasing and

$$x_k=y_{k-1}+\varphi^s_d(x_{k-1})-(x_{k-1}+y_{k-1}+\varphi^s_d(x_{k-1}))^d\le\varphi^s_d(x_{k-1})-x_{k-1}^d<2x_{k-1}^d,$$

and inductively we get

$$x_k<2^{\left(\frac{d^k-1}{d-1}\right)}x_0^{d^k}<(2^{1/(d-1)}x_0)^{d^k}. \tag{34}$$

Note that since $2^{1/(d-1)}x_0 < 2^{1/(d-1)}\rho_2 < 1/2$ then $x_k \to 0$. We have

$$G(x,0) = \varphi_d^s(x) - 2(x+\varphi_d^s(x))^d - \varphi_d^s(\varphi_d^s(x) - (x+\varphi_d^s(x))^d) = 0$$

by the invariance equation (30), and

$$\begin{aligned} G_1(x) :=& \frac{\partial G}{\partial y}(x,0) = 1 - 2d(x+\varphi_d^s(x))^{d-1} - (\varphi_d^s)'(F(x,0))(1-d(x+\varphi_d^s(x))^{d-1}) \\ =& 1 - 2dx^{d-1} + \dots \end{aligned} \tag{35}$$

so that

$$G(x,y) = G_1(x)y + G_2(x,y) \qquad \text{with} \quad G_2(x,y) = \mathcal{O}(y^2).$$

There exists $\rho_4 \in (0,\rho_3]$ such that

$$G_1(x) > 1 - \nu x^{d-1} \qquad \text{and} \qquad |G_2(x,y)| < M|y|^2, \quad x \in (0,\rho_4), \quad (x,y) \in \widehat{\Omega}_0^-,$$

for some $\nu > 2d$ and $M > 0$. Then, taking an iterate (x_k, y_k) such that $x_k < \rho_4$ and relabeling it by (x_0, y_0) and, starting again the iteration, we have

$$\begin{aligned} y_{k+1} = G(x_k,y_k) \leq& (1-\nu x_k^{d-1})y_k + My_k^2 \\ <& (1-\nu x_k^{d-1} - M\varphi_d^s(x_k))y_k \leq (1-bx_k^{d-1})y_k, \end{aligned} \tag{36}$$

where $b = \nu + 3\rho_4 M$. Iterating (36) we obtain

$$y_k < \prod_{j=0}^{k-1}(1-bx_j^{d-1})y_0 = y_0 \exp\sum_{j=0}^{k-1}\log(1-bx_j^{d-1}).$$

The series $\sum\log(1-bx_j^{d-1})$ is convergent since bx_j^{d-1} tends to zero and $\log(1+x) > (2\log 2)x$ if $x \in (-1/2, 0)$. Then, $y_k < y_0\exp(S_0)$ where $S_0 = \sum_{j=0}^{\infty}\log(1-bx_j^{d-1})$. This means that y_k is less than some negative number so that y_k cannot converge to 0 and therefore $(x_0,y_0) \notin \mathcal{A}_d(0)$.

If $(x_0,y_0) \in \Omega_0^-$, assume that all its iterates stay in $\mathcal{T}_{R_d}$. Then the sequences $\{x_k\}$ and $\{y_k\}$ are decreasing and there exists $m \geq 0$ such that $x_m < \rho_4$ and, by the previous estimates, $(x_m,y_m) \notin \mathcal{A}_d(0)$. □

Proof of statement (a) of the Main Theorem. Let Ω_r be the closure of the bounded domain whose boundary is the simple closed curve formed by the concatenation of Γ and $J_r := \{(x,y) \mid \ x=y,\ 0<x<p\}$ (the meaning of r is *right*, in contrast of the later notation J_ℓ for *left*). See Figure 4. The domain Ω_r is invariant by T_d since the iterates cannot jump across the boundary. Moreover, there exists $m \geq 1$ such that $T_d^m(\Omega_r) \subset \Omega_0^+$. Then $\Omega_r \subset \mathcal{A}_d(0)$. By Lemma 4.3, to obtain $\mathcal{A}_d(0)$ we only need to take one preimage of Ω_r by T_d.

In the light of Lemma 2.3(a) we write

$$\Gamma_\pm := T_{\pm,d}^{-1}(\Gamma)\,, \qquad \Omega_\pm := T_{\pm,d}^{-1}(\Omega_r)\,.$$

Clearly, the sets $\Omega_\pm$ are contained in $\mathcal{A}_d(0)$. The boundaries of $\Omega_\pm$ are the images of the boundaries of Ω_r by $T_{\pm,d}^{-1}$. Consequently, we have

$$\partial\Omega_+ = \Gamma_+ \cup J_\ell \qquad \text{and} \qquad \partial\Omega_- = \Gamma_- \cup J_\ell,$$

where $\Gamma_+ := T_{+,d}^{-1}(\Gamma)$ is a curve contained in $\{y \geq -x\}$ which joints $(0,0)$ with $(-p,p)$, $\Gamma_- := T_{-,d}^{-1}(\Gamma)$ is a curve contained in $\{y \leq -x\}$ which joints $(0,0)$ with

$(-p,p)$, and $J_\ell := T_{+,d}^{-1}(J_r) = \{(x,y) \mid \ y=-x,\ -p<x<0\}$. Notice that every point in $\Gamma \setminus \{(0,0)\cup(p,p)\}$ has two preimages while

$$T_{+,d}^{-1}(0,0) = T_{-,d}^{-1}(0,0) = (0,0) \qquad \text{and} \qquad T_{+,d}^{-1}(p,p) = T_{-,d}^{-1}(p,p) = (-p,p)\,.$$

See Figure 4. Accordingly, the curves $\Gamma_\pm$ joint the points $(0,0)$ and $(-p,p)$, they are mapped bijectively onto Γ by T_d and determine the boundary of the basin of attraction of the origin. That is,

$$\mathcal{A}_d(0) = \Omega_+ \cup \Omega_- \cup \Gamma_+ \cup \Gamma_-.$$

In Figure 4 we draw Γ_- in green and Γ_+ in blue. This finishes the proof of statement (a) of the Main Theorem. □

We can add some extra information about the geometry of $W_d^s(0)$. See Figure 4. On the one hand, direct computations from (8) imply that if $u=(u_1,u_2)$ is the tangent vector of $W_d^s(0)$ at the point (p,p) then

$$DT_{\pm,d}^{-1}(p,p)(u) = \begin{pmatrix} \infty \\ 2u_1-u_2 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{since} \quad DT_{\pm,d}^{-1}(p,p) = \begin{pmatrix} \infty & \infty \\ 2 & -1 \end{pmatrix},$$

where here ∞ has to be understood as a limit. Concretely, the tangent vector of $W_d^s(0)$ at the point $(-p,p)$ is horizontal.

Moreover, taking into account the symmetry (24), the points with highest value of y in $W_d^s(0)$ should be symmetric. Actually, they coincide with the two points $q^\pm = \left(q_x^\pm, q_y^\pm\right)$ which are mapped by T_d to a point $q=(q_x,q_y)\in\Gamma$ whose tangent vector has slope $m=2$.

5. Proof of statement (b) of the main Theorem: The case d odd and $a=1$.

For the whole section we assume that $d\geq 3$ is odd and $a=1$. The proof of statement (b) of the Main Theorem is quite long and therefore we will split it into several lemmas and propositions. Roughly speaking the strategy is as follows. First we will see that $\mathcal{A}_d(0)$ is open, simply connected and that $[-1/2,0]\times\{0\}\subset\mathcal{A}_d(0)$ (Proposition 5.1). Second we will show that there exists a hyperbolic two-cycle of saddle type whose unstable manifold intersects $[-1/2,0]\times\{0\}$. From this we will show that the two-cycle as well as its stable manifold belong to $\partial\mathcal{A}_d(0)$ (Proposition 5.6). And finally we will see that $\partial\mathcal{A}_d(0)$ is unbounded (Proposition 5.7).

Let $b\in(0,1/2]$. We denote by $Q_b\subset\mathbb{R}^2$ be the compact convex polygon bounded by the straight segments

$$A_b := \{(2b^d,y)\in\mathbb{R}^2 \mid \ y\in[0,2b^d]\}, \qquad B_b := \{(x,2b^d)\in\mathbb{R}^2 \mid \ x\in[0,2b^d]\},$$

$$C_b := \{(x,2x+2b^d)\in\mathbb{R}^2 \mid \ x\in[-b^d,0]\}, \qquad D_b := \{(-b^d,y)\in\mathbb{R}^2 \mid \ y\in[-b^d,0]\},$$

$$E_b := \{(x,-b^d)\in\mathbb{R}^2 \mid \ x\in[-b^d,0]\}, \qquad F_b := \{(x,\tfrac{1}{2}x-b^d)\in\mathbb{R}^2 \mid \ x\in[0,2b^d]\}.$$

We denote $Q^\star := Q_{1/2}$.

Proposition 5.1. *We have that $Q^\star\subset\mathcal{A}_d(0)$. In particular, $[-1/2,0]\times\{0\}\subset\mathcal{A}_d(0)$. Moreover, $\mathcal{A}_d(0)$ is open and simply connected.*

Proof. From Lemma 3.1 the origin is asymptotically stable and therefore $\mathcal{A}_d(0)$ is open (see also Figure 3). The family $\{Q_b\}_{b\in(0,1/2]}$ is a neighbourhood basis of the origin.

We claim that $T_d(Q_b)\subset \operatorname{int}(Q_b)$, $b\in(0,1/2]$. See Figure 5 for a sketch of Q_b and its image.

Assume the claim is true. This implies that $Q^\star \subset \mathcal{A}(0)$. Since the image of $[-1/2,0]\times\{0\}$ by T_d is the segment $\{(x,2x)\in\mathbb{R}^2 \mid x\in[0,(1/2)^d]\}\subset Q^\star$ we conclude that $[-1/2,0]\times\{0\}\subset\mathcal{A}_d(0)$ as desired. Moreover, since there exists an open simply connected neighborhood $Q^\star$ containing $(0,0)$ and contained in $\mathcal{A}_d(0)$, the origin is asymptotically stable (our proof demonstrates again that the origin is asymptotically stable). Finally, $\mathcal{A}_d(0)=\bigcup_{k\geq 0}T_d^{-k}(Q^\star)$. Since T_d is one to one, $T_d^{-k}(Q^\star)$ is also open and simply connected, for all k. Furthermore, since $T_d^{-k-1}(Q^\star)\supset T_d^{-k}(Q^\star)$, we conclude that $\mathcal{A}_d(0)$ is open and simply connected as well.

