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Construction of extremal mixed graphs of diameter two

Nacho López [a,*], Hebert Pérez-Rosés [b,c], Jordi Pujolàs [a], Mária Ždímalová [d]

[a] *Departament de Matemàtica, Universitat de Lleida, C/ Jaume II 69, 25001 Lleida, Spain*
[b] *Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, Tarragona, Spain*
[c] *Conjoint fellow, Department of Software Engineering and Computer Science, University of Newcastle, Australia*
[d] *Department of Mathematics and Constructive Geometry, Slovak University of Technology, Bratislava, Slovakia*

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ABSTRACT

Graphs and digraphs with maximum order allowed by its degree and diameter have been widely studied in the context of the Degree/Diameter problem. This problem turns out to be very interesting in the mixed case, where many open problems arise, specially when the diameter is two and the order of the graph achieves the largest theoretical value given by the *mixed Moore bound*. These extremal graphs are known as *mixed Moore graphs*. In this paper we construct by voltage assignment some infinite families of mixed graphs of diameter two and order approaching the Moore bound. One of these families, in particular, yields most of the known mixed Moore graphs. We also present other families which are the result of the first known extension of the paradigmatic McKay-Miller-Širáň construction (McKay et al., 1998).

1. Introduction

A *mixed graph* G may contain (undirected) *edges* as well as directed edges (also known as *arcs*). From this point of view, a *graph* [resp. *directed graph* or *digraph*] has all its edges undirected [resp. directed]. The *undirected degree* of a vertex v, denoted by $d(v)$ is the number of edges incident to v. The *out-degree* [resp. *in-degree*] of v, denoted by $d^+(v)$ [resp. $d^-(v)$], is the number of arcs emanating from [resp. to] v. If $d^+(v) = d^-(v) = z$ and $d(v) = r$, for all $v \in V$, then G is said to be *totally regular* of degree d, where $d = r+z$. A *walk* of length $\ell \geq 0$ from u to v is a sequence of $\ell+1$ vertices, $u_0u_1 \dots u_{\ell-1}u_\ell$, such that $u = u_0$, $v = u_\ell$ and each pair $u_{i-1}u_i$, for $i = 1, \dots, \ell$, is either an edge or an arc of G. A walk whose vertices are all different is called a *path*. The length of a shortest path from u to v is the *distance* from u to v, and it is denoted by $\mathrm{dist}(u, v)$. Note that $\mathrm{dist}(u, v)$ may be different from $\mathrm{dist}(v, u)$, when shortest paths between u and v involve arcs. The maximum distance between any pair of vertices is the *diameter* k of G.

The *Degree/Diameter problem* is a classic extremal problem in network design. In particular, the *Degree/Diameter problem for mixed graphs* asks for the largest possible number of vertices $n(r, z, k)$ in a mixed graph with maximum undirected degree r, maximum directed out-degree z and diameter k. This problem has been extensively studied for purely undirected and directed graphs, but little is known for mixed graphs. A natural upper bound for $n(r, z, k)$ is derived just by counting the number of vertices at every distance from any given vertex v in a mixed graph with given maximum undirected degree r, maximum directed out-degree z, and diameter k. This bound is known as the *Moore bound* for mixed graphs (see [9,11]). Here we focus on the case of diameter 2, where this bound is easily derived as

$$M(r, z, 2) = 1 + z + (r + z)^2. \tag{1}$$

* Corresponding author.
E-mail address: nlopez@matematica.udl.es (N. López).

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In this context, we deal with mixed graphs containing at least one edge and one arc. Mixed graphs of diameter 2, maximum undirected degree $r \geq 1$, maximum out-degree $z \geq 1$ and order $1+z+(r+z)^2$ are called *mixed Moore graphs*. Such extremal mixed graphs are totally regular of degree $d = r + z$ and they have the property that for any ordered pair (u, v) of vertices there is a unique path of length at most 2 between them. Bosák gave the following necessary condition for the existence of a mixed Moore graph of diameter 2:

Theorem 1 (*[4]*). *Let G be a (proper) mixed graph of diameter two. Then, G is totally regular with directed degree $z \geq 1$ and undirected degree $r \geq 1$. Moreover, there must exist a positive odd integer c such that*

$$r = \frac{1}{4}(c^2 + 3) \text{ and } c|(4z - 3)(4z + 5). \tag{2}$$

There are infinitely many pairs (r, z) satisfying this necessary condition for which the existence of a mixed Moore graph is not yet known. There is a unique mixed Moore graph of order 18, corresponding to $(r, z) = (3, 1)$ (see [24]), and there are at least two non-isomorphic mixed Moore graphs of order 108 (corresponding to $(r, z) = (3, 7)$) [13]. Recently it has been proved that there is no mixed Moore graph for the (r, z) pairs $(3, 3)$, $(3, 4)$, $(7, 2)$ satisfying such necessary condition (see [15]). The family of Kautz digraphs of diameter 2 are also mixed Moore graphs for $r = 1$ and any $z \geq 1$.

Besides the general approach to this problem given above, researchers have been also interested in some particular versions of the problem, namely when the graphs are restricted to a certain class, such as the class of mixed bipartite graphs [10] or mixed Cayley graphs [17,18,25], among others.