The rest of the proof is devoted to prove the claim. Hereafter we remove from the notation the dependence of the whole construction with respect to the parameter b, unless strictly necessary. The proof consists in studying the image of each side of the boundary of Q by T_d. We will get that the image of the boundary of Q is contained in $\operatorname{int}(Q)$ and therefore $T_d(Q)\subset\operatorname{int}(Q)$.

We denote by Γ_A the image of the segment A under T_d and similarly for the other pieces of the boundary. Next, we prove that each image is contained in $\operatorname{int}(Q)$. See Figure 5.

The image $\Gamma_A = T_d(A)$. We parametrize Γ_A as follows

$$\Gamma_A = T_d(A)=\left\{\left(\Psi_1(y):=y-(2b^d+y)^d,\ \Psi_2(y):=y-2(2b^d+y)^d\right),\, y\in[0,2b^d]\right\}.$$

We check that $\Gamma_A\subset\operatorname{int}Q\cap\{y<x\}$. The condition $y<x$ is equivalent to $\Psi_2(y)<\Psi_1(y)$ for $y\in[0,2b^d]$ which is clearly true. The condition $y>\frac{1}{2}x-b^d$ is equivalent to $\Psi_2(y)>\frac{1}{2}\Psi_1(y)-b^d$ for $y\in[0,2b^d]$ which can be written as $\chi_1(y):=\frac{1}{2}y+b^d-\frac{3}{2}(2b^d+y)^d>0$ for $y\in[0,2b^d]$. We have that $\chi_1(0)=b^d(1-\frac{3}{2}2^db^{d^2-d})>0$ and $\chi_1(2b^d)=2b^d(1-\frac{3}{4}4^db^{d^2-d})>0$ since $b\in(0,1/2]$ and $d\geq 3$. Also $\chi_1''(y)=-d(d-1)\frac{3}{2}(2b^d+y)^{d-2}<0$, therefore $\chi_1(y)>0$. Finally, $\Psi_1(y)<y\leq 2b^d$ and $\Psi_2(y)\geq -b^d$. The first claim is immediate. For the second we consider the auxiliary function $\chi_2(y):=y+b^d-2(2b^d+y)^d$. We have $\chi_2(0)=b^d(1-2^{d+1}b^{d^2-d})>0$ and $\chi_2(2b^d)=3b^d(1-\frac{1}{3}2^{2d+1}b^{d^2-d})>0$ since $d\geq 3$. Moreover, $\chi_2''(y)=-2d(d-1)(2b^d+y)^{d-2}<0$ and hence $\chi_2(y)>0$.

The image $\Gamma_B = T_d(B)$. We parametrize Γ_B by x:

$$\Gamma_B=T_d(B)=\left\{\left(\Psi_1(x):=2b^d-(2b^d+x)^d,\ \Psi_2(x):=2b^d-2(2b^d+x)^d\right),\, x\in[0,2b^d]\right\}.$$

It is immediate to see that

$$\Psi_1'(x)=-d(2b^d+x)^{d-1}<0 \qquad\text{and}\qquad \Psi_2'(x)=-2d(2b^d+x)^{d-1}<0.$$

Therefore,

$$\Gamma_B\subset[\Psi_1(2b^d),\Psi_1(0)]\times[\Psi_2(2b^d),\Psi_2(0)]\subset(0,2b^d)\times[0,2b^d)\subset\operatorname{int}(Q).$$

The image $\Gamma_C = T_d(C)$. We parametrize Γ_C by x:

$$\begin{aligned}\Gamma_C=\Big\{&\Big(\Psi_1(x):=2x+2b^d-(3x+2b^d)^d,\ \Psi_2(x):=2x+2b^d-2(3x+2b^d)^d\Big),\\ &x\in[-b^d,0]\Big\}.\end{aligned}$$

Similarly as before

$$\Psi_1'(x)=2-3d(3x+2b^d)^{d-1}>2-3d2^{d-1}b^{d^2-d}>0$$

and

$$\Psi_2'(x)=2-6d(3x+2b^d)^{d-1}>2-6d2^{d-1}b^{d^2-d}>0.$$

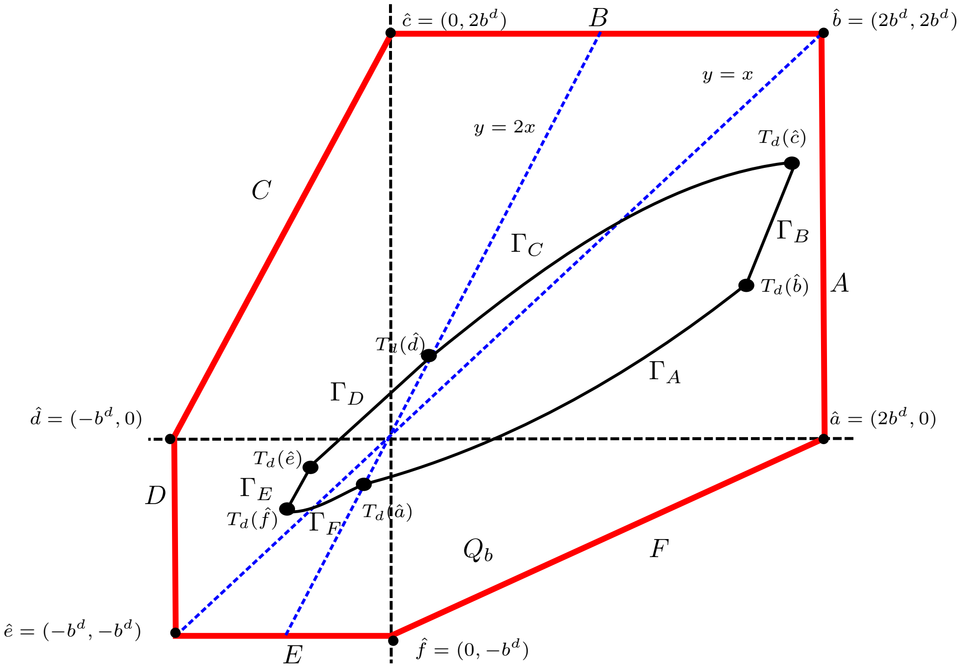


FIGURE 5. Sketch of the closed set Q_b and its image $T_d(Q_b)$ for $d \geq 5$.

Then,

$$\Gamma_C \subset [\Psi_1(-b^d), \Psi_1(0)] \times [\Psi_2(-b^d), \Psi_2(0)] \subset (0, 2b^d) \times (0, 2b^d) \subset \operatorname{int}(Q).$$

The image $\Gamma_D = T_d(D)$. We parametrize Γ_D by y: Clearly,

$$\Gamma_D = \left\{ \left(\Psi_1(y) := y - (-b^d + y)^d,\ \Psi_2(y) := y - 2(-b^d + y)^d \right),\ y \in [-b^d, 0] \right\}.$$

First, we check that $\Gamma_D \subset [-b^d, 2b^d] \times [-b^d, 2b^d]$. Indeed,

$$\Psi_1'(y) = 1 - d(-b^d + y)^{d-1} > 0 \qquad \text{and} \qquad \Psi_2'(y) = 1 - 2d(-b^d + y)^{d-1} > 0.$$

Then

$$-b^d < \Psi_1(-b^d) \leq \Psi_1(y) \leq \Psi_1(0) < 2b^d,$$
$$-b^d < \Psi_2(-b^d) \leq \Psi_2(y) \leq \Psi_2(0) < 2b^d.$$

The condition $\psi_2(y) < 2\psi_1(y) + 2b^d$ reads $y - 2(-b^d + y)^d < 2(y - (-b^d + y)^d) + 2b^d$ which is satisfied if $y + 2b^d > 0$ which is the case.

The condition $\psi_2(y) > \frac{1}{2}\psi_1(y) - b^d$ reads $y - 2(-b^d + y)^d > \frac{1}{2}(y - (-b^d + y)^d) - b^d$ which is satisfied if $\chi_3(y) := \frac{1}{2}y - \frac{3}{2}(-b^d + y)^d + b^d > 0$. This is indeed true because $\chi_3(-b^d) = \frac{1}{2}b^d + \frac{3}{2}(2b^d)^d > 0$ and $\chi_3'(y) = \frac{1}{2} - \frac{3}{2}d(-b^d + y)^{d-1} > 0$ (since $d \geq 3$ and $b \in (0, 1/2]$).

The image $\Gamma_E = T_d(E)$. We parametrize Γ_E by x:

$$\Gamma_E = \left\{ \left(\Psi_1(x) := -b^d - (-b^d + x)^d,\ \Psi_2(x) := -b^d - 2(-b^d + x)^d \right),\ x \in [-b^d, 0] \right\}.$$

In this case we will check that $\Gamma_E \subset (-b^d,0)\times(-b^d,0) \subset \operatorname{int}(Q)$. Indeed, directly from the expression of the parametrization we have, $\Psi_1(x) > -b^d$, $\Psi_2(x) > -b^d$, $\Psi_1(x) < -b^d-(-2b^d)^d = -b^d(1-2^d b^{d^2-d}) < 0$ and $\Psi_2(x) < -b^d(1-2^{d+1}b^{d^2-d}) < 0$.

The image $\Gamma_F = T_d(F)$. We parametrize Γ_F by x:

$$\Gamma_F = \left\{ \left(\Psi_1(x) := \frac{1}{2}x - b^d - \left(\frac{3}{2}x - b^d\right)^d,\ \Psi_2(x) := \frac{1}{2}x - b^d - 2\left(\frac{3}{2}x - b^d\right)^d \right),\ x \in [0, 2b^d] \right\}.$$

In this case we will also check that $\Gamma_F \subset (-b^d,0)\times(-b^d,0) \subset \operatorname{int}(Q)$. We have that

$$\Psi_1(0) = -b^d - (-b^d)^d > -b^d, \qquad \Psi_1(2b^d) = -(2b^d)^d < 0.$$

Moreover, since $\Psi_1'(x) = \frac{1}{2} - \frac{3}{2}d\left(\frac{3}{2}x - b^d\right)^{d-1}$ which is positive because

$$\frac{3}{2}d\left(\frac{3}{2}x - b^d\right)^{d-1} \le \frac{3}{2}d2^{d-1}b^{d(d-1)} \le \frac{3}{2}d2^{-d^2+2d-1} \le 9/32 < 1/2,$$

we get $\Psi_1(x) \in (-b^d, 0)$.

Concerning Ψ_2, $\Psi_2(0) = -b^d + 2\left(b^d\right)^d > -b^d$ and $\Psi_2(2b^d) = -2(2b^d)^d < 0$. However, it is not (always) monotone. Depending on b and d it may have a maximum at some $x_c \in (0, b^d)$. The value of x_c is obtained from the condition $\Psi_2'(x_c) = 0$. It is the positive solution of

$$\left(\frac{3}{2}x_c - b^d\right)^{d-1} = \frac{1}{6d}.$$

In case x_c belongs to the interval $(0, 2b^d)$ we have that $\Psi_2(x_c) = \frac{1}{2}x_c - b^d - \left(\frac{3}{2}x_c - b^d\right)^d = \frac{1}{2}x_c - b^d - \left(\frac{1}{6d}\right)^{d/(d-1)} < -\left(\frac{1}{6d}\right)^{d/(d-1)} < 0$. Thus, $\Psi_2(x) < 0$ for $x \in [0, 2b^d]$. This finishes the proof. $\square$

Following the steps of the strategy of the proof (of the statement (b) of the Main Theorem) described at the beginning of this section, we start checking that $\{p_0 = (0,1), p_1 = (0,-1)\}$ forms a hyperbolic two-cycle.

Since $DT_d(p_0) = DT_d(p_1)$, the chain rule implies that

$$DT_d^2(p_0) = DT_d^2(p_1) = DT_d(p_0)DT_d(p_1) = \begin{pmatrix} 3d^2-2d & 3d^2-4d+1 \\ 6d^2-2d & 6d^2-6d+1 \end{pmatrix}. \tag{37}$$

A direct computation shows that the eigenvalues and eigenvectors of $DT_d^2(p_j)$, $j = 0, 1$, are given by

$$\lambda_d^{\pm} = \frac{1}{2}\left(9d^2 - 8d + 1 \pm (3d-1)\sqrt{9d^2-10d+1}\right)$$

and

$$(1, m_d^{\pm}) = \left(1,\ \frac{4d}{1 - d \pm \sqrt{9d^2-10d+1}}\right), \tag{38}$$

respectively. Finally, it is straightforward to check that both eigenvalues are strictly positive. Moreover, λ_d^- is strictly decreasing and λ_d^+ is strictly increasing, with respect to the parameter d. We also have

$$\lim_{d\to\infty} \lambda_d^- = 1/9 \quad \text{and} \quad 1/9 < \lambda_d^- \le \lambda_3^- = 29 - 8\sqrt{13} \approx 0.1556,$$
$$\lim_{d\to\infty} \lambda_d^+ = \infty \quad \text{and} \quad \lambda_d^+ \ge \lambda_3^+ = 29 + 8\sqrt{13} \approx 57.8444.$$

On the other hand, m_d^- is negative and strictly increasing while m_d^+ is positive and strictly decreasing (both with respect to the parameter d). We also have

$$\lim_{d\to\infty} m_d^- = -1 \quad \text{and} \quad -1.3028 \approx \tfrac{-6}{1+\sqrt{13}} = m_3^- \leq m_d^- < -1,$$
$$\lim_{d\to\infty} m_d^+ = 2 \quad \text{and} \quad 2 < m_d^+ \leq m_3^+ = \tfrac{6}{\sqrt{13}-1} \approx 2.3028.$$

Therefore, the two cycle $\{p_0, p_1\}$ is a hyperbolic saddle point. In what follows we will denote by $W^s := W^s_{\{p_0,p_1\}}$ and $W^u := W^u_{\{p_0,p_1\}}$ the (global) stable and unstable manifolds of the periodic orbit $\{p_0, p_1\}$, respectively. We split $W^u = W^u_{p_0} \cup W^u_{p_1}$ where $W^u_{p_j}$ is the (global) unstable manifold of the fixed point p_j for the map T_d^2, $j = 1, 2$. Similarly, $W^s = W^s_{p_0} \cup W^s_{p_1}$ for the stable manifold. Consequently, we remark that W^s and W^u refer to the manifolds associated to the hyperbolic periodic orbit $\{p_0, p_1\}$, and hence they are not the manifolds associated to the origin (with a similar notation) studied and considered in Sections 3 and 4.

To simplify the notation, unless strictly necessary, we drop the dependence of $\lambda_d^\pm$ and $m_d^\pm$ with respect to the parameter d. Thus, we will write

$$\lambda^\pm := \lambda_d^\pm \qquad \text{and} \qquad m^\pm := m_d^\pm.$$

We introduce $m^\star = \frac{7}{2}$.

Next lemma gives a precise description of the geometry of $W^u_{p_0}$ that we will use to prove that $\{p_0, p_1\} \subset \partial\mathcal{A}_d(0)$ and finally to prove that $W^s \subset \partial\mathcal{A}_d(0)$.

Lemma 5.2. *Let $\mathcal{D}$ be the closed triangle determined by the vertices*

$$p_1 = (0, -1), \qquad \left(\frac{1}{m^+ + 1}, \frac{-1}{m^+ + 1}\right) \qquad \text{and} \qquad \left(\frac{1}{m^\star + 1}, \frac{-1}{m^\star + 1}\right).$$

Then, there is a local piece of $W^u_{p_0}$ (attached to p_0) tangent to the line $y = 1 + m^+ x$ contained in $T_d(\mathcal{D})$. Moreover, if we parametrize $W^u_{p_0} \cap \{y \leq 1\}$ as $W^u_{p_0} := \{\varphi(t) \mid t \geq 0\}$, with $\varphi(0) = (0, 1)$ and $\varphi(t) \subset \operatorname{int}(T_d(\mathcal{D}))$ for $t \in (0, t_0)$ and $\varphi(t_0) \in \partial T_d(\mathcal{D})$ then

$$\varphi(t_0) \subset \partial T_d(\mathcal{D}) \cap \{y = x\} = \left\{(s, s) \mid \frac{1}{m^+ + 1} \leq s \leq \frac{1}{m^\star + 1}\right\}.$$

See Figure 6 (right).

Proof. The triangle $\mathcal{D}$ can also be represented as

$$\mathcal{D} = \{(t, -1 + mt) \mid t \in [0, 1/(m+1)],\ m \in [m^+, m^\star]\}.$$

The proof of this lemma will follow from an accurate description of the sets $T_d(\mathcal{D})$ and $T_d^{-1}(\mathcal{D})$, their relative position and geometry in the plane, and the behaviour of the map $T_d^{-2} : T_d(\mathcal{D}) \to T_d^{-1}(\mathcal{D})$.

The shape of $T_d(\mathcal{D})$. We consider the decomposition of $\mathcal{D}$ into the segments

$$\ell_m = \{(t, -1 + mt),\ t \in [0, 1/(m+1)]\} \qquad \text{with} \qquad m \in [m^+, m^\star]. \tag{39}$$

If we write $\gamma(t) := \gamma_m(t) = T_d(\ell_m) := (x_m(t), y_m(t)) =: (x(t), y(t))$ we have

$$x(t) = mt - 1 - ((m+1)t - 1)^d \qquad \text{and} \qquad y(t) = mt - 1 - 2((m+1)t - 1)^d. \tag{40}$$

Thus, the first derivatives of $x(t)$ and $y(t)$ are given by

$$x'(t) = m - d(m+1)((m+1)t - 1)^{d-1} \quad \text{and} \quad y'(t) = m - 2d(m+1)((m+1)t - 1)^{d-1}.$$

Easy computations show that $x'(t)$ and $y'(t)$ vanish at the points

$$r^\pm = \frac{1}{m+1}\left[1 \pm \left(\frac{m}{d(m+1)}\right)^{1/(d-1)}\right] \quad \text{and} \quad s^\pm = \frac{1}{m+1}\left[1 \pm \left(\frac{m}{2d(m+1)}\right)^{1/(d-1)}\right],$$

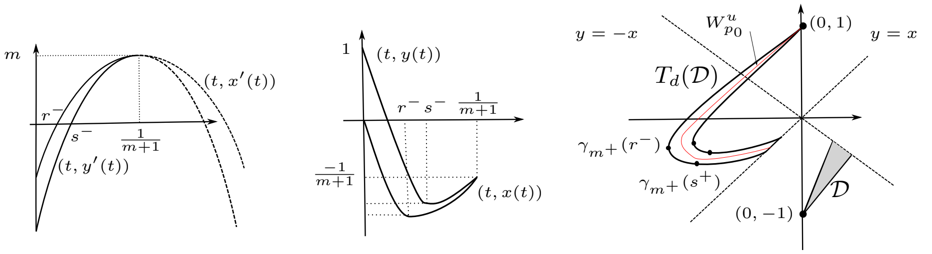


FIGURE 6. Left figure shows the graphs of the curves $x'(t)$ and $y'(t)$. The center picture shows the graphs of the curves $x(t)$ and $y(t)$. Finally, the right picture displays a sketch of the triangle $\mathcal{D}$ and its image $T_d(\mathcal{D})$.

respectively. Moreover,

$$0 < r^- < s^- < \frac{1}{m+1} < s^+ < r^+ \quad \text{and} \quad x\left(\frac{1}{m+1}\right) = y\left(\frac{1}{m+1}\right) = \frac{-1}{m+1},$$

where $t = 1/(m+1)$ corresponds to the common maximum of $x'(t)$ and $y'(t)$. See Figure 6 (left). In summary, the components $x(t)$ and $y(t)$ of the curve $\gamma(t)$ are polynomial functions in t, having a unique minimum in the interval $[0, 1/(m+1)]$ located at $t = r^-$ and $t = s^-$, respectively, and sharing the same negative value, $-1/(m+1)$, at $t = 1/(m+1)$. See the middle picture in Figure 6.