Voltage assignments and lifts

Let G be a directed graph (which may have directed loops and/or parallel arcs) with vertex set $V = V(G)$ and arc multiset $E = E(G)$. Then, given a group Γ, a *voltage assignment* of G is a mapping $\alpha : E \to \Gamma$. The *lift* G^α is the directed graph with vertex $V(G^\alpha) = V \times \Gamma$ and arc set $E(G^\alpha) = E \times \Gamma$, where there is an arc from (u, g) to (v, h) if and only if $uv \in E$ and $h = g\alpha(uv)$. Here, we consider that a pair of mutually reverse arcs in G^α gives rise to an edge. Hence, we see G^α as a mixed graph. In particular, any Cayley digraph is a lift of a bouquet digraph (a one-vertex graph with directed loops). Let $u_0u_1 \ldots u_{\ell-1}u_\ell$ be a walk of G, if α is a voltage assignment on G, then the *net voltage* of this walk is defined as the product $\alpha(u_0u_1)\alpha(u_1u_2)\cdots\alpha(u_{\ell-1}u_\ell)$. For more details about the voltage assignment technique see [6,12]. The following result is well known:

Lemma 1 (*[3]*). *Let α be a voltage assignment on a directed graph G in a group Γ. Then, diam$(G^\alpha) \leq k$ if and only if for each ordered pair of vertices u, v (possibly $u = v$) of G and for each $g \in \Gamma$ there exists a walk from u to v of length $\leq k$ of net voltage g.*

2. Mixed Moore graphs as voltage graphs

Cayley graphs have been used extensively to construct large graphs in the directed and the undirected case. In our context, some mixed Moore graphs are also mixed Cayley graphs. For instance, the Bosák graph is a mixed Cayley graph both for $S_3 \times \mathbb{Z}_3$ and $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ (see [17]). The two mixed Moore graphs on 108 vertices found by Jørgensen [13] are both mixed Cayley graphs (for groups number 15 and 17 from the GAP catalogue of groups of order 108). The vertices of the *Kautz digraph* Ka(d, k), $d \geq 2$, $k \geq 1$, are words of length k on an alphabet S of $d + 1$ letters without two consecutive identical letters. There is an arc from vertex $(v_0, v_1, \ldots, v_{k-1})$ to vertices $(v_1, \ldots, v_{k-1}, x)$, where $x \in S \setminus \{v_{k-1}\}$. It is known that Ka$(d, k)$ has order $(d + 1)d^{k-1}$ and diameter k. Kautz digraph Ka$(d, 2)$ is the unique mixed Moore graph for every $z \geq 1$ and $r = 1$ (if every oriented digon is replaced by an edge) where $d = r + z$, as noted in [24]. It is well known that *Ka*$(d, 2)$ is Cayley if and only if $d + 1$ is a power of a prime number (see [8]). Hence not every mixed Moore graph can be constructed as a mixed Cayley graph. Since mixed Cayley graphs can be seen as a particular case of a voltage graph, it is natural to consider which mixed Moore graphs are voltage graphs. In other words, we want to characterize those voltage assignments that give a mixed Moore graph in the corresponding lifted graphs. Mixed Moore graphs have the property that for any ordered pair of vertices (u, v) there is a unique path of length at most 2 between them, hence by Section 1 we have the following:

Proposition 1. *Let G be a digraph and let $\alpha : E \to \Gamma$ be a voltage assignment. Then the lift graph G^α is a mixed Moore graph of diameter 2 and undirected degree r if and only if the following conditions are satisfied:*

(a) *For every pair of distinct vertices u and v there is a unique walk from u to v of length ≤ 2 with net voltage g, for all $g \in \Gamma$.*
(b) *For every vertex u there is a unique closed walk at u of length ≤ 2 with net voltage g, for all $g \in \Gamma \setminus \{1_\Gamma\}$, and there are exactly r closed walks of length 2 with net voltage 1_Γ.*

Next we provide a construction of a family of voltage graphs that in particular gives known mixed Moore graphs for several cases: Let us take $\mathbb{Z}_t = \{0, 1, \ldots, t-1\}$, $t \geq 3$ as the voltage group (in additive notation) and let $G_{t,s}$ be the complete multipartite digraph with one directed loop at every vertex, containing $t - 1$ partite sets of order $s \geq 1$ each one. The vertex set of $G_{t,s}$ is therefore $V = (\mathbb{Z}_t \setminus \{0\}) \times \{1, 2, \ldots, s\}$. Now the voltage assignment on $G_{t,s}$ is as follows: For every arc emanating from (u, v) to (u', v') we assign voltage u if $v = v'$, and 0 otherwise (see Fig. 1).

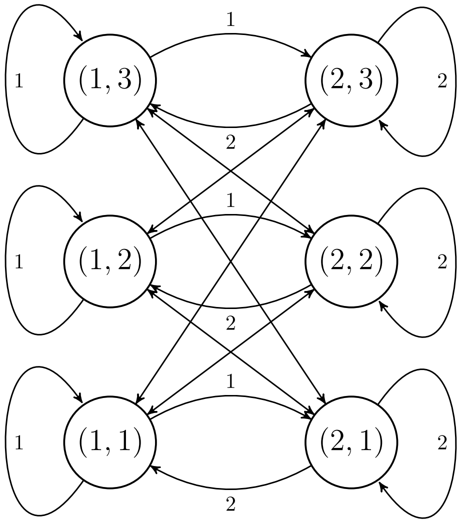

Fig. 1. The construction of the family $G^{\alpha}_{t,s}$ of mixed graphs of diameter two for $t = s = 3$ (voltages over the digons are 0, but they have been not depicted). In this case, $G^{\alpha}_{3,3}$ is the mixed Moore graph with parameters $r = 3$ and $z = 1$ (the Bosák graph).

Proposition 2. *Let $G_{t,s}$ be the digraph and $\alpha : E \to \mathbb{Z}_t$ a voltage assignment as defined above. Then $G^{\alpha}_{t,s}$ is a mixed graph of order $t(t-1)s$, undirected degree $r = s + (s-1)(t-3)$ and directed degree $z = t-2$. Moreover,* $\mathrm{diam}(G^{\alpha}_{t,2}) = 3$ *and* $\mathrm{diam}(G^{\alpha}_{t,s}) = 2$ *for $s \neq 2$.*

Proof. There are $(t-1)s$ vertices in $G_{t,s}$ and since the voltage group is $\mathbb{Z}_t$ we have that $G^{\alpha}_{t,s}$ has order $t(t-1)s$. Now we show that the diameter of $G^{\alpha}_{t,s}$ is 2 whenever $s \neq 2$. To this end, we verify that for any pair of vertices (u, v) and (u', v') (not necessarily distinct) of $G_{t,s}$, there exists at least one walk of length ≤ 2 with net voltage g, for every $g \in \mathbb{Z}_t$. We have the following cases for $s \geq 1$:

(a) $u' = u$ and $v' = v$. For every $g \neq u$ and $g \neq 2u$ the walk $(u, v) \to (g-u, v) \to (u, v)$ has net voltage g. The remaining voltages u and $2u$ can be given using the loop in (u, v) once and twice, respectively.
(b) $u' \neq u$ and $v' = v$. For every $g \neq u$ the walk $(u, v) \to (g-u, v) \to (u', v)$ has net voltage g. For $g = u$ the direct arc $(u, v) \to (u', v)$ does the job.