To conclude the description of the shape of γ, see Figure 6 (right). Let us prove that its image can be represented as the union of two graphs with respect to the variable x (i.e., it admits a piecewise parametrization of graphs with respect to x). We write

$$\gamma(x) = (x, \gamma^{(2)}(x)) := (x, y\,(t(x)))\,,$$

where $t(x) := x^{-1}(t)$ is one of the two branches of the inverse of $x(t)$. Some direct computations show that

$$\begin{aligned}
\frac{d\gamma^{(2)}}{dx} &= \frac{dy}{dt}\left(\frac{dx}{dt}\right)^{-1}, \\
\frac{d^2\gamma^{(2)}}{dx^2} &= \left(\frac{dx}{dt}\right)^{-3}\left(\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{dy}{dt}\frac{d^2x}{dt^2}\right) \qquad (41)\\
&= \left(\frac{dx}{dt}\right)^{-3}\left(d(d-1)m(m+1)^2\left((m+1)t-1\right)\right)^{d-2},
\end{aligned}$$

everything evaluated at the corresponding branch of $t = t(x)$. We denote $\gamma^{(2)}_{\mathrm{u}} := \gamma^{(2)}_{\mathrm{u},m}$ and $\gamma^{(2)}_{\ell} := \gamma^{(2)}_{\ell,m}$ the functions corresponding to the upper (concave) and lower (convex) graphs, respectively.

From the previous discussion, dx/dt has a unique zero at $t = r^-$ and it is monotone in the whole interval $[0, 1/(m+1)]$, see Figure 6 (left). For the upper branch, corresponding to $0 \le t < r^-$, we have $dx/dt < 0$ and therefore $\gamma^{(2)}_{\mathrm{u}}(x)$ is increasing and concave (see (41)) and for the lower branch, $r^- < t \le 1/(m+1)$, we have $dx/dt > 0$ and therefore $\gamma^{(2)}_{\ell}(x)$ is convex having a minimum at $x(s^-)$ (see again (41)). See Figure 6 (right), we have drawn (qualitatively) the curve γ for the values $m = m^+$ and $m = m^\star$. We remark that $\gamma^{(2)}_{\mathrm{u,m^+}}(x)$ is tangent at p_0 to the line $y = m^+x + 1$ since it is the image of the side of $\mathcal{D}$ tangent to $W^u_{p_1}$. Since T_d sends the line $\{y = -x\}$ to $\{y = x\}$ the images of all the curves γ_m end up at $\{y = x\}$.

All together determines the shape of $T_d(\mathcal{D})$. Moreover, in the light of the above arguments we have that

$$\partial T_d(\mathcal{D}) = \gamma^{(2)}_{\mathrm{u},m^+} \cup \gamma^{(2)}_{\ell,m^+} \cup \gamma^{(2)}_{\mathrm{u},m^\star} \cup \gamma^{(2)}_{\ell,m^\star} \cup \{(x,x) \mid \frac{-1}{m^+ + 1} < x < \frac{-1}{m^\star + 1}\}.$$

Hereafter we will refer to

$$\gamma^{(2)}_{\mathrm{u},m^+} \cup \gamma^{(2)}_{\mathrm{l},m^+} \qquad \text{and} \qquad \gamma^{(2)}_{\mathrm{u},m^\star} \cup \gamma^{(2)}_{\mathrm{l},m^\star}$$

as the left and right the boundaries of $T_d(\mathcal{D})$, respectively. See Figure 6 (right) and Figure 7.

The shape of $T_d^{-1}(\mathcal{D})$. We consider the same decomposition of $\mathcal{D}$ into the segments ℓ_m as in (39). We denote $\Gamma(t) := \Gamma_m(t) = T_d^{-1}(\ell_m) := (\alpha_m(t), \beta_m(t)) =: (\alpha(t), \beta(t))$. We have

$$\alpha(t) = (m-2)t - 1 + ((1-m)t+1)^{1/d}, \quad \beta(t) = (2-m)t + 1, \quad t \in [0, 1/(m+1)]. \tag{42}$$

Therefore, the first and second derivatives of $\alpha(t)$ and $\beta(t)$ are given by

$$\begin{aligned} \alpha'(t) &= m - 2 + \frac{1-m}{d}((1-m)t+1)^{(1-d)/d}, & \beta'(t) &= 2 - m, \\ \alpha''(t) &= \frac{1-d}{d^2}(1-m)^2((1-m)t+1)^{(1-2d)/d} < 0, & \beta''(t) &= 0. \end{aligned}$$

Clearly $\beta'(t) < 0$ since $m \geq m^+ > 2$. Next, we focus the attention on $\alpha'(t)$. The line $t = 1/(m-1)$ is a vertical asymptote (outside the domain $(0, 1/(m+1))$) and simple computations show that $\alpha'(t) = 0$ if and only if $t = t^{\pm}$ where

$$t^{\pm} := t^{\pm}_m = \frac{1}{m-1} \pm \left(\frac{m-1}{d^d(m-2)^d}\right)^{1/(d-1)}.$$

Some further computations show that

$$t^-_{m^+} < 0, \qquad t^-_{m^\star} > \frac{1}{m^\star + 1} \qquad \text{and} \qquad t^+_m > \frac{1}{m-1}, \qquad \forall m \in [m^+, m^\star]. \tag{43}$$

Since $\alpha'_{m^+}(0) = 4(d-1)\left(1 - d + \sqrt{1 - 10d + 9d^2}\right)^{-1} + 1/d - 2 < 0$ and $\alpha'_{m^\star}(0) > \frac{1}{2}(3 - 5/d) > 0$, it follows from the previous arguments and (43) that $\alpha_{m^+}(t)$ is monotonically decreasing and $\alpha_{m^\star}(t)$ is monotonically increasing in the considered domain. Consequently, Γ_{m^+} and $\Gamma_{m^\star}$ can be expressed as graphs of monotone functions of the form

$$\Gamma(x) = (x, \Gamma^{(2)}(x)) := (x, \beta(t(x))),$$

where $t(x) := \alpha^{-1}(t)$ for $m = m^+$ and $m = m^\star$, respectively. We have

$$\begin{aligned} \frac{d\Gamma^{(2)}}{dx} &= \frac{d\beta}{dt}\left(\frac{d\alpha}{dt}\right)^{-1}, \\ \frac{d^2\Gamma^{(2)}}{dx^2} &= -\left(\frac{d\alpha}{dt}\right)^{-3}\frac{d\beta}{dt}\frac{d^2\alpha}{dt^2} = -\left(\frac{d\alpha}{dt}\right)^{-3}\left(\frac{d-1}{d^2}\frac{(m-2)(m-1)^2}{((1-m)t+1)^{(2d-1)/d}}\right), \end{aligned} \tag{44}$$

everything evaluated at $t = t(x)$. Indeed, taking into account (44), when $m = m^+$, $\Gamma^{(2)}$ is increasing and convex, while when $m = m^\star$, $\Gamma^{(2)}$ is decreasing and concave. From (42) we conclude that

$$\beta(t) > \beta(1/(m+1)) = 3/(m+1) > 3/(m^\star + 1) = 2/3,$$

and then the preimage $T^{-1}(\mathcal{D})$ is above the line $\{y = 2/3\}$. Finally we notice that the image by T_d^{-1} of the segment $\mathcal{D} \cap \{y = -x\}$ is contained in (the graph of)

$$x = \phi(y) := -y + \left(\frac{2}{3}y\right)^{1/d}.$$

Then

$$\phi'(y) = -1 + \frac{2}{3d}\left(\frac{2}{3}y\right)^{(1-d)/d} < -1 + \frac{3}{2d} < -\frac{1}{2},$$

and the function $y = \phi^{-1}(x)$ is decreasing in the corresponding domain.

In particular the curve $\Gamma_{m^+}(x)$ belongs to the second quadrant, while $\Gamma_{m^\star}(x)$ belongs to the first one. In Figure 7 we display $T_d(\mathcal{D})$ and $T_d^{-1}(\mathcal{D})$ and we can see the relative position of these two sets and the initial triangle $\mathcal{D}$. We emphasize that, from arguments above, $\gamma_{m^+}(x)$ and $\Gamma_{m^+}(x)$ are both tangent to the line $y = m^+x+1$, but, using the convexity properties, $\gamma_{m^+}(x)$ is below this line while $\Gamma_{m^+}(x)$ is above it, hence their relative position illustrated in Figure 7 is the right one.

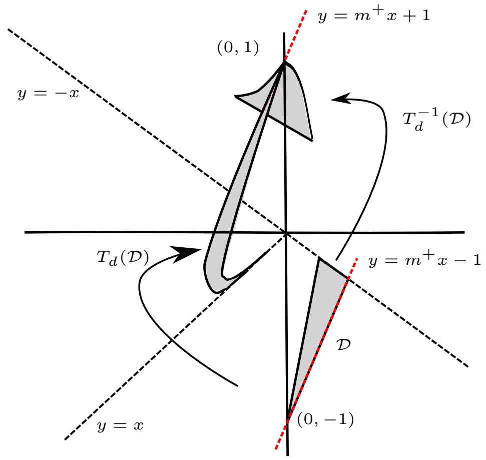


FIGURE 7. A qualitative representation of the triangle $\mathcal{D}$ and the sets $T_d(\mathcal{D})$ and $T_d^{-1}(\mathcal{D})$.

Consider now $T_d^{-2} : T_d(\mathcal{D}) \to T_d^{-1}(\mathcal{D})$. From the stable/unstable manifold theorem and the relative position and geometry of the sets $T_d(\mathcal{D})$ and $T_d^{-1}(\mathcal{D})$, we can conclude that, locally, $W^u_{p_0}$ exists, it is tangent to $y = m^+x + 1$, and it is contained in $T_d(\mathcal{D})$. In particular we also conclude that there is a local piece of $W^u_{p_1}$ attached to p_1 belonging to $\mathcal{D}$.

We claim that $W^u_{p_0}$ may only *leave* $T_d(\mathcal{D})$ through the piece of the boundary given by $\partial T_d(\mathcal{D}) \cap \{y = x\}$ (later we will see that $W^u_{p_0}$ does leave $T_d(\mathcal{D})$ through this boundary). To check the claim we first observe that $W^u_{p_0} \cap \{y \le 1\}$ can be parametrized $W^u_{p_0} := \{\varphi(t) \mid t \ge 0\}$, with $\varphi(0) = (0, 1)$ (see [6]). Second, we suppose it leaves $T_d(\mathcal{D})$ either for the left or the right boundaries of $T_d(\mathcal{D})$, and we get a contradiction.