Hence $\mathrm{diam}(G^{\alpha}_{t,1}) = 2$. In addition, when $s \geq 2$ we have to take into account the following cases:

(c) $u' = u$ and $v' \neq v$. For every $g \neq u$ and $g \neq 0$ the walk $(u, v) \to (g, v') \to (u, v')$ has net voltage g. For $g = u$, we take the walk $(u, v) \to (u+1, v) \to (u, v')$ and for $g = 0$ we can take the walk $(u, v) \to (u'', v'') \to (u, v')$ for any $u'' \neq u$ and $v'' \neq v, v'$.
(d) $u' \neq u$ and $v' \neq v$. The direct arc $(u, v) \to (u', v')$ has net voltage 0. The walk $(u, v) \to (u', v) \to (u', v')$ has net voltage u. For the remaining voltages, we take the walk $(u, v) \to (g, v') \to (u', v')$.

Let us observe that in case (c) we need at least three different elements v, v', v'' from $\{1, 2, \ldots, s\}$ to guarantee the existence of the given walk with net voltage 0 and length 2. That is, our reasoning works if $s \geq 3$. When $s = 2$, we can complete the proof with the walk $(u, v) \to (-u, v') \to (u, v') \to (u, v')$ of length 3. Hence $\mathrm{diam}(G^{\alpha}_{t,2}) = 3$ and $\mathrm{diam}(G^{\alpha}_{t,s}) = 2$ for $s \neq 2$. To see that the undirected degree is $r = s + (s-1)(t-3)$ it is sufficient to check that there are r walks of length 2 and net voltage 0 at each vertex. Indeed, for any (u, v) all s vertices with coordinates $(-u, v')$ satisfy this condition, and so do the $(s-1)$ vertices belonging to the remaining $(t-3)$ partite sets. The directed degree $z = t-2$ can be computed as the total number of arcs emanating from any vertex, that is $s(t-2)+1$, minus the undirected degree r. □

We have constructed a family of mixed graphs $G^{\alpha}_{t,s}$ of diameter 2, directed out-degree $z \geq 1$, undirected degree $r = z(s-1)+1$ for all $s \geq 1$ ($s \neq 2$) and order $(z+2)(z+1)s$, by using voltage assignment. Both the Kautz and Bosák graphs are unique mixed Moore graphs for different values of r and z. Hence our construction recovers Kautz and Bosák's as particular instances.

Corollary 1. *$G^{\alpha}_{t,s}$ is a mixed Moore graph of diameter 2 if and only if $s = 1$ or $(t, s) = (3, 3)$. Moreover, $G^{\alpha}_{3,3}$ is the Bosák graph and $G^{\alpha}_{t,1}$ is the Kautz digraph for every $t \geq 3$.*

Proof. Following the ideas behind the proof of Proposition 2, it is not difficult to see that $G^{\alpha}_{3,3}$ and $G^{\alpha}_{t,1}$ satisfy the conditions given in 1 for $r = 3$ and $r = 1$, respectively. Another approach is to consider mixed Moore graphs as mixed graphs with order attaining the Moore bound (1). This bound for $r = s + (s-1)(t-3)$ and $z = t-2$ is $s^2(t-2)^2 + 2s(t-2) + t$. Taking into account that the order of $G^{\alpha}_{t,s}$ is $t(t-1)s$, we have that the equality

$$s^2(t-2)^2 + 2s(t-2) + t = t(t-1)s$$

holds if and only if either $s = 1$ or $(t, s) = (3, 3)$. So, in these cases we obtain mixed graphs with order equal to the Moore bound. Moreover, for $s = 1$ we have $r = 1$ and since mixed Moore graphs in this case are unique [24], we have that $G^{\alpha}_{t,1}$ must be the family of Kautz digraphs. When $(t, s) = (3, 3)$ (see Fig. 1) we have that $G^{\alpha}_{3,3}$ has order 18 and there is a unique mixed Moore graph of order 18 [24]. □

It remains to consider Jørgensen graphs as voltage graphs (other than lifts of bouquets). We discuss that open problem in more detail in Section 4.

3. Constructing large mixed graphs with order approaching the Moore bound

Another important line of research in the Degree/Diameter problem is the construction of graphs with order ever closer to the Moore bound. Many general techniques and ad-hoc constructions have been developed for purely directed or undirected graphs. Thus, it is quite natural to attempt the extension of those techniques to the mixed case. Among the many techniques that have proved successful in the purely directed or undirected case, voltage assignment occupies again an outstanding place. The idea of using voltage assignment to obtain large graphs dates back to 1986 (see [1]), but at that time they were not called 'voltage graphs'. Later, Baskoro et al. [3] popularized this method and it quickly gathered momentum [5,6,23,27]. In undirected graphs, voltage assignment is currently responsible for about 60% of the largest known graphs of the table of record graphs.[1]

Combination with computer search has given us a glimpse of the full potential of voltage assignment. In 2008 Loz and Širáň published a large number of new record graphs found with the aid of voltage assignment and computer search [21]. Their algorithm explores a search space consisting of lifts of simple base graphs (bouquets, dipoles, etc.) by non-abelian groups, namely semidirect products of cyclic groups, such as $\mathbb{Z}_m \rtimes_r \mathbb{Z}_n$, $(\mathbb{Z}_m \times \mathbb{Z}_m) \rtimes_{\psi} \mathbb{Z}_n$, and $(\mathbb{Z}_m \rtimes_r \mathbb{Z}_n) \rtimes (\mathbb{Z}_m \rtimes_r \mathbb{Z}_n)$. Those same groups were used later in [20] to enlarge the table of the largest known graphs up to degree 20. Other recent applications of voltage assignment to the construction of large graphs are [19,22].

Not much work has been done in this direction for the mixed case. Araujo et al. [2] construct a family of mixed graphs of diameter 2 with parameters $r = q + 2t$, $z = \frac{q-1}{2} - 2t$ and $2q^2$ vertices, where $q \geq 3$ is an odd prime power and t belongs to a set of integers depending on q. Such a construction is based on biaffine planes, and it yields optimal mixed graphs for some cases. Besides, Šiagiová [25] approaches the mixed Moore bound for diameter 2 using Cayley mixed graphs, by extending a known construction for the undirected case.

In our case, the mixed graph $G^{\alpha}_{t,s}$ of diameter 2, directed out-degree $z \geq 1$ and undirected degree $r = z(s-1) + 1$ described in Section 2 has order $(z+2)(z+1)s$ for all $s \geq 1$ $(s \neq 2)$. The Moore bound in this case is $z^2s^2 + 2zs + z + 2$. Hence, as the degree z increases we see

$$\lim_{z\to\infty} \frac{|G^{\alpha}_{t,s}|}{M(r, z, 2)} = \frac{(z+2)(z+1)s}{z^2s^2 + 2zs + z + 2} = \frac{1}{s}$$

For $s = 1$ we achieve the Moore bound because $G_{t,1}$ is the family of Kautz digraphs, while for $s \geq 3$ we have a construction that approaches the Moore bound in a function value depending only on s. Our goal next is to give a new family of mixed graphs of diameter 2 with order closer to the Moore bound.

McKay, Miller and Širáň describe a family of large vertex-transitive graphs with degree $(3q-1)/2$, where q is a prime power $q \equiv 1 \pmod 4$, (eventhough they contend that the construction generalizes to all prime powers [23]). The base graphs used in [23] are complete bipartite graphs with loops. Later, Šiagiová showed that the McKay-Miller-Širáň graphs can also be obtained as lifts of dipoles [26]. The McKay-Miller-Širáň graphs still stand as the best family of graphs of diameter 2, asymptotically.

We now focus on Jana Šiagiová's reformulation of the McKay-Miller-Širáň construction. As we mentioned before, the base graph is a dipole, i.e. a directed pseudograph with two vertices, u and v, and multiple arcs from u to v, as well as multiple loops at each vertex. Voltages are taken from $\mathbb{F}^+_q \times \mathbb{F}^+_q$, where $\mathbb{F}^+_q$ denotes the additive group of the finite field $\mathbb{F}_q$, where $q \equiv 1 \pmod 4$. The dipole has exactly $(q-1)/4$ loops at each vertex, and q arcs from u to v. Let ξ be a primitive root of $\mathbb{F}_q$. The voltages are defined as follows:

1. Every arc emanating from u to v has voltage (g, g^2), for all $g \in \mathbb{F}^+_q$.
2. Loops at vertex u have voltage $(0, \xi^{2i+1})$, for all $0 \leq i < \frac{q-1}{4}$.
3. Loops at vertex v have voltage $(0, \xi^{2i})$, for all $0 \leq i < \frac{q-1}{4}$.

[1] http://combinatoricswiki.org/wiki/The_Degree/Diameter_Problem.

The lift defined by the above voltage assignment is a directed graph. It is proved in [26] that, if the orientation of the arcs is removed, we obtain a vertex-transitive graph of degree $\frac{3q-1}{2}$ and diameter 2.

Next we proceed to give a construction of a family of mixed graphs with order approaching the Moore bound. This construction is a modification of the one given by Jana Šiagiová. Again, let $\mathbb{F}_q$ denote a finite field of prime power order $q \equiv 1 \pmod 4$. The base group is again $\mathbb{F}_q^+ \times \mathbb{F}_q^+$. Now, our base graph is the dipole graph D_q^* on two vertices u and v, containing q arcs both from u to v and from v to u, and $\frac{q-1}{2}$ loops at each vertex (see Fig. 2). As above, let ξ be a primitive root of $\mathbb{F}_q$. The voltages α_1 over D_q^* are defined as follows:

1. Every arc emanating from u to v has voltage (g, g^2), for all $g \in \mathbb{F}_q^+$.
2. Every arc emanating from v to u has voltage $(g, -g^2-1)$, for all $g \in \mathbb{F}_q^+$.
3. Loops at vertex u have voltage $(0, \xi^{2i+1})$, for all $0 \le i < \frac{q-1}{2}$.
4. Loops at vertex v have voltage $(0, \xi^{2i})$, for all $0 \le i < \frac{q-1}{2}$.

Theorem 2. *For any prime power $q \equiv 1 \pmod 4$ the lift $(D_q^*)^{\alpha_1}$ is a mixed graph on $2q^2$ vertices of diameter 2, undirected degree $r = \frac{q-1}{2}$ and directed out-degree $z = q$.*

Proof. Since $|\mathbb{F}_q^+ \times \mathbb{F}_q^+| = q^2$ and D_q^* has two vertices, the lift graph $(D_q^*)^{\alpha_1}$ has order $2q^2$. To show that $r = \frac{q-1}{2}$ and $z = q$, we note that none of the voltages defined on the q arcs from u to v has an inverse in a voltage defined on an arc from v to u. Indeed, we would have $(g, g^2) + (h, -h^2-1) = (0, 0)$ for some $g, h \in \mathbb{F}_q^+$ otherwise, and this is impossible. Besides, the sets $X_1 = \{\xi^{2i+1} \mid 0 \le i < \frac{q-1}{2}\}$ and $X_2 = \{\xi^{2i} \mid 0 \le i < \frac{q-1}{2}\}$ have the property that $X_1 \cup X_2 = \mathbb{F}_q^+ \setminus \{0\}$, $X_1 \cap X_2 = \emptyset$ and (since $q \equiv 1 \pmod 4$) they are respectively closed under additive inverses. In fact, $X_1 = \{\xi^{2i+1}, -\xi^{2i+1} \mid 0 \le i < \frac{q-1}{4}\}$ and $X_2 = \{\xi^{2i}, -\xi^{2i} \mid 0 \le i < \frac{q-1}{4}\}$. This means that for any loop emanating from u (or v), there is another one with the inverse voltage, that is, both generate two closed walks of length 2 and net voltage $(0, 0)$ in $(D_q^*)^{\alpha_1}$. Hence $r = \frac{q-1}{2}$ and $z = q$. To prove that $\operatorname{diam}((D_q^*)^{\alpha_1}) = 2$ we only have to check that for any pair of vertices u and v of D_q^* (not necessary distinct) there exists at least one walk W of length ≤ 2 with net voltage (g, h), for every $(g, h) \in \mathbb{F}_q^+ \times \mathbb{F}_q^+$.