Let $p = \varphi(t_0)$ for some $t_0 > 0$ such that $\{\varphi(t),\ t \in (0, t_0)\} \subset \operatorname{int}(T_d(\mathcal{D}))$ for all $t \in (0, t_0)$ and $p \in \partial T_d(\mathcal{D}) \setminus \{x = y\}$ (that is, it leaves $T_d(\mathcal{D})$ through the left or right boundaries of it). Consider $q := T_d^{-2}(p)$. Since $W^u_{p_0}$ is invariant by T_d^{-2} we have that $q = \varphi(t_q)$ for some $t_q \in (0, t_p)$. However, we also have

$$q \in \partial\left(T_d^{-1}(\mathcal{D})\right) \setminus T_d^{-1}\left(\partial\mathcal{D} \cap \{y = -x\}\right),$$

which provides a contradiction (see Figure 7). $\square$

Let $\mathcal{E}$ be the closed triangle determined by the vertices $\tau_0 = (0, 1)$, $\tau_1 = (-1/2, 0)$ and $\tau_2 = (-1/3, 0)$.

Next two lemmas refer to the set $T_d(\mathcal{D}) \cap \{y \geq 0\}$. First we show that this set belongs to the triangular region $\mathcal{E}$ and, second, we will give relevant information of the dynamics of $T^{-2}|_{\mathcal{E}}$ (and therefore in $T_d(\mathcal{D}) \cap \{y \geq 0\}$). All together implies two main properties for $W^u_{p_0}$. On the one hand, we will prove that $W^u_{p_0} \cap [-1/2, 0] \times \{0\} \neq \emptyset$ and, on the other hand, we will prove that $W^u_{p_0} \cap \{y = x\} \neq \emptyset$ (such intersection happens on the fourth quadrant). See Figure 8.

Lemma 5.3. *We have*

$$T_d(\mathcal{D}) \cap \{y \geq 0\} \subset \mathcal{E}.$$

Proof. We will check that the left and the right boundaries of $T_d(\mathcal{D}) \cap \{y \geq 0\}$, given by pieces of the curves γ_{m^+} and $\gamma_{m^\star}$, respectively, are contained in $\mathcal{E}$. By the discussion we did when analyzing the shape of $T_d(\mathcal{D})$ we know that both curves can be written as graphs of concave functions that only intersect at the point $p_0 = (0, 1)$ (see Lemma 5.2). Moreover the slope s_{m^+} of γ_{m^+} at $t = 0$ is $m^+ > 2$ (the slope of the left side of $\mathcal{E}$). Indeed,

$$2 < s_{m^+} \leq 6/(\sqrt{13} - 1) \lessapprox 2.3028.$$

All together implies that the statement of the lemma follows from proving that the (only) intersection of $\gamma_{m^\star}$ with $y = 0$ happens at a point $x < -1/3$.

Consider the second component $y(t)$ of $\gamma_{m^\star}$ defined for $t \in [0, 1/(m^\star + 1)]$. Since $y(0) = 1$, $y(1/(m^\star + 1)) = -1/(m^\star + 1) < 0$ and $y''(t) > 0$ there exists a unique $t_1 \in [0, 1/(m^\star + 1)]$ such that $y(t_1) = 0$. See Figure 6 (center).

When $d = 3$ we can localize t_1 with some precision. Let $t_1^- = 1/18$ and $t_1^+ = 1/16$. Both values belong to $[0, r^-]$ where the functions $x(t)$ and $y(t)$ are decreasing. See (40). We have

$$y(t_1^-) = -\frac{29}{36} + 2\left(\frac{27}{36}\right)^3 > \frac{1}{30} \qquad \text{and} \qquad y(t_1^+) = -\frac{25}{32} + 2\left(\frac{23}{32}\right)^3 < \frac{-1}{30}.$$

This means that $t_1 \in (1/18, 1/16)$ and that, since $\gamma_{m^\star}$ is the graph of a concave function, $x(t_1) < x(t_1^-) = -\frac{29}{36} + \left(\frac{27}{36}\right)^3 < -\frac{1}{3}$. By concavity we get that $\gamma_{m^\star} \cap \{y \geq 0\} \subset \mathcal{E}$.

To deal with the general value of d odd, $d \geq 3$, we will see that if we consider the intersection point $(x(t_1), 0)$ as a function of d, it is decreasing so that for $d \geq 3$, $x(t_1) < -\frac{1}{3}$. Indeed, we compute the derivative of $x(t_1)$ with respect to d.

We write $y = y(t, d)$. Let $t_1(d)$ be the parameter such that $y(t_1(d), d) = 0$. We want to compute $(x(t_1(d), d))'$, where prime stands for the derivative with respect to d.

Derivating impliticly (and simplifying notation) we have

$$(t_1(d))' = -\frac{\partial y}{\partial d}(t_1(d), d) / \frac{\partial y}{\partial t}(t_1(d), d) =: -\left(\frac{\partial y}{\partial d} / \frac{\partial y}{\partial t}\right)|_{(t_1(d),d)} =: -\left(\frac{\partial y}{\partial d} / \frac{\partial y}{\partial t}\right).$$

Then,

$$\begin{aligned}(x(t_1(d),d))' &= \frac{\partial x}{\partial t}\times(t_1(d))' + \frac{\partial x}{\partial d}\\ &= \frac{\partial x}{\partial t}\times\Big(-\frac{\partial y}{\partial d}/\frac{\partial y}{\partial t}\Big) + \frac{\partial x}{\partial d}\\ &= \Big(1/\frac{\partial y}{\partial t}\Big)\times\Big(\frac{\partial x}{\partial d}\times\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\times\frac{\partial y}{\partial d}\Big).\end{aligned}$$

Taking the corresponding derivatives from equation (40) and simplifying we get

$$(x(t_1(d),d))' = -\Big(1/\frac{\partial y}{\partial t}\Big)m^\star(1-(m^\star+1)t)^d\log(1-(m^\star+1)t) < 0.$$

Indeed, we are evaluating the right side of the above equation at the point $(t_1(d), d)$ with $0 < t_1(d) < r^-(d)$. Thus, we have

$$\frac{\partial y}{\partial t}(t_1(d),d) < 0 \qquad \text{and} \qquad 0 < 1-(m^\star+1)t < 1.$$

□

Next lemma tell us that, while the iterates by T_d^{-2} remain in $\mathcal{E}$, the sequence of their second coordinates of them is strictly increasing. See Figure 8.

Lemma 5.4. *Let* $(f(x,y), g(x,y)) := T_d^{-2}(x,y)$. *If* $(x,y)\in\mathcal{E}$ *then* $g(x,y)\geq y$ *and the equality only holds when* $(x,y) = (0,1)$.

Proof. From (9) we have that $g(x,y) = 3y-6x+2(x-y)^{1/d}$, and then $g(x,y)\geq y$ in $\mathcal{E}$ if and only if

$$G(x,y) := 2y-6x+2(x-y)^{1/d} > 0, \qquad \forall (x,y)\in\mathcal{E}\setminus\{(0,1)\}.$$

To prove this inequality we will show that G restricted to $\mathcal{E}$ has a global minimum $G=0$ at $(0,0)$ which is only attained at $(0,0)$. A direct computation shows that the partial derivatives of G cannot vanish simultaneously, therefore the minimum has to be attained at the boundary of $\mathcal{E}$. It is clear that the restriction of the function G on each of the three segments of $\partial\mathcal{E}$ is given by

$$\begin{aligned}\chi_1(x) :=& G(x,0) = 2\left(-3x+x^{1/d}\right), && x\in[-1/2,-1/3],\\ \chi_2(x) :=& G(x,2x+1) = 2\left(1-x-(1+x)^{1/d}\right), && x\in[-1/2,0],\\ \chi_3(x) :=& G(x,3x+1) = 2\left(1-(1+2x)^{1/d}\right), && x\in[-1/3,0].\end{aligned}$$

Using elementary methods we can check that indeed $\chi_1(x)>0$, $\chi_2(x)\geq 0$ and $\chi_3(x)\geq 0$ in the indicated intervals and that $\chi_2(x)=0$, $\chi_3(x)=0$ only hold when $x=0$. □

Lemma 5.5. *The unstable manifold* $W^u_{p_0}$ *crosses the interval* $I_0 := T_d(\mathcal{D})\cap\{y=x\}$ *at some point* $(\widehat{p},\widehat{p})$ *such that* $\frac{1}{m^+ +1} < \widehat{p} < \frac{1}{m^\star+1}$. *Moreover, the piece of* $W^u_{p_0}$ *from* $(0,0)$ *to* $(\widehat{p},\widehat{p})$ *is contained in* $T_d(\mathcal{D})$. *We also have that this piece of* $W^u_{p_0}$ *cuts the segment* $(-1/2,-1/3)\times\{0\}\subset\mathbb{R}^2$.

Proof. We first prove the existence of the point $(\widehat{p},\widehat{p})$. A completely analogous procedure will be used in the proof of Proposition 5.7 and in Section 6. Let

$$I_0 := T_d(\mathcal{D})\cap\{y=x\} = \left\{(s,s) \mid \frac{1}{m^+v+1}\leq s\leq\frac{1}{m^\star+1}\right\}.$$

The image $T_d^{-2}(I_0)$ is a curve, which is a piece of the boundary of $T_d^{-1}(\mathcal{D})$ that, be previous arguments, has to cross the left and right boundaries of $T_d(\mathcal{D})$. Actually, it can be parametrized as $s \mapsto (-3s + (2s)^{1/d}, 3s)$. In the study of the shape of $T_d^{-1}(\mathcal{D})$ we have seen that $T_d^{-1}(\mathcal{D}) \subset \{y > \frac{1}{m^\star+1} = \frac{2}{3}\}$. We define

$$I_1 = T_d^2(T_d^{-2}(I_0) \cap T_d(\mathcal{D})) \subset I_0$$

and, in general,

$$I_n = T_d^{2n}(T_d^{-2n}(I_{n-1}) \cap T_d(\mathcal{D})), \qquad n \geq 1.$$

It is clear that $I_n \subset I_{n-1}$ for all $n \geq 1$.