(a) Walks W from u to u with net voltage (g, h): For $(g, h) = (0, 0)$ the walk W consists of two loops with voltages $(0, \xi^{2i+1})$ and $(0, -\xi^{2i+1})$ for any $0 \le i < \frac{q-1}{4}$. If $g = 0$ and $h \in X_1$, then the walk W consists of a single loop with voltage $(0, h)$. If $h \notin X_1$, $h \ne 0$, then also $-h \notin X_1$ and, since $0 \notin X_1$ and $h \notin h + X_1$, we see that X_1 and $h + X_1$ have a nonempty intersection. Indeed, assume that $X_1 \cap (h + X_1) = \emptyset$. Then, since $|h + X_1| = |X_1| = \frac{q-1}{2}$, we derive that $|X_1| + |h + X_1| = q - 1$, but this is impossible because $0 \notin X_1$ and $h \notin X_1$. Hence, there exist $x, y \in X_1$ such that $x = h + y$. In this case W is composed of two loops with voltages $(0, x)$ and $(0, -y)$. Now, for $g \ne 0$, we show that there exists a $u \to v \to u$ walk of any given voltage (g, h). There are $q(q-1)$ such pairs (g, h). Notice that we achieve $q^2 - q + 1$ different voltages through $u \to v \to u$ walks consisting of two distinct arcs in D_q^*. Indeed, let W_1 be a $u \to v \to u$ walk of net voltage $(i, i^2) + (j, -j^2 - 1) = (i + j, i^2 - j^2 - 1)$ and let W_2 another $u \to v \to u$ walk of net voltage $(m + n, m^2 - n^2 - 1)$. The equality $(i + j, i^2 - j^2 - 1) = (m + n, m^2 - n^2 - 1)$ implies either $i = m, j = n$ or $i + j = m + n = 0$. Hence, for any (i, i^2) there exists just one voltage $(-i, -i^2 - 1)$ that yields $(0, -1)$. This means that the voltage $(0, -1)$ is achieved walking q different $u \to v \to u$ paths, while there is just one $u \to v \to u$ walk with voltage (g, h) for any $g \ne 0$. The arguments for walks from v to v are similar, using the set X_2.

(b) Walks W from u to v with net voltage (g, h): For $(g, h) = (0, 0)$ the walk W consists on the single arc $u \to v$ with voltage (i, i^2), for $i = 0$. If $g = 0$ and $h \ne 0$, then the walk W consists of the arc with voltage $(0, 0)$ and a loop at vertex u for $h \in X_1$ or at vertex v for $h \in X_2$. Let $g \ne 0$. If $h = g^2$, then there is a $u \to v$ walk W of length one with net voltage (g, g^2). If $h \ne g^2$ and $h - g^2 \in X_1$ then W is composed by the loop at vertex u of voltage $(0, h - g^2)$ and the arc of the voltage (g, g^2). If $h \ne g^2$ and $h - g^2 \in X_2$ then we take first the arc with voltage (g, g^2) and next the loop at vertex v of voltage $(0, h - g^2)$.

(c) Walks W from v to u with net voltage (g, h): For $(g, h) = (0, -1)$ the walk W consists on the single arc $v \to u$ with voltage $(i, -i^2 - 1)$, for $i = 0$. If $g = 0$ and $h \ne -1$, then $h + 1 \ne 0$ and the walk W consists of the arc with voltage $(0, -1)$ and a loop at vertex u when $h + 1 \in X_1$ or at vertex v when $h + 1 \in X_2$. Let $g \ne 0$. If $h = -g^2 - 1$, then there is a $u \to v$ walk W of length one with net voltage $(g, -g^2 - 1)$. If $h \ne -g^2 - 1$ and $h + g^2 + 1 \in X_1$, then W is composed by the arc of the voltage $(g, -g^2 - 1)$ and the loop at vertex u of voltage $(0, h + g^2 + 1)$. If $h \ne -g^2 - 1$ and $h + g^2 + 1 \in X_2$, then we take first the loop at vertex v of voltage $(0, h + g^2 + 1)$ and next the arc with voltage $(g, -g^2 - 1)$. □

Corollary 2. *For any prime power $q \equiv 1 \pmod 4$ there is a family of mixed graphs of diameter 2, undirected degree $r = \frac{q-1}{2}$ and directed out-degree $z = q$ approaching the Moore bound in the constant factor $\frac{8}{9}$ when q tends to infinity.*

Proof. The Moore bound $M(r, z, 2)$ for undirected degree $r = \frac{q-1}{2}$ and directed out-degree $z = q$ is $\frac{9}{4}q^2 - \frac{1}{2}q + \frac{5}{4}$. Taking into account that $(D_q^*)^{\alpha_1}$ has order $2q^2$, we have that

$$\lim_{q\to\infty} \frac{|(D_q^*)^{\alpha_1}|}{M(r, z, 2)} = \frac{8}{9}. \quad \square$$