Then, $I_\infty = \cap_{n\geq 0} I_n$ is compact and contains the points in $T_d(\mathcal{D})$ such that all their negative iterates by T_d^2 are in $T_d(\mathcal{D})$. Moreover, by Lemma 5.4, the sequence of the second components of these iterates is increasing and has to converge to 1. Then, those points must belong to $W^u_{p_0}$ and therefore there exists $(\widehat{p}, \widehat{p}) \in I_0$ such that

$$(\widehat{p}, \widehat{p}) \subset W^u_{p_0} \cap \{y = x\} \subset T_d(\mathcal{D}) \cap \{y = x\} \neq \emptyset.$$

From Lemma 5.2 the piece of $W^u_{p_0}$ from $(0,0)$ to $(\widehat{p}, \widehat{p})$ must be contained in $T_d(\mathcal{D})$. Hence Lemma 5.3 implies that $W^u_{p_0}$ cuts the segment $(-1/2, -1/3) \times \{0\} \subset \mathbb{R}^2$. $\square$

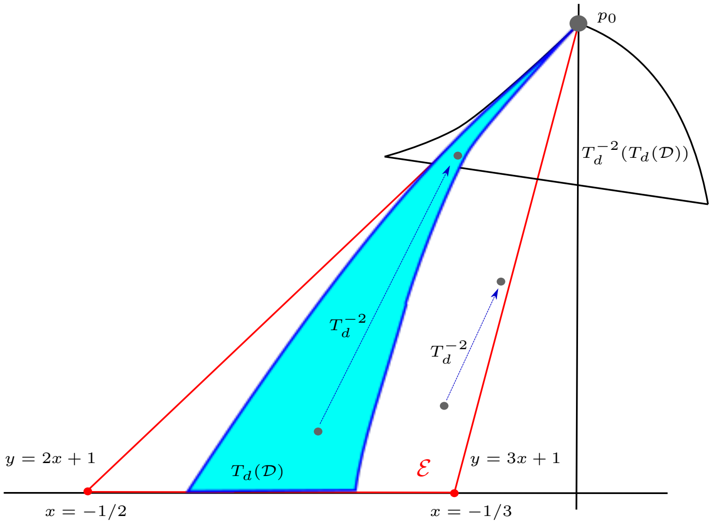


FIGURE 8. $T_d(\mathcal{D}) \cap \{y \geq 0\} \subset \mathcal{E}$. Points in $\mathcal{E}$ are mapped "up" by T_d^{-2}.

Next propositions are devoted to show the two further properties of $\mathcal{A}_d(0)$ claimed in statement (b) of the Main Theorem. First we show that the stable manifold of the periodic orbit belongs to $\partial\mathcal{A}_d(0)$ and second we show that $\mathcal{A}_d(0)$ is unbounded, which follows because the stable manifold is unbounded.

Proposition 5.6. *Let $d \geq 3$ odd. Then, $W^s \subset \partial\mathcal{A}_d(0)$.*

Proof. From Proposition 5.1 we know that $W^u_{p_0}$ crosses the interval $(-1/2, -1/3) \times \{0\} \subset \mathbb{R}^2$. Let $(x_0, 0) \in W^u_{p_0} \cap (-1/2, -1/3) \times \{0\}$ be the first intersection point of

$W^u_{p_0}$ with the segment. From Proposition 5.1 we also have that $[x_0, 0] \times \{0\} \subset \mathcal{A}_d(0)$. We have (recall that $T_d^{-1}(p_0) = p_1$ and that when d is odd T_d is one-to-one)

$$\bigcup_{n=0}^{\infty} T_d^{-n}\left([x_0,0]\times\{0\}\right) \subset \mathcal{A}_d(0) \qquad \text{and} \qquad p_j \in \mathrm{Acc}\left(\{T_d^{-n}(x_0)\}_{n\geq 0}\right), \quad j=0,1,$$

where $\mathrm{Acc}(X)$ denotes the set of accumulation points of X. Since, of course, $\{p_0, p_1\} \notin \mathcal{A}_d(0)$ we conclude that $p_j \in \partial\mathcal{A}_d(0)$, $j = 0, 1$. Now, let q be any point in $W^s_{p_0}$ and U a small disc centered at q and let $\Sigma \subset U$ be a transversal segment to $W^s_{p_0}$ through q. On the one hand, $W^s_{p_0} \cap U$ is not contained in $\mathcal{A}_d(0)$. On the other hand, by the λ-Lemma [15], the iterates by T_d^2 of the points in Σ (close enough to $W^s_{p_0}$) accumulate to $W^u_{p_0}$. Therefore, we would have

$$(x_0, 0) \in \mathrm{Acc}\left(\{T_d^n(\Sigma)\}_{n\geq 0}\right).$$

Since $(x_0, 0) \in \mathcal{A}_d(0)$ and $\mathcal{A}_d(0)$ is open we conclude that U contains points of $\mathcal{A}_d(0)$. If $q \in W^s_{p_1}$ then $T_d(q) \in W^s_{p_0}$ and the conclusion is the same. All together implies that $q \in \partial\mathcal{A}_d(0)$, as desired. $\square$

Proposition 5.7. *$\mathcal{A}_d(0)$ is unbounded.*

Proof. From the previous lemma it is enough to see that the stable manifold W^s of the hyperbolic two-cycle $\{p_0, p_1\}$ is unbounded. We start introducing some notation. See Figure 9. Let $Q_2^\star$ and $Q_4^\star$ be the closed unbounded subsets of the second and fourth quadrant defined as follows

$$Q_2^\star := \{(x,y) \mid x \leq 0,\ y \geq 1\} \qquad \text{and} \qquad Q_4^\star := \{(x,y) \mid x \geq 0,\ y \leq -1\}.$$

Next we split the above sets into three pieces. Concretely,

$$Q_2^\star = \bigcup_{j=1}^{3} E_j \qquad \text{and} \qquad Q_4^\star = \bigcup_{j=1}^{3} D_j,$$

where

$$\begin{aligned}
E_1 &= \{x \leq \left(\tfrac{y+1}{2}\right)^{\frac{1}{d}} - y \mid\ y \geq 1\}, & D_1 &= \{0 \leq x \leq y^{\frac{1}{d}} - y \mid\ y \leq -1\},\\
E_2 &= \{\left(\tfrac{y+1}{2}\right)^{\frac{1}{d}} - y \leq x \leq y^{\frac{1}{d}} - y \mid\ y \geq 1\}, & D_2 &= \{y^{\frac{1}{d}} - y \leq x \leq \left(\tfrac{y-1}{2}\right)^{\frac{1}{d}} - y \mid\ y \leq -1\},\\
E_3 &= \{y^{\frac{1}{d}} - y \leq x \leq 0 \mid\ y \geq 1\}, & D_3 &= \{x \geq \left(\tfrac{y-1}{2}\right)^{\frac{1}{d}} - y \mid\ y \leq -1\}.
\end{aligned}$$

We denote by $\{\mathcal{J}_\ell, \mathcal{I}_\ell\}$ with $\ell = 1, 2$ the straight boundaries of the above sets. That is,

$$\begin{aligned}
\mathcal{J}_1 &= \{(x,1) \mid\ x \leq 0\}, & \mathcal{I}_1 &= \{(x,-1) \mid\ x \geq 0\},\\
\mathcal{J}_2 &= \{(0,y) \mid\ y \geq 1\}, & \mathcal{I}_2 &= \{(0,y) \mid\ y \leq -1\}.
\end{aligned}$$

Finally, we denote by $\{\gamma^\pm, \sigma^\pm\}$ the other boundaries of the sets E_j and D_j. That is,

$$\gamma^+ = E_2 \cap E_3 = \left\{\left(y^{\frac{1}{d}} - y, y\right) \mid\ y \geq 1\right\} = \{y = (x+y)^d \mid\ x \leq 0,\ y \geq 1\},$$

$$\gamma^- = E_1 \cap E_2 = \left\{\left(\left(\frac{y+1}{2}\right)^{\frac{1}{d}} - y, y\right) \mid\ y \geq 1\right\} = \{y = -1 + 2(x+y)^d \mid\ x \leq 0,\ y \geq 1\},$$

$$\begin{aligned}\sigma^- &= D_1 \cap D_2 = \left\{ \left(y^{\frac{1}{d}} - y, y \right) \mid \ y \le -1 \right\} = \{ y = (x+y)^d \mid \ x \ge 0, \ y \le -1 \},\\ \sigma^+ &= D_2 \cap D_3 = \left\{ \left(\left(\frac{y-1}{2} \right)^{\frac{1}{d}} - y, y \right) \mid \ y \le -1 \right\} \\ &= \{ y = 1 + 2(x+y)^d \mid \ x \ge 0, \ y \le -1 \}.\end{aligned} \tag{45}$$

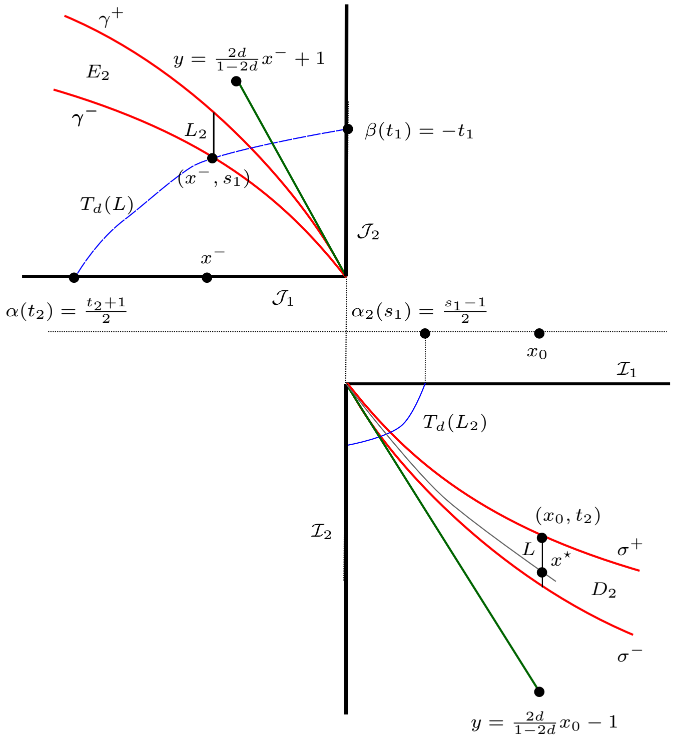


FIGURE 9. The sets and curves used in the proof of Proposition 5.7.