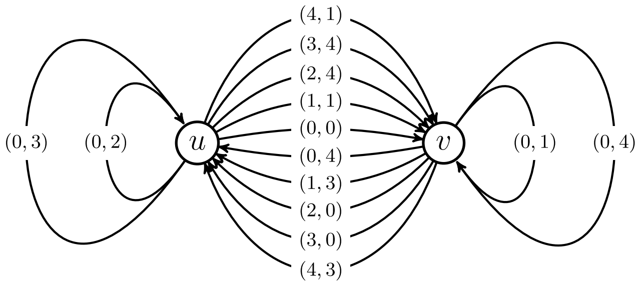

Fig. 2. The dipole graph and its corresponding voltages as defined above for the case $q = 5$. Here the base group is $\mathbb{Z}_5 \times \mathbb{Z}_5$ and we take $\xi = 2$ as a primitive root of $\mathbb{Z}_5$, where $\mathbb{Z}_5 = \mathbb{F}_5^+$. The lifted graph $(D_5^*)^{\alpha_1}$ has diameter 2, undirected degree $r = 2$, directed out-degree $z = 5$ and order 50. The Moore bound for these parameters is $M(2, 5, 2) = 55$.

Let us construct another family of mixed graphs of diameter 2 using voltage assignment. We start from the dipole graph D_q^* but with q arcs from u to v and $\frac{q-1}{2}$ loops at each vertex. Now, we define the following voltages α_2 over D_q^*:

1. Every arc emanating from u to v has voltage (g, g^2), for all $g \in \mathbb{F}_q^+$.
2. Loops at vertex u have voltage $(0, \xi^{2i+1})$, for all $0 \le i < \frac{q-1}{4}$, and voltage $(0, \xi^{2i})$ for all $\frac{q-1}{4} \le i < \frac{q-1}{2}$.
3. Loops at vertex v have voltage $(0, \xi^{2i})$, for all $0 \le i < \frac{q-1}{4}$, and voltage $(0, \xi^{2i+1})$ for all $\frac{q-1}{4} \le i < \frac{q-1}{2}$.

In addition to the voltages, we transform the arcs from u to v into undirected edges in the lifted graph. Notice that the set $X_1 = \{\xi^{2i+1} \mid 0 \le i < \frac{q-1}{4}\} \cup \{\xi^{2i} \mid \frac{q-1}{4} \le i < \frac{q-1}{2}\}$ has cardinality $\frac{q-1}{2}$ and has no pair of additive inverses. In the same manner, the set $X_2 = \{\xi^{2i} \mid 0 \le i < \frac{q-1}{4}\} \cup \{\xi^{2i+1} \mid \frac{q-1}{4} \le i < \frac{q-1}{2}\}$ does not contain any pair of additive inverses. In fact, $X_1 = -X_2$. Therefore, the loops at u and v generate directed arcs instead of undirected edges in the lifted graph $(D_q^*)^{\alpha_2}$.

Note that the fiber over the arcs from u to v forms a bipartite subgraph of $(D_q^*)^{\alpha_2}$. Recall that it is an undirected subgraph because we have removed the orientation of the arcs in order to form edges. Since $\mathbb{F}_q^*$ is cyclic, the fiber over the loops at u and v forms a set of $2q$ directed *circulant graphs*, each with q vertices, which are then joined by the edges mentioned above. Obviously, all the circulant digraphs arising from u and its loops are isomorphic, so we may denote them generically by C_u. Similarly, we may denote the circulants arising from v by C_v. It is not hard to show that C_u and C_v are isomorphic, as we will show in Theorem 3.

It is easy to describe the mixed graph $(D_q^*)^{\alpha_2}$ without using voltages as follows: The vertex set V is $\{0, 1\} \times \mathbb{F}_q^+ \times \mathbb{F}_q^+$, and each vertex (r, i, j) is adjacent to $(1 - r, i + (-1)^r g, j + (-1)^r g^2)$, for every $g \in \mathbb{F}_q^+$ (edges), and it is also adjacent to $(r, i, j + (-1)^r x)$, for every $x \in X_1$ (arcs). Note that $(1, i, j) \to (0, i, j + x_2)$ for every $x_2 \in X_2$, since $X_2 = -X_1$. The main properties of $(D_q^*)^{\alpha_2}$ are summarized in the following theorem:

Theorem 3. *For any prime power $q \equiv 1 \pmod 4$ the lift $(D_q^*)^{\alpha_2}$ is a vertex-transitive mixed graph on $2q^2$ vertices, with diameter 2, undirected degree $r = q$ and directed out-degree $z = \frac{q-1}{2}$.*

Proof. The assertion about the diameter of $(D_q^*)^{\alpha_2}$ is proved with the same ideas as in Theorem 2.

In order to prove that $(D_q^*)^{\alpha_2}$ is vertex-transitive, we will identify a subgroup Γ of $\mathrm{Aut}((D_q^*)^{\alpha_2})$ that acts transitively on the set of vertices V.

Let us begin by checking that the bijective function $\phi_s(r, i, j) = (r, i, j + s)$ is an automorphism of $(D_q^*)^{\alpha_2}$ for all $s \in \mathbb{F}_q^+$. The function ϕ simply rotates the vertices within each circulant on both sides, i.e. the copies of C_u and C_v. We just have to check that ϕ preserves the two types of adjacency in $(D_q^*)^{\alpha_2}$, that is, edges and arcs. This verification can be done visually in the following diagrams, where the double arrow $\Longleftrightarrow$ represents an edge, and $\Longrightarrow$ represents and arc:

$$\begin{array}{ccc} (r,i,j) & \Longleftrightarrow & (1-r, i+(-1)^r g, j+(-1)^r g^2) \\ \phi_s \downarrow & & \downarrow \phi_s \\ (r,i,j+s) & \Longleftrightarrow & (1-r, i+(-1)^r g, j+s+(-1)^r g^2) \end{array} \qquad \begin{array}{ccc} (r,i,j) & \Longrightarrow & (r,i,j+(-1)^r x) \\ \phi_s \downarrow & & \downarrow \phi_s \\ (r,i,j+s) & \Longrightarrow & (r,i,j+s+(-1)^r x) \end{array}$$