See Figure 9 for a qualitative picture and the relative position of all curves and sets. One can check that by construction we have $T_d(\gamma^-) = \mathcal{I}_1$, $T_d(\gamma^+) = \mathcal{I}_2$, $T_d(\sigma^-) = \mathcal{J}_2$ and $T_d(\sigma^+) = \mathcal{J}_1$. Consequently,

$$T_d(D_2) = \bigcup_{j=1}^{3} E_j \qquad \text{and} \qquad T_d(E_2) = \bigcup_{j=1}^{3} D_j.$$

We also notice that the curves $\gamma^\pm(y)$ and $\sigma^\pm(y)$ are graphs of monotonically decreasing functions of y. Indeed, for instance, if we write $\gamma^\pm = \{ \left(\gamma_1^\pm(y), y \right) \mid \ y \ge 1 \}$ then

$$\frac{d\gamma_1^+}{dy}(y) = \frac{1}{d} \left(\frac{1}{y} \right)^{\frac{d-1}{d}} - 1 \le \frac{1}{d} - 1 < 0 \quad \text{and} \quad \frac{d\gamma_1^-}{dy}(y) = \frac{1}{2d} \left(\frac{2}{y+1} \right)^{\frac{d-1}{d}} - 1 \le \frac{1}{2d} - 1 < 0.$$

Let

$$\Omega := T_d^{-1}(E_2).$$

According to the previous discussion it is clear that $\Omega \subset D_2$ (remember that $\partial E_2 = \gamma^+ \cup \gamma^-$). We also claim that $\partial \Omega$ is given by two curves contained in D_2

which can be written as graphs of monotone functions (of y as well as of x). Of course $\partial\Omega = T_d^{-1}(\gamma^-) \cup T_d^{-1}(\gamma^+)$. Using (9) and (45) we have

$$T_d^{-1}\left(\gamma^-(y)\right) = \begin{pmatrix} \xi_1(y) \\ \xi_2(y) \end{pmatrix} := \begin{pmatrix} 3y - 2^{\frac{d-1}{d}}(y+1)^{\frac{1}{d}} + \left[2^{-\frac{1}{d}}(y+1)^{\frac{1}{d}} - 2y\right]^{\frac{1}{d}} \\ -3y + 2^{\frac{d-1}{d}}(y+1)^{\frac{1}{d}} \end{pmatrix},$$

with $y \geq 1$. Thus, we have

$$\begin{aligned} \frac{d\xi_1}{dy}(y) &= 3 - \frac{1}{d}\left(\frac{y+1}{2}\right)^{\frac{1-d}{d}} + \frac{1}{d}\left[\left(\frac{y+1}{2}\right)^{\frac{1}{d}} - 2y\right]^{\frac{1-d}{d}}\left(\frac{1}{2d}\left(\frac{y+1}{2}\right)^{\frac{1-d}{d}} - 2\right) \\ &\geq 3 - \frac{1}{d} > 0 \end{aligned}$$

and

$$\frac{d\xi_2}{dy}(y) = -3 + \frac{1}{d}\left(\frac{y+1}{2}\right)^{\frac{1-d}{d}} \leq -3 + \frac{1}{d} < 0.$$

Therefore, using the same formulas as the ones in (44), $T_d^{-1}(\gamma^-)$ can be written as a graph of a monotonically decreasing function (with respect to y as well as x). Similar computations lead to the same conclusion for $T_d^{-1}(\gamma^+)$.

Claim 1. *Let* $\lambda := d^2/(1-2d)^2$. *Let* $x_0 > 0$ *and* $(x_0, y_0) \in \Omega$. *We denote* $(x_2, y_2) = T_d^2(x_0, y_0)$. *Then,*

$$0 \leq x_2 < \lambda\, x_0.$$

Given $x_0 > 0$, let $L := \{x = x_0\} \cap D_2 = \{(x_0, t) \mid t_1 = t_1(x_0) \leq t \leq t_2(x_0) = t_2\}$, where $(x_0, t_1) \in \sigma^-$ and $(x_0, t_2) \in \sigma^+$ and hence

$$t_1 = (x_0 + t_1)^d \qquad \text{and} \qquad t_2 = 2(x_0 + t_2)^d + 1.$$

The image of L by T_d can be represented by

$$\Gamma_1(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} := T_d \begin{pmatrix} x_0 \\ t \end{pmatrix} = \begin{pmatrix} t - (x_0 + t)^d \\ t - 2(x_0 + t)^d \end{pmatrix}, \qquad t \in [t_1, t_2].$$

Since $d - 1$ is even and the fact that if $(x, y) \in D_2$ we have $y > 2(x+y)^d + 1$ and $x + y < 0$,

$$\alpha'(t) = 1 - d\,(x_0 + t)^{d-1} \leq 1 - d\left(\frac{t-1}{2}\right)^{(d-1)/d} \leq 1 - d < 0. \tag{46}$$

This means that $\alpha(t)$ is strictly decreasing in t with

$$\alpha(t_2) = (t_2 + 1)/2 \leq \alpha(t) \leq \alpha(t_1) = 0. \tag{47}$$

Similarly we have $\beta'(t) = 1 - 2d\,(x_0 + t)^{d-1} \leq 1 - 2d < 0$ and $\beta(t_2) = 1 \leq \beta(t) \leq \beta(t_1) = -t_1$.

This implies that $\Gamma_1(t)$ can be seen as the graph of an increasing function joining $\mathcal{J}_1$ with $\mathcal{J}_2$. Therefore, it crosses transversally the boundary of E_2. Let $(x^+, y^+) \in \gamma^+$ and $(x^-, y^-) \in \gamma^-$ be the corresponding intersections. From (47) and (46) we have

$$(t_2 + 1)/2 \leq x^- \leq x \qquad \text{for all} \quad (x, y) \in \Gamma_1 \cap E_2. \tag{48}$$

Now, given $\xi \in [x^-, 0]$ we consider the new vertical segment in E_2,

$$L_2 := \{x = \xi\} \cap E_2.$$

By its definition $T_d(L_2)$ is a curve joining $\mathcal{I}_1$ and $\mathcal{I}_2$, parametrized by

$$\Gamma_2(s) = \begin{pmatrix} \alpha_2(s) \\ \beta_2(s) \end{pmatrix} := T_d \begin{pmatrix} \xi \\ s \end{pmatrix} = \begin{pmatrix} s - (\xi + s)^d \\ s - 2(\xi + s)^d \end{pmatrix}, \qquad s \in [s_1, s_2],$$

where $s_1 = s_1(\xi)$, $s_2 = s_2(\xi)$ and

$$s_1 = 2(\xi + s_1)^d - 1 > 1 \qquad \text{and} \qquad s_2 = (\xi + s_2)^d.$$

A similar computation to the one in (46) gives that $\beta_2'(s) < \alpha_2'(s) \leq 1 - d < 0$ and then

$$\alpha_2(s_2) \leq \alpha_2(s) \leq \alpha_2(s_1) = \frac{s_1 - 1}{2}, \qquad s \in [s_1, s_2].$$

The claim will follow from $\alpha_2(s_1(\xi)) = \frac{s_1(\xi)-1}{2} \leq \lambda x_0$ for all $\xi \in [x^-, 0]$.

Clearly σ^+ is above its tangent line at the point $(0, -1)$ which is given by $y = \frac{2d}{1-2d}x - 1$ (to get the slope of the line we can use implicit derivation to $y = 1 + 2(x + y)^d$ at $(x, y) = (0, -1)$). As a consequence, since $(x_0, t_2) \in \sigma^+$,

$$t_2 + 1 > \frac{2d}{1 - 2d} x_0. \tag{49}$$

Similarly, γ^- is below its tangent line at the point $(0, 1)$ given by $y = \frac{2d}{1-2d}x + 1$. Consequently, since $(\xi, s_1(\xi)) \in \gamma^-$,

$$s_1(\xi) - 1 < \frac{2d}{1 - 2d}\xi. \tag{50}$$

Using (48), (49) and (50) we have that

$$\alpha_2(s_1(\xi)) = \frac{s_1(\xi) - 1}{2} < \frac{d}{1 - 2d}\xi \leq \frac{d}{1 - 2d}x^- \leq \frac{d}{1 - 2d}\frac{t_2 + 1}{2} \leq \left(\frac{d}{1 - 2d}\right)^2 x_0 = \lambda\, x_0.$$

Claim 2. *Let $x = x_0 > 0$. Then, there exists a point $(x_0, y) \in \Omega$ such that $(x_0, y) \in W^s$. In particular, from Proposition 5.6, we conclude that $\partial\mathcal{A}(0)$ is unbounded.*

Since $(0, -1)$ is hyperbolic we already know that $W^s_{p_1}$ exists and consists of the points such that their ω-limit with respect to T_d^2 is $(0, -1)$.

We recall that $\Omega \subset T_d^2(\Omega) = Q_4^\star$. Thus,

$$T_d^{-2}(\Omega) \subset \Omega. \tag{51}$$

Let $K_0 := \{x = x_0\} \cap \Omega$. Clearly, $T_d(K_0)$ is a curve connecting γ^- and γ^+ and so, by construction, $T_d^2(K_0)$ is a curve connecting $\mathcal{I}_1$ and $\mathcal{I}_2$ and crossing $\partial\Omega$ at exactly two points (remember that T_d is one-to-one): one in $T_d^{-1}(\gamma^-)$ and the other in $T_d^{-1}(\gamma^+)$. We write $K_1 = T_d^{-2}(T_d^2(K_0) \cap \Omega) \subset K_0 \subset \Omega$.

Repeating this procedure we can define recursively

$$K_j = T_d^{-2j}(T_d^{2j}(K_{j-1}) \cap \Omega) \subset K_{j-1}, \qquad j \geq 1.$$

Therefore, $\{K_j\}_{j\geq 0}$ is a sequence of nested compact sets and therefore

$$\bigcap_{j\geq 0} K_j \neq \emptyset.$$

Now, we check that if $(x_0, y_0) \in \bigcap_{j\geq 0} K_j$, then $(x_0, y_0) \in W^s$. Indeed, let $(x_0, y_0) \in \bigcap_{j\geq 0} K_j$. By the definition of K_j,

$$(x_{2j}, y_{2j}) = T_d^{2j}(x_0, y_0) \in T_d^{2j}(K_{j-1}) \cap \Omega \subset T_d^{2j}(K_0) \cap \Omega, \qquad j \geq 0.$$

We can prove by induction that $x_{2j} < \lambda^j x_0$ for all $j \geq 1$. Since $(x_0, y_0) \in K_0 \cap \Omega$, by Claim 1, $x_2 < \lambda x_0$. Assuming the statement is true for $j-1$, since $(x_{2j-2}, y_{2j-2}) \in T_d^{2j-2}(K_0) \cap \Omega$, then $(x_{2j}, y_{2j}) = T_d^2(x_{2j-2}, y_{2j-2})$ satisfies $x_{2j} < \lambda x_{2j-2}$. We conclude that $x_{2j} \to 0$. Since $(x_{2j}, y_{2j}) \in \Omega$ we also have $y_{2j} \to -1$. Since x_0 is arbitrarily large we obtain that the invariant manifold is unbounded. □

6. Proof of statement (c) of the main Theorem: The case d odd and $a = -1$.

According to Remark 3.2, under the parameter values d odd and $a = -1$, the dynamics on the center manifold of the origin is repelling and therefore the only points tending to the origin under iteration are the ones of the stable manifold of (0,0). Hence, it remains to show that the stable manifold is unbounded.

It follows from (5) that for d odd and $a = -1$ the map T_d is a homeomorphism and we have

$$T_d \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y + (x+y)^d \\ y + 2(x+y)^d \end{pmatrix} \quad \text{and} \quad T_d^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x + y - (x-y)^{1/d} \\ 2x - y \end{pmatrix}. \tag{52}$$

In a similar way as in the proof of Proposition 5.7 we introduce a domain, which we expect to contain W^s, and we prove that contains points, arbitrarily far away, such that all their iterates are in the domain and moreover tend to the origin so that indeed it contains W^s.

We will take this domain in the fourth quadrant $Q_4 := \{x \geq 0,\ y \leq 0\}$. We define $D^0 \subset Q_4$ by the condition $T_d^2(D^0) = Q_4$. Since T_d^2 is a homeomorphism the boundary of D^0 is obtained by taking the preimage of the boundary of Q_4 with respect to T_d^2.

Consequently, the boundary of D^0 is the union of the images of the curves

$$\begin{aligned} \sigma_0^+(t) &:= (\alpha_0^+(t), \beta_0^+(t)) = T_d^{-2}(t, 0) = \Big(6t + 2t^{1/d} + (4t + t^{1/d})^{1/d}, -6t - 2t^{1/d}\Big), \\ \sigma_0^-(t) &:= (\alpha_0^-(t), \beta_0^-(t)) = T_d^{-2}(0, -t) = \Big(3t + 2t^{1/d} + (2t + t^{1/d})^{1/d}, -3t - 2t^{1/d}\Big), \end{aligned} \tag{53}$$

with $t \geq 0$. We have that $(\alpha_0^\pm)'(t) > 0$ and $(\beta_0^\pm)'(t) < 0$. Therefore, the curves $\sigma_0^\pm$ are graphs of well defined decreasing functions $h_0^\pm = \beta_0^\pm \circ (\alpha_0^\pm)^{-1}$ from $[0, \infty)$ onto $(-\infty, 0]$.

By construction, the set $D^1 := T_d(D^0) = T_d^{-1}(Q_0)$ is the domain limited by the curves $\sigma_1^\pm := T_d(\sigma_0^\pm(t))$, $t \geq 0$. Concretely, these curves are

$$\begin{aligned} \sigma_1^+(t) &= (\alpha_1^+(t), \beta_1^+(t)) = T_d(\sigma_0^+(t)) = \big(-2t - t^{1/d}, 2t\big), \\ \sigma_1^-(t) &= (\alpha_1^-(t), \beta_1^-(t)) = T_d(\sigma_0^-(t)) = \big(-t - t^{1/d}, t\big), \end{aligned} \tag{54}$$

with $t \geq 0$. Similarly, since $(\alpha_1^\pm)'(t) < 0$ and $(\beta_1^\pm)'(t) > 0$, we have that $\sigma_1^\pm$ are graphs of decreasing functions $h_1^\pm = \beta_1^\pm \circ (\alpha_1^\pm)^{-1}$ from $(-\infty, 0]$ onto $[0, \infty)$.

Finally, $D^2 := T_d(D^1)$ is the full closed fourth quadrant Q_4. For notational convenience we also define

$$\begin{aligned} \sigma_2^+(t) &:= T_d(\sigma_1^+(t)) = \big(t, 0\big), \\ \sigma_2^-(t) &:= T_d(\sigma_1^-(t)) = \big(0, -t\big), \end{aligned}$$

with $t \geq 0$. Since T_d is a homeomorphism, the curves σ_1^+ and σ_1^- are the only preimages of the curves σ_2^+ and σ_2^-, respectively. They only intersect at the origin. The same happens with σ_0^+ and σ_0^-. Moreover, σ_1^+ is above σ_1^- and σ_0^+ is above σ_0^-. Also, we will use that σ_1^+ is below $\{y = -x\}$ and σ_0^- is above $\{y = -x\}$. Indeed, these claims can be checked from (53) and (54) after some computations.

Lemma 6.1. *If* $(x_0,y_0)\in D^0$ *then* $(x_2,y_2):=T_d^2(x_0,y_0)\in Q_4$ *and* $x_2\le x_0/2$.

Proof. The first part of the statement follows from the previous construction. We have to prove the inequality. We define

$$D^0_\rho=\{(x,y)\in D^0\mid x\le\rho\},\quad D^1_\rho=\{(x,y)\in D^1\mid x\ge-\rho\},\quad D^2_\rho=\{(x,y)\in D^2\mid x\le\rho\}.$$

We will prove that, for any $\rho>0$,

$$T_d(D^0_\rho)\subset D^1_{\rho/2}\qquad\text{and}\qquad T_d(D^1_{\rho/2})\subset D^2_{\rho/2}.$$

For the first inclusion we consider the segments $\{x=r\}\cap D^0_\rho$ with $0\le r\le\rho$, parametrized by $s\in[s_-,s_+]\subset[-r,0]$, with s_- and s_+ such that $(r,s_-)\in\{\text{image}\,\sigma_0^-\}$ and $(r,s_+)\in\{\text{image}\,\sigma_0^+\}$. In particular, we have that there exists $t_1\ge0$ such that

$$\sigma_0^-(t_1)=\left(3t_1+2t_1^{1/d}+(2t_1+t_1^{1/d})^{1/d},-3t_1-2t_1^{1/d}\right)=(r,s_-).$$

The image of the segment can be represented by

$$\tau_1(s)=\big(\tau_1^x(s),\tau_1^y(s)\big):=T_d(r,s)=\big(s+(r+s)^d,s+2(r+s)^d\big),\qquad s\in[s_-,s_+].$$

Since $d-1$ is even, $(\tau_1^x)'(s)$ and $(\tau_1^y)'(s)$ are positive. This implies that the minimum of $\tau_1^x(s)$ is attained at the value $s=s_-$. This point is sent by T_d to $(-t_1-t_1^{1/d},\,t_1)$ and we have

$$-t_1-t_1^{1/d}=\frac12(-3t_1-2t_1^{1/d})+\frac12 t_1\ge\frac12 s_-\ge-\frac12 r\ge-\frac12\rho.$$

Now let $\widetilde r$ be such that $-\rho/2\le\widetilde r\le0$ and we consider the image of the segment $\{x=\widetilde r\}\cap D^1_\rho$, parametrized by $\widetilde s\in[\widetilde s_-,\widetilde s_+]\subset[0,-\widetilde r]$. We write

$$\tau_2(\widetilde s)=\big(\tau_2^x(\widetilde s),\tau_2^y(\widetilde s)\big):=T_d(\widetilde r,\widetilde s)=\big(\widetilde s+(\widetilde r+\widetilde s)^d,\widetilde s+2(\widetilde r+\widetilde s)^d\big).$$

Since σ_1^+ is below $\{y=-x\}$, $\widetilde r+\widetilde s<0$. In this case we also have that $(\tau_2^x)'(\widetilde s)$ and $(\tau_2^y)'(\widetilde s)$ are positive. Then, a bound of the maximum of $\tau_2^x(\widetilde s)$ is obtained from

$$\tau_2^x(\widetilde s)\le\tau_2^x(-\widetilde r)=-\widetilde r\le\rho/2.$$

This implies $T_d(D^1_{\rho/2})\subset D^0_{\rho/2}$. □

Now we take $\rho>0$ arbitrary and define

$$I^0=D^0\cap\{x=\rho\}.$$

Its image by T_d^2 is a curve in Q_4 that joints a point in $\{\text{image}\,\sigma_2^-\}$ and a point in $\{\text{image}\,\sigma_2^+\}$. Then this curve has to cross $\{\text{image}\,\sigma_0^-\}$ and $\{\text{image}\,\sigma_0^+\}$. The set $I^1=T_d^{-2}(T_d^2(I^0)\cap D^0)\subset I^0$ contains points such that they, together with their second iterates, belong to I^0. Repeating this procedure we define, as in the final part of the proof of Proposition 5.7,

$$I^k=T_d^{-2k}(T_d^{2k}(I^{k-1})\cap D^0).$$

Clearly, $I^k\subset I^{k-1}$ so that I^k is a sequence of nested compact sets as well.

Then $I^\infty=\cap_{n\ge0}I^n\ne\emptyset$. By this construction, if $(x_0,y_0)\in I^\infty$, $(x_{2k},y_{2k})=T_d^{2k}(x_0,y_0)\in T_d^{2k}(I^{k-1})\cap D^0\subset D^0$ for all $k\ge0$. Then, by Lemma 6.1,

$$0<x_{2k}<\left(\frac12\right)^k x_0$$

and as $(x_{2k},y_{2k})\in D^0$, the iterates (x_{2k},y_{2k}) converge to $(0,0)$ which implies that $(x_0,y_0)\in W_0^s$. Since ρ is arbitrary, W_0^s is unbounded.

Remark 6.2. Since the curves that determine th e boundary of D^0 are very close they provide a very good approximation for the stable manifold, even far away from the origin.

Conclusions. The understanding of the dynamical plane of numerical methods has been extensively studied in order to improve the associated root-finding algorithms. In this paper we study the topology and geometry of the basin of attraction of a fixed point for a model of the critical three periodic orbit of the secant map. Although the link between the model and the secant method is, a priory, only valid near the cycle the results capture the essences of the whole basin of attraction of the three cycle.

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