Similarly, the bijection $\theta_t(r, i, j) = (r, i + t, j)$ is an automorphism of $(D_q^*)^{\alpha_2}$ for all $t \in \mathbb{F}_q^+$. The function θ_t permutes different copies of the circulant C_u, and different copies of C_v. Actually, this automorphism is the proof that all the C_u are isomorphic, and that all the C_v are isomorphic.

$$\begin{array}{ccc} (r,i,j) & \Longleftrightarrow & (1-r, i+(-1)^r g, j+(-1)^r g^2) \\ \theta_t \downarrow & & \downarrow \theta_t \\ (r,i+t,j) & \Longleftrightarrow & (1-r, i+t+(-1)^r g, j+(-1)^r g^2) \end{array} \qquad \begin{array}{ccc} (r,i,j) & \Longrightarrow & (r,i,j+(-1)^r x) \\ \theta_t \downarrow & & \downarrow \theta_t \\ (r,i+t,j) & \Longrightarrow & (r,i+t,j+(-1)^r x) \end{array}$$

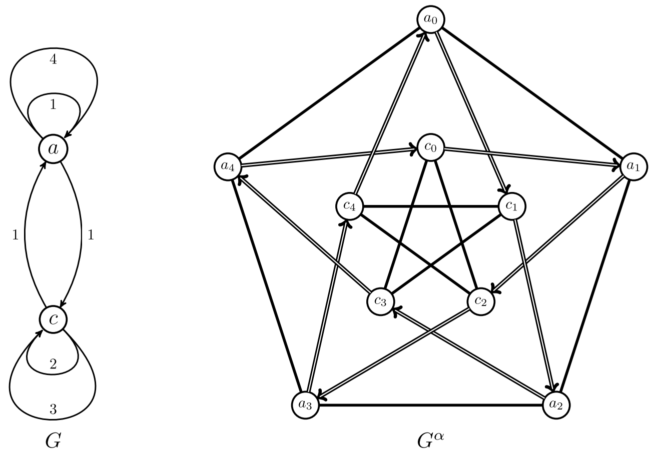

Fig. 3. A voltage graph G over $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ and its corresponding lifted graph G^α. In this case, we obtain a mixed graph with largest order for $r = 2$ and $z = 1$.

Finally, we will check that the function $\psi(r, i, j) = (1 - r, -i, -j)$ is also an automorphism of $(D_q^*)^{\alpha_2}$. The bijection ψ interchanges the fiber over u with the fiber over v.

$$\begin{array}{ccc} (r, i, j) & \Longleftrightarrow & (1-r, i+(-1)^r g, j+(-1)^r g^2) \\ \psi\downarrow & & \downarrow\psi \\ (1-r, -i, -j) & \Longleftrightarrow & (r, -i-(-1)^r g, -j-(-1)^r g^2) \end{array} \qquad \begin{array}{ccc} (r, i, j) & \Longrightarrow & (r, i, j+(-1)^r x) \\ \psi\downarrow & & \downarrow\psi \\ (1-r, -i, -j) & \Longrightarrow & (r, -i, -j-(-1)^r x) \end{array}$$

Clearly, the group $\langle \phi_t, \theta_s, \psi \rangle$ acts transitively on V, and therefore $(D_q^*)^{\alpha_2}$ is vertex-transitive. □

Although the parameters r and z are different for $(D_q^*)^{\alpha_1}$ and $(D_q^*)^{\alpha_2}$, they have the same asymptotic behavior as q tends to infinity. However, a computer check shows that $(D_q^*)^{\alpha_1}$ is not vertex transitive in general. The family $(D_q^*)^{\alpha_2}$ of mixed graphs of diameter two is related to the construction given by Araujo-Pardo et al. ([2], Section 3.2), for the case $t = 0$. In fact, they are isomorphic for $q = 5$ but may not be so for other combinations of the parameters. These two families $(D_q^*)^{\alpha_1}$ and $(D_q^*)^{\alpha_2}$ can be generalized for more pairs (r, z), as the next corollary states:

Corollary 3. *For any prime power $q \equiv 1 \pmod 4$,*

(a) *there exists a family of mixed graphs on $2q^2$ vertices, with diameter 2 and undirected degree $r = q + 2l$ and directed out-degree $z = \frac{q-1}{2} - l$, for any $0 \le l < \frac{q-1}{2}$. And*
(b) *there exists a family of mixed graphs on $2q^2$ vertices, with diameter 2 and undirected degree $r = \frac{q-1}{2} + 2l$ and directed out-degree $z = q - l$, for any $0 \le l < q$.*

Proof. The family described in (a) is $(D_q^*)^{\alpha_2}$ when $l = 0$. In general, $(D_q^*)^{\alpha_2}$ contains an induced subgraph D which is a regular digraph of degree z (the digraph obtained from $(D_q^*)^{\alpha_2}$ by removing all its edges), hence by Hall's theorem (see, for instance, [7]) D contains a z' factor (that is, a regular digraph of degree z') for any $z' < z$. Then, the claimed mixed graph (a) is obtained by replacing all the arcs of this $z' = \frac{q-1}{2} - l$ factor by edges. The same idea applies to $(D_q^*)^{\alpha_1}$ thus obtaining (b). □

4. Open problems

The Moore bound could be approached even closer for particular cases of r and z, using voltage assignment. For instance, the Moore bound $M(2, 1, 2) = 11$ is unattainable since $r = 2$ and $z = 1$ do not satisfy Eq. (2). Besides, the mixed graph G^α depicted in Fig. 3 has 10 vertices and is totally regular with undirected degree $r = 2$ and directed degree $z = 1$. Moreover, it is easy to check that G^α has diameter 2. Since $M(2, 1, 2) - 1 = 10$, we have that G^α is a mixed graph with largest order in this case, and moreover, it is unique. This graph can be constructed as a voltage graph of the dipole over $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$, as it appears in Fig. 3. These extremal mixed graphs with order $M(r, z, 2) - 1$ were first studied in [14]. Graphs attaining the bound $M(r, z, 2) - \delta$ could be non-regular, even for relatively small values of $\delta > 1$ (see [16]), hence other techniques should be used instead to construct these graphs.

Problem 1. Find mixed graphs of diameter 2 with largest order for particular parameters r and z.

As explained before, the Jørgensen graphs of order 108 are Cayley graphs, and they can be obtained as lifts of bouquets using the groups 15 and 17 of order 108, from the GAP 'small groups' library, as voltage groups. An interesting problem is to represent those graphs as voltage graphs over other base graphs and other voltage groups. In principle, if a group G is a split extension of another group H, we can obtain the Cayley graph of G in two steps, as a lift of the Cayley graph of H [3]. This would be the reverse process of the one described in [26]. In [13] it is mentioned that the group 17 of order 108 is an extension of $S_3 \times S_3$, so we might be tempted to construct the Cayley graph of $S_3 \times S_3$, and then use it as a base graph to obtain one of Jørgensen's graphs by voltage assignment. Unfortunately, the group 17 of order 108 is a non-split extension of $S_3 \times S_3$, hence the method described in [3] does not apply. It may still be possible to represent the group 17 of order 108 as a split extension of another smaller group, and thus get the voltage assignment construction that we are looking for, but this looks like a laborious task. The same applies to the group number 15, of order 108, which also has a Cayley graph isomorphic to one of Jørgensen's graphs. In any case, all this discussion leads to another question, which may have farther-reaching consequences: Is it possible to extend the method described in [3] to non-split extensions of groups?

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References

[1] I. Alegre, M.A. Fiol, J.L.A. Yebra, Some large graphs with given degree and diameter, J. Graph Theory 10 (2) (1986) 219–224.
[2] C. Araujo-Pardo, C. Balbuena, M. Miller, M. Ždímalová, A construction of dense mixed graphs of diameter 2, Electron. Notes Discrete Math. 54 (2016) 235–240.
[3] E. Baskoro, L. Branković, M. Miller, J. Plesník, J. Ryan, J. Širáň, Large digraphs with small diameter: A voltage assignment approach, J. Comb. Math. Comb. Comput. (24) (1997) 161–176.
[4] J. Bosák, Partially directed Moore graphs, Math. Slovaca (29) (1979) 181–196.
[5] L. Branković, M. Miller, J. Plesník, J. Ryan, J. Širáň, Large graphs with small diameter: A voltage assignment approach, Australas. J. Combin. (18) (1998) 65–76.
[6] L. Branković, M. Miller, J. Plesník, J. Ryan, J. Širáň, A note on constructing large Cayley graphs of given degree and diameter by voltage assignments, Electron. J. Combin. (5) (1998).
[7] R. Brualdi, Introductory Combinatorics, North-Holland, Amsterdam, 1977.
[8] J. Brunat, M. Espona, M. Fiol, O. Serra, On cayley line digraphs, in: 14th British Combinatorial Conference, Discrete Math. 138 (1–3) (1995) 147–159.
[9] D. Buset, M. El Amiri, G. Erskine, M. Miller, H. Pérez-Rosés, A revised Moore bound for mixed graphs, Discrete Math. 339 (2016) 2066–2069.
[10] C. Dalfó, M. Fiol, N. López, On bipartite mixed graphs, arXiv preprints, 2016.
[11] C. Dalfó, M. Fiol, N. López, Sequence mixed graphs, Discrete Appl. Math. 219 (2017) 110–116.
[12] J.L. Gross, T.W. Tucker, Topological Graph Theory, Courier Corporation, 1987.
[13] L.K. Jørgensen, New mixed Moore graphs and directed strongly regular graphs, Discrete Math. 338 (6) (2015) 1011–1016.
[14] N. López, J. Miret, On mixed almost moore graphs of diameter two, Electron. J. Combin. 2 (23) (2016) 1–14.
[15] N. López, J.M. Miret, C. Fernández, Non existence of some mixed Moore graphs of diameter 2 using SAT, Discrete Math. 339 (2) (2016) 589–596.
[16] N. López, Q. Monserrat, El problema de la regularidad de los grafos cuasi de moore mezclados, in: Proceedings of the 10th Andalusian Meeting on Discrete Mathematics, 2017, pp. 151–154.
[17] N. López, H. Pérez-Rosés, J. Pujolàs, Cayley mixed Moore graphs, Electron. Notes Discrete Math. 46 (2014) 193–200.
[18] N. López, H. Pérez-Rosés, J. Pujolàs, The Degree/Diameter problem for mixed abelian Cayley graphs, in: Algorithmic Graph Theory on the Adriatic Coast, Discrete Appl. Math. 231 (Supplement C) (2017) 190–197.
[19] E. Loz, M. Miller, J. Šiagiová, J. Širáň, J. Tomanová, Small vertex-transitive and cayley graphs of girth six and given degree: an algebraic approach, J. Graph Theory 68 (4) (2011) 265–284.
[20] E. Loz, G. Pineda-Villavicencio, New benchmarks for large-scale networks with given maximum degree and diameter, Comput. J. (2009).
[21] E. Loz, J. Širáň, New record graphs in the degree/diameter problem, Australas. J. Combin. (41) (2008) 63–80.
[22] H. Macbeth, J. Šiagiová, J. Širáň, T. Vetrík, Large cayley graphs and vertex-transitive non-cayley graphs of given degree and diameter, J. Graph Theory 64 (2) (2010) 87–98.
[23] B.D. McKay, M. Miller, J. Širáň, A note on large graphs of diameter two and given maximum degree, J. Combin. Theory Ser. B 74 (1) (1998) 110–118.
[24] M.H. Nguyen, M. Miller, J. Gimbert, On mixed Moore graphs, Discrete Math. 307 (7–8) (2007) 964–970.
[25] J. Šiagiová, Approaching the mixed Moore bound for diameter two by Cayley graphs, Australas. J. Combin. 61 (2015) 73–81.
[26] J. Šiagiová, A note on the McKay—Miller—Širáň graphs, J. Combin. Theory Ser. B 81 (2) (2001) 205–208.
[27] J. Zlatoš, The diameter of lifted digraphs, Australas. J. Combin. (19) (1999) 73–82